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CHARACTERIZATION OF ARITHMETICALLY BUCHSBAUM SUBSCHEMES OF CODIMENSION 2 IN $\mathbb{P}^n$

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In some sense, the simplest subvarieties are the complete intersections. In codimension 2, two varieties are said to be *linked* if their union is a complete intersection and they have no common component. The resulting equivalence relation was studied extensively by Rao [12] [13], who showed two curves in a projective 3-space $\mathbb{P}^3$ are in the same linkage class if and only if they have the "same" Hartshorne-Rao module $\bigoplus H^1(\mathscr{I}_Y(k))$. For higher dimension, besides the same intermediate cohomologies $\bigoplus_k H^p(\mathscr{I}_Y(k))$, certain extension elements as defined by Horrocks [8] are also needed. From the cohomological viewpoint, the simplest subvarieties after complete intersections are the arithmetically Cohen-Macaulay ones, whose intermediate cohomologies $\bigoplus_k H^p(\mathscr{I}_Y(k))$ are trivial. In fact, for $n \geq 6$, arithmetically Cohen-Macaulay varieties of codimension 2 are complete intersections. This lends support to a conjecture of Hartshorne which states that any subvariety of small codimension in higher dimensional projective space is a complete intersection.

The next simplest class is the class of arithmetically Buchsbaum varieties which have trivial module structures for the intermediate cohomologies $\bigoplus H^p(\mathscr{I}_{Y\cap M}(k))$, where M is any linear space. This is equivalent to the property that their homogeneous coordinate rings are Buchsbaum rings.

The topic of Buchsbaum rings has been under intense investigation in recent years. For a comprehensive introduction to the subject with extensive references, see the recent book by Stückrad and Vogel [14]. Over the past couple of years there has been a surge of activity in the area of arithmetically Buchsbaum projective varieties, especially curves in $\mathbb{P}^3$.

The main purpose of this article is to prove the following characterization which leads to a complete classification of arithmetically Buchsbaum codimension 2 subschemes of $\mathbb{P}^n$ for $n \geq 3$.

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Theorem. *Let Y be a codimension 2 subscheme of $\mathbb{P}^n$ for $n \geq 3$. Then Y is arithmetically Buchsbaum if and only if there exists an exact sequence*

$$0 \to \bigoplus \mathscr{O}(-a_i) \xrightarrow{\varphi} \bigoplus_j l_j \Omega^{p_j}(-k_j) \bigoplus_s \mathscr{O}(-c_s) \to \mathscr{I}_Y \to 0$$

where $p_j \neq 0$.

(See Definition 0.8 for a discussion.)

The easy case $n = 3$ was independently obtained by Amasaki and implicitly contained in [1].

Corollary. *Let Y be as above. Then $\mathscr{I}_Y$ has a free (maybe nonminimal) resolution (see Remark* 0.8.4)

$$\begin{aligned} 0 &\to \bigoplus_j l_j \binom{n+1}{p_j+n} \mathscr{O}(-p_j - n - k_j) \to \cdots \\ &\to \bigoplus_j l_j \binom{n+1}{p_j+3} \mathscr{O}(-p_j - 3 - k_j) \\ &\to \bigoplus_j l_j \binom{n+1}{p_j+2} \mathscr{O}(-p_j - 2 - k_j) \oplus \mathscr{O}(-a_i) \\ &\to \bigoplus_j l_j \binom{n+1}{p_j+1} \mathscr{O}(-p_j - 1 - k_j) \oplus \mathscr{O}(-c_s) \to \mathscr{I}_Y \to 0. \end{aligned}$$

The theorem coupled with a general result [4] concerning smoothness or reducedness of the dependency locus of a map such as φ above allows us to give an essentially complete classification of (smooth) codimension 2 arithmetically Buchsbaum subvarieties, [5] except for one aspect which is not clear in our approach, namely whether the scheme is irreducible or not (see Theorem 2.3 and Remark 2.3.1). This also allows us to construct infinitely many families of nonsingular dependency loci of maps between bundles E and F, such that $E^\vee \otimes F$ is not generated by global sections.

Next we give several consequences of our classification. To begin with, by just reading off some of the numerical invariants involved, we generalize (see Theorem 2.4 and Remarks 2.6, 2.8) to arbitrary dimension some results in [3], [6] and [2]. Then with some computation we are able to give some analogues of results of [10], bounding such invariants as the degrees and number of minimal generators of the homogeneous ideal and the regularity in terms of the degree of the variety (see Theorem 2.5 and Remark 2.7). In a different direction, we prove the nonexistence of nonsingular codimension 2 arithmetically Buchsbaum subvariety of $\mathbb{P}^n$ for $n \geq 6$ (see Theorem 2.2), other than complete intersection.

Due to their simple nature, Buchsbaum curves have been used as test cases for various theories of curves in $\mathbb{P}^3$, e.g. how far to shift to have a given module as the Hartshorne-Rao module of a nonsingular curve, and whether every even liaison class has the Lazarsfeld-Rao property. The Lazarsfeld-Rao property [9] is that there is a minimal curve Y_0 in the class such that every other curve in the class is the deformation, within the same Hartshorne-Rao module, of the union of Y_0 and complete intersection curves. For the answers to these questions see Remarks 2.4.2 and 2.3.3.

Throughout this paper we work over an algebraically closed field of arbitrary characteristic. Our schemes are always projective and nondegenerate. A Buchsbaum scheme always means arithmetically Buchsbaum. It will be clear from the context that sometimes we exclude projectively Cohen-Macaulay subvarieties from the set of Buchsbaum subvarieties without so specifying. Our notation is standard, as in [7].

0. Preliminaries and notation

Let Ω^p be the pth exterior power of the cotangent bundle Ω of $\mathbb{P}^n$. We have the following facts [11]:

Fact 0.1. The Euler sequence and its pth exterior power

$$0 \to \Omega^p(p) \to \binom{n+1}{p}\mathcal{O} \to \Omega^{p-1}(p) \to 0.$$

(We set $\Omega^0 = \mathcal{O}$.)

Fact 0.2 (Bott formula). $H^0(\Omega^p(p+1))$ is the first nonzero H^0-cohomology, has dimension $\binom{n+1}{p+1}$ and generates $\Omega^p(p+1)$. $h^p(\Omega^p) = 1$ is the only nonzero intermediate cohomology for $\Omega^p(*)$.

Fact 0.3. $\Omega^p|_H \simeq \Omega^p_H \oplus \Omega^{p-1}_H(-1)$ for a hyperplane H.

Fact 0.4. $\operatorname{Hom}(\Omega^p(p), \mathcal{O}) = \bigwedge^p V$, where $V = H^0(\mathcal{O}_{\mathbb{P}_n}(1))$. It clearly follows that a nonzero $v \in \operatorname{Hom}(\Omega^p(p), \mathcal{O})$ induces a nonzero $v_1 \in \operatorname{Hom}(\Omega^1_M(1), \mathcal{O}_M)$, where M is a general linear subspace of codimension $p-1$.

