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CRITICAL POINTS OF YANG-MILLS FOR NONCOMMUTATIVE TWO-TORI

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In [5] A. Connes and the author described the moduli spaces for the minima of the Yang-Mills function for the case of connections on projective modules over noncommutative two-tori, in the setting of the noncommutative differential geometry initiated by Connes in [4]. The main purpose of the present note is to describe the critical points of the Yang-Mills function for the same case, and also the moduli spaces for these critical points. It turns out that the critical points coincide with certain connections which were used in [3] to construct actions of the Heisenberg Lie group on noncommutative tori. (In fact, we will make crucial use of one of the arguments from [3].) We will find that the moduli spaces for the critical points are finite products of the kinds of spaces which were obtained in [5] as moduli spaces for the minima.

1. The Yang-Mills equations

We begin by recalling briefly the setting of [5]. Let G be a Lie group, and let α be an action of G as automorphisms of a C^*-algebra A. We let A^∞ be the dense $*$-subalgebra of A consisting of the C^∞-vectors for α. Then the infinitesimal form of α gives an action, δ, of the Lie algebra, L, of G, as derivations of A^∞. Every finitely generated projective right A-module Ξ has a C^∞-version Ξ^∞. Since we will never work with A or Ξ, but only with A^∞ and Ξ^∞, we will for notational simplicity denote the latter by A and Ξ from now on. Also, for brevity we will say "projective" when we mean "finitely generated projective".

We can and will assume that Ξ is equipped with a Hermitian metric, $\langle\, ,\, \rangle_A$, that is, an A-valued inner product for which it is self-dual. The effect of the choice of Hermitian metric on what follows is discussed in [5, p. 241]. In the role of Riemannian metric for A we assume that L is equipped

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with an ordinary real inner product. This will define a bilinear form on various spaces of alternating forms on L. In particular, if $E = \mathrm{End}_A(\Xi)$, then we obtain an E-valued bilinear form, denoted $\{\ ,\ \}$, on the space of alternating E-valued two-forms on L. For computational purposes this form is conveniently given by

$$\{\Phi, \Psi\} = \sum_{i<j} \Phi(Z_i \wedge Z_j)\Psi(Z_i \wedge Z_j),$$

where $\{Z_i\}$ is an orthonormal basis for L. But it is, of course, independent of the choice of orthonormal basis.

Given a connection ∇ on Ξ, that is, a linear map from Ξ to $\Xi \otimes L^*$ such that

$$\nabla_X(\xi a) = (\nabla_X \xi)a + \xi(\delta_X(a))$$

for $X \in L$, $\xi \in \Xi$ and $a \in A$, its curvature, Θ_∇, is defined by

$$\Theta_\nabla(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

One finds that Θ_∇ is an E-valued alternating two-form on L. If ∇ is compatible with the Hermitian metric on Ξ, in the sense that

$$\delta_X(\langle \xi, \eta \rangle_A) = \langle \nabla_X \xi, \eta \rangle_A + \langle \xi, \nabla_X \eta \rangle_A,$$

then Θ_∇ has values in the space E_s of skew-adjoint elements of E. The space of all compatible connections on Ξ is denoted $\mathrm{CC}(\Xi)$. It is an affine space over the vector space of linear maps from L to E_s.

We assume that A has a faithful α-invariant trace, τ. From τ we obtain a faithful trace, τ_E, on E, determined by

$$\tau_E(\langle \xi, \eta \rangle_E) = \tau(\langle \eta, \xi \rangle_A),$$

where $\langle \xi, \eta \rangle_E$ is defined by

$$\langle \xi, \eta \rangle_E \zeta = \xi \langle \eta, \zeta \rangle_A.$$

The Yang-Mills function, YM, on $\mathrm{CC}(\Xi)$ is defined in the present setting by

$$\mathrm{YM}(\nabla) = -\tau_E(\{\Theta_\nabla, \Theta_\nabla\}).$$

The Yang-Mills problem is that of determining the nature of the set of critical points for YM. The Yang-Mills equation is the Euler-Lagrange equation for the critical points of YM. We derive it here in the standard way, since we need it and it was not derived in [5]. (A derivation has also very recently been given in [11].) Given $\nabla \in \mathrm{CC}(\Xi)$, any other compatible connection is of the form $\nabla + \mu$, where μ is a linear map from L to E_s. Then ∇ will be a critical point for YM if for all μ we have

