J. DIFFERENTIAL GEOMETRY
31 (1990) 365-385

MODIFIED DEFECT RELATIONS FOR THE GAUSS MAP OF MINIMAL SURFACES. II

HIROTAKA FUJIMOTO

1. Introduction

Let $x = (x_1, \cdots, x_m)\colon M \to \mathbf{R}^m$ be a (connected, oriented) minimal surface immersed in a Euclidean m-space $\mathbf{R}^m$ ($m \geq 3$). We denote the set of all oriented 2-planes in $\mathbf{R}^m$ by Π. For each $P \in \Pi$ taking a positive orthonormal basis (X, Y) of P and setting $Z := (X - iY)/2$ in a complex number m-space $\mathbf{C}^m$, we assign the point $\Phi(P) := \pi(Z)$, where π denotes the canonical projection of $\mathbf{C}^m - \{0\}$ onto the complex projective space $P^{m-1}(\mathbf{C})$. Then the map $\Phi\colon \Pi \to P^{m-1}(\mathbf{C})$ maps Π bijectively onto the quadric

$$Q_{m-2}(\mathbf{C}) := \{(w_1 : \cdots : w_m); w_1^2 + \cdots + w_m^2 = 0\}.$$

For a point $p \in M$ the tangent plane $T_p(M)$ of M at p is considered an oriented 2-plane in $\mathbf{R}^m$, where $T_p(\mathbf{R}^m)$ is identified with $\mathbf{R}^m$ by the parallel translation which maps p to the origin. By definition, the (generalized) Gauss map of M is the map $G\colon M \to Q_{m-2}(\mathbf{C})$ ($\subset P^n(\mathbf{C})$) which maps each point $p \in M$ to the point $\Phi(T_p(M))$, where $n = m - 1$. The metric induced from $\mathbf{R}^m$ gives a conformal structure on M, and M is considered a Riemann surface. By the assumption of minimality of M, G is a holomorphic map of M into $P^n(\mathbf{C})$. In the case $m = 3$, $Q_1(\mathbf{C})$ can be identified with the Riemann sphere, and G is considered a meromorphic function, whose conjugate is the classical Gauss map of M.

In 1981, F. Xavier showed that the Gauss map of a nonflat complete minimal surface in $\mathbf{R}^3$ could not omit 7 points of the sphere [13]. Afterwards, as a generalization of this, the author proved that, if the Gauss map G of a complete minimal surface M in $\mathbf{R}^m$ is nondegenerate, namely, $G(M)$ is not contained in any hyperplane in $P^{m-1}(\mathbf{C})$, then it can omit at most m^2 hyperplanes in general position [4]. Moreover, in [5] and [6] he gave several improvements of this result. Recently, the author has improved F. Xavier's result by showing that the Gauss map of a nonflat complete

Received April 25, 1988.

minimal surface can omit at most 4 points of the sphere [7]. Moreover, in the previous paper [8] he introduced some new types of modified defects for a meromorphic function on an open Riemann surface and gave a modified defect relation for the Gauss map of a minimal surface in $\mathbf{R}^3$ which is similar to the defect relation in Nevanlinna theory of value distribution of meromorphic functions.

The purpose of this paper is to generalize some results of [8] to complete minimal surfaces in $\mathbf{R}^m$ ($m \geq 3$). We shall give a modified defect relation for a holomorphic map of a Riemann surface into $P^n(\mathbf{C})$ under some conditions, which will be stated in §2 and proved in §5 after giving some preliminary results in §§3 and 4. As a special case of it we shall give the following.

Theorem 1.1. *Let M be a complete minimal surface in $\mathbf{R}^m$ and assume that the Gauss map G of M is nondegenerate. Then G can omit at most $m(m+1)/2$ hyperplanes in $P^{m-1}(\mathbf{C})$ located in general position.*

For the case $m = 3$, the number $m(m+1)/2 = 6$ in Theorem 1.1 is best-possible (cf. [4, p. 280]). It is an open problem whether the same is true for the case $m \geq 4$.

We shall give another application of the above-mentioned modified defect relation. Let M be a Riemann surface holomorphically immersed in $\mathbf{C}^m$. The complex Gauss map is defined to be the map which maps each point $p \in M$ to the point in $P^{m-1}(\mathbf{C})$ corresponding to the complex tangent line of M at p. We shall show the following.

Theorem 1.2. *Let M be a Riemann surface holomorphically immersed in $\mathbf{C}^m$ which is complete with respect to the metric induced from $\mathbf{C}^m$. If M is not contained in any affine hyperplane in $\mathbf{C}^m$, then the complex Gauss map of M can omit at most $m(m+1)/2$ hyperplanes in $P^{m-1}(\mathbf{C})$ located in general position.*

This is an improvement of [6, Theorem 7.4] for a special case where M is of dimension one. The number $m(m+1)/2$ in Theorem 1.2 is best-possible for arbitrary odd numbers m. It seems likely that the same is true for all even numbers. Some examples are given in §6.

2. Statement of Main Theorem

Let M be an open Riemann surface, and f a nondegenerate holomorphic map of M into $P^n(\mathbf{C})$. For an arbitrarily fixed homogeneous coordinate system $(w_0 \colon \cdots \colon w_n)$ we represent f as $f = (f_0 \colon \cdots \colon f_n)$ with holomorphic functions $f_0, \cdots, f_n$ on M without common zeros. In the following sections, such a representation of f is referred to as a reduced representation of f. Now, set $\|f\|^2 = |f_0|^2 + \cdots + |f_n|^2$ and, for a hyperplane H: $a_0 w_0 + \cdots + a_n w_n = 0$ in $P^n(\mathbf{C})$, define the function $F(H) := a_0 f_0 + \cdots + a_n f_n$.

As in the previous papers, we give

Definition 2.1. We define the *S-defect* of H for f as

$$\delta_f^S(H) := 1 - \inf\{\eta \geq 0; \eta \text{ satisfies condition } (*)_S\}.$$

Here, condition $(*)_S$ means that there exists a $[-\infty, \infty)$-valued continuous subharmonic function u ($\not\equiv -\infty$) on M satisfying the following conditions:

(D1) $e^u \leq \|f\|^\eta$,

(D2) for each $\zeta \in f^{-1}(H)$ there exists the limit

$$\lim_{z \to \zeta}(u(z) - \min(\nu_{F(H)}(\zeta), n) \log |z - \zeta|) \in [-\infty, \infty),$$

where z is a holomorphic local coordinate around ζ, and $\nu_{F(H)}(\zeta)$ denotes the order of the holomorphic function $F(H)$ at ζ.

Definition 2.2. The *H-defect* of H for f is defined by

$$\delta_f^H(H) := 1 - \inf\{\eta \geq 0; \eta \text{ satisfies condition } (*)_H\}.$$

Here, condition $(*)_H$ means that there exists a $[-\infty, \infty)$-valued continuous function u on M which is harmonic on $M - f^{-1}(H)$ and satisfies conditions (D1) and (D2).

These modified defects have the following properties.

Proposition 2.3 (cf. [8, §1]). (i) $0 \leq \delta_f^H(H) \leq \delta_f^S(H) \leq 1$.

(ii) *If there exists a bounded nonzero holomorphic function g such that $\nu_g = \min(\nu_{F(H)}, n)$, then $\delta_f^H(H) = \delta_f^S(H) = 1$.*

(iii) *If $F(H)$ has no zero of order less than m ($> n$), then $\delta_f^H(H) \geq 1 - n/m$.*

Assertion (i) is obvious because condition $(*)_H$ implies condition $(*)_S$. To see (ii) we may assume that $|g| \leq 1$. Then the function $u = \log |g|$ satisfies conditions (D1), (D2) for $\eta = 0$. This gives (ii). Assertion (iii) is true because the function $u = \frac{n}{m} \log |F(H)|$ satisfies conditions (D1), (D2) for $\eta = n/m$.