Lemma 0.5. $\operatorname{Ext}^1(\Omega^p, \Omega^q(k)) = 0$ *if either* $q > p+1$, *or* $q \le p+1$, *and* $k \ne q-p-1,\ q-p-2$.

Proof. Tensoring $\Omega^q(k+r)$ with the dual of the sequence in Fact 1, we have

$$0 \to T^{r-1} \otimes \Omega^q(k) \to \bigoplus \Omega^q(k+r) \to T^r \otimes \Omega^q(k) \to 0.$$

The hypothesis on p, q and k and Fact 2 give, for $r = p, p-1, \cdots, 1$,

$$H^{p+1-r}(\Omega^q(k+r)) = H^{p+2-r}(\Omega^q(k+r)) = 0,$$

which implies $H^{p+1-r}(T^r \otimes \Omega^q(k)) = H^{p+2-r}(T^{r-1} \otimes \Omega^q(k))$ for $r = p, \cdots, 1$. In particular, $H^1(T^p \otimes \Omega^q(k)) = H^{p+1}(\Omega^q(k)) = 0$.

Fact 0.6. Let $Q_p = \Omega^p(p+1)$ if $p > 0$ and $Q_0 = 0$, $c_p = c_1(Q_p)$, $c_{2,p} = c_2(Q_p)$, $\bar{c}_p = c_1(Q_p(k))$, $\bar{c}_{2,p} = c_2(Q_p(k))$ and $r_p = \operatorname{rank} Q_p = \binom{n}{p}$. Then $\bar{c}_p = c_p + r_p k$, $\bar{c}_{2,p} = c_{2,p} + (r_p - 1)kc_p + k^2\binom{r_p}{2}$, and $c_{2,p} = \frac{1}{2}c_p^2 + c_p(p/(n-1) - \frac{1}{2})$. The last formula is obtained by applying Fact 3 repeatedly, and computing the Chern classes on $\mathbb{P}^2$. (For $i = 1, 2$, $c_i(\Omega^p(p+1)) = c_i(\binom{n-2}{p-1}\Omega^1_{\mathbb{P}_2}(2) \oplus \binom{n-2}{p}\mathscr{O}_{\mathbb{P}_2}(1))$.)

Fact 0.7. The Koszul resolution for $\mathbb{P}^n$ gives

$$0 \to \mathscr{O}(-n-1) \to (n+1)\mathscr{O}(-n)$$
$$\to \cdots \to \binom{n+1}{p+2}\mathscr{O}(-p-2) \to \binom{n+1}{p+1}\mathscr{O}(-p-1) \to \Omega^p \to 0.$$

Remark 0.7.1. By Fact 0.7, it is easy to see that the theorem implies the corollary.

Definition 0.8. A codimension 2 subscheme Y is said to have an Ω-resolution if there exists an exact sequence

$$(1) \qquad 0 \to \bigoplus \mathscr{O}(-a_i) \xrightarrow{\varphi} \bigoplus l_j \Omega^{p_j}(-k_j) \to \mathscr{I}_Y \to 0,$$

where $0 \le p_j \le n-2$, (p_j, k_j) are all distinct ordered pairs, and there is no line bundle L appearing in both the middle and left terms, such that the map $L \to L$ induced by φ is nonzero (i.e., an isomorphism).

Remark 0.8.1. For $n-2 \ge p_j \ge 1$, $h^{p_j}(\mathscr{I}_Y(k_j)) = l_j$ are the only nonzero intermediate cohomologies for $\mathscr{I}_Y$. So the Ω-resolution is unique.

Remark 0.8.2. The integers a_i appearing in (1) are the degrees of the minimal generators of $\bigoplus \operatorname{Ext}^1(\mathscr{I}_Y, \mathscr{O}(-k))$, because

$$\bigoplus \operatorname{Hom}(\bigoplus \mathscr{O}(-a_i), \mathscr{O}(-k))$$

maps surjectively to it.

Remark 0.8.3. If Y has an Ω-resolution, then by restricting exact sequence (1) on a hyperplane H, and replacing Ω_H^{n-2} by $0 \to \mathscr{O}_H(-n) \to n\mathscr{O}_H(-n+1)$, we see that $Y \cap H$ also has an Ω-resolution.

Remark 0.8.4. Let $L \to \bigoplus \Omega^p(k) \to \mathscr{I}_Y$ and $\cdots \to L_1 \to L_0 \to \bigoplus \Omega^p(k)$ be the Ω-resolution of $\mathscr{I}_Y$ and the minimal resolution of $\bigoplus \Omega^p(k)$. Then the line bundles appearing both in L and L_0 get cancelled out in the minimal resolution of $\mathscr{I}_Y$, because there is an induced injection $H^0(L(*)) \to H^0(L_0(*))$.

Proposition 0.9 [14]. *Let Y be a subscheme in $\mathbb{P}^n$. Then Y is Buchsbaum if and only if*

$$(*) \qquad H^p(\mathscr{I}_{Y\cap M}(*)) \cdot \mathfrak{M} = 0$$

for any linear subspace M of dimension m and $p = 1, \cdots, m-2$.

Remark 0.9.1. It is easy to see that if Y has an Ω-resolution, then $(*)$ holds, i.e., Y is Buchsbaum. This establishes the "if" part of the main theorem.

1. Proof of the structure theorem

In the previous section we have seen that a subvariety Y with an Ω-resolution is Buchsbaum. To prove the converse, first recall that

$$\mathrm{Ext}^1(\mathcal{I}_Y, \mathcal{O}(-k)) \simeq \omega_Y(n+1-k).$$

The minimal generators of $\bigoplus H^0(\omega_Y(n+1-k)) \simeq \bigoplus \mathrm{Ext}^1(\mathcal{I}_Y, \mathcal{O}(-k))$ give the extension

$$0 \to \bigoplus \mathcal{O}(-a_i) \to E \to \mathcal{I}_Y \to 0,$$

where E is a vector bundle with $H^{n-1}(E(*)) = 0$, and we have

$$\begin{array}{ccc} H^p(E_M(k-1)) & \xrightarrow{\sim} & H^p(\mathcal{I}_{Y\cap M,M}(k-1)) \\ {\scriptstyle x}\downarrow & & {\scriptstyle x}\downarrow \\ H^p(E_M(k)) & \xrightarrow{\sim} & H^p(\mathcal{I}_{Y\cap M,M}(k)) \end{array}$$

for any linear subspace M of codimension $m \leq n-3$, and any $1 \leq p \leq n-m-2$. Now our theorem follows from the following general criteria (take $i = n-2$) of a vector bundle being a sum of Ω^p's.

Theorem 1.1. *Let E be a vector bundle on $\mathbb{P}^n$ with the following properties*:

(i) $H^{i+1}(E(*)) = \cdots = H^{n-1}(E(*)) = 0$.

(ii) $H^p(E_M(*))\cdot\mathfrak{m} = 0$ *for any linear subspace M of* codim $0 \leq m \leq i-1$, *and any* $1 \leq p \leq i-m$.