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{YM}(\nabla + t\mu) = 0.$$

But

$$\frac{d}{dt}\mathrm{YM}(\nabla + t\mu) = -\frac{d}{dt}\tau_E(\{\Theta_{\nabla+t\mu}, \Theta_{\nabla+t\mu}\})$$
$$= -2\tau_E\left(\left\{\frac{d}{dt}(\Theta_{\nabla+t\mu}), \Theta_{\nabla+t\mu}\right\}\right).$$

A simple calculation shows that

$$\left.\frac{d}{dt}\right|_{t=0} \Theta_{\nabla+t\mu}(X,Y) = [\nabla_X, \mu_Y] - [\nabla_Y, \mu_X] - \mu_{[X,Y]}.$$

But this latter expression is what is commonly denoted by $(\hat{\nabla}\mu)(X,Y)$, where here $\hat{\nabla}$ is the extension to E-valued 1-forms of the $\hat{\delta}$ defined near the bottom of p. 243 of [5] for 0-forms. Using this notation, we see that a compatible connection ∇ is a critical point exactly when

$$\tau_E(\{\hat{\nabla}\mu, \Theta_\nabla\}) = 0$$

for all linear maps μ from L to E_s. Define an ordinary real inner product on E_s-valued 2-forms by

$$\langle\Phi, \Psi\rangle = -\tau_E(\{\Phi, \Psi\}),$$

and on 1-forms by

$$\langle\mu, \nu\rangle = -\tau_E\left(\sum \mu_{Z_j}\nu_{Z_j}\right).$$

Let $\{c_{ij}^k\}$ denote the structure constants of L for the basis $\{Z_j\}$. Then straightforward calculations show that $\hat{\nabla}$ has a formal adjoint, $\hat{\nabla}^*$, taking E_s-valued 2-forms to 1-forms, and determined by

$$(\hat{\nabla}^*\Phi)(Z_i) = \sum_j [\nabla_{Z_j}, \Phi(Z_i \wedge Z_j)] - \sum_{j<k} c_{jk}^i \Phi(Z_j \wedge Z_k).$$

Then the condition that ∇ be a critical point can be rewritten as $\langle\mu, \hat{\nabla}^*(\Theta_\nabla)\rangle = 0$. Since μ is arbitrary, we see that we have obtained

1.1 Theorem. *A compatible connection ∇ is a critical point of* YM *exactly when it satisfies the Yang-Mills equation $\hat{\nabla}^*(\Theta_\nabla) = 0$.*

The Bianchi identity, which holds for all connections, is $\hat{\nabla}\Theta_\nabla = 0$, where $\hat{\nabla}$ has been extended to E-valued 2-forms. It often appears as the companion of the Yang-Mills equation.

We will need later:

1.2 Proposition. *Let notation be as above, let Ξ_1 and Ξ_2 be two projective A-modules with Hermitian metrics, and let $\nabla_j \in \mathrm{CC}(\Xi_j)$ for $j = 1, 2$. Let $\nabla = \nabla_1 \oplus \nabla_2$ on $\Xi = \Xi_1 \oplus \Xi_2$, so that $\nabla \in \mathrm{CC}(\Xi)$. Then ∇ is a critical point for* YM *on Ξ if and only if ∇_1 and ∇_2 are critical points for* YM *on Ξ_1 and Ξ_2 respectively.*

Proof. With the evident meaning, we view E_1 and E_2 as subalgebras of E, and denote their identity elements by e_1 and e_2, so that e_1 and e_2 are projections in E such that $e_1e_2 = 0$ and $e_1 + e_2$ is the identity element of E. With the evident notation, it is easy to verify that $\tau_{E_j} = \tau_E|_{E_j}$ and $\Theta_{\nabla_j} = e_j\Theta_\nabla e_j$ for $j = 1, 2$. Using the trace property of τ_E, it then follows that for any linear map μ from L to E we have

$$\tau_E(\{\hat{\nabla}\mu, \Theta_\nabla\}) = \tau_{E_1}(\{\hat{\nabla}_1(e_1\mu e_1), \Theta_{\nabla_1}\}) + \tau_{E_2}(\{\hat{\nabla}_2(e_2\mu e_2), \Theta_{\nabla_2}\}).$$

Since $e_i\mu e_i$ can be an arbitrary linear map from L to E_i, the desired conclusion follows. q.e.d.

