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ASYMPTOTIC BEHAVIOR FOR SINGULARITIES OF THE MEAN CURVATURE FLOW

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Let M^n, $n \geq 1$, be a compact n-dimensional manifold without boundary and assume that $\mathbf{F}_0 \colon M^n \to \mathbb{R}^{n+1}$ smoothly immerses M^n as a hypersurface in a Euclidean $(n+1)$-space $\mathbb{R}^{n+1}$. We say that $M_0 = \mathbf{F}_0(M^n)$ is moved along its mean curvature vector if there is a whole family $\mathbf{F}(\bullet, t)$ of smooth immersions with corresponding hypersurfaces $M_t = \mathbf{F}(\bullet, t)(M^n)$ such that

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{F}(p,t) &= \mathbf{H}(p,t), \qquad p \in M^n, \\ \mathbf{F}(\cdot, 0) &= \mathbf{F}_0 \end{aligned} \tag{1}$$

is satisfied. Here $\mathbf{H}(p,t)$ is the mean curvature vector of the hypersurface M_t at $\mathbf{F}(p,t)$. We saw in [7] that (1) is a quasilinear parabolic system with a smooth solution at least on some short time interval. Moreover, it was shown that for convex initial data M_0 the surfaces M_t contract smoothly to a single point in finite time and become spherical at the end of the contraction.

Here we want to study the singularities of (1) which can occur for non-convex initial data. Our aim is to characterize the asymptotic behavior of M_t near a singularity using rescaling techniques. These methods have been used in the theory of minimal surfaces and more recently in the study of semilinear heat equations [3], [4]. An important tool of this approach is a monotonicity formula, which we establish in §3. Assuming then a natural upper bound for the growth rate of the curvature we show that after appropriate rescaling near the singularity the surfaces M_t approach a selfsimilar solution of (1). In §4 we consider surfaces M_t, $n \geq 2$, of positive mean curvature and show that in this case the only compact selfsimilar solutions of (1) are spheres. Finally, in §5 we study the model-problem of a rotationally symmetric shrinking neck. We prove that the natural growth rate estimate is valid in this case and that the rescaled solution asymptotically approaches a cylinder.

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1. Preliminaries

We will use the notation of [7]. In particular, $\boldsymbol{\nu}$ is the (outer) unit normal to M and H is the mean curvature, such that the mean curvature vector of M is given by $\mathbf{H} = -H\boldsymbol{\nu}$. We write $g = (g_{ij})$ and $A = (h_{ij})$ for the induced metric and the second fundamental form on M.

It was shown in [7, Theorem 8.1] that for smooth compact initial data (1) has a smooth solution on a finite maximal time interval, and it can only develop a singularity if the curvature blows up:

1.1 Theorem. *The evolution equation* (1) *has a smooth solution on a maximal time interval* $0 \leq t < T < \infty$, *and* $\max_{M_t} |A|^2$ *becomes unbounded as* $t \to T$.

In a first step we prove a lower bound for the blow-up rate of the curvature.

1.2 Lemma. *The function* $U(t) = \max_{M_t} |A|^2$ *is Lipschitz continuous and satisfies* $U(t) \geq 1/2(T-t)$.

Proof. From [7, Corollary 3.5] we have the evolution equation

$$\frac{d}{dt}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4.$$

It easily follows that U is Lipschitz continuous as long as $|A|^2$ is bounded and

$$\frac{d}{dt}U(t) \leq 2(U(t))^2,$$

or, equivalently,

$$\frac{d}{dt}(U^{-1}(t)) \geq -2.$$

Since T is the blow-up time, we have $U^{-1}(t) \to 0$ as $t \to T$ and obtain after integration from t to T the desired inequality

$$U(t) \geq \frac{1}{2(T-t)}.$$

In the following we will *assume* that the blow-up rate of the curvature also satisfies an upper bound of the form

$$U(t) = \max_{M_t} |A|^2 \leq \frac{C_0}{2(T-t)}. \tag{2}$$

This is the blow-up rate of convex surfaces and cylinders, and we conjecture that it is valid for singularities of arbitrary *embedded* surfaces, at least in low dimensions. Estimate (2) cannot be expected to hold for all immersed surfaces: The curvature of an immersed curve developing a cusp will blow up at a faster rate. In §5 we prove that (2) is valid for the model problem of a rotationally symmetric shrinking neck.

2. Rescaling the singularity

We want to rescale M_t near a singular point as $t \to T$, such that the curvature of the rescaled surfaces remains uniformly bounded.

First notice that in view of (2) we have

$$|\mathbf{F}(p,t) - \mathbf{F}(p,s)| \leq \int_s^t |H(p,\tau)|\, d\tau \leq C[(T-s)^{1/2} - (T-t)^{1/2}]$$

for all $p \in M^n$ and $0 \leq s < t < T$. Thus $\mathbf{F}(\cdot, t)$ converges uniformly as $t \to T$, and we are led to the following definition.

2.1 Definition. We say that $\mathbf{x} \in \mathbb{R}^{n+1}$ is a blow-up point if there is $p \in M^n$ such that $\mathbf{F}(p,t) \to \mathbf{x}$ as $t \to T$ and $|A|(p,t)$ becomes unbounded as $t \to T$.