Then we have an isomorphism $E \simeq \bigoplus l_j\Omega^{p_j}(-k_j)$, where $0 \leq p_j \leq i$, and $h^{p_j}(E(k_j)) = l_j$ are the only nonzero cohomologies for $0 < p_j < n$.

As the conditions in Theorem 1.1 are sometimes difficult to verify, we will state a special case whose conditions are, although more restrictive, easier to verify (compare [14]).

Corollary 1.2. *In Theorem* 1.1 *above we may set $i = n-1$ and replace conditions* (i) *and* (ii) *by*

(i)′ $H^p(E(*))\cdot\mathfrak{M} = 0$ *for* $p = 1, \cdots, n-1$.

(ii)′ *If* $H^p(E(k)) \neq 0 \neq H^q(E(h))$ *and* $1 \leq p < q \leq n-1$, *then* $(p+k)-(q+h) \neq 1$.

Proof. Let H be a hyperplane. Then (i)′ gives the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^p(E(k)) & \longrightarrow & H^p(E_H(k)) & \longrightarrow & H^{p+1}(E(k-1)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^p(E(k+1)) & \longrightarrow & H^p(E_H(k+1)) & \longrightarrow & H^{p+1}(E(k)) & \longrightarrow & 0 \end{array}$$

(ii)′ implies that either $H^{p+1}(E(k-1)) = 0$ or $H^p(E(k+1)) = 0$. So the map in the middle is 0.

Remark 1.2.1. To see that (i)′ alone is not sufficient to draw our conclusion, we define E on $\mathbb{P}^4$ by

$$0 \to \mathscr{O} \to \Omega^2(3) \to E^\vee \to 0.$$

E satisfies condition (i)′, if $h^1(E) = h^2(E(-2)) = 1$ are the only nonzero intermediate cohomologies, i.e., we need to check $H^1(E(1)) = 0$ which is clear under the following identification:

$$\begin{array}{ccccccc} H^0(E(1)) & \to & H^0(\Omega^2(3)) & \to & H^0(\mathscr{O}(1)) & \to & H^1(E(1)) \to 0 \\ & & \| & & \| & & \\ & & \bigwedge^2 V & & \bigwedge^4 V & & \end{array}$$

E cannot be a sum of Ω^p's by rank consideration.

Similarly define E' on $\mathbb{P}^4$ by

$$0 \to E' \to \Omega^3(3) \oplus \mathscr{O}(-1) \to \mathscr{O} \to 0.$$

Then $h^1(E') = h^3(E'(-3)) = 1$ imply $h^1(E'_{\mathbb{P}^3}) = h^2(E'_{\mathbb{P}^3}(-2)) = 1$ are the only nonzero intermediate cohomologies. However $E'_{\mathbb{P}^2}$ does not satisfy condition (ii).

We begin the proof with two technical propositions.

Proposition 1.3. *Let F be a vector bundle on $\mathbb{P}^n$ with $H^1(F(*)) = 0$, and suppose there is an exact sequence*

$$0 \to F \to L \to K \to 0$$

where $K = \bigoplus_{p_j \geq 1} \Omega^{p_j}(k_j)$ and L is a sum of line bundles, which are not summands of F. Then $F = \bigoplus \Omega^{p_j+1}(k_j)$.

Proof. We do induction on $m(K) :=$ the number of distinct $(p_j - k_j)$'s in $K = \bigoplus_{p_j \geq 1} \Omega^{p_j}(k_j)$. We may assume that $K = \bigoplus n_q \Omega^q(q+1) \oplus \Omega^{p_j}(k_j)$, where $k_j \leq p_j$, and that L has no direct summand of positive degree. Let $N = h^0(K)$. Then the N sections of K lift to N copies of $\mathscr{O}$ in L, since

$H^1(F) = 0$. So we have the following diagram:

$$\begin{array}{ccccccc} & & & & \bigoplus n_q\Omega^{q+1}(q+1) & & \\ & & & & \downarrow & & \\ & & N\mathcal{O} & = & N\mathcal{O} & & \\ & & \downarrow & & \downarrow & & \\ 0 \to & F & \to L'\oplus N\mathcal{O} & \to & \bigoplus n_q\Omega^q(q+1)\oplus\Omega^{p_j}(k_j) & \to 0 \\ & \| & \downarrow & & \downarrow & & \\ & F & \to L' & \to & \bigoplus\Omega^{p_j}(k_j) & \to 0 \\ & & & & \| & & \\ & & & & K' & & \end{array}$$

Let F' be the kernel of $L' \to K' := \bigoplus \Omega^{p_j}(k_j)$. Then $m(K') = m(K) - 1$ and by induction $F' = \bigoplus \Omega^{p_j+1}(k_j)$ if $H^1(F'(*)) = 0$. To see $H^1(F'(*)) = 0$, we look at the sequence from the snake lemma

$$0 \to \bigoplus n_q\Omega^{q+1}(q+1) \to F \to F' \to 0.$$

All we need to check is $H^1(F'(-2)) = 0$, i.e., when $q = 1$. We have

$$0 = H^1(F(-2)) \to H^1(F'(-2)) \to H^2(n_1\Omega^2) \to H^2(F(-2)) \to H^2(F'(-2)).$$

Since

$$h^2(F(-2)) = h^1\left(\left(\bigoplus n_q\Omega^q(q+1)\oplus\Omega^{p_j}(k_j)\right)(-2)\right) = n_1 = h^2(n_1\Omega^2),$$

to see $H^1(F'(-2)) = 0$, it suffices to check that $H^2(F'(-2)) = 0$. This follows from the sequence $0 \to F' \to L' \to \bigoplus\Omega^{p_j}(k_j) \to 0$ and $h^1(\Omega^{p_j}(k_j-2)) = 0$ when $p_j = 1$ because $k_j \leq p_j$. Finally the extension $0 \to \bigoplus n_q\Omega^{q+1}(q+1) \to F \to \bigoplus\Omega^{p_j+1}(k_j) \to 0$ splits because of Lemma 0.5.

In the initial step, $K' = 0$. $0 \to \bigoplus n_q\Omega^{q+1}(q+1) \to F \to L' \to 0$ splits because $q \geq 1$ and $\mathrm{Ext}^1(L', \bigoplus n_q\Omega^{q+1}(q+1)) = 0$.

Proposition 1.4. *Let E, G, J be vector bundles in the following sequence over $\mathbb{P}^n$,*

$$0 \to E \to \Omega^1(1)\oplus G \to l\mathcal{O}\oplus J \to 0,$$

where $H^0(G) = 0$, $\mathrm{Hom}(\Omega^1(1), J) = 0$ and the map $H^1(E(-1)) \xrightarrow{x} H^1(E)$ is trivial for all x. Then $E = \Omega^1(1)\oplus E'$, where E' is the kernel of $G \to l\mathcal{O}\oplus J \to 0$.