By similar arguments it is very easy to verify:

1.3 Proposition. *With notation as above, let $\nabla_j \in \mathrm{CC}(\Xi_j)$ for $j = 1, 2$. Then*

$$\mathrm{YM}(\nabla_1 \oplus \nabla_2) = \mathrm{YM}(\nabla_1) + \mathrm{YM}(\nabla_2).$$

Suppose now that G, and so L, is Abelian, as happens for the noncommutative tori [10]. Then the structure constants for L are all zero, so that the Yang-Mills equation becomes

$$0 = \sum[\nabla_{Z_j}, [\nabla_{Z_j}, \nabla_X]]$$

for all $X \in L$. For the situation which will be pertinent to noncommutative two-tori we then clearly obtain:

1.4 Proposition. *Let notation be as above, and suppose that L is Abelian and of dimension two. Then ∇ is a critical point for* YM *if and only if $\Theta_\nabla(X, Y)$ commutes with ∇_Z for all $X, Y, Z \in L$. Thus either $\Theta_\nabla = 0$ (so that ∇ is a flat connection, clearly minimizing* YM*), or the range of ∇ generates a* 3*-dimensional Heisenberg Lie algebra.*

2. The case of noncommutative two-tori

We will use the notation of [5, §3]. Thus θ is a real number, $\lambda = \exp(2\pi i\theta)$, and A_θ is the universal C^*-algebra generated by two unitary operators, U_1 and U_2, subject to the relation $U_2U_1 = \lambda U_1U_2$. The group $G = T^2$ acts on A_θ by the dual action and, following our earlier convention, we will from now on let A_θ denote the dense $*$-subalgebra consisting of C^∞ functions for the dual action, so that the elements of A_θ are of the form $\sum f(m, n)U_1^mU_2^n$, where f is a complex-valued Schwartz function on Z^2. The Lie algebra, L, of G is two-dimensional Abelian, and we take as a basis for L the derivations δ_1 and δ_2 of A_θ determined by

$$\delta_k(U_k) = 2\pi i U_k, \qquad \delta_k(U_j) = 0 \quad \text{for } j \neq k.$$

Accordingly, to define connections we only need to specify them on this basis, and we will write ∇^1 and ∇^2 for the corresponding operators. The curvature of the corresponding connection is then determined by $[\nabla^1, \nabla^2]$.

Let Ξ be a projective A_θ-module with Hermitian metric, and suppose that Ξ has been expressed as the finite direct sum of a family $\{\Xi_k\}$ of projective A_θ-modules (which then inherit their Hermitian metrics from Ξ). By Proposition 5.6 of [5], each Ξ_k has a compatible connection ∇_k with constant curvature, that is, such that $[\nabla^1, \nabla^2]$ is a scalar multiple of the identity operator on Ξ. But then ∇_k is a minimum for YM on Ξ_k by Theorem 2.1 of [5], and so, in particular, is a critical point. By Proposition 1.2 it follows that $\bigoplus \nabla_k$ is a critical point for YM on Ξ. Our main result in this section is that, conversely, every critical point for YM on Ξ is of this form, that is:

2.1 Theorem. *Let Ξ be a projective A_θ-module, and let $\nabla \in \mathrm{CC}(\Xi)$. Then ∇ is a critical point for* YM *if and only if there is a finite direct sum decomposition $\Xi = \bigoplus \Xi_k$, and on each Ξ_k a connection ∇_k with constant curvature such that $\nabla = \bigoplus \nabla_k$.*

Proof. We have given above the proof of one direction. For the other direction, suppose that ∇ is a critical point. We first reduce to the case of free modules. Accordingly, let Ξ' be a projective A_θ-module such that $\Xi \oplus \Xi'$ is free, say $\cong A_\theta^n$ for some n. By arguments similar to those on p. 241 of [5], we can adjust the isomorphism so that the Hermitian metric on $\Xi \oplus \Xi'$ corresponds to the standard Hermitian metric on A_θ^n. By Proposition 5.6 of [5] we can choose a ∇' in $\mathrm{CC}(\Xi')$ with constant curvature, which is thus a critical point for YM. Then $\nabla \oplus \nabla'$ is a critical point for YM on A_θ^n by Proposition 1.2. Let e denote the projection of A_θ^n onto Ξ, so that e commutes with $\nabla \oplus \nabla'$ and $\nabla = e(\nabla \oplus \nabla')e$.