Let us assume now that $\mathbf{O} \in \mathbb{R}^{n+1}$ is a blow-up point. Then we define the rescaled immersions $\tilde{\mathbf{F}}(p,s)$ by

$$\tilde{\mathbf{F}}(p,s) = (2(T-t))^{-1/2}\mathbf{F}(p,t), \qquad s(t) = -\frac{1}{2}\log(T-t). \tag{3}$$

The surfaces $\tilde{M}_s = \tilde{\mathbf{F}}(\cdot, s)(M^n)$ are therefore defined for $-\frac{1}{2}\log T \leq s < \infty$ and satisfy the equation

$$\frac{d}{ds}\tilde{\mathbf{F}}(p,s) = \tilde{\mathbf{H}}(p,s) + \tilde{\mathbf{F}}(p,s), \tag{4}$$

where $\tilde{\mathbf{H}}$ is the mean curvature vector of $\tilde{M}_s$.

Similarly we derive as in [7, Lemma 9.1] evolution equations for all rescaled curvature quantities: Let P and Q be expressions formed from g and A, and let $\tilde{P}$ and $\tilde{Q}$ be the corresponding rescaled quantities. We say that P has degree α if $\tilde{P} = (2(T-t))^{-\alpha/2}P$. We have

2.2 Lemma. *Suppose P satisfies $dP/dt = \Delta P + Q$ for the original evolution equation* (1) *and P has degree α. Then Q has degree* $(\alpha - 2)$ *and* $d\tilde{P}/ds = \tilde{\Delta}\tilde{P} + \tilde{Q} + \alpha\tilde{P}$.

In view of assumption (2) we know that the curvature on $\tilde{M}_s$ is uniformly bounded: $|\tilde{A}|^2 \leq C_0$. We show now that all higher derivatives of the curvature on $\tilde{M}_s$ are bounded as well.

2.3 Proposition. *For each $m \geq 0$ there is $C(m) < \infty$ such that $|\tilde{\nabla}^m \tilde{A}|^2 \leq C(m)$ holds on $\tilde{M}_s$ uniformly in s, where $C(m)$ depends on n, m, C_0 and M_0.*

Proof. We have degree($|\nabla^m A|^2$) $= -2(m+1)$; then [7, Theorem 7.1] and Lemma 2.2 imply that

$$\begin{aligned}\frac{d}{ds}|\tilde\nabla^m \tilde A|^2 &\le \tilde\Delta|\tilde\nabla^m \tilde A|^2 - 2|\tilde\nabla^{m+1}\tilde A|^2 \\ &\quad + C(n,m)\sum_{i+j+k=m}|\tilde\nabla^i \tilde A|\,|\tilde\nabla^j \tilde A|\,|\tilde\nabla^k \tilde A|\,|\tilde\nabla^m \tilde A|.\end{aligned} \tag{5}$$

We know that the estimate is valid for $m = 0$, and we proceed by induction on m. Suppose the estimate is valid up to order $m-1$. Then there is a constant B, depending on m, n, C_0 and M_0 such that

$$\frac{d}{ds}|\tilde\nabla^m \tilde A|^2 \le \tilde\Delta|\tilde\nabla^m \tilde A|^2 + B(1+|\tilde\nabla^m \tilde A|^2).$$

We now add enough of the evolution equation of $|\tilde\nabla^{m-1}\tilde A|^2$ to control the right-hand side. We have from (5)

$$\frac{d}{ds}(|\tilde\nabla^m \tilde A|^2 + B|\tilde\nabla^{m-1}\tilde A|^2) \le \tilde\Delta(|\tilde\nabla^m \tilde A|^2 + B|\tilde\nabla^{m-1}\tilde A|^2) - B|\tilde\nabla^m \tilde A|^2 + B_1,$$

where B_1 depends on B and $C(l)$, $0 \le l \le m-1$. Since $|\tilde\nabla^{m-1}\tilde A|^2$ is already bounded, this inequality clearly implies that $|\tilde\nabla^m \tilde A|^2$ can be estimated uniformly in s by a constant depending on its initial data and on B and B_1. This completes the proof of Proposition 2.3.

3. Monotonicity and selfsimilar solutions

In this section we prove a general monotonicity formula for surfaces moving along their mean curvature vector. We then use the monotonicity result to show that singularities satisfying the growth rate estimate (2) are asymptotically selfsimilar.

Let $\rho(\mathbf{x}, t)$ be the backward heat kernel at $(0, t_0)$, i.e.,

$$\rho(\mathbf{x}, t) = \frac{1}{(4\pi(t_0 - t))^{n/2}} \cdot \exp\left(-\frac{|\mathbf{x}|^2}{4(t_0 - t)}\right), \qquad t < t_0.$$

3.1 Theorem. *If M_t is a surface satisfying* (1) *for $t < t_0$, then we have the formula*

$$\frac{d}{dt}\int_{M_t}\rho(\mathbf{x}, t)\,d\mu_t = -\int_{M_t}\rho(\mathbf{x}, t)\left|\mathbf{H} + \frac{1}{2\tau}\mathbf{F}^\perp\right|^2 d\mu_t,$$

where $\mathbf{F}^\perp$ is the normal component of $\mathbf{F}$ and $\tau = (t_0 - t)$.