Proof. It suffices to show that $\Omega^1(1) \xrightarrow{v} l\mathcal{O}$ is 0. Assuming the contrary, v induces a nonzero map $V^\vee\otimes\mathcal{O} \to l\mathcal{O}$ and we have the following diagram:

$$\begin{array}{cccccc} 0 \to & E & \to & \Omega^1(1)\oplus G & \to l\mathcal{O}\oplus J & \to 0 \\ & \downarrow & & \downarrow & \| & \\ 0 \to & F & \xrightarrow{\beta} & V^\vee\otimes\mathcal{O}\oplus G & \to l\mathcal{O}\oplus J & \to 0 \\ & {\scriptstyle\alpha}\downarrow & & \downarrow & & \\ & \mathcal{O}(1) & = & \mathcal{O}(1) & & \end{array}$$

where F by definition is the kernel. The left column gives

$$\begin{array}{ccccccc} & 0 & \to & H^0(\mathcal{O}) & \to H^1(E(-1)) & \to & H^1(F(-1)) \\ & & & \downarrow \tilde{x} & \downarrow x & & \\ 0 \to & H^0(F) & \xrightarrow{\alpha} & H^0(\mathcal{O}(1)) \to & H^1(E) & \to & H^1(F) \end{array}$$

The map given by x is 0, for all x is equivalent to $\operatorname{im}\tilde{x} \subset \operatorname{im}\alpha$, i.e., $\operatorname{im}\alpha = H^0(\mathcal{O}(1))$. However the map α factors through $H^0(F) \xrightarrow{\beta} H^0(V^\vee \otimes \mathcal{O} \oplus G) = V^\vee$ which is not surjective because the map $V^\vee \xrightarrow{v} H^0(l\mathcal{O} \oplus J)$ is nonzero.

Proof of Theorem 1.1. We use induction on $i = i(E)$ such that $H^p(E(*)) = 0$ for $i+1 \le p \le n-1$. When $i = 0$, E is a sum of line bundles.

$H^1(E(*))$ gives

$$0 \to E \to F \to L \to 0, \tag{2}$$

where L is a sum of line bundles and F satisfies the hypotheses in the theorem plus $H^1(F(*)) = 0$. Now the minimal generators of $F^\vee$ give

$$0 \to F \to L' \to K \to 0, \tag{3}$$

where $H^{n-1}(K(*)) = 0$ and we have:

$$\begin{array}{ccc} H^p(K_M(k-1)) & \xrightarrow{\sim} & H^{p+1}(F_M(k-1)) \\ \downarrow x & & \downarrow x \\ H^p(K_M(k)) & \xrightarrow{\sim} & H^{p+1}(F_M(k)) \end{array}$$

So K satisfies the induction hypotheses with $i(K) = i(E) - 1$. Hence K is a sum of Ω^p's. Applying Proposition 1.3 to sequence (3), we see that F is also a sum of Ω^p's. Now, in sequence (2), we may assume that $L = \bigoplus_{i=1}^{s} \gamma_i\mathcal{O}(-c_i)$, where $0 = c_s < \cdots < c_1$, and that $F = \bigoplus \Omega^{p_j}(k_j)$, where $k_j \le p_j$ if $p_j \ge 2$ and $k_j < 0$ if $p_j = 0$.

We do induction on s.

Case 1. $k_j < p_j$ for all j. Sequence (2) gives:

$$\begin{array}{ccccc} & & H^0\left(\bigoplus_{i=1}^{s-1} \gamma_i\mathcal{O}(-c_i) \oplus \gamma_s\mathcal{O}\right) & \longrightarrow & H^1(E) \\ & & \downarrow & & \downarrow x \\ H^0(F(1)) & \longrightarrow & H^0\left(\bigoplus \gamma_i\mathcal{O}(-c_i+1) \oplus \gamma_s\mathcal{O}(1)\right) & \longrightarrow & H^1(E(1)) \end{array}$$

The map given by x is 0, for all x implies that the map $H^0(F(1)) \to H^0(\gamma_s\mathcal{O}(1))$ is surjective. $H^0(\bigoplus \Omega^{p_j}(k_j+1)) = 0$ for $p_j \ge 2$ implies that

$F(1)$ has $N := h^0(\gamma_s\mathscr{O}(1))$ copies of $\mathscr{O}$. So we have

$$\begin{array}{ccccccc}
 & & & & \gamma_s\Omega^1(1) & & \\
 & & & & \downarrow & & \\
 & & N\mathscr{O} & = & N\mathscr{O} & & \\
 & & \downarrow & & \downarrow & & \\
0 \to E(1) & \to & \bigoplus\Omega^{p_j}(k_j+1)\oplus N\mathscr{O} & \to & \bigoplus_{i=1}^{s-1}\gamma_i\mathscr{O}(-c_i+1)\oplus\gamma_s\mathscr{O}(1) & \to & 0 \\
\| & & \downarrow & & \downarrow & & \\
E(1) & \to & \bigoplus\Omega^{p_j}(k_j+1) & \to & \bigoplus_{i=1}^{s-1}\gamma_i\mathscr{O}(-c_i+1) & \to & 0
\end{array}$$

Let $\tilde{E}(1)$ be the kernel of $\bigoplus\Omega^{p_j}(k_j+1) \to \bigoplus_{i=1}^{s-1}\gamma_i\mathscr{O}(-c_i+1)$ in the last row. Induction on s implies that

$$\tilde{E} = \bigoplus_{i=1}^{s-1}\gamma_i\Omega^1(-c_i) \oplus \bigoplus_{p_j\neq 1}\Omega^{p_j}(k_j).$$

Using Lemma 0.5, we have $\operatorname{Ext}^1(\tilde{E}, \gamma_s\Omega^1) = 0$. So $E = \bigoplus\gamma_i\Omega^1(-c_i)\oplus\bigoplus\Omega^{p_j}(k_j)$ as claimed.

Case 2. Some $k_j = p_j$, i.e., $F = \Omega^q(q)\oplus F_0'$. Let M be a general linear subspace of codimension $q-1$. Then $\Omega^q(q)|_M \simeq \Omega^q(q)\oplus\Omega^{q-1}(q-1)\oplus\cdots\oplus\Omega^1(1)$ and sequence (2)$|_M$ is of the form $0 \to E_M \to \Omega^1_M(1)\oplus G \to \gamma_s\mathscr{O}_M\oplus J \to 0$. Proposition 1.4 implies that the map $\Omega^1_M(1) \to \gamma_s\mathscr{O}_M$ is 0. Hence the map $\Omega^q(q) \to \gamma_s\mathscr{O}$ is 0, i.e., $\Omega^q(q) \to L$ is 0 and $\Omega^q(q)$ is a direct summand of E. Repeat this process until $E = \bigoplus n_q\Omega^q(q)\oplus E'$, where $h^p(E'(-p)) = 0$ for all $1 \le p \le n-1$, i.e., we have the sequence $0 \to E' \to F' \to L \to 0$, where $F' = \bigoplus\Omega^{p_j}(k_j)$ and $\Omega^{p_j}(k_j)$'s are all the summands of F with $k_j < p_j$ if $p_j \ge 2$. We reduce the proof to Case 1.