Consider the case $M = \mathbf{C}$. By a coordinate change, we may assume $f(0) \notin H$. The order function of f, the counting function for H and the classical Nevanlinna defect (truncated by n) are defined respectively by

$$T^f(r) := \frac{1}{2\pi}\int_0^{2\pi} \log \|f(re^{i\theta})\|\, d\theta - \log\|f(0)\|,$$
$$N_H^f(r) := \int_0^r \sum_{|z|\le t} \min(\nu_{F(H)}(z), n)\frac{1}{t}\, dt,$$
$$\delta_f(H) := 1 - \limsup_{r\to\infty} \frac{N_H^f(r)}{T^f(r)}.$$

We easily see

$$0 \le \delta_f^S(H) \le \delta_f(H)$$

(cf. [8, §1]). The classical defect relation in value distribution theory of meromorphic functions is stated as follows.

Theorem 2.4. *Let f be a nondegenerate holomorphic map of* $\mathbf{C}$ *into* $P^n(\mathbf{C})$. *Then*

$$\sum_{1\le j\le q} \delta_f(H_j) \le n+1$$

for arbitrary hyperplanes $H_1, \cdots, H_q$ in general position.

To state our Main Theorem, we give

Definition 2.5. Let M be an open Riemann surface with a conformal metric ds^2. For a number ρ (> 0), a nondegenerate holomorphic map $f\colon M \to P^n(\mathbf{C})$ is said to satisfy *condition* (C_ρ^*) if there exist a harmonic function h and a nowhere zero holomorphic one-form ω on M such that

$$\lambda e^h \le \|f\|^\rho, \tag{2.6}$$

where λ is a function on M with $ds^2 = \lambda^2|\omega|^2$.

Now, we state the

Main Theorem. *Let M be an open Riemann surface with a complete conformal metric ds^2, and let $f\colon M \to P^n(\mathbf{C})$ be a nondegenerate holomorphic map satisfying condition* (C_ρ^*). *Then, for arbitrary hyperplanes $H_1, \cdots, H_q$ in $P^n(\mathbf{C})$ located in general position,*

$$\sum_{1\le j\le q} \delta_f^H(H_j) \le n+1+\frac{\rho n(n+1)}{2}. \tag{2.7}$$

Remark. In the previous papers [5] and [6], under somewhat weaker conditions it was shown that

$$\sum_{1\le j\le q} \delta_f^S(H_j) \le n+1+\rho n(n+1).$$

We now consider a minimal surface $x = (x_1, \cdots, x_m): M \to \mathbf{R}^m$. By associating a holomorphic local coordinate $z = u + iv$ with each positive isothermal local coordinates u, v, we may consider M as a Riemann surface. The Gauss map G is given by $G = \pi \cdot (\partial x/\partial z)$ locally, where $\pi: \mathbf{C}^m - \{0\} \to P^{m-1}(\mathbf{C})$ is the canonical projection. If we set $f_i = (\partial/\partial x_i)z$ $(0 \leq i \leq n)$, then we have $G = (f_0 : \cdots : f_n)$. This is a reduced representation since x is an immersion. On the other hand, the metric ds^2 on M induced from the standard metric on $\mathbf{R}^m$ is given by

$$ds^2 = 2\|f\|^2 |dz|^2.$$

This shows that the map $G: M \to P^n(\mathbf{C})$ satisfies condition (C_1^*). We can conclude from the Main Theorem the following:

Theorem 2.8. *Let M be a complete minimal surface in $\mathbf{R}^m$, and G be the Gauss map of M. If G is nondegenerate, then*

$$\sum_{1 \leq j \leq q} \delta_f^H(H_j) \leq \frac{m(m+1)}{2}$$

for arbitrary hyperplanes $H_1, \cdots, H_q$ in general position.

Theorem 1.1 stated in §1 is an immediate consequence of Theorem 2.8 in view of Proposition 2.3(ii).

For the case $m = 3$, $Q_1(\mathbf{C})$ is biholomorphic with $P^1(\mathbf{C})$ by the map which maps $(w_1 : w_2 : w_3)$ to $(w_3 : w_1 - iw_2)$ (cf. [11]). Instead of G we consider the map $g := (f_3 : f_1 - if_2): M \to P^1(\mathbf{C})$. Take a reduced representation $g = (g_1 : g_2)$. Then the metric of M is given by

$$ds^2 = \frac{(|g_1|^2 + |g_2|^2)^2 |h|^2}{|g_2|^2} |dz|^2$$

(cf. [5, §6]), where h is a nonzero holomorphic function. This shows that g satisfies condition (C_2^*). Therefore, the Main Theorem implies the following result of the previous paper [8, Theorem I].

Theorem 2.9. *Let $x: M \to \mathbf{R}^3$ be a nonflat complete minimal surface, and let $g: M \to P^1(\mathbf{C})$ be the Gauss map. Then, for arbitrary distinct points $\alpha_1, \cdots, \alpha_q$ in $P^1(\mathbf{C})$,*

$$\sum_{1 \leq j \leq q} \delta_g^H(\alpha_j) \leq 4.$$

We consider next a Riemann surface immersed in $\mathbf{C}^m$ by a holomorphic map $f = (f_1, \cdots, f_m): M \to \mathbf{C}^m$. To each point $p \in M$ we assign the complex tangent line $T_p(M)$ $(\subset T_p(\mathbf{C}^m))$ of M at p. On the other hand, $T_p(\mathbf{C}^m)$ is identified with $\mathbf{C}^m$ by the parallel translation which maps p to the origin, and the totality of all complex lines in $\mathbf{C}^m$ constitutes the complex projective space $P^{m-1}(\mathbf{C})$. The *complex Gauss map* G of M is defined to be the map which maps each point $p \in M$ to the point in $P^{m-1}(\mathbf{C})$ corresponding to $T_p(M)$. We can represent G as

$$G = (f_1' : \cdots : f_m')$$

locally, and the induced metric on M is given by

$$ds^2 = |dw_1|^2 + \cdots + |dw_m|^2 = (|f_1'|^2 + \cdots + |f_m'|^2)|dz|^2,$$

where f_i' denotes the derivative of f_i with respect to a holomorphic local coordinate z. This shows that the map $G \colon M \to P^{m-1}(\mathbf{C})$ satisfies condition (C_1^*). Moreover, it is easily seen that G is nondegenerate if and only if M is not contained in any affine hyperplane in $\mathbf{C}^m$. We can conclude from the Main Theorem the following.

Theorem 2.10. *Let M be a complete Riemann surface holomorphically immersed in $\mathbf{C}^m$, which is not contained in any affine hyperplane, and let G be the complex Gauss map of M. Then*

$$\sum_{1 \le j \le q} \delta_G^H(H_j) \le \frac{m(m+1)}{2}$$

for arbitrary hyperplanes $H_1, \cdots, H_q$ in general position.

Theorem 1.2 stated in §1 is an immediate consequence of Theorem 2.10 by Proposition 2.3(ii).

3. Some properties of the derived curves

To prove the Main Theorem, we shall recall some known results on the derived curves of a holomorphic curve in $P^n(\mathbf{C})$.

Let f be a nondegenerate holomorphic map of $\Delta_R := \{z; |z| < R\}$ $(\subset \mathbf{C})$ into $P^n(\mathbf{C})$, where $0 < R \le +\infty$. Take a reduced representation $f = (f_0 : \cdots : f_n)$ and set $\|f\| = (\sum_{0 \le i \le n} |f_i|^2)^{1/2}$, $F = (f_0, \cdots, f_n)$. We define $F^{(l)} = (f_0^{(l)}, \cdots, f_n^{(l)})$ for each $l = 0, 1, \cdots$, and

$$F_k := F^{(0)} \wedge F^{(1)} \wedge \cdots \wedge F^{(k)} \colon \Delta_R \to \bigwedge^{k+1} \mathbf{C}^{n+1}.$$

Let $G(n, k)$ denote the set of all $(k+1)$-dimensional vector subspaces of $\mathbf{C}^{n+1}$. By Plücker imbedding $G(n, k)$ is regarded as a complex submanifold of $P^N(\mathbf{C})$, where $N = \binom{n+1}{k+1} - 1$. Let $\pi \colon \bigwedge^{k+1} \mathbf{C}^{n+1} - \{0\} \to P^N(\mathbf{C})$ denote the canonical projection map. The map $f_k := \pi \circ F_k$ is called the kth *derived curve* of f.