Proof. From [7, Corollary 3.6] we have $\frac{d}{dt}\mu_t = -H^2\mu_t$, so we derive from (1) that

$$\begin{aligned}\frac{d}{dt}\int_{M_t}\rho\,d\mu_t &= -\int_{M_t}\rho\left\{H^2 - \frac{n}{2\tau} + \frac{1}{2\tau}\langle\mathbf{F},\mathbf{H}\rangle + \frac{|\mathbf{F}|^2}{4\tau^2}\right\}d\mu \\ &= -\int_{M_t}\rho\left|\mathbf{H} + \frac{1}{2\tau}\mathbf{F}\right|^2 d\mu + \int_{M_t}\rho\frac{1}{2\tau}\langle\mathbf{F},\mathbf{H}\rangle\,d\mu + \int_{M_t}\frac{n}{2\tau}\rho\,d\mu.\end{aligned}$$

Now we use the first variation formula

$$\int_M \operatorname{div}_M \mathbf{Y}\,d\mu = -\int_M \langle\mathbf{H},\mathbf{Y}\rangle\,d\mu$$

with $\mathbf{Y} = \frac{1}{2}\rho\mathbf{F}/\tau$ to obtain

$$\int_{M_t}\rho\frac{1}{2\tau}\langle\mathbf{F},\mathbf{H}\rangle\,d\mu = \int_{M_t}\rho\left\{-\frac{n}{2\tau} + \frac{|\mathbf{F}^T|^2}{4\tau^2}\right\}d\mu.$$

So we conclude finally

$$\begin{aligned}\frac{d}{dt}\int_{M_t}\rho\,d\mu &= -\int_{M_t}\rho\left|\mathbf{H} + \frac{1}{2\tau}\mathbf{F}\right|^2 d\mu_t + \int_{M_t}\rho\frac{1}{4\tau^2}|\mathbf{F}^T|^2\,d\mu_t \\ &= -\int_{M_t}\rho\left|\mathbf{H} + \frac{1}{2\tau}\mathbf{F}^\perp\right|^2 d\mu_t.\end{aligned}$$

Remarks. (i) A corresponding monotonicity formula holds if ρ is centered at an arbitrary point $\mathbf{x}_0 \in \mathbb{R}^{n+1}$. (ii) In a similar fashion weighted energy estimates were used by Y. Giga and R. Kohn in their study of semilinear heat equations [3], [4]. Moreover, having finished this article, the author recently learned that M. Struwe used the backward heat kernel to study the heatflow for harmonic maps [9].

In the rescaled setting of §2 we obtain a monotonicity formula if we define a weighting function $\tilde{\rho}$ by

$$\tilde{\rho}(\mathbf{x}) = \exp\left(-\frac{1}{2}|\mathbf{x}|^2\right).$$

3.2 Corollary. *If the surfaces $\tilde{M}_s$ satisfy the rescaled evolution equation* (3), *then we have*

$$\frac{d}{ds}\int_{\tilde{M}_t}\tilde{\rho}\,d\tilde{\mu}_s = -\int_{\tilde{M}_s}\tilde{\rho}|\tilde{\mathbf{H}} + \tilde{\mathbf{F}}^\perp|^2\,d\tilde{\mu}_s.$$

Proof. The identity follows from $d\tilde{\mu}_s/ds = (n-H^2)\tilde{\mu}_s$ and a calculation analogous to the proof of Theorem 3.1.

We want to use Corollary 3.2 to study the behavior of $\tilde{M}_s$ as $s \to \infty$. As a first step notice that $\tilde{M}_s$ cannot disappear at infinity.

3.3 Lemma. *Let* $\mathbf{O} \in \mathbb{R}^{n+1}$ *be a blow-up point of* M, *and assume that the growth rate assumption* (2) *is valid. Then there is* $p \in M^n$ *such that* $\tilde{\mathbf{F}}(p,s)$ *as defined in* (3) *remains bounded for* $s \to \infty$.

Proof. Since $\mathbf{O} \in \mathbb{R}^{n+1}$ is a blow-up point, there is $p \in M^n$ such that $\mathbf{F}(p,t) \to \mathbf{O}$ as $t \to T$. Thus we have in view of (2)

$$|\mathbf{F}(p,t)| \leq \int_t^T |H(p,\tau)|\,d\tau \leq \int_t^T \frac{n^{1/2}\cdot C_0}{(2(T-\tau))^{1/2}}\,d\tau \leq C(2(T-t))^{1/2},$$

yielding the desired estimate.

We can now use a compactness argument to obtain a nontrivial limiting surface.

3.4 Proposition. *Suppose that assumption* (2) *is satisfied. Then for each sequence* $s_j \to \infty$ *there is a subsequence* s_{jk} *such that* $\tilde{M}_{s_{jk}}$ *converges smoothly to an immersed nonempty limiting surface* $\tilde{M}_\infty$.