2. Applications

We start this section with an immediate consequence of the theorem.

Proposition 2.1. *Let Y be a codimension 2 Buchsbaum subscheme of $\mathbb{P}^n$, $n \ge 3$. If Y is a zero set of a rank 2 vector bundle E, then $n = 3$ and E is (a twist of) the null-correlation bundle.*

Proof. The assumptions give the following diagram

$$\begin{array}{ccccccccc}
 & & \bigoplus_{i=1}^{m-1}\mathscr{O}(-a_i) & = & \bigoplus_{i=1}^{m-1}\mathscr{O}(-a_i) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \bigoplus_{i=1}^{m}\mathscr{O}(-a_i) & \to & \bigoplus_j l_j\Omega^{p_j}(-k_j) & \to & \mathscr{I}_Y & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathscr{O}(-a_m) & \to & E & \to & \mathscr{I}_Y & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where the a_i's appearing in the middle column are the degrees of the minimal generators of $H^{n-1}(E(*))$. So $m-1 =$ the number of minimal generators of $H^{n-1}(E(*)) = \sum_{p_j=1} l_j$. The last equality is by Serre duality. By observing the ranks in the middle column, we have that

$$\sum_{p_j=1} l_j + 2 = \sum_{p_j=1} l_j \binom{n}{1} + \sum_{p_j\neq 1} l_j \binom{n}{p_j}.$$

So $\sum_{p_j\neq 1} l_j\binom{n}{p_j} = 0$, $n = 3$ and the sequence is

$$0 \to \mathscr{O}(-a_1) \to \Omega^1(-k) \to E \to 0.$$

Tensoring the above sequence by $\mathscr{O}(k+2)$, and letting $b = k+2-a_1$, we have that $0 = c_3(E(k+2)) = -b(2+(b-2)b)$, and hence $0 \to \mathscr{O} \to \Omega^1(2) \to E(k+2) \to 0$.

Theorem 2.2. *There is no nonsingular Buchsbaum subvariety of codimension 2 in $\mathbb{P}^n$ for $n \geq 6$.*

Proof. Any nonsingular subvariety of codimension 2 in $\mathbb{P}^n$ for $n \geq 6$ comes from a rank 2 vector bundle.

The next theorem which follows from some general smoothing criteria [4] allows us to construct (nonsingular) Buchsbaum subvarieties of codimension 2.

Theorem 2.3. *Suppose Y has an Ω-resolution* (1), *φ is sufficiently general and $a_1 \leq \cdots \leq a_r$. We replace each copy of $\Omega^p(-k)$ by $\binom{n}{p}$ copies of $\mathscr{O}(-p-1-k)$ if $p > 0$, and denote the new direct sum of line bundles by $\bigoplus_{i=0}^{r}\mathscr{O}(-b_i)$ with $b_0 \leq \cdots \leq b_r$. Then the following hold.*

(a) *$a_i \geq b_i$ for all i if and only if Y is of codimension* 2.

(b) *If $a_i \geq b_{i+\alpha}$ for all i, then Y is nonsingular except for a subset of codimension* $\geq \min\{2\alpha+1, 4\}$. *Moreover when $n \leq 5$, the converse is also true.*

Remark 2.3.1. If there is no $\alpha > 0$ such that $a_i \geq b_{i+\alpha}$, then Y is reducible.

Remark 2.3.2. For $n = 3, 4$ (resp. 5), we need $\alpha \geq 1$ (resp. $\alpha \geq 2$) to have Y nonsingular.

Remark 2.3.3. To see that every even liaison class has the Lazarsfeld-Rao property, we observe that for Buchsbaum curves X and Y with the resolutions

$$0 \to \bigoplus \mathscr{O}(-a_i) \to E \to \mathscr{I}_X \to 0,$$
$$0 \to \bigoplus \mathscr{O}(-b_i) \to E \to \mathscr{I}_Y(c) \to 0,$$

we can define Z to be the dependency locus with the resolution

$$0 \to \bigoplus \mathscr{O}(-c_i) \to E \to \mathscr{I}_Z(\alpha) \to 0,$$

where $c_i = \min\{a_i, b_i\}$. Then Z is a nonsingular Buchsbaum curve by Theorem 2.3, and the argument of Proposition 1.4 in [9] implies that Z is "smaller" than X and Y.

By a simple calculation we have:

Theorem 2.4. *Let $\{(p_j, k_j, l_j)\}$ be a finite set of ordered triples of integers such that $k_j \geq 0$, $l_j > 0$, $1 \leq p_j \leq n-2$ for all j and $k_1 + p_1 \leq k_j + p_j \leq k_m + p_m$. Then there is a Buchsbaum scheme Y in $\mathbb{P}^n$ with $h^{p_j}(\mathscr{I}_Y(k_j)) = l_j$ if and only if $k_1 + p_1 + 1 \geq \sum_{j=1}^m l_j \binom{n-1}{p_j}$. Moreover a nonsingular such a Y exists if and only if either $n = 3$ or 4 and*

$$2(k_1 + p_1 + 1) \geq k_m + p_m + 1 + \sum_{j=1}^{m} l_j \binom{n-1}{p_j},$$

or $n = 5$ and

$$3(k_1 + p_1 + 1) \geq 2(k_m + p_m + 1) + \sum_{j=1}^{m} l_j \binom{n-1}{p_j}.$$

Proof. Such a Y exists if and only if there is an Ω-resolution

$$0 \to \bigoplus_{i=1}^{r} \mathscr{O}(-a_i) \to \bigoplus l_j \Omega^{p_j}(-k_j) \oplus_s \mathscr{O}(-c_s) \to \mathscr{I}_Y \to 0.$$

Comparing the first Chern classes and using the notations as in Theorem 2.3, we have

$$\begin{aligned} &\sum c_s + \sum_j l_j \left(\binom{n}{p_j} (k_j + p_j + 1) - \binom{n-1}{p_j} \right) \\ &= \sum_{i=1}^{r-\alpha} a_i + \sum_{r-\alpha+1}^{r} a_i \geq \sum_{i=\alpha+1}^{r} b_i + \sum_{r-\alpha+1}^{r} a_i \\ &= \sum c_s + \sum_j l_j \binom{n}{p_j} (k_j + p_j + 1) - \sum_{i=0}^{\alpha} b_i + \sum_{r-\alpha+1}^{r} a_i, \end{aligned}$$

i.e.,

$$\sum_{i=0}^{\alpha} b_i \geq \sum_j l_j \binom{n-1}{p_j} + \sum_{r-\alpha+1}^{r} a_i,$$

where $\alpha = 0$, or 1 (resp. 2) if Y is nonsingular and $n = 3, 4$ (resp. 5). Combining with the facts that $b_\alpha \leq k_1 + p_1 + 1$ (since $\alpha \leq 2$) and $a_{r-\alpha+1} \geq b_r \geq k_m + p_m + 1$, we have

$$(\alpha+1)(k_1 + p_1 + 1) \geq \sum_j l_j \binom{n-1}{p_j} + \alpha(k_m + p_m + 1).$$

Conversely, given the inequalities, it is easy to reverse the process and choose the α_i's.