For holomorphic functions $f_0, \cdots, f_k$ we denote the Wronskian of $f_0, \cdots, f_k$ by $W(f_0, \cdots, f_k)$, namely

$$W(f_0, \cdots, f_k) = \det(f_i^{(j)}; 0 \leq i, j \leq k).$$

We define

$$|F_k| := \left(\sum_{0 \leq i_0 < \cdots < i_k \leq n} |W(f_{i_0}, \cdots, f_{i_k})|^2 \right)^{1/2},$$

and set

$$\Omega_k := dd^c \log |F_k|^2,$$

where $d^c = (\sqrt{-1}/(4\pi))(\overline{\partial} - \partial)$. For $k = n$, since F_n is holomorphic, we have $\Omega_n = 0$. For the sake of convenience, we set $|F_{-1}| = 1$.

Lemma 3.1. *Set $\Omega_k = h_k \, dz \wedge d^c z$. Then*

$$h_k = \frac{|F_{k-1}|^2 |F_{k+1}|^2}{|F_k|^4}.$$

For the proof, see [3, Lemma 4.16, p. 118] or [12, Lemma, p. 108].

Take a hyperplane H in $P^n(\mathbf{C})$. Choosing a vector $a = (a_0, \cdots, a_n)$ in $\mathbf{C}^{n+1}$ with $\|a\| = (\sum_i |a_i|^2)^{1/2} = 1$, we represent H as

$$H\colon a_0 w_0 + \cdots + a_n w_n = 0.$$

Set $F(H) := a_0 f_0 + \cdots + a_n f_n$ and

$$\varphi_k(H)(z) = \frac{|F_k(H)(z)|^2}{|F_k(z)|^2},$$

where

$$|F_k(H)|^2 = \sum_{0 \leq i_1 < \cdots < i_k \leq n} \left| \sum_{j \neq i_1, \cdots, i_k} a_j W(f_j, f_{i_1}, \cdots, f_{i_k}) \right|^2.$$

For $k = 0$ we have $\varphi_0(H) = |F(H)|^2 / \|f\|^2$ and $\varphi_n(H) = 1$.

Lemma 3.2. (i) $d\varphi_k \wedge d^c \varphi_k = (\varphi_{k+1} - \varphi_k)(\varphi_k - \varphi_{k-1})\Omega_k$.

(ii) $dd^c \log \varphi_k = \frac{\varphi_{k-1}\varphi_{k+1} - \varphi_k^2}{\varphi_k^2} \Omega_k$.

For the proof, see [3, Lemma 5.16] for (i) and [3, Lemma 5.17] for (ii), or [12, pp. 116–120].

Lemma 3.3. *For an arbitrarily given $\varepsilon > 0$ there exists some $\mu_0(\varepsilon)$ (≥ 1) such that for every $\mu \geq \mu_0(\varepsilon)$ and a hyperplane H in $P^n(\mathbf{C})$*

$$dd^c \log \frac{1}{\log^2(\mu/\varphi_k(H))} \geq \frac{2\varphi_{k+1}(H)}{\varphi_k(H) \log^2(\mu/\varphi_k(H))} \Omega_k - \varepsilon \Omega_k.$$

For the proof, see [3, p. 129] or [12, p. 122].

We shall also need the following:

Lemma 3.4 (*Sums into products*). *Let $H_1, \cdots, H_q$ be hyperplanes in $P^n(\mathbf{C})$ located in general position and set*

$$\Phi_{jk} := \frac{\varphi_{k+1}(H_j)}{\varphi_k(H_j)\log^2(\mu/\varphi_k(H_j))}.$$

Then there exists a positive constant c_k depending only on k and H_j ($1 \leq j \leq q$) such that

$$\sum_{1\leq j\leq q} \Phi_{jk} \geq c_k \prod_{1\leq j\leq q} \Phi_{jk}^{1/(n-k)}$$

on $\Delta_R - \bigcup_{1\leq j\leq q}\{z; \varphi_k(H_j)(z) = 0\}$.

For the proof, see [3, p. 134] or [12, p. 124].

Now we give the following proposition, which is fundamental for the proof of the Main Theorem.

Proposition 3.5. *For every $\varepsilon > 0$ there exist some positive numbers μ (> 1) and C depending only on ε and H_j ($1 \leq j \leq q$) such that*

(3.6)

$$dd^c \log \frac{|F_0|^{2\varepsilon}|F_1|^{2\varepsilon}\cdots|F_{n-1}|^{2\varepsilon}}{\prod_{1\leq j\leq q}|F(H_j)|^2(\prod_{0\leq k\leq n-1}\log^2\frac{\mu}{\varphi_k(H_j)})}$$
$$\geq C\left(\frac{\|f\|^{2(q-n-1)}|F_n|^2}{\prod_{1\leq j\leq q}|F(H_j)|^2(\prod_{0\leq k\leq n-1}\log^2\frac{\mu}{\varphi_k(H_j)})}\right)^{\frac{2}{n(n+1)}} dz\wedge d^c z.$$

For the proof, we use the following elementary inequality.

(3.7) *For all positive numbers $x_1, \cdots, x_n$ and $a_1, \cdots, a_n$,*

$$a_1x_1 + \cdots + a_nx_n \geq (a_1 + \cdots + a_n)(x_1^{a_1}\cdots x_n^{a_n})^{1/(a_1+\cdots+a_n)}.$$

Proof of Proposition 3.5. We denote the left-hand side of (3.6) by A. Then, by the definition of Ω_k, it is rewritten as

$$A = \varepsilon \sum_{0\leq k\leq n-1}\Omega_k + \sum_{1\leq j\leq q}\sum_{0\leq k\leq n-1} dd^c\log\left(\frac{1}{\log^2(\mu/\varphi_k(H_j))}\right).$$

Choose a positive number $\mu_0(\varepsilon/q)$ with the property as in Lemma 3.3. For an arbitrarily fixed $\mu \geq \mu_0(\varepsilon/q)$ we obtain

$$A \geq \varepsilon\sum_{0\leq k\leq n-1}\Omega_k + \sum_{1\leq j\leq q}\sum_{0\leq k\leq n-1}\left(\frac{2\varphi_{k+1}(H_j)}{\varphi_k(H_j)\log^2(\mu/\varphi_k(H_j))}\Omega_k - \frac{\varepsilon}{q}\Omega_k\right)$$
$$= \sum_{0\leq k\leq n-1} 2\left(\sum_{1\leq j\leq q}\Phi_{jk}\right)\Omega_k,$$

where Φ_{jk} is the quantity defined in Lemma 3.4. By the help of Lemma 3.4, we conclude

$$\begin{aligned} A &\geq \sum_{0\leq k\leq n-1} c_k \left(\prod_{1\leq j\leq q} \Phi_{jk}^{1/(n-k)} \right) \Omega_k \\ &= \sum_{0\leq k\leq n-1} c_k \left(\prod_{1\leq j\leq q} \Phi_{jk} h_k^{n-k} \right)^{1/(n-k)} dz \wedge d^c z, \end{aligned}$$

where c_k are some positive constants, and h_k are the quantities defined in Lemma 3.1. Now applying inequality (3.7) to $a_k := n-k$ and $x_k := \prod_{1\leq j\leq q} \Phi_{jk} h_k^{n-k}$ yields

$$A \geq C \left(\prod_{0\leq k\leq n-1} \left(\prod_{1\leq j\leq q} \Phi_{jk} \right) h_k^{n-k} \right)^{2/n(n+1)} dz \wedge d^c z$$