Proof. From Corollary 3.2 we see that for each $R > 0$ there is a uniform bound on $\mathscr{H}^n(\tilde{M}_n \cap B_R(0))$. For $q \in M^n$ let $U^s_{r,q}$ be the q-component of $\tilde{\mathbf{F}}^{-1}(\cdot,s)(B_r(\tilde{\mathbf{F}}(q,s)))$. Since $|\tilde{A}|^2 \leq C_0$ uniformly in s, there is a number $r_0 > 0$ such that for any $q \in M^n$ it is true that $\tilde{\mathbf{F}}(\cdot,s)(U_{r_0},q)$ can be written as the graph of a C^∞-function f over the tangent plane to $\tilde{M}_s$ in $B_{r_0}(\tilde{\mathbf{F}}(q,s))$. Moreover, in view of Proposition 2.3 there is a constant C_α bounding all derivatives of f up to order α. Both r_0 and C_α are independent of s. We may then follow the method in [8] to see that a subsequence of the $\tilde{M}_{s_j} \cap B_R(0)$ converges smoothly to an immersed limiting hypersurface in $B_R(0)$. In view of Lemma 3.3 the limit will be nonempty if R is chosen large enough. We can do this for every R, and after picking a diagonal sequence we obtain a smooth limit $\tilde{M}_\infty \subset \mathbb{R}^{n+1}$. Note that a subsequence of the $\tilde{\mathbf{F}}(\cdot,s)$ does not necessarily converge to a limiting immersion; it will be necessary to "reparametrize" $\tilde{\mathbf{F}}(\cdot,s)$ (see [8] for the details).

The monotonicity formula then yields a characterization of the limiting hypersurface.

3.5 Theorem. *Each limiting hypersurface* $\tilde{M}_\infty$ *as obtained in Proposition* 3.4 *satisfies the equation*

$$H = \langle \mathbf{x}, \boldsymbol{\nu} \rangle, \tag{6}$$

where $\mathbf{x}$ *is the position vector,* H *is the mean curvature and* $\boldsymbol{\nu}$ *is the unit normal such that the mean curvature vector is given by* $\mathbf{H} = -H\boldsymbol{\nu}$.

Proof. From the monotonicity formula we have

$$\int_{s_0}^\infty \int_{\tilde{M}_s} \tilde{\rho}|\tilde{\mathbf{H}} + \tilde{\mathbf{F}}^\perp|^2\,d\tilde{\mu}_s\,ds < \infty,$$

and the theorem follows immediately in view of the uniform estimates in Proposition 2.3.

Suppose the initial hypersurface M_0 satisfies the equation $H_0 = \langle \mathbf{F}_0, \boldsymbol{\nu} \rangle$. Then the homothetic deformation

$$\mathbf{F}(p,t) = (2(T-t))^{1/2}\mathbf{F}(p,0)$$

satisfies

$$\left(\frac{d}{dt}\mathbf{F}(p,t)\right)^{\perp} = \frac{1}{(2(T-t))^{1/2}} H_0 \cdot \boldsymbol{\nu} = -H\boldsymbol{\nu}.$$

So up to a tangential deformation the mean curvature flow is realized by homotheties for these initial surfaces.

Then Theorem 3.5 states that singularities of the mean curvature flow satisfying the growth rate estimate (2) are asymptotically selfsimilar.

Natural open questions are concerned with the uniqueness of the limit in Proposition 3.4 and the number of solutions to equation (6). In the last two sections we address these questions in some special cases.

4. Surfaces of positive mean curvature

If the mean curvature H is positive on the initial hypersurface M_0, it will stay positive on M_t as long as a solution of (1) exists (see [7, Corollary 3.5(i)]). Thus it is natural to try to classify solutions of the selfsimilarity condition (6) in this special case. In case $n = 1$ it was shown by Abresch and Langer in [1] that there is a 2-parameter family of closed immersed curves in $\mathbb{R}^2$ of positive geodesic curvature which are selfsimilar solutions of (1). We prove that in higher dimensions the sphere is the only compact hypersurface of positive mean curvature moving under selfsimilarities.

4.1 Theorem. *If M^n, $n \geq 2$, is compact with nonnegative mean curvature H and satisfies the equation $H = \langle \mathbf{x}, \boldsymbol{\nu} \rangle$, then M^n is a sphere of radius $\sqrt{n}$.*

Proof. We differentiate the equation $H = \langle \mathbf{x}, \boldsymbol{\nu} \rangle$ in an orthonormal frame $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ on M^n and obtain

$$\nabla_i H = \langle \mathbf{x}, \mathbf{e}_l \rangle h_{li}, \tag{7}$$

$$\nabla_i \nabla_j H = h_{ij} - H h_{il} h_{lj} + \langle \mathbf{x}, \mathbf{e}_l \rangle \nabla_l h_{ij}. \tag{8}$$

Here we used again $H = \langle \mathbf{x}, \boldsymbol{\nu} \rangle$ and the Codazzi equation. Contracting now (8) with g_{ij} and h_{ij} respectively we derive

$$\Delta H = H - H|A|^2 + \langle \mathbf{x}, \mathbf{e}_l \rangle \nabla_l H, \tag{9}$$

$$h_{ij} \nabla_i \nabla_j H = |A|^2 - H \operatorname{tr}(A^3) + \langle \mathbf{x}, \mathbf{e}_l \rangle \nabla_l h_{ij} \cdot h_{ij}. \tag{10}$$