For convenience, we will now denote $\nabla \oplus \nabla'$ by ∇, that is, we let ∇ be a critical point for YM on A_θ^n, and we let e be a projection operator in E ($= \mathrm{End}_{A_\theta}(A_\theta^n)$) which commutes with ∇, so that eventually we will consider $e\nabla$ on eA_θ^n. Now ∇ is determined by its two components, ∇^1 and ∇^2, while its curvature is determined by the operator $C = [\nabla^1, \nabla^2]$ in E. If $C = 0$, then ∇ already has constant curvature, and there is nothing more to show. So we suppose that $C \neq 0$. Because ∇ is a critical point, it follows from Proposition 1.4 that C commutes with ∇^1 and ∇^2, so that ∇^1, ∇^2 and C generate a Lie algebra isomorphic to the Heisenberg Lie algebra. Now because we are using the standard Hermitian metric on A_θ, the pair δ_1 and δ_2, extended from A_θ to A_θ^n in the evident way, represents an element of $\mathrm{CC}(A_\theta^n)$, and so

$$\nabla^j = \delta_j + S_j$$

for some $S_j \in E_s$, for $j = 1, 2$. But the dual action of T^2 on A_θ extends to A_θ^n in the evident way, and the corresponding action of L is determined by the extended δ_j's. Let us put on A_θ^n the Hilbert space inner product coming from the invariant trace on A_θ. Then, much as in the proof of Theorem 2.5 of [3] or the part of the proof of Proposition 3.1 on [5, pp. 249–250], we find that we are exactly in a position to apply Proposition 3.1 of [2], which in turn is based on Theorem 9.9c of [6]. We conclude that the Lie algebra spanned by ∇^1, ∇^2 and C exponentiates to a unitary representation of the Heisenberg Lie group, which will carry the space A_θ^n of C^∞-vectors into itself. At this point we now need the following crucial argument from [3]. Let $H = \delta_1 - i\delta_2$ and let

$$K = \nabla^1 - i\nabla^2 = H + (S_1 - iS_2).$$

It is easy to see that the kernel of H is finite dimensional, and that if P is the projection onto the kernel, then $(H + P)^{-1}$ is compact. The argument for this is given just before Lemma 2.6 of [3] for the case of A_θ rather than A_θ^n. But K, as a bounded perturbation of H, will then have finite dimensional kernel and cokernel by Lemma 2.6 of [3]. Then C must have finite spectrum by Lemma 2.7 of [3].

Let $F_1, \cdots, F_n$ be the eigenprojections of C, for the (pure imaginary) eigenvalues $\lambda_1, \cdots, \lambda_n$. Now C commutes with ∇^1 and ∇^2, and each F_j is a polynomial in C. Thus each F_j commutes with ∇^1 and ∇^2. For each F_k let $\Xi_k = F_k A_\theta^n$. Then the connection ∇ carries each Ξ_k into itself, and if we let ∇_k denote the restriction of ∇ to Ξ_k, then ∇_k is a connection on Ξ_k whose curvature, θ_k, is determined by

$$\Theta_k(\delta_1, \delta_2) = \lambda_k F_k.$$

That is, ∇_j has constant curvature. Thus the theorem is established for the case of free modules. But now we can cut down by the projection e of the beginning of the proof, to obtain the general case.

3. The values of YM on critical points

Let us consider briefly the potential for using YM as a gauge-equivariant Morse function on $\mathrm{CC}(\Xi)$, along the lines described in [1], though we will not undertake here to actually apply Morse theory. If θ is rational, then A_θ is strongly Morita equivalent to $C(T^2)$, and so matters behave much as they do for $C(T^2)$. Thus we will concentrate here on the case in which θ is irrational. For convenience we assume that $\theta > 0$.

From the work of Pimsner and Voiculescu [7] we know that $K_0(A_\theta) \cong Z^2$, and that (for θ irrational) the homomorphism, τ, of $K_0(A_\theta)$ into the real numbers, R, given by the standard trace, τ, on A_θ is an isomorphism of $K_0(A_\theta)$ onto the dense subgroup $Z + Z\theta$ of R. Combining this with the results of [8], one knows that under τ the positive cone of $K_0(A_\theta)$ corresponds to the positive numbers in $Z + Z\theta$. But the main result of [9] is that projective A_θ-modules which represent the same element of $K_0(A_\theta)$ are isomorphic. (A somewhat different view of this result is given in [10].) Thus we see that the isomorphism classes of projective A_θ-modules are exactly labeled by the positive numbers of form $p + q\theta$ for $p, q \in Z$ (these positive numbers being the 0th Chern characters, as explained in [4]). We will indicate this labeling by expressions such as $\tau(\Xi) = p + q\theta$.