for some positive constant C. On the other hand, we have

$$\begin{aligned} \prod_{0\leq k\leq n-1} \Phi_{jk} &= \prod_{0\leq k\leq n-1} \frac{\varphi_{k+1}(H_j)}{\varphi_k(H_j)} \frac{1}{\log^2(\mu/\varphi_k(H_j))} \\ &= \frac{\|f\|^2}{|F(H_j)|^2} \prod_{0\leq k\leq n-1} \frac{1}{\log^2(\mu/\varphi_k(H_j))}, \end{aligned}$$

$$\prod_{0\leq k\leq n-1} h_k^{n-k} = \prod_{0\leq k\leq n-1} \left(\frac{|F_{k-1}|^2 |F_{k+1}|^2}{|F_k|^4} \right)^{n-k} = \frac{|F_n|^2}{|F_0|^{2(n+1)}},$$

because $\varphi_0(H_j) = |F(H_j)|^2/\|f\|^2$, $\varphi_n(H_j) = 1$ and the products telescope. Therefore,

$$A \geq C \left(\frac{\|f\|^{2(q-n-1)} |F_n|^2|}{\prod_{1\leq j\leq q} |F(H_j)|^2 (\prod_{0\leq k\leq n-1} \log h^2(\mu/\varphi_k(H_j)))} \right)^{2/n(n+1)} dz \wedge d^c z,$$

which gives Proposition 3.5.

We shall prove here another proposition.

Proposition 3.8. *Set $A_n := n(n+1)/2$ and $B_n := \sum_{k=1}^n A_k$. Then*

$$\begin{aligned} &dd^c \log |F_0|^2 |F_1|^2 \cdots |F_{n-1}|^2 \\ &\quad \geq \frac{B_n}{A_n} \left(\frac{|F_0|^2 \cdots |F_{n-1}|^2 |F_n|^2}{|F_0|^{2A_{n+1}}} \right)^{1/B_n} dz \wedge d^c z. \end{aligned}$$

Proof. Since $dd^c \log |F_k|^2$ $(0 \le k \le n-1)$ are nonnegative, by the aid of (3.7) we can conclude from Lemma 3.1 that

$$\begin{aligned}
&A_n dd^c \log |F_0|^2 \cdots |F_{n-1}|^2 \\
&\quad \ge \left(A_n \frac{|F_1|^2}{|F_0|^4} + A_{n-1} \frac{|F_0|^2|F_2|^2}{|F_1|^4} + \cdots + A_1 \frac{|F_{n-2}|^2|F_n|^2}{|F_{n-1}|^4} \right) dz \wedge d^c z \\
&\quad \ge B_n \left(\left(\frac{|F_1|^2}{|F_0|^4} \right)^{A_n} \left(\frac{|F_0|^2|F_2|^2}{|F_1|^4} \right)^{A_{n-1}} \cdots \left(\frac{|F_{n-2}|^2|F_n|^2}{|F_{n-1}|^4} \right)^{A_1} \right)^{B_n} dz \wedge d^c z \\
&\quad = B_n \left(\frac{|F_1|^2 \cdots |F_{n-1}|^2 |F_n|^2}{|F_0|^{n^2+3n}} \right)^{B_n} dz \wedge d^c z,
\end{aligned}$$

which implies Proposition 3.8.

4. A result of the generalized Schwarz lemma

Let f be a nondegenerate holomorphic map of Δ_R into $P^n(\mathbf{C})$, and let $H_1, \cdots, H_q$ be hyperplanes in $P^n(\mathbf{C})$ located in general position. We use the same notation as in the previous section. Suppose that there exist nonnegative numbers $\eta_1, \cdots, \eta_q$ and $[-\infty, \infty)$-valued continuous subharmonic functions $u_1, \cdots, u_q$ such that

(C1) $\gamma := q - \eta_1 - \eta_2 - \cdots - \eta_q - n - 1 > 0$,

(C2) $e^{u_j} \le \|f\|^{\eta_j}$ for $j = 1, \cdots, q$,

(C3) for each $\zeta \in f^{-1}(H_j)$ $(1 \le j \le q)$ the limit

$$\lim_{z \to \zeta} (u_j(z) - \min(\nu_{F(H_j)}(\zeta), n) \log |z - \zeta|) \in [-\infty, \infty)$$

exists, where $\|f\| = (|f_0|^2 + \cdots + |f_n|^2)^{1/2}$ for a reduced representation $f = (f_0 : \cdots : f_n)$.

Set $A_n = n(n+1)/2$ and $B_n = \sum_{k=1}^n A_k$ as in Proposition 3.8.

Lemma 4.1. *For positive constants ε, C and μ (> 1), define the function*

$$\eta_\varepsilon := C \frac{\|f\|^{\gamma - A_{n+1}\varepsilon} e^{u_1 + \cdots + u_q} |F_0|^\varepsilon \cdots |F_{n-1}|^\varepsilon |F_n|^{1+\varepsilon}}{\prod_{1 \le j \le q} |F(H_j)| (\prod_{0 \le k \le n-1} \log(\mu / \varphi_k(H_j)))}.$$

If we choose suitable C and μ depending only on ε and H_j, then

$$dd^c \log \eta_\varepsilon^2 \ge \eta_\varepsilon^{2/(A_n + B_n \varepsilon)} dz \wedge d^c z.$$

Proof. Since the functions u_j and $\log|F_k|^2$ are subharmonic and F_n is holomorphic, Proposition 3.5 implies that, for $\mu \geq \mu_0(\frac{\varepsilon}{2q})$,

$$\begin{aligned} dd^c \log \eta_\varepsilon^2 &\geq \frac{\varepsilon}{2} dd^c \log |F_0|^2 \cdots |F_{n-1}|^2 \\ &\quad + dd^c \log \frac{|F_0|^\varepsilon |F_1|^\varepsilon \cdots |F_{n-1}|^\varepsilon}{\prod_{1\leq j\leq q} |F(H_j)|^2 (\prod_{0\leq k\leq n-1} \log^2(\mu/\varphi_k(H_j)))} \\ &\geq \frac{\varepsilon}{2} dd^c \log |F_0|^2 \cdots |F_{n-1}|^2 \\ &\quad + C_0 \left(\frac{\|f\|^{2(q-n-1)} |F_n|^2}{\prod_{1\leq j\leq q} |F(H_j)|^2 (\prod_{0\leq k\leq n-1} \log^2 \frac{\mu}{\varphi_k(H)})} \right)^{\frac{2}{n(n+1)}} dz \wedge d^c z, \end{aligned}$$

where C_0 is a constant depending only on ε and H_j. Applying Proposition 3.8 to the first term of the right-hand side of the above inequality we obtain

$$\begin{aligned} dd^c \log \eta_\varepsilon^2 &\geq \frac{\varepsilon B_n}{2A_n} \left(\frac{|F_0|^2 \cdots |F_n|^2}{|F_0|^{2A_{n+1}}} \right)^{1/B_n} dz \wedge d^c z \\ &\quad + C_0 \left(\frac{\|f\|^{2(q-n-1)} |F_n|^2}{\prod_{1\leq j\leq q} |F(H_j)|^2 (\prod_{0\leq k\leq n-1} \log^2 \frac{\mu}{\varphi_k(H)})} \right)^{\frac{2}{n(n+1)}} dz \wedge d^c z. \end{aligned}$$