Simons' identity states that (see e.g. [7, Lemma 2.1])

$$\Delta|A|^2 = 2h_{ij}\nabla_i\nabla_j H + 2|\nabla A|^2 + 2H\,\mathrm{tr}(A^3) - 2|A|^4. \tag{11}$$

So we derive from (10) that

$$\Delta|A|^2 = 2|\nabla A|^2 + 2|A|^2 - 2|A|^4 + 2\langle \mathbf{x}, \mathbf{e}_l\rangle \nabla_l h_{ij} h_{ij}.$$

Now notice that in view of (9) and the maximum principle, H satisfies the strict inequality $H > 0$. We are then ready to compute

$$\Delta\left(\frac{|A|^2}{H^2}\right) = \frac{\Delta|A|^2}{H^2} - \frac{2|A|^2}{H^3}\Delta H - \frac{4}{H^3}\nabla_i|A|^2\nabla_i H + \frac{6|A|^2}{H^4}|\nabla H|^2.$$

From (9) and (11) we obtain

$$\Delta\left(\frac{|A|^2}{H^2}\right) = \frac{2}{H^4}\Big(H^2|\nabla A|^2 + \frac{1}{2}\langle \mathbf{x}, \mathbf{e}_l\rangle\nabla_l|A|^2 H^2 - H|A|^2\langle \mathbf{x}, \mathbf{e}_l\rangle\nabla_l H - 2H\nabla_i|A|^2\nabla_i H + 3|A|^2|\nabla H|^2\Big).$$

The right-hand side thus equals

$$\frac{2}{H^4}|h_{ij}\nabla_l H - \nabla_l h_{ij} H|^2 + \frac{2}{H^4}\Big\{2|A|^2|\nabla H|^2 - H\nabla_i|A|^2\nabla_i H + \frac{1}{2}H^2\langle \mathbf{x}, \mathbf{e}_l\rangle\nabla_l|A|^2 - |A|^2\langle \mathbf{x}, \mathbf{e}_l\rangle H\nabla_l H\Big\}$$

since

$$|h_{ij}\nabla_l H - \nabla_l h_{ij} H|^2 = |A|^2|\nabla H|^2 + |\nabla A|^2 H^2 - H\nabla_l H\nabla_l|A|^2.$$

Now notice that

$$\nabla_i\left(\frac{|A|^2}{H^2}\right) = \frac{1}{H^2}\nabla_i|A|^2 - \frac{2|A|^2}{H^3}\nabla_i H,$$

such that finally

$$\Delta\left(\frac{|A|^2}{H^2}\right) = \frac{2}{H^4}|h_{ij}\nabla_l H - \nabla_l h_{ij} H|^2 - \frac{2}{H}\nabla_i H\nabla_i\left(\frac{|A|^2}{H^2}\right) + \langle \mathbf{x}, \mathbf{e}_i\rangle\nabla_i\left(\frac{|A|^2}{H^2}\right). \tag{12}$$

Since M is compact, the maximum principle then implies that $|A|^2 = \alpha H^2$ with a fixed constant α and also

$$|h_{ij}\nabla_l H - \nabla_l h_{ij} H|^2 \equiv 0 \quad \text{on } M^n. \tag{13}$$

We split the tensor $h_{ij}\nabla_l H - \nabla_l h_{ij} H$ into its symmetric and antisymmetric parts and obtain from (13) and Codazzi's equation

$$|h_{ij}\nabla_l H - h_{il}\nabla_j H|^2 \equiv 0. \tag{14}$$

At a given point of M^n we now rotate $\mathbf{e}_1, \cdots, \mathbf{e}_n$ such that $\mathbf{e}_1 = \nabla H/|\nabla H|$ points in the direction of the gradient of the mean curvature; then

$$0 = |h_{ij}\nabla_l H - h_{il}\nabla_j H|^2 = 2|\nabla H|^2\left(|A|^2 - \sum_{i=1}^n h_{1i}^2\right).$$

Thus at each point of M^n we have either $|\nabla H|^2 = 0$ or $|A|^2 = \sum_{i=1}^n h_{1i}^2$. If $|\nabla H|^2 \equiv 0$ it follows immediately that M^n is a sphere and we are done.

So suppose there is a point in M where $|A|^2 = \sum_{i=1}^n h_{1i}^2$. Since

$$|A|^2 = h_{11}^2 + 2\sum_{i=1}^n h_{1i}^2 + \sum_{i,j\neq 1}^n h_{ij}^2,$$

this is only possible if $h_{ij} = 0$ unless $i = j = 1$. Then we have $|A|^2 = H^2$ at this point and therefore everywhere on M. Now we integrate (9) and obtain after integration by parts

$$\begin{aligned}\int_M H^3\, d\mu &= \int_M H\, d\mu + \int_M \langle \mathbf{x}, \mathbf{e}_l\rangle \nabla_l H\, d\mu \\ &= \int_M H\, d\mu - n\int_M H\, d\mu + \int_M \langle \mathbf{x}, \boldsymbol{\nu}\rangle H^2\, d\mu.\end{aligned}$$

Since $\langle \mathbf{x}, \boldsymbol{\nu}\rangle = H$, we derive $(n-1)\int_M H\, d\mu = 0$, which is a contradiction for $n \geq 2$. This completes the proof of Theorem 4.1.

