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THE ADIABATIC LIMIT, HODGE COHOMOLOGY AND LERAY'S SPECTRAL SEQUENCE FOR A FIBRATION

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0. Introduction

Consider a fibration of a compact manifold, M, with fibers modeled differentially on the compact manifold F:

$$\begin{array}{ccc} F & \longrightarrow & M \\ & & \downarrow{\scriptstyle \pi} \\ & & Y \end{array}$$

Here, and throughout, we work with $\mathscr{C}^\infty$ maps and spaces. Let h be a Riemannian metric on the base space Y. Suppose that $g_\infty \in \mathscr{C}^\infty(M; S^2M)$ is a $\mathscr{C}^\infty$ symmetric 2-cotensor on M which is positive definite on the fibers. In particular g_∞ induces a Riemannian structure on each fiber. Combining these two forms gives a 1-parameter family of metrics on M:

$$g_x = g_\infty + x^{-2}\pi^* h \qquad (x > 0). \tag{1}$$

Let Δ_x be the Laplace-Beltrami operator of this metric and let $\mathscr{H}_x$ be the null space of Δ_x, the space of g_x-harmonic forms on M. We are interested in the behavior of $\mathscr{H}_x$ as $x \downarrow 0$, the so-called 'adiabatic limit', as discussed recently—with somewhat different objectives—by Bismut [1], Bismut and Freed [3], Cheeger [5] and Bismut and Cheeger [2] (see also Witten [11]). We shall show that $\mathscr{H}_x$ has a basis which extends to be smooth for $x \in [0, \infty)$. This basis remains independent at $x = 0$, spanning $\mathscr{H}_0 = H^k_{\mathrm{HL}}(M)$ (which we call the Hodge-Leray cohomology), and this limiting space represents the cohomology of M. The determination of which forms on M lie in the limiting space $H^k_{\mathrm{HL}}(M)$ can be carried out by a Taylor series analysis. We show that this approach gives a Hodge theoretic version of Leray's spectral sequence for the cohomology of M. An attractive treatment of the spectral sequence can be found in [4].

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To discuss the Taylor series computation we first consider the decomposition of forms on M. The tensor g_∞ fixes a horizontal space, or normal space to the fiber, at each point of M:

$$T^h_p M = \{v \in T_p M; g_\infty(v, w) = 0 \ \forall w \in T_p F_y,\ F_y = \pi^{-1}(y),\ y = \pi(p)\}$$
$$\cong T_y Y.$$

Since $\pi^* h$ vanishes on the tangent space to each fiber, this splitting is orthogonal with respect to g_x

$$T^h_p M \perp T_p F_y \quad \text{with respect to } g_x \ \forall x > 0,\ y = \pi(p).$$

There is an associated splitting of the form bundles

$$\Lambda^k_p M = \bigoplus_{j=0}^{k} \Lambda^{k,j}_p M, \qquad \Lambda^{k,j}_p M = \Lambda^j_p F_y \otimes \Lambda^{k-j}_y Y. \tag{2}$$

Forms on M depending on the parameter x can be regarded as sections of the pull-back to

$$X = [0, \infty) \times M$$

of the bundle $\Lambda^* M$. We shall rescale this pull-back in a manner related to the splitting (2) and suggested by the form of the metric g_x in (1). Thus let

$$ {}^{\mathscr{V}}\Lambda^k_{(x,p)} M = \bigoplus_{j=0}^{k} {}^{\mathscr{V}}\Lambda^{k,j}_{(x,p)} M = \Lambda^j_p F_y \otimes x^{-(k-j)} \Lambda^{k-j}_y Y, \qquad (x, p) \in X. \tag{3}$$

This means that

$$u \in \mathscr{C}^\infty(X; {}^{\mathscr{V}}\Lambda^k M) \leftrightarrow u = \sum_{j=0}^{k} x^{-(k-j)} u_j \qquad \text{with } u_j \in \mathscr{C}^\infty(X; \Lambda^{k,j} M). \tag{4}$$

The invariant significance of this rescaling, especially down to $x = 0$, is discussed in §1. Analytically the important property of these bundles comes from the fact that the action of Δ_x on them is more readily decomposed than is the action on $\mathscr{C}^\infty(X; \Lambda^k M)$.

Observe that the norm induced on such forms by g_x does not become singular at $x = 0$. Because of this the Laplacian Δ_x of g_x maps $C^\infty(X; {}^{\mathscr{V}}\Lambda^k)$ into itself (see Lemma (26)). Now define a nested sequence of spaces of forms over $M_0 = \{0\} \times M = \partial X$:

$$E^k_N \subset \mathscr{C}^\infty(M_0; {}^{\mathscr{V}}\Lambda^k M),$$
$$E^k_N = \{u; \exists \tilde{u} \in \mathscr{C}^\infty(X; {}^{\mathscr{V}}\Lambda^k M) \text{ with } \tilde{u}_{|\partial X} = u$$
$$\text{and } \Delta_x \tilde{u} = x^{N+1} v,\ v \in \mathscr{C}^\infty(X; {}^{\mathscr{V}}\Lambda^k M)\}, \qquad N \in \mathbf{N}, \tag{5}$$
$${}^{\mathscr{V}}H^k_{\mathrm{HL}}(M) = E^k_\infty = \bigcap_{N=1}^{\infty} E^k_N.$$

The spaces E_N^k are Hodge theoretic analogues of the successive terms in the Leray spectral sequence for the cohomology of M in terms of the cohomology of the base and the fibers. For this reason we call ${}^{\mathscr{V}}H^k_{\mathrm{KL}}(M)$ the ($\mathscr{V}$)-Hodge-Leray cohomology of M with respect to the metric family g_x in (1). Let $\mathscr{L}$ be the ring of Laurent series in the parameter x:

$$a \in \mathscr{L} \Leftrightarrow a = \sum_{j \geq p} a_j x^j \quad \text{for some } p \in \mathbf{Z}. \tag{6}$$

(7) Theorem. *If N is large enough, then ${}^{\mathscr{V}}H^k_{\mathrm