Set $\varepsilon' = \varepsilon B_n / A_n$. It then follows from (3.7) that

$$\begin{aligned} dd^c \log \eta_\varepsilon^2 &\geq C_1 \left(\frac{|F_0|^2 \cdots |F_n|^2}{|F_0|^{2A_{n+1}}} \right)^{\varepsilon/B_n(1+\varepsilon')} \\ &\quad \times \left(\frac{|F_0|^{2(q-n-1)} |F_n|^2}{\prod_{1\leq j\leq q} |F(H_j)|^2 (\prod_{0\leq k\leq n-1} \log^2 \frac{\mu}{\varphi_k(H_j)})} \right)^{\frac{1}{A_n(1+\varepsilon')}} dz \wedge d^c z \\ &\leq C_2 \left(\frac{|F_0|^{2\gamma'} (|F_0|^2 \cdots |F_{n-1}|^2)^\varepsilon |F_n|^{2(1+\varepsilon)}}{\prod_{1\leq j\leq q} |F(H_j)|^2 (\prod_{0\leq k\leq n-1} \log^2 \frac{\mu}{\varphi_k(H_j)})} \right)^{\frac{1}{A_n(1+\varepsilon')}} dz \wedge d^c z, \end{aligned}$$

where C_i are some constants and $\gamma' = q - n - 1 - (A_{n+1}A_n/B_n)\varepsilon' = q - n - 1 - A_{n+1}\varepsilon$. By the assumption, since each u_j satisfies condition (C2), we get

$$\begin{aligned} \|f\|^{\gamma'} &= \|f\|^{q-n-1-\eta_1-\cdots-\eta_q-A_{n+1}\varepsilon} \prod_{1\leq j\leq q} \|f\|^{\eta_j} \\ &\geq \|f\|^{\gamma - A_{n+1}\varepsilon} e^{u_1+\cdots+u_q}, \end{aligned}$$

and therefore, for a positive constant C,

$$dd^c \log \eta_\varepsilon^2 \geq C \eta_\varepsilon^{2/(A_n+B_n\varepsilon)} dz \wedge d^c z,$$

which concludes Lemma 4.1.

Lemma 4.2. *Let η_ε be a function defined as in Lemma* 4.1. *Set*

$$v := C\eta_\varepsilon^{1/(A_n+B_n\varepsilon)}$$

on $\Delta_R - (\bigcup_{1\le j\le q} f^{-1}(H_j) \cup (\bigcup_{1\le k\le n-1}\{\varphi_k(H_j)=0\}))$ *and* $v := 0$ *elsewhere. If we choose a suitable* C, v *is continuous on* Δ_R *and satisfies the condition*

$$\Delta \log v \ge v^2 \tag{4.3}$$

in distribution sense.

Proof. It suffices to show that v is continuous on Δ_R. In fact, the inequality (4.3) is an immediate consequence of Lemma 4.1. Obviously, v is continuous on $\Delta_R - \bigcup_{1\le j\le q} f^{-1}(H_j)$. Set

$$\chi := \frac{W(f_0,\cdots,f_n)}{F(H_1)\cdots F(H_q)}.$$

Then, for every point ζ in Δ_R, the order of poles of χ at ζ is not larger than $L := \sum_{1\le j\le q}\min(\nu_{F(H_j)}(\zeta), n)$. In fact, for each $\zeta\in\Delta_F$, if we choose indices $i_0,\cdots,i_n$ such that $F(H_j)(\zeta)\ne 0$ for $j\ne i_0,\cdots,i_n$, we can rewrite

$$\begin{aligned}\chi &= C\frac{W(F(H_{i_0}),\cdots,F(H_{i_n}))}{F(H_{i_0})\cdots F(H_{i_n})}h\\ &= \det(F_{i_m}^{(l)}/F_{i_m}; 0\le l, m\le n)h,\end{aligned}$$

with a nowhere vanishing holomorphic function h, and so χ has no pole of order larger than L at ζ (cf. [1]). On the other hand, since each u_j satisfies condition (C3), if we take a holomorphic function φ in a neighborhood U of ζ such that

$$\nu_\varphi(z) = \sum_{1\le j\le q}\min(\nu_{f(H_j)}(z), n)$$

on U, $\varphi\chi$ is holomorphic on U and $w := u_1+\cdots+u_q - \log|\varphi|$ is continuous on U as a $[-\infty,\infty)$-valued function. Therefore, the function

$$e^{u_1+\cdots+u_q}\frac{|F_n|}{\prod_{1\le j\le q}|F(H_j)|} = e^{w+\log|\varphi\chi|}$$

is continuous. From this fact we can easily show that the function η_ε is continuous. Hence Lemma 4.2 is proved.

We recall here the following generalized Schwarz lemma.

Lemma 4.4. *Let v be a nonnegative real-valued continuous subharmonic function on* Δ_R. *If v satisfies the inequality* $\Delta\log v \ge v^2$ *in distribution sense, then*

$$v(z) \le \lambda_R(z) := \frac{2R}{R^2-|\zeta|^2}.$$

For the proof, see [8, Lemma 2.5].

We now give the following.

Main Lemma. *Let $f\colon \Delta_R \to P^n(\mathbf{C})$ be a nondegenerate holomorphic map, and let H_j $(1 \le j \le q)$ be hyperplanes in general position. Suppose that there are positive numbers η_j $(1 \le j \le q)$ and $[-\infty, \infty)$-valued continuous subharmonic functions u_j satisfying conditions* (C1), (C2) *and* (C3). *Then, for an arbitrarily given $\varepsilon > 0$, there exists some positive constant C such that*

$$\frac{\|f\|^{\gamma - A_{n+1}\varepsilon} e^{u_1+\cdots+u_q} (\prod_{0\le k\le n-1} (\prod_{1\le j\le q} |F_k(H_j)|))^{\varepsilon/q} |F_n|^{1+\varepsilon}}{\prod_{1\le j\le q} |F(H_j)|} \le C \left(\frac{2R}{R^2 - |z|^2}\right)^{A_n + B_n\varepsilon},$$

where $A_n = n(n+1)/2$ and $B_n = \sum_{1\le k\le n} A_k$.

Proof. By virtue of Lemmas 4.2 and 4.4, we see

$$\frac{\|f\|^{\gamma - A_{n+1}\varepsilon} e^{u_1+\cdots+u_q} |F_0|^{\varepsilon} \cdots |F_{n-1}|^{\varepsilon} |F_n|^{1+\varepsilon}}{\prod_{1\le j\le q} |F(H_j)| (\prod_{0\le k\le n-1} \log(\mu/\varphi_k(H_j)))} \le C \left(\frac{2R}{R^2 - |z|^2}\right)^{A_n + B_n\varepsilon}$$

for a suitable positive constant C. Set

$$K := \sup_{0<x\le 1} x^{\varepsilon/q} \log^2 \frac{\mu}{x} \quad (< \infty).$$

Since $\varphi_k(H_j) \le 1$ for all k and j, we have

$$\frac{1}{\log^2(\mu/\varphi_k(H_j))} \ge \frac{1}{K} \varphi_k(H_j)^{\varepsilon/q} = \frac{1}{K} \frac{|F_k(H_j)|^{2\varepsilon/q}}{|F_k|^{2\varepsilon/q}}.$$

Substituting this in the above inequality, we obtain the desired conclusion.

5. Proof of the Main Theorem

As in the Main Theorem, let M be an open Riemann surface with a complete conformal metric ds^2 and $f\colon M \to P^n(\mathbf{C})$ a nondegenerate holomorphic map, and assume that f satisfies condition (C_p^*). Take q hyperplanes $H_1, \cdots, H_q$ in $P^n(\mathbf{C})$ located in general position. The purpose of this section is to show inequality (2.7). Take the universal covering $\pi\colon \widetilde{M} \to M$ of M. Then $\widetilde{M}$ has a complete conformal metric $\pi^* ds^2$, and $\tilde{f} := f \cdot \pi$ satisfies condition (C_p^*). Moreover, we easily see $\delta_f^H(H_j) \le \delta_{\tilde{f}}^H(H_j)$ for all $j = 1, 2, \cdots, q$. Therefore, it suffices to show (2.7) for the holomorphic map $\tilde{f}\colon \widetilde{M} \to P^n(\mathbf{C})$. On the other hand, by Koebe's uniformization theorem $\widetilde{M}$ is biholomorphic with $\mathbf{C}$ or the unit disc Δ. For the case $\widetilde{M} = \mathbf{C}$, the Main Theorem is true by Theorem 2.4. For our purpose it suffices to consider the case $\widetilde{M} = \Delta$. In the following, we assume that M itself is equal to Δ.