Remarks. (i) The assumption $H \geq 0$ seems to be necessary: The author was told by M. Grayson that there is numerical evidence for the existence of an imbedded torus in $\mathbb{R}^3$ satisfying (6). (ii) In the noncompact case we expect for $n = 2$ cylinders to be the only imbedded surfaces satisfying (6) (see also §5).

5. The rotationally symmetric shrinking neck

In this section we consider a two-dimensional rotationally symmetric hypersurface M_0 with positive mean curvature. We prove that in this case all singularities satisfy the natural blow-up estimate (2) and behave asymptotically like cylinders. The rotationally symmetric case was first studied by R. Hamilton (oral communication), who observed Lemmas 5.1 and 5.2 of this section.

Let $y_0\colon [a, b] \to \mathbb{R}$ be a smooth positive function on the bounded interval $[a, b]$ with $y_0'(a) = y_0'(b) = 0$, and consider the 2-dimensional hypersurface M_0 in $\mathbb{R}^3$ generated by rotating graph y around the x_1-axis. To compute the evolution equations on M_t in this situation, let $\iota_1, \iota_2, \iota_3$ be the standard basis in $\mathbb{R}^3$, and $\mathbf{e}_1, \mathbf{e}_2$ a local orthonormal frame on M such that

$$\langle \mathbf{e}_2, \iota_1 \rangle = 0, \qquad \langle \mathbf{e}_1, \iota_1 \rangle > 0.$$

Again let $\boldsymbol{\nu}$ be the outward unit normal and introduce the notation

$$p = \langle \mathbf{e}_1, \iota_1 \rangle y^{-1}, \qquad q = \langle \boldsymbol{\nu}, \iota_1 \rangle y^{-1},$$

such that in particular

$$(15) \qquad p^2 + q^2 = y^{-2}, \qquad \nabla_1 y = -qy.$$

The second fundamental form has one eigenvalue equal to p and one eigenvalue equal to

$$k = \langle \nabla_{\mathbf{e}_1} \boldsymbol{\nu}, \mathbf{e}_1 \rangle = -y''(1 + y'^2)^{-3/2}.$$

Now we evolve M_0 along its mean curvature vector subject to Neumann-boundary conditions at $x_1 = a$ and $x_1 = b$. (Equivalently we could consider the evolution of a periodic surface defined along the whole x_1-axis.) Then the position vector $\mathbf{F} = \mathbf{x}$ of the hypersurface satisfies the evolution equation

$$(16) \qquad \frac{d}{dt}\mathbf{x} = \Delta \mathbf{x} = \mathbf{H} = -H\boldsymbol{\nu},$$

and the function $y = (|\mathbf{x}|^2 - |\langle \mathbf{x}, \iota_1 \rangle|^2)^{1/2}$ agrees with y_0 at time $t = 0$. M_t stays rotationally symmetric and we get the following evolution equations.

5.1 Lemma. *As long as $y > 0$, we have*

(i) $\frac{d}{dt}\langle \mathbf{x}, \iota_1 \rangle = \Delta \langle \mathbf{x}, \iota_1 \rangle$,
(ii) $\frac{d}{dt} y = \Delta y - y^{-1}$,
(iii) $\frac{d}{dt} q = \Delta q + |A|^2 q + q(p^2 - q^2 - 2kp)$,
(iv) $\frac{d}{dt} p = \Delta p + |A|^2 p + 2q^2(k - p)$,
(v) $\frac{d}{dt} k = \Delta k + |A|^2 k - 2q^2(k - p)$,
(vi) $\frac{d}{dt} H = \Delta H + H|A|^2$.

Proof. The first identity is immediate from (16), to obtain (ii) we compute

$$\frac{d}{dt} y = y^{-1}\{\langle \mathbf{x}, \mathbf{H} \rangle - \langle \mathbf{x}, \iota_1 \rangle \langle \mathbf{H}, \iota_1 \rangle\},$$
$$\Delta y = y^{-1}\{2 + \langle \mathbf{x}, \mathbf{H} \rangle - |\langle \mathbf{e}_1, \iota_1 \rangle|^2 - \langle \mathbf{x}, \iota_1 \rangle \langle \mathbf{H}, \iota_1 \rangle - |\nabla y|^2\},$$

and the conclusion follows from (15), since

$$|\langle \mathbf{e}_1, \iota_1 \rangle|^2 = y^2 p^2 = 1 - y^2 q^2 = 1 - |\nabla y|^2.$$