Let Ξ be a projective A_θ-module, and let ∇ be a connection of constant curvature on Ξ, so that it is a minimum for YM. If Ξ is free, then the curvature of ∇ is 0. If Ξ is not free, then Ξ is isomorphic to one of the Heisenberg A_θ-modules first introduced in [4] and studied further in [5]. If $\tau(\Xi) = p + q\theta$, then it is shown in [4], with slightly different sign conventions, that the curvature, Θ, of ∇, is determined up to a sign by

$$\Theta_{12} = \Theta(\delta_1, \delta_2) = 2\pi i q/(p + q\theta).$$

By the definition of YM, we find then that

$$\mathrm{YM}(\nabla) = -(2\pi i q/(p + q\theta))^2 \tau(\Xi) = 4\pi^2 q^2/(p + q\theta).$$

Notice that this expression is always nonnegative, and that earlier ambiguities about signs do not matter because of taking squares. For notational simplicity we will omit the factor $4\pi^2$ from now on.

Suppose instead that we are given a decomposition of Ξ as a direct sum of a finite family $\{\Xi_k\}$ of submodules, and that we have in mind choosing on each Ξ_k a connection of constant curvature, so as to construct a critical point for YM on Ξ. Then we should combine submodules for which the connections of constant curvature have the same curvature, since the combined connection will again have constant curvature. But any $p + q\theta$ can be written as $m(p' + q'\theta)$ where $(p', q') = 1$, i.e., p' and q' are relatively prime, and m is positive (where if $q' = 0$ then relatively prime means $p' = 1$, and similarly if $p' = 0$). Furthermore, the expression $2\pi i q/(p + q\theta)$ for the constant curvature shows that the curvature depends only on p' and q' and not on m. Thus, to reflect this combining of modules with connections of the same curvature, we should require that if $\tau(\Xi_k) = m_k(p_k + q_k\theta)$ with $(p_k, q_k) = 1$, then all the $p_k + q_k\theta$ are distinct for different

k. Correspondingly, if $d = \tau(\Xi)$, then by a *partition* of d we will mean a family $\{(m_k, p_k, q_k)\}$ where $m_k, p_k, q_k \in Z$, with $m_k > 0$ and $p_k + q_k\theta > 0$, with $(p_k, q_k) = 1$, and with the $p_k + q_k\theta$ all distinct, such that

$$\sum m_k(p_k + q_k\theta) = d.$$

From the results mentioned above, to each partition of d we can associate, in many ways, a decomposition $\{\Xi_k\}$ of Ξ such that $\tau(\Xi_k) = m_k(p_k + q_k\theta)$ for every k. Then for each such decomposition we can choose connections of constant curvature in many ways, whose direct sum will be a critical point for YM. But for a fixed partition, YM will have, by Proposition 1.3, the same value on all these connections, independent of the choices made, the value being

$$\sum m_k q_k^2/(p_k + q_k\theta).$$

Thus we have obtained:

3.1 Proposition. *Let Ξ be a projective A_θ-module with $d = \tau(\Xi)$. Then the possible values of* YM *at its critical points in* CC(Ξ) *are exactly all numbers of the form*

$$\sum m_k q_k^2/(p_k + q_k\theta)$$

as $\{(m_k, p_k, q_k)\}$ ranges over all partitions of d, as defined above.

Since we know that the minimum of YM is taken exactly at connections of constant curvature on Ξ, on which YM will have value $q^2/(p+q\theta)$ where $\tau(\Xi) = p + q\theta$, we obtain the following amusing corollary:

3.2 Corollary. *Let $\{(m_k, p_k, q_k)\}$ be any partition of $p + q\theta$. Then*

$$\sum m_k q_k^2/(p_k + q_k\theta) \geq q^2/(p + q\theta).$$

We did not seek the elementary proof that should exist for this, but the referee reported that one of his colleagues found that it consists of applying the Cauchy-Schwarz inequality to the vectors a and b where

$$a_k = q_k(m_k/(p_k + q_k\theta))^{1/2}, \qquad b_k = (m_k(p_k + q_k\theta))^{1/2}.$$