Now, suppose that (2.7) does not hold, namely,

$$\sum_{1\le j\le q} \delta_f^H(H_j) > n+1+\frac{\rho n(n+1)}{2}.$$

Then, by Definition 2.2, there exist positive numbers η_j $(1 \le j \le q)$ and $[-\infty, \infty)$-valued continuous subharmonic functions u_j which are harmonic on $M - f^{-1}(H_j)$ such that they satisfy the condition

$$\text{(C1)}' \qquad \gamma = q - \eta_1 - \eta_2 - \cdots - \eta_q - n - 1 > \rho n(n+1)/2$$

and conditions (C2), (C3) in §4. Moreover, by Definition 2.5, there exists a harmonic function h on M satisfying condition (2.6).

Let $f = (f_0 \colon f_1 \colon \cdots \colon f_n)$ be a reduced representation of f, and let H_j be given by

$$H_j\colon a_{j0}w_0 + \cdots + a_{jn}w_n = 0 \qquad (1 \le j \le q).$$

We use the same notation as in the previous sections. Since f is nondegenerate, none of $F_k(H_j)$ $(1 \le j \le q,\ 0 \le k \le n-1)$ vanishes identically. We can find some $i_1, \cdots, i_k$ such that

$$\psi_{jk} := \sum_{l \ne i_1, \cdots, i_k} a_{jl} W(f_l, f_{i_1}, \cdots, f_{i_k})$$

does not vanish identically, where we set $\psi_{j0} = F(H_j)$ and $\psi_{jn} = F_n$ for the sake of convenience. As in the previous sections, we set $A_n = n(n+1)/2$ and $B_n = \sum_{k=1}^{n} A_k$. Consider the numbers

$$\text{(5.1)} \qquad p = \frac{\rho(A_n + B_n\varepsilon)}{\gamma - A_{n+1}\varepsilon}, \qquad p^* = \frac{\rho}{(1-p)(\gamma - A_{n+1}\varepsilon)}.$$

Choose some ε with

$$\frac{\gamma - \rho A_n}{A_{n+1} + \rho B_n} > \varepsilon > \frac{\gamma - \rho A_n}{\rho/q + A_{n+1} + \rho B_n},$$

so that

$$\text{(5.2)} \qquad 0 < p < 1, \qquad \frac{\varepsilon p^*}{q} > 1.$$

Consider the open subset $M' = M - \bigcup_{1\le j\le q, 0\le k\le n}\{\psi_{jk} = 0\}$ of M, and define the function

$$(5.3)\qquad v = \left(\frac{\prod_{1\le j\le q}|F(H_j)|}{e^{u_1+\cdots+u_q+\tilde{h}}|F_n|^{1+\varepsilon}\prod_{1\le j\le q,0\le k\le n-1}|\psi_{jk}|^{\varepsilon/q}}\right)^{p^*}$$

on M', where $\tilde{h} = ((\gamma - A_{n+1}\varepsilon)/\rho)h$. Let $\pi\colon \tilde{M}' \to M'$ be the universal covering of M'. Since $\log v \cdot \pi$ is harmonic on M' by the assumption, we can take a holomorphic function φ on M' such that $|\varphi| = v\cdot\pi$. Without loss of generality, we may assume that M' contains the origin o of $\mathbf{C}$. As in the previous papers [7] and [8], for each point $\tilde{p}$ of $\widetilde{M}'$ we take a continuous curve $\gamma_{\tilde{p}}\colon [0,1] \to M'$ with $\gamma_{\tilde{p}}(0) = o$ and $\gamma_{\tilde{p}}(1) = \pi(\tilde{p})$, which corresponds to the homotopy class of $\tilde{p}$. Let $\tilde{o}$ denote the point corresponding to the constant curve o. Set

$$w = F(\tilde{p}) = \int_{\gamma_{\tilde{p}}}\varphi(z)\,dz,$$

where z denotes the holomorphic coordinate on $\widetilde{M}'$ induced from the holomorphic global coordinate on M' by π. Then F is a single-valued holomorphic function on $\widetilde{M}'$ satisfying the conditions $F(\tilde{o}) = 0$ and $dF(\tilde{p}) \neq 0$ for every $\tilde{p} \in \widetilde{M}'$. Choose the largest R $(\le +\infty)$ such that F maps an open neighborhood U of $\tilde{o}$ biholomorphically onto an open disc $\Delta_R = \{z; |z| < R\}$ in $\mathbf{C}$, and consider the map $\Phi = \pi\cdot(F|U)^{-1}\colon \Delta_F \to M'$. By the Liouville theorem it is impossible that $R = \infty$.

For each point $a \in \partial\Delta$ consider the curve

$$L_a\colon w = ta, \qquad 0 \le t < 1,$$

and the image Γ_a of L_a by Φ. We shall show that there exists a point a_0 in $\partial\Delta_R$ such that Γ_{a_0} tends to the boundary of M. To this end, we assume the contrary. Then, for each $a \in \partial\Delta_R$, there is a sequence $\{t_\nu; \nu = 1,2,\cdot\}$ such that $\lim_{\nu\to\infty} t_\nu = 1$, and $z_0 = \lim_{\nu\to\infty}\Phi(t_\nu a)$ exists in M. Suppose that $z_0 \notin M'$. Then by the same argument as in the proof of Lemma 4.2 we can easily show that

$$\liminf_{z\to z_0}|F_n|^{\varepsilon p^*}\left(\prod_{\substack{1\le j\le q\\ 1\le k\le n-1}}|\psi_{jk}|^{\varepsilon p^*/q}v\right) > 0.$$

Set $\delta_0 := \varepsilon p^*/q$ $(\le \varepsilon p^*)$. If $F_n(z_0) = 0$ or $\psi_{jk}(z_0) = 0$, then we can find a positive constant C such that

$$v \ge \frac{C}{|z - z_0|^{\delta_0}}$$

in a neighborhood of z_0. By virtue of (5.2), we obtain

$$R = \int_{L_a} |dw| = \int_{L_a} \left|\frac{dw}{dz}\right| |dz| = \int v(z)\, |dz|$$
$$\geq C \int_{\Gamma_a} \frac{1}{|z - z_0|^{\delta_0}} |dz| = \infty.$$

Since this is a contradiction, we have $z_0 \in M'$.

Take a simply connected neighborhood V of z_0, which is relatively compact in M'. Set $C' = \min_{z \in \overline{V}} v(z) > 0$. Then $\Phi(ta) \in V$ $(t_0 < t < 1)$ for some t_0. In fact, if not, Γ_a goes and returns infinitely many times from ∂V to a sufficiently small neighborhood of z_0, and so we get an absurd conclusion:

$$R = \int_{L_a} |dw| \geq C' \int_{\Gamma_a} |dz| = \infty.$$

By the same argument, we can easily see that $\lim_{t \to 1} \Phi(ta) = z_0$. Since π maps each connected component of $\pi^{-1}(V)$ biholomorphically onto V, there exists the limit

$$\tilde{p}_0 = \lim_{t \to 1} (F|U)^{-1}(ta) \in \widetilde{M}'.$$

Thus $(F|U)^{-1}$ has a biholomorphic extension to a neighborhood of a. Since a is arbitrarily chosen, F maps an open neighborhood of $\overline{U}$ biholomorphically onto an open neighborhood of $\overline{\Delta}_R$. This contradicts the property of R. In conclusion, there exists a point $a_0 \in \partial \Delta_R$ such that Γ_{a_0} tends to the boundary of M.