To derive (iii) we infer from [7, Lemma 3.3] that

$$\frac{d}{dt}\langle \boldsymbol{\nu}, \boldsymbol{\iota}_1\rangle = \langle \nabla H, \boldsymbol{\iota}_1\rangle = \Delta\langle \boldsymbol{\nu}, \boldsymbol{\iota}_1\rangle + |A|^2\langle \boldsymbol{\nu}, \boldsymbol{\iota}_1\rangle,$$

the last identity being a consequence of Codazzi's equation. Combining now this equation with (ii) we get

$$\frac{d}{dt}q = \Delta q + 2y^{-2}\nabla_i y \nabla_i \langle \boldsymbol{\nu}, \boldsymbol{\iota}_1\rangle - 2y^{-3}\langle \boldsymbol{\nu}, \boldsymbol{\iota}_1\rangle |\nabla y|^2 + y^{-3}\langle \boldsymbol{\nu}, \boldsymbol{\iota}_1\rangle + |A|^2 q,$$

and therefore the desired identity in consequence of (15) and

$$\nabla_i y = -\delta_{i1} q y, \qquad \nabla_1 \langle \boldsymbol{\nu}, \boldsymbol{\iota}_1\rangle = kpy.$$

From (ii) and (iii) we can now compute

$$\begin{aligned}\frac{d}{dt}p &= \frac{d}{dt}(y^2 - q^2)^{1/2} \\ &= \Delta p + p^{-1}|\nabla p|^2 + p^{-1}|\nabla q|^2 - 3p^{-1}y^{-4}|\nabla y|^2 + p^{-1}y^{-4} \\ &\quad - qp^{-1}(|A|^2 q + q(p^2 - q^2 - 2kp)).\end{aligned}$$

Equation (iv) then follows if we take the relations

$$\begin{aligned}\nabla_i p &= \delta_{i1} q(p-k), \qquad \nabla_i q = \delta_{i1}(q^2 + kp), \\ |A|^2 &= k^2 + p^2, \qquad y^{-4} = p^4 + 2p^2q^2 + q^4\end{aligned}$$

and (15) into account. The evolution equation for H was derived in [7, Corollary 3.5], and (v) is then a consequence of (vi), (iv) and the fact that $H = k + p$.

From Lemma 5.1(vi) we obtain that the mean curvature stays strictly positive as long as the solution of (16) exists. We now show that $y' = -q/p$ stays uniformly bounded.

5.2 Lemma. *There is a constant C_1 depending only on the initial hypersurface, such that $|q/p| \le C_1$, independent of time.*

Proof. As a first step observe that

$$|A|^2 = k^2 + p^2 \le C_2^2 H^2 \tag{17}$$

with a uniform constant C_2. Indeed, from [7, Lemma 5.2] we have the evolution equation

$$\frac{d}{dt}\frac{|A|^2}{H^2} \le \Delta\left(\frac{|A|^2}{H^2}\right) + \frac{2}{H}\nabla_i H \nabla_i \left(\frac{|A|^2}{H^2}\right),$$

and $|A|^2/H^2$ is bounded by the maximum of its initial values. Furthermore, we calculate from Lemma 5.1

$$\frac{d}{dt}\left(\frac{q}{H}\right) = \Delta\left(\frac{q}{H}\right) + \frac{2}{H}\nabla_i H \nabla_i \left(\frac{q}{H}\right) + \frac{q}{H}(p^2 - q^2 - 2kp).$$

In view of (17) the last term on the right-hand side is negative if $q/H \geq 2C_2$, and positive if $q/H \leq -2C_2$. Thus

$$(18) \qquad |q| \leq C_3 H$$

with a constant C_3 depending only on C_2 and the maximum of $|q|/H$ at time $t = 0$. Finally we consider the evolution equation

$$\frac{d}{dt}\left(\frac{k}{p}\right) = \Delta\left(\frac{k}{p}\right) + \frac{2}{p}\nabla_i p \nabla_i \left(\frac{k}{p}\right) + 2\frac{q^2}{p^2}(p^2 - k^2),$$

which follows from Lemma 5.1. It implies that

$$(19) \qquad k/p \leq \max\left(1, \max_{M_0} k/p\right).$$

Combining (18) and (19) we get

$$(20) \qquad |q| \leq C_3 H = C_3(p + k) \leq C_4 p$$

as desired.

We are now ready to prove the crucial estimate for the blow-up rate.

5.3 Proposition. *If T is the blow-up time, the second fundamental form satisfies*

$$\max_{M_t} |A|^2 \leq C_5 \frac{1}{(T - t)}$$

for all $t < T$.

Proof. From $\frac{d}{dt}\mathbf{x} = \Delta\mathbf{x}$ we compute

$$\frac{d}{dt} y^{-1} = -y^{-3}\langle \mathbf{H}, \mathbf{x} - \langle \mathbf{x}, \iota_1\rangle \iota_1\rangle = Hpy^{-1}.$$

From Lemma 5.2 it follows that

$$(21) \qquad y^{-2} = p^2 + q^2 \leq (1 + C_1^2)p^2$$

and moreover that

$$(22) \qquad p^2 \leq |A|^2 \leq C_2^2 H^2.$$

Hence we have

$$\frac{d}{dt}(y^{-1}) \geq \varepsilon y^{-3}$$

with a fixed $\varepsilon > 0$ for all $t < T$, and derive for $U(t) = \max_{M_t} y^{-1}$ the inequality

$$\frac{d}{dt} U(t) \geq \varepsilon U^3(t) \Leftrightarrow \frac{d}{dt} U^{-2}(t) \leq -2\varepsilon.$$

Since $U^{-2}(t)$ tends to zero as $t \to T$, we obtain after integration from t to T the estimate

$$\max_{M_t} y^{-1} \leq \frac{\sqrt{\varepsilon^{-1}}}{\sqrt{2(T-t)}}.$$

This implies the result, since from (15) and (20) we have $H \leq cp \leq cy^{-1}$, and also $|A|^2 \leq C_2^2 H^2$.