If YM is to be a possible candidate for a useful Morse function, then its set of values on critical points should be discrete. This is indeed the case:

3.3 Proposition. *With notation as above, for any constant c there is only a finite set of partitions of d such that the value of* YM *on any corresponding critical point connection is less than c.*

Proof. Suppose that $\{(m_k, p_k, q_k)\}$ is a partition of d such that

$$\sum m_k q_k^2/(p_k + q_k\theta) \leq c.$$

Then for each k we must have

$$m_k q_k^2 \leq c(p_k + q_k\theta).$$

But

$$\sum m_k(p_k + q_k\theta) = d,$$

and so $m_k(p_k + q_k\theta) \leq d$ for each k. Combining these two facts, we see that $m_k q_k^2 \leq cd$. Since $m_k \neq 0$, it follows that $q_k^2 \leq cd$. If $q_k \neq 0$, we also see that $m_k \leq cd$. But if $q_k = 0$, then $p_k > 0$ and we must have $m_k p_k \leq d$. Putting all this together, we see that m_k and q_k can range over only a finite number of integers. But $0 < m_k(p_k + q_k\theta) \leq d$ for each k, so we see that the p_k's also can range over only a finite set of integers. q.e.d.

It would be nice if YM took distinct values on distinct partitions (where partitions which differ only by a permutation are not viewed as distinct). But this is not in general the case. The simplest example is perhaps given by the following two partitions of $10 + 4\sqrt{2}$ for $\theta = \sqrt{2}$:

$$6(0 + \sqrt{2}) + 2(2 - \sqrt{2}) = 6(1 + 0\sqrt{2}) + (1 + \sqrt{2}) + 3(-1 + \sqrt{2}).$$

A simple calculation shows that YM has the same value for both. The general state of affairs is described by:

3.4 Proposition. *If θ is algebraic over the rationals, then there always exist projective A_θ-modules with distinct partitions for which* YM *takes the same value. But if θ is transcendental over the rationals, then* YM *never takes the same value for distinct partitions.*

Proof. Suppose first that θ is algebraic, so that the field F it generates over the rational numbers has finite dimension, say n. Any expression of form $q^2/(p + q\theta)$ will lie in this field. Choose $2n + 1$ distinct relatively prime pairs (p_k, q_k) such that $p_k + q_k\theta > 0$. Then the set of $2n + 1$ vectors $(p_k + q_k\theta, q_k^2/(p_k + q_k\theta))$ in F^2 must be linearly dependent over the rational numbers, so that we can find integers m_k, not all zero, such that

$$\sum m_k(p_k + q_k\theta) = 0,$$
$$\sum m_k q_k^2/(p_k + q_k\theta) = 0.$$

Since all the $p_k + q_k\theta$ are strictly positive, there must be some m_k's which are strictly positive and some which are strictly negative (while some may be 0). Then in the two equations above we can move all the terms with negative m_k's to the right-hand side. This gives two distinct partitions of some positive element of $Z + Z\theta$ such that YM for the two corresponding decompositions of the corresponding projective module has the same value.

Suppose now that θ is transcendental. Then we claim that for any finite collection $\{(p_k, q_k)\}$ of distinct relatively prime pairs for which $p_k + q_k\theta > 0$, the set of numbers $(p_k+q_k\theta)^{-1}$ is linearly independent over the rationals. For if a linear combination, say with integer coefficients $\{m_k\}$, is zero, then on clearing denominators we find that θ is a root of the polynomial

$$\sum m_k \prod_{j\neq k}(p_j + q_j x)$$

with integer coefficients. But this polynomial is not the zero polynomial, since by the distinctness assumption and the assumption that $p_k+q_k\theta > 0$, it is easily seen that no $-p_k/q_k$ is a root of it (with a little extra argument if $q_k = 0$ for some k). But if YM had the same value on two distinct partitions, then on combining them we would obtain an expression of form

$$\sum m_k q_k^2/(p_k + q_k\theta) = 0,$$

contradicting the linear independence shown above. q.e.d.

Nevertheless it is clear from Proposition 3.3 that even when θ is algebraic, for any given critical value of YM there will be only a finite number of distinct partitions for which YM will have that value. In particular, to study the set of connections which are critical points at which YM takes some fixed value, it suffices to study separately the sets of critical point connections corresponding to each partition for which YM has that value. We undertake that study next.