By the definition of $w = F(z)$ we have

$$\left|\frac{dw}{dz}\right| = |\varphi|^{1-p} \left|\frac{dw}{dz}\right|^p$$
$$= \left(\frac{\prod_{1 \leq j \leq q} |F(H_j)|}{e^{u_1 + \cdots + u_q + \tilde{h}} |F_n|^{1+\varepsilon} \prod_{1 \leq j \leq q, 0 \leq k \leq n-1} |\psi_{jk}|^{\varepsilon/q}} \right)^{\rho/(\gamma - A_{n+1}\varepsilon)} \left|\frac{dw}{dz}\right|^p.$$

Set $g = f \cdot \Phi$, $g_0 = f_0 \cdot \Phi, \cdots, g_n = f_n \cdot \Phi$, and abbreviate $u_j \cdot \Phi$ and $\tilde{h} \cdot \Phi$ to u_j and $\tilde{h}$ respectively. Define also

$$G(H_j) := a_{j0} g_0 + \cdots + a_{jn} g_n,$$
$$G_n = W(g_0, \cdots, g_n),$$
$$\varphi_{jk} := \sum_{l \neq i_1, \cdots, i_k} a_{jl} W(g_l, g_{i_1}, \cdots, g_{i_k}),$$

where the Wronskians are given by differentiation with respect to w. Then

$$G_n = (F_n \cdot \Phi) \left(\frac{dz}{dw}\right)^{A_n}, \qquad \varphi_{jk} = (\psi_{jk} \cdot \Phi) \left(\frac{dz}{dw}\right)^{A_k}.$$

Since $A_n(1+\varepsilon)+\sum_{j,k}(\varepsilon/q)A_k = A_n + B_n\varepsilon$, we have easily by (5.1)

$$\left|\frac{dw}{dz}\right| = \left(\frac{\prod_{j=1}^q |G(H_j)|}{e^{u_1+\cdots+u_q+\tilde{h}}|G_n|^{1+\varepsilon}\prod_{1\le j\le q, 0\le k\le n-1}|\varphi_{jk}|^{\varepsilon/q}}\right)^{\rho/(\gamma - A_{n+1}\varepsilon)}.$$

On the other hand, the metric in Δ_R induced from $ds^2 = \lambda^2|dz|^2$ through Φ is given by

$$\Phi^* ds^2 = (\lambda\cdot\Phi)^2\left|\frac{dz}{dw}\right|^2.$$

Let G_k and $G_k(H_j)$ be the functions defined in the same manner as the definition of the functions F_k and $F_k(H_j)$ for the map g. Since $|\varphi_{jk}| \le |G_k(H_j)|$, we obtain

$$\begin{aligned}\Phi^* ds &= \lambda\left(\frac{e^{u_1+\cdots+u_q+\tilde{h}}|G_n|^{1+\varepsilon}\prod_{1\le j\le q,0\le k\le n-1}|\varphi_{jk}|^{\varepsilon/q}}{\prod_{1\le j\le q}|G(H_j)|}\right)^{\rho/(\gamma-A_{n+1}\varepsilon)}\\ &\le \lambda e^h\left(\frac{e^{u_1+\cdots+u_q}(\prod_{1\le j\le q,0\le k\le n-1}|G_k(H_j)|^{\varepsilon/q})|G_n|^{1+\varepsilon}}{\prod_{1\le j\le q}|G(H_j)|}\right)^{\rho/(\gamma-A_{n+1}\varepsilon)}.\end{aligned}$$

On the other hand, $\lambda e^h \le \|g\|^\rho$ by the assumption. It then follows that

$$\Phi^* ds \le \left(\frac{\|g\|^{\gamma - A_{n+1}}e^{u_1+\cdots+u_q}(\prod_{1\le j\le q,0\le k\le n-1}|G_k(H_j)|^{\frac{\varepsilon}{q}})|G_n|^{1+\varepsilon}}{\prod_{1\le j\le q}|G(H_j)|}\right)^{\frac{\rho}{\gamma-A_{n+1}\varepsilon}}.$$

By the use of the Main Lemma we conclude

$$\Phi^* ds \le C\left(\frac{2R}{R^2-|z|^2}\right)^p,$$

where C is a positive constant. Thus

$$d(0) \le \int_{\Gamma_{a_0}} ds = \int_{L_{a_0}} \Phi^*\, ds \le C^p \int_0^R \left(\frac{2R}{R^2-|z|^2}\right)^p |dw| < +\infty,$$

which contradicts the assumption of completeness of M. Hence the proof of the Main Theorem is completed.

6. Some examples

We shall give in this section some examples of complete Riemann surfaces holomorphically immersed in $\mathbf{C}^m$, whose Gauss maps omit $m(m+1)/2$ hyperplanes in $P^{m-1}(\mathbf{C})$ located in general position.

Taking m distinct numbers $a_1, a_2, \cdots, a_m$ in $\mathbf{C}$, we set

$$M := \mathbf{C} - \{a_1, a_2, \cdots, a_m\},$$

and let $\pi \colon \widetilde{M} \to M$ be the universal covering of M. We consider the functions

$$w_i(z) := \int_{z_0}^{z} \frac{d\zeta}{\zeta - a_i} \qquad (1 \leq i \leq m)$$

on $\widetilde{M}$, and define a holomorphic immersion $w := (w_1, w_2, \cdots, w_m)$ of $\widetilde{M}$ into $\mathbf{C}^m$, where z_0 is an arbitrarily fixed point in $\widetilde{M}$. Then, the Gauss map of $w \colon \tilde{M} \to \mathbf{C}^m$ is given by

$$G = \left(\frac{1}{z - a_1} : \cdots : \frac{1}{z - a_m} \right).$$

The map G may be rewritten as $G = (f_1(z) : \cdots : f_m(z))$ with polynomials

$$g_i(z) = (z - a_1) \cdots (z - a_{i-1})(z - a_{i+1}) \cdots (z - a_m) \qquad (1 \leq i \leq m).$$

Obviously, $g_1, \cdots, g_m$ are linearly independent, and so w is nondegenerate. On the other hand, the metric on M induced from $\mathbf{C}^m$ is given by

$$ds^2 = \frac{|g_1|^2 + \cdots + |g_m|^2}{(|z - a_1|\,|z - a_2| \cdots |z - a_m|)^2} |dz|^2,$$

and by

$$ds^2 = \frac{\sum_{1 \leq i \leq m}(|1 - a_1\zeta| \cdots |1 - a_{i-1}\zeta|\,|1 - a_{i+1}\zeta| \cdots |1 - a_m\zeta|)^2}{(|1 - a_1\zeta|\,|1 - a_2\zeta| \cdots |1 - a_m\zeta|)^2} \frac{|d\zeta|^2}{|\zeta|^2}$$

around the point ∞ if we take a holomorphic local coordinate $\zeta = 1/z$. The Riemann surface with this metric is complete. In fact, if there is a piecewise smooth curve $\gamma(t)$ $(0 \leq t < 1)$ in $\widetilde{M}$ with finite length, which tends to the boundary of $\widetilde{M}$, then the curve $\tilde{\gamma} := \pi\gamma$ in M tends to one of the points $a_1, a_2, \cdots, a_m$ and ∞. This is impossible as is easily seen by the above representations of ds^2.

We now prove the following.

Proposition 6.1. *The complex Gauss map G of the above surface $w \colon \widetilde{M} \to \mathbf{C}^m$ omits $m(m+1)/2$ hyperplanes in $P^{m-1}(\mathbf{C})$ located in general position for each odd number m.*

To this end, we show first

Lemma 6.2. *For an arbitrarily given odd number m (≥ 3) set $n := m - 1$ and $t_0 := n/2$, and consider $m(m+1)/2$ polynomials*

$$\begin{aligned} f_i(z) &:= (z - a_0)^{n-i} && (0 \leq i \leq n), \\ f_{n+1+i}(z) &:= (z - a_1)^{n-i}(z - b_1)^i && (0 \leq i \leq n), \\ f_{t_0(n+1)+i}(z) &:= (z - a_{t_0})^{n-i}(z - b_{t_0})^i && (0 \leq i \leq n), \end{aligned}$$

where a_σ, b_τ are distinct complex numbers. If we take a_σ and b_τ $(0 \leq \sigma \leq t_0$, $1 \leq \tau \leq t_0)$ suitably, then arbitrarily chosen m polynomials among them are linearly independent.