Remark 2.4.1. Let Y be a curve in $\mathbb{P}^3$ with only one nonzero $h^1(\mathcal{J}_Y(k)) = n$ (i.e., in the class L_n), nonsingular and of maximal rank. Then

$$2n^2 + 2ln + \binom{l}{2} \leq \deg Y \leq 2n^2 + 2ln + l^2,$$

where $l = k + 2 - 2n$. Conversely, for any d in the range above, there is a Y, with all the properties above, of degree d.

Proof. Under the hypotheses, the smallest t such that $h^0(\mathcal{J}_Y(t)) \neq 0$ is either $k+1$ or $k+2$. Use the Ω-resolution form in Theorem 2.4 and choose the suitable c_s's.

Remark 2.4.2. A graded module M with a trivial multiplication map and k_1 (respectively k_m) being the smallest (respectively largest) grading of nonzero component is a Rao-module of a nonsingular Buchsbaum curve after being shifted by $k_m - 2k_1 - 2 + 2N$, where N is the sum of the lengths of nonzero components of M.

Let Y be a subvariety in $\mathbb{P}^n$; we need the following notation:

$$t = \min\{n | H^0(\mathcal{J}_Y(n)) \neq 0\},$$
$$e = \max\{n | H^0(\omega_Y(-n)) \neq 0\},$$
$$\mu = \#\{\text{minimal generators of } I(Y)\},$$
$$s = \max\{\text{degrees of minimal generators of } I(Y)\},$$
$$N = \sum_{1 \leq p_j \leq n-2} h^{p_j}(\mathcal{J}_Y(k_j)).$$

Theorem 2.5. *Let Y be a Buchsbaum subscheme of codimension 2 in $\mathbb{P}^n$. Then we have the following sharp bounds*:

(i) $\mu \leq 1 + (\sqrt{8(n-1)^2 d + (n-3)^2} + n - 3)n/4(n-1)$;

(ii) $N \leq (\sqrt{8(n-1)^2 d + (n-3)^2} + n - 3)/2(n-1)^2$;

(iii) $s \leq \max\{d - c_{2,1}, s'\}$;

(iv) $s \leq d - t^2/2 + 3t/2 - (t-1)/(n-1) - 1$, *where* $t = \min\{$*degrees of the minimal generators*$\}$;

(v) *If* $k \geq \max\{d - c_{2,1} + 1, s'\} := \tau'$, *then* $\mathcal{I}_Y$ *is* k*-regular*;

(vi) $e \leq \tau' - n - 1$;

(vii) $t \leq \frac{1}{2} - 1/(n-1) + \sqrt{(\frac{1}{2} - 1/(n-1))^2 + 2d}$,

where

$$c_{2,1} = \frac{n^2 - 3n + 4}{2}, \quad c_p = \binom{n-1}{p}, \quad s' = \max\left(\frac{d}{c_p} + \frac{c_p + 1}{2} - \frac{p}{n-1}\right).$$

Proof. We use a twist of the Ω-resolution

$$0 \to \bigoplus_{i=1}^{r} \mathcal{O}(a_i) \to \bigoplus_{j=1}^{m} Q_{p_j}(k_j) \to \mathcal{I}_Y(t) \to 0.$$

Here Q_p (resp. $Q_p(k)$) has Chern classes c_p, $c_{2,p}$ (resp. $\bar{c}_p$, $\bar{c}_{2,p}$) and rank r_p as in §0.

Theorem 2.3 implies that $a_i \leq b_i$, where $a_1 \geq \cdots \geq a_r$, and $0 = b_0 \geq b_1 \geq \cdots \geq b_r$. (Recall that the ordered set $\{b_i\}_{i=0}^{r}$ is derived from the ordered set $\{k_j\}_{j=1}^{m}$ by duplicating each k_j r_{p_j} times.) Let $c_i = b_i - a_i$ for $i = 1, \cdots, r$. Then

$$\sum b_i = \sum r_{p_j} k_j,$$

$$\sum_{i<j} b_i b_j = \sum_{i<j} r_{p_i} r_{p_j} k_i k_j + \sum_i \binom{r_{p_i}}{2} k_i^2,$$

$$\sum_{i<j} b_i b_j - \sum_{i<j} a_i a_j = \sum_{i \neq j} a_i c_j + \sum_{i<j} c_i c_j,$$

$$t = \sum c_i + \sum_j c_{p_j}.$$

In each case, we will fix μ, N, s respectively, and find a low bound of d:

$$d = c_2(\mathcal{I}_Y(t)) = \sum_j \bar{c}_{2,p_j} + \sum_{i<j} \bar{c}_{p_i} \bar{c}_{p_j} - \sum_{i<j} a_i a_j - t \sum_i a_i.$$

Using Fact 0.6 and the formulas above, we simplify d and obtain

(4)

$$d = \sum_j c_{2,p_j} + \sum_{i<j} c_{p_i} c_{p_j} + \sum_j c_{p_j}\left(-k_j + \sum c_i\right) + \sum c_i \left(\sum_{j>i} c_j - a_i\right).$$

For cases (i) and (ii), we first fix the ordered set $\{p_j\}_{j=1}^{m}$. Since

$$-k_j + \sum c_i \geq 0 \quad \text{and} \quad \sum_{j>i} c_j - a_i \geq 0,$$

d reaches minimal when $-k_j = 0$ for all j; $c_i = 0$, $-a_i = 0$ if $p_i > 0$; $c_i = 1$, $-a_i = 1$ if $p_i = 0$; the Ω-resolution is

$$0 \to \bigoplus \mathscr{O} \oplus n_0\mathscr{O}(-1) \to \bigoplus_{p=1}^{n-2} n_p Q_p \oplus n_0\mathscr{O} \to \mathscr{I}_Y(t) \to 0,$$

$$d = \sum_{p=1}^{n-2} n_p c_{2,p} + \binom{n_p}{2} c_p^2 + \sum_{p<q} n_p n_q c_p c_q + n_0 \sum n_p c_p + \binom{n_0}{2} + n_0.$$

Case (i). $\mu = n_0 + \sum_{p>0} n_p(h_p - r_p) + 1$ is constant. Treating d as a function of real variables $n_0, n_1, \cdots, n_{n-2}$, subject to the constraint $n_0 + \sum n_p(h_p - r_p) + 1 = \mu$, the Lagrange multiplier gives

$$\begin{aligned}\lambda(h_p - r_p) &= c_{2,p} - \frac{1}{2}c_p^2 + \sum_q n_q c_p c_q + n_0 c_p \\ &= c_p\left(\frac{p}{n-1} - \frac{1}{2} + \sum n_q c_q + n_0\right),\end{aligned}$$

while $\lambda = \sum n_p c_p + n_0 + \frac{1}{2}$. So $\lambda > 0$ and

$$\lambda \frac{h_p - r_p}{c_p} - \frac{p}{n-1} = \lambda \frac{h_q - r_q}{c_q} - \frac{q}{n-1}$$

for $p, q > 0$, where $(h_p - r_p)/c_p = n/(p+1)$. Solving λ gives

$$\lambda = \frac{p-q}{n-1}\left(\frac{h_p - r_p}{c_p} - \frac{h_q - r_q}{c_q}\right)^{-1} < 0$$

which is a contradiction. Therefore there is only one nonzero n_p for $p > 0$. Similarly, we have $n_0 = 0$.