4. The moduli spaces

We now fix a projective module Ξ, and we fix a partition $\pi = \{(m_k, p_k, q_k)\}$ of $d = \tau(\Xi)$. We will say that a decomposition $\{\Xi_k\}$ of Ξ is of type π if $\tau(\Xi_k) = m_k(p_k + q_k\theta)$ for all k. We will say that a critical point connection ∇ is associated with a decomposition $\{\Xi_k\}$ of Ξ if there are connections ∇_k of constant curvature on each Ξ_k, such that $\nabla = \bigoplus \nabla_k$. We let $\mathrm{CP}(\pi)$ denote the set of critical point connections of type π, that is, associated with a decomposition of type π.

As described in [5], the gauge group UE, consisting of all unitary elements of $E = \mathrm{End}_{A_\theta}(\Xi)$, acts by conjugation on $\mathrm{CC}(\Xi)$, leaving YM invariant. It also acts in the evident way on decompositions of Ξ, and clearly carries decompositions of type π to decompositions of type π. It easily follows that UE carries $\mathrm{CP}(\pi)$ into itself for each π. We wish to describe the space of orbits, $\mathrm{CP}(\pi)/\mathrm{UE}$, for this action. We will call this space of orbits the moduli space of type π.

Let us fix now a decomposition $\{\Xi_k\}$ of Ξ of type π. Let $\{\Xi'_k\}$ be any other decomposition of Ξ of type π. Since $\tau(\Xi'_k) = \tau(\Xi_k)$ for each k, the main result of [9] mentioned earlier implies that $\Xi'_k \cong \Xi_k$ for each k. By taking polar decompositions, one can adjust the isomorphisms to be unitary. (Although E is not complete, it is closed under the holomorphic functional calculus for the reasons given in [5, p. 241], so there is no problem in forming polar decompositions of invertible elements in E.) Combining these unitary isomorphisms, we obtain a $U \in \mathrm{UE}$ which carries $\{\Xi'_k\}$ onto $\{\Xi_k\}$, in the obvious sense. Then it is easily seen that conjugation by U carries any element of $\mathrm{CP}(\pi)$ which is associated to $\{\Xi'_k\}$ onto one associated with $\{\Xi_k\}$. That is, if we let $\mathrm{CP}(\{\Xi_k\})$ denote the set of elements of $\mathrm{CP}(\pi)$ which are associated with $\{\Xi_k\}$, then $\mathrm{CP}(\{\Xi_k\})$ meets every orbit of $\mathrm{CP}(\pi)$ for the action of UE. Thus what remains to be done is to determine when two elements of $\mathrm{CP}(\{\Xi_k\})$ are in the same orbit for UE.

Suppose now that we are given a $\nabla \in \mathrm{CP}(\{\Xi_k\})$, so that $\nabla = \bigoplus \nabla_k$ where ∇_k is a connection with constant curvature on Ξ_k. Let $U \in \mathrm{UE}$ conjugate ∇ to another element of $\mathrm{CP}(\{\Xi_k\})$. Since the curvature of any connection on Ξ_k with constant curvature must be $2\pi i q_k/(p_k + q_k\theta)$, and since these numbers are all distinct by the definition of a partition, and since conjugation of connections of constant curvature does not change the curvature, we see that U must carry each Ξ_k to itself. That is, $U = \bigoplus U_k$ where $U_k \in \mathrm{UE}(\Xi_k)$. It follows that the set of equivalence classes in $\mathrm{CP}(\{\Xi_k\})$ under conjugation by UE is just the Cartesian product over k of the moduli spaces of the space $MC(\Xi_k)$ of connections of constant curvature. But the main result of [5] implies that the moduli space for $MC(\Xi_k)$ just looks like $(T^2)^{m_k}/\sum_{m_k}$, where T^2 is the ordinary two-torus, and $\sum_{m_k}$ is the group of permutations of m_k objects, acting on $(T^2)^{m_k}$ in the evident way by permuting entries. Putting all of the above together, we obtain:

4.1 Theorem. *Let Ξ be a projective A_θ-module, for θ irrational, and let $\pi = \{(m_k, p_k, q_k)\}$ be a partition of $\tau(\Xi)$. Then the moduli space for the set of critical points of* YM *on* $\mathrm{CC}(\Xi)$ *of type π looks like*

$$\prod_k (T^2)^{m_k}/\sum\nolimits_{m_k}.$$

We remark that some closely related moduli spaces are described in §2.9 of [3].

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