Proof. We shall show that arbitrarily chosen m polynomials among $f_0, \cdots, f_{t(n+1)+n}$ are linearly independent by induction on t, where $t \leq t_0$. It is trivial for the case $t = 0$. Suppose that Lemma 6.2 is true in the case where t is replaced by a number $\leq t-1$ for suitably chosen a_σ, b_τ $(0 \leq \sigma \leq t-1,\ 1 \leq \tau \leq t-1)$. We shall show that m polynomials $f_{i_0}, f_{i_1}, \cdots, f_{i_n}$ among f_j $(0 \leq j \leq t(n+1)+n)$ are linearly independent. We may assume

$$i_0 < i_1 < \cdots < i_k \leq t(n+1) - 1 < i_{k+1} < \cdots < i_n,$$

where it may be supposed that $k < n$ because of the induction hypothesis. For brevity, set $g_r := f_{i_r}$ $(0 \leq r \leq n)$. Then the Wronskian $W(g_0, \cdots, g_k)$ does not vanish identically by the induction hypothesis. We can choose a point c with $W(f_{j_0}, \cdots, f_{j_l})(c) \neq 0$ whenever $1 \leq j_0 < \cdots < j_l \leq t(n+1)+n$ $(1 \leq l \leq n)$. Replacing the coordinate z by $z + c$, we may assume that $c = 0$. Set

$$g_r(z) = \sum_{0 \leq s \leq n} A_{rs} z^s \qquad (0 \leq r \leq n),$$

where A_{rs} may be considered as polynomials in a_σ and b_τ $(0 \leq \sigma, \tau \leq t)$. It suffices to show that $F := \det(A_{rs}; 0 \leq r, s \leq n)$ does not vanish identically as a function of a_σ and b_τ. Let $b_t = 0$. Then $g_{k+1}, \cdots, g_n$ can be written as

$$g_r(z) = (z - a_t)^{l_r} z^{n-l_r} \qquad (k+1 \leq r \leq n-k),$$

and so $A_{rs} = \binom{l_r}{s-n+l_r}(-a_t)^{n-s}$ for $k+1 \leq r \leq n$ and $0 \leq s \leq n$, where $\binom{l}{s}$ denotes the number of combinations of l elements taken s at a time, and we set $\binom{l}{s} = 0$ if $s < 0$. On the other hand, the A_{rs} are independent of a_t for $0 \leq r \leq k$. We apply the Laplace expansion theorem on the determinant to the first $k+1$ columns and the last $n-k$ columns of $(A_{rs}; 0 \leq r, s \leq n)$. As is easily seen, F has no nonzero term of degree $< (n-k)(n-k-1)/2$, and the coefficient of the term of degree $(n-k)(n-k-1)/2$ of F with respect to a_t is given by

$$B := \det(A_{rs}; 0 \leq r \leq k, 0 \leq s \leq k) \times \det\left(\binom{l_r}{s-n+l_r}; k+1 \leq r, s \leq n\right).$$

The first term equals $W(g_0, g_1, \cdots, g_k)(0)$, and the second term equals $(l_r^{n-s}; k+1 \leq r, s \leq n)$ up to a nonzero constant multiple. Therefore, we conclude $B \neq 0$, and the proof of Lemma 6.2 is complete.

Proof of Proposition 6.1. Take polynomials $f_{i-1}(z)$ $(1 \leq i \leq q := m(m+1)/2)$ given in Lemma 6.2. Since $g_1(z), \cdots, g_m(z)$ are linearly independent and so give a basis of the vector space of all polynomials of degree $\leq m-1$, we can find some constants c_{ij} such that

$$f_{i-1}(z) = \sum_{0 \leq j \leq n} c_{ij} g_j(z) \qquad (1 \leq i \leq q).$$

Now consider q hyperplanes

$$H_i \colon c_{i0}w_0 + c_{i1}w_1 + \cdots + c_{in}w_n = 0 \qquad (1 \leq i \leq q),$$

which are located in general position by Lemma 6.2. Moreover, we see $f^{-1}(H_i) = \varnothing$ for $1 \leq i \leq q$ because $F(H_i)(z) = f_{i-1}(z)$ vanish nowhere on $\widetilde{M}$. Hence the proof of Proposition 6.1 is complete.

For the case where m is an even number, we give the following.

Conjecture. *For an arbitrarily given even number* m (≥ 2) *set* $t := m/2$ *and consider* $3t$ *polynomials*

$$\begin{aligned} f_i(z) &:= z^{i-1} && (1 \leq i \leq t), \\ f_i(z) &:= (z-1)^{i-1} && (t+1 \leq i \leq 2t), \\ f_i(z) &:= z^{i-t-1}(z-1)^{m-i+t} && (2t+1 \leq i \leq 3t). \end{aligned}$$

Then m *arbitrarily chosen polynomials among them are linearly independent.*

If the above conjecture is true for an even number m, then we can find m distinct constants $a_i := 0, b_1 := 1, a_2, b_2, \cdots, a_t, b_t$ such that for the above polynomials $f_i(z)$ $(1 \leq i \leq 3t)$ and

$$\begin{aligned} f_{3t+i}(z) &:= (z-a_2)^{m-i}(z-b_2)^{i-1} \qquad (1 \leq i \leq m), \\ f_{3t+2t(t-2)+i}(z) &:= (z-a_t)^{m-i}(z-b_t)^{i-1} (1 \leq i \leq m), \end{aligned}$$

any m polynomials among them are linearly independent, which we can prove in the same manner as in the proof of Lemma 6.1 by induction on t. So, the same conclusion as in Proposition 6.1 holds for this number m. The author could verify the above conjecture for the case $m \leq 16$ by the help of a computer. Concludingly, the number $m(m+1)/2$ in Theorem 1.2 is best-possible for all odd numbers m and for even numbers with $2 \leq m \leq 16$.

References

[1] H. Cartan, *Sur les zèros combinaisons linéaires de p fonctions holomorphes données*, Mathematica **7** (1933) 5–31.

[2] S. S. Chern & R. Osserman, *Complete minimal surfaces in euclidean n-space*, J. Analyse Math. **19** (1967) 15–34.

[3] M. J. Cowen & P. A. Griffiths, *Holomorphic curves and metrics of negative curvature*, J. Analyse Math. **29** (1976) 93–153.
[4] H. Fujimoto, *On the Gauss map of a complete minimal surface in* $\mathbf{R}^m$, J. Math. Soc. Japan **35** (1983) 279–288.
[5] ——, *Value distribution of the Gauss map of complete minimal surfaces in* $\mathbf{R}^m$, J. Math. Soc. Japan **35** (1983) 663–681.
[6] ——, *Non-integrated defect relation for meromorphic maps of complete Kähler manifolds into* $P^{N_1}(\mathbf{C}) \times \cdots \times P^{N_k}(\mathbf{C})$, Japan. J. Math. **11** (1985) 233–264.
[7] ——, *On the number of exceptional values of the Gauss map of minimal surfaces*, J. Math. Soc. Japan, **40** (1988) 235–247.
[8] ——, *Modified defect relations for the Gauss map of minimal surfaces*, J. Differential Geometry, **29** (1989) 245–262.
[9] R. Osserman, *Minimal surfaces in the large*, Comment. Math. Helv. **35** (1961) 65–76.
[10] ——, *Global properties of minimal surfaces in* E^3 *and* E^n, Ann. of Math. (2) **80** (1964) 340–364.
[11] ——, *A survey of minimal surfaces*, 2nd ed., Dover, New York, 1986.
[12] B. V. Shabat, *Distribution of values of holomorphic mappings*, Transl. Math. Monographs, Vol. 61, Amer. Math. Soc., Providence, RI, 1985.
[13] F. Xavier, *The Gauss map of a complete non-flat minimal surface cannot omit 7 points of the sphere*, Ann. of Math. (2) **113** (1981) 211–214.

KANAZAWA UNIVERSITY, JAPAN