Let $\bar{\mu} = \mu - 1$. Substituting $n_p = \bar{\mu}/(h_p - r_p)$ into $d = n_p c_{2,p} + \binom{n_p}{2} c_p^2$ and simplifying, we have

$$d = \bar{\mu}\frac{c_p}{h_p - r_p}\left(\frac{p}{n-1} - \frac{1}{2}\right) + \frac{\bar{\mu}^2}{2}\frac{c_p^2}{(h_p - r_p)^2},$$

which is an increasing function of p. So $p = 1$ gives the minimal

$$d = \frac{3-n}{n(n-1)}\bar{\mu} + \frac{\bar{\mu}^2}{2}\frac{4}{n^2}.$$

This bound is obtained by the Ω-resolution

$$0 \to \bigoplus \mathscr{O} \to \bar{\mu}\Omega^1(2) \to \mathscr{I}_Y(\bar{\mu}(n-1)) \to 0$$

which corresponds to the minimal resolution

$$0 \to \bar{\mu}\mathscr{O}(-n+1) \to \cdots \to \bar{\mu}\binom{n+1}{3}\mathscr{O}(-1) \to \mu\mathscr{O} \to \mathscr{I}_Y(\bar{\mu}(n-1)) \to 0$$

where $\tilde{\mu} = 2(\mu - 1)/(n^2 - n)$.

Case (ii). $N = \sum_{1 \le p \le n/2} n_p$ is constant. Here we assume $n_p = 0$ for $p \ge (n-1)/2$, because for $p' := n - 1 - p > p$, we have $c_{p'} = c_p$ and $c_{2,p'} > c_{2,p}$.

Again there is only one nonzero n_p for $p > 0$. The reason is the same as in Case (i). Here

$$\lambda = \frac{p-q}{n-1}\left(\frac{1}{c_p} - \frac{1}{c_q}\right)^{-1} < 0$$

because $0 < q < p < n/2$. So the minimal d occurs when $p = 1$,

$$d = N(n-1)\left(\frac{1}{n-1} - \frac{1}{2}\right) + \frac{N^2}{2}(n-1)^2,$$

and the minimal resolution is

$$0 \to N\mathscr{O}(-n+1) \to \cdots \to N\binom{n+1}{3}\mathscr{O}(-1)$$
$$\to \left(N\binom{n}{2} + 1\right)\mathscr{O} \to \mathscr{I}_Y(N(n-1)) \to 0.$$

Case (iii). $s = t - b_r = t - k_m$ is constant. Formula (4) is

$$d = \sum c_{2,p_j} + \sum_{i<j<m} c_{p_i}c_{p_j} + c_{p_m}(s - c_{p_m}) + \sum_{j<m} c_{p_j}\left(-k_j + \sum_{i=1}^{r-1} c_i\right)$$
$$+ \sum_{i=1}^{r-1} c_i \left(\sum_{j>i} c_j - a_i\right) + \sum_{j=1}^{m-1} c_{p_j}c_r + c_r(-b_r + c_r).$$

Note that all terms are nonnegative, and at least some $p_j > 0$. Using

$$c_r(-b_r + c_r) = c_r\left(s - \sum_1^m c_{p_j} - \sum_1^{r-1} c_i\right) \quad \text{and} \quad \sum_1^{r-1} c_i \sum_{j>i} c_j \ge c_r \sum_1^{r-1} c_i,$$

we have

$$d \ge c_{2,1} + c_{p_m}(s - c_{p_m}) + c_r(s - c_{p_m}),$$

since c_{p_m} and c_r correspond to the same summand, either $c_{p_m} = 0$, $c_r \ge 1$ and $d \ge c_{2,1} + s$, or $c_{p_m} \ne 0$, $c_r \ge 0$ and $d \ge c_{2,1} + c_{p_m}(s - c_{p_m})$. The bound $d = c_{2,1} + s$ is obtained by taking $m = 2$, $p_1 = 1$ and $p_2 = 0$. In this case the Ω-resolution is

$$0 \to (n-1)\mathscr{O} \oplus \mathscr{O}(n-1-s) \to \Omega^1(2) \oplus \mathscr{O}(n-s) \to \mathscr{I}_Y(n) \to 0,$$

and the minimal resolution is

$$0 \to \mathcal{O}(-2n+1) \to \dots \to \binom{n+1}{3}\mathcal{O}(-n-1) \oplus \mathcal{O}(-1-s)$$
$$\to \frac{n^2-n+2}{2}\mathcal{O}(-n) \oplus \mathcal{O}(-s) \to \mathcal{I}_Y \to 0.$$

Case (iv). The constants, the minimal and the maximal degrees of the generators are $t = \sum_l^m c_{p_j} + \sum_l^r c_i$ and $s = t - k_m = t - b_r$. Therefore

$$d = \frac{t^2}{2} + \sum_1^m c_{p_j}\left(\frac{p_j}{n-1} - \frac{1}{2} - k_j\right) + \sum_1^r c_i\left(\frac{c_i}{2} - b_i\right).$$

Since $-k_j \geq 0$, $-b_i \geq 0$, $\sum_1^r c_i^2/2 \geq \sum_1^r c_i/2$ and $c_p(p/(n-1)-1) = -\binom{n-2}{p}$,

$$d \geq \frac{t^2}{2} + \frac{t}{2} - \sum_{p_j \neq 0}\binom{n-2}{p_j} + (s-t)(c_{p_m} + c_r).$$

We need to maximize $\sum_{p_j\neq 0}\binom{n-2}{p_j}$ subject to $\sum_1^m c_{p_j} + \sum_1^r c_i = t$. So we maximize $\sum_1^m \binom{n-2}{p_j}$ subject to either $t = \sum_1^m \binom{n-1}{p_j}$ when $p_m > 0$ or $t-1 = \sum_1^m \binom{n-1}{p_j}$ when $p_m = 0$. If $p_m > 0$, we have

$$d \geq \frac{t^2}{2} + \frac{t}{2} - t\frac{(n-2)}{n-1} + (s-t)(n-1).$$

If $p_m = 0$, then

$$d \geq \frac{t^2}{2} + \frac{t}{2} - (t-1)\frac{(n-2)}{n-1} + (s-t).$$

Hence

$$d \geq \frac{t^2}{2} + s - \frac{3t}{2} + \frac{t-1}{n-1} + 1$$

always, and equality holds for $0 \to (ln-1)\mathcal{O} \oplus \mathcal{O}(t-s-1) \to l\Omega^1(2) \oplus \mathcal{O}(t-s) \to \mathcal{I}_Y(t) \to 0$ where $l = (t-1)/(n-1)$, with minimal resolution

$$0 \to l\mathcal{O}(-n+1-t) \to \cdots \to l\binom{n+1}{3}\mathcal{O}(-t-1) \oplus \mathcal{O}(-s-1)$$
$$\to \left(\frac{n^2-n}{2}l + 1\right)\mathcal{O}(-t) \oplus \mathcal{O}(-s) \to \mathcal{I}_Y \to 0.$$