Now assume again without loss of generality that the origin is a blow-up point. In view of Proposition 5.3 we may apply the results of §3 to the rescaled hypersurfaces $\tilde{M}_s$ defined in (3). Thus we obtain from Proposition 3.4 and Theorem 3.5 that for each sequence $s_j \to \infty$ a subsequence of the $\tilde{M}_{s_j}$ converges to a smooth limiting hypersurface $\tilde{M}_\infty$ satisfying the equation $H = \langle \mathbf{x}, \nu \rangle$. The surface $\tilde{M}_\infty$ is again rotationally symmetric and is defined on the whole x_1-axis. We now prove that $\tilde{M}_\infty$ is unique, namely a cylinder of radius 1.

5.4 Proposition. *Let M be a two-dimensional rotationally symmetric hypersurface defined by a graph along the whole x_1-axis. If M has nonnegative mean curvature and satisfies the equation $H = \langle \mathbf{x}, \nu \rangle$, then M is a cylinder of radius* 1.

Proof. We saw in §4 that the equation $H = \langle \mathbf{x}, \nu \rangle$ implies the relations

$$\Delta H = H - H|A|^2 + \langle \mathbf{x}, \nabla H \rangle,$$
$$\Delta\left(\frac{|A|^2}{H^2}\right) \geq \left\langle \mathbf{x}, \nabla\left(\frac{|A|^2}{H^2}\right)\right\rangle - \frac{2}{H}\nabla_i H \nabla_i\left(\frac{|A|^2}{H^2}\right).$$

From the equation for H and the strong maximum principle we see that the mean curvature must be strictly positive everywhere. Similarly, the second relation implies that the quotient $|A|^2/H^2$ cannot attain an interior maximum, unless it is constant. If $|A|^2/H^2$ is constant, it is easy to see that the constant has to be one and M is a cylinder of radius 1.

Now suppose M is not a cylinder and consider two cases:

(i) $|A|^2 \leq H^2$ everywhere. Since $|A|^2 - H^2 = -2kp$, this is equivalent to $k \geq 0$. But k is the curvature of graph y with respect to the upper normal, so $y > 0$ is a concave function on all of $\mathbb{R}$. Clearly this is impossible unless y is a constant and then $|A|^2 \equiv H^2$.

(ii) There is at least one point $x_0 \in \mathbb{R}$ where $|A|^2 > H^2$. (We consider $|A|^2$, H^2, etc. as functions on $\mathbb{R}$ here.) Without loss of generality we may assume that $|A|^2/H^2$ is nondecreasing at x_0, otherwise change x_1 to $-x_1$. Since $|A|^2/H^2$ has no interior maxima, $|A|^2/H^2$ remains monotonically increasing for all $x \geq x_0$, and in particular $|A|^2 \geq (1+\varepsilon)H^2$ for all $x \geq x_0$

with a fixed $\varepsilon > 0$. This is equivalent to

$$-k \geq \frac{\varepsilon}{2} p^{-1} H^2 \quad \forall x \geq x_0,$$

so graph y is convex for $x \geq x_0$. Then $y' = -q/p$ is monotonically increasing for $x \geq x_0$ and we claim that y' becomes arbitrarily large. Indeed, if $-q/p$ was bounded by some constant C_6, we had from $y^{-2} = p^2 + q^2$ the inequalities

$$(1 + C_6^2)^{-1/2} y^{-1} \leq p \leq y^{-1},$$
$$H^2 \geq (1+\varepsilon)^{-1}|A|^2 \geq (1+\varepsilon)^{-1} p^2 \geq C(\varepsilon, C_6) y^{-2},$$

and therefore

$$y'' \geq y'(1 + y'^2)^{-3/2} = -k \geq \tilde{\varepsilon} y^{-1}.$$

Moreover, y could at most grow linearly if y' was bounded. But then the right-hand side of the last inequality is not integrable, yielding a contradiction. Thus y' becomes arbitrarily large as x increases. (Note that this would already contradict Lemma 5.2 if we restrict our attention to surfaces arising from a blow-up.) The equation $H = \langle \mathbf{x}, \boldsymbol{\nu} \rangle$ is equivalent to

$$y'' = (1 + y'^2) y^{-1} - y(1 + y'^2) + x y'(1 + y'^2),$$

so since $y' \to \infty$, for sufficiently large x we have

$$y'' \geq (y')^3.$$

Then y' would become infinite for a finite x, yielding a contradiction and proving Proposition 5.4.

Summing up the results of this section, we have

5.5 Theorem. *If the initial hypersurface M_0^2 in $\mathbb{R}^3$ as above is rotationally symmetric and has positive mean curvature, then the solution of the mean curvature flow develops a singularity in finite time with blow-up rate equal to $(T - t)^{-1/2}$, and at any blow-up point the rescaled surfaces $\tilde{M}_s$ converge to a cylinder of radius* 1.

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