Case (v). I_Y is a k-regular if $k \geq \max\{t - \bar{b}_r, t - a_r\}$, where $\bar{b}_r$ is the smallest b_i corresponding to $p > 0$. Let $\tau = t - a_r$ be constant, i.e., in (iii) replace s by $\tau - c_r$. Then $d \geq c_{2,1} + (c_r + c_{p_m})(\tau - c_r - c_{p_m}) \geq c_{2,1} + \tau - 1$. We have seen that $t - \bar{b}_r \leq s'$.

Case (vi). Let τ be as in Case (v). Then $e = \tau - n - 1$.

Case (vii). As in Case (iv), we have

$$d \geq \frac{t^2}{2} + \frac{t}{2} - \sum_{p_j \neq 0} \binom{n-2}{p_j}.$$

An extremal example is $\bigoplus \mathscr{O} \to (t/(n-1))\Omega^1(2) \to \mathscr{I}_Y(t)$.

Remark 2.5.1. For projectively Cohen-Macaulay varieties, the bounds are $\mu \leq \sqrt{2d + \frac{1}{4}} + \frac{1}{2}$, $s \leq d-1$, $s \leq d - t^2/2 + t/2$, $k \geq d$, $e \leq d - n - 1$, and $t \leq -\frac{1}{2} + \sqrt{d + \frac{1}{4}}$ with resolutions $(\mu - 1)\mathscr{O}(-1) \to \mu\mathscr{O} \to \mathscr{I}_Y$, $\mathscr{O}(-3) \oplus \mathscr{O}(-s-1) \to 2\mathscr{O}(-2) \oplus \mathscr{O}(-s) \to \mathscr{I}_Y$, and $(t-1)\mathscr{O}(-t-1) \oplus \mathscr{O}(-s-1) \to t\mathscr{O}(-t) \oplus \mathscr{O}(-s) \to \mathscr{I}_Y$.

Remark 2.6. In [6] the authors establish bounds involving various numerical invariants of Buchsbaum curves in $\mathbb{P}^3$. The following result, which is an evident reinterpretation of Theorem 2.3, generalizes to higher dimension most of the results of [6], including 1.3, 1.6, 2.10–2.12, 3.2, 3.3 and 3.6. To get the theorems there, one sets $n = 3$, $p = \bar{p} = q = 1$, $m = r$, $t = \alpha$, $\mu = \nu$, $N_1 = N$, $\bar{h} = 2N + h - 2$, $\alpha = 1$ for nonsingular curves and $\alpha = 0$ otherwise.

Let $N_p = \sum_{j=1}^{m_p} l_{p,j}$, where m_p is the diameter of H^p, and $h^p(\mathscr{I}_Y(k_j)) = l_{p,j}$ are the only nonzero cohomologies for $1 \leq p \leq n-2$. Denote

$$m = m_q = \max_p \{m_p\},$$

$$\bar{h} = \min\{n | h^{\bar{p}}(\mathscr{I}_Y(n)) \neq 0 \text{ for some } 1 \leq \bar{p} \leq n-2\}.$$

Then

(i) $t \geq \alpha(m-1) + \sum_{p=1}^{n-2} N_p \binom{n-1}{p}$.

(ii) $\mu \leq t + 1 - \alpha(m-1) + \sum N_p \binom{n-1}{p+1}$. If $t \leq \sum N_p \binom{n-1}{p}$, then $\mu = 1 + \sum N_p \binom{n}{p+1}$.

(iii) $e \geq \bar{h} + \bar{p} - n + m - 1$.

(iv) $t + m - n - 2 \leq e \leq t + 2\big(\bar{h} - \sum N_p \binom{n-1}{p}\big) + 2\bar{p} - (2\alpha - 1) \cdot (m-1) - n + 1$, if we use notation as in Theorem 2.5 and assume $a_i \leq b_{i+\alpha}$ for $i = 1, \cdots, r - \alpha$.

The equalities are obtained, e.g. when the middle term in the Ω-resolution is

$$\left(\bigoplus_{p \neq q} N_p \Omega^p(p+1) \bigoplus_{k=0}^{m-2} \Omega^q(q+1+k) \oplus (N_q - m + 1)\Omega^q(q+1) \right)(-m+1),$$

while $a_i = b_{i+\alpha}$ for $i = 1, \cdots, r - \alpha$, and $a_i = -m + 1$ for $i \geq r - \alpha + 1$.

Remark 2.7. Let α be as in Theorem 2.3. Then some of the bounds in 2.5, 2.5.1 are obtained by varieties Y with $\operatorname{codim}_Y \operatorname{sing} Y \geq 4$, while the others can be improved in terms of α.

For projectively Cohen-Macaulay varieties we have

$$s \le \frac{d}{\alpha+1} + \frac{\alpha}{2}, \quad k \ge \tau = \frac{d}{\alpha+1} - \frac{\alpha}{2} + 1, \quad e \le \tau - n - 1.$$

(Note that setting $\alpha = 0$ does not give bounds in 2.5.1, because the complete intersections are excluded there.)

For nonprojectively Cohen-Macaulay Buchsbaum varieties with $\alpha > 0$ we have

$$s \le \frac{d}{n-1} + \frac{(n-2)(n+1)}{2(n-1)},$$
$$k \ge \tau' = \frac{d}{n-1} + \frac{(n-2)(n+1)}{2(n-1)}.$$

Equalities are obtained by

$$(\alpha+1)\mathscr{O}(-\beta) \to (\alpha+1)\mathscr{O}(-\beta+1) \oplus \mathscr{O}(-\alpha-1) \to \mathscr{J}_Y$$

or

$$n\mathscr{O}(-\gamma) \to \mathscr{O}(-n+1) \oplus \Omega^1(2-\gamma) \to \mathscr{J}_Y,$$

where $\beta = s+1$ or τ, and $\gamma = s$ or τ'.

Remark 2.8. Let $i(Y) = \sum_{p=1}^{n-2} \binom{n-3}{p-1} N_p$ be the Buchsbaum invariant, where $N_p = \sum_k h^p(\mathscr{J}_Y(k))$. Then in Theorem 2.5 the formula of t which is the minimal degree of the generators of $I(Y)$, gives

$$t \ge \begin{cases} \dfrac{4(n-2)}{n-1} i(Y) & \text{if } n \text{ is odd,} \\ \dfrac{4(n-1)}{n} i(Y) & \text{if } n \text{ is even.} \end{cases}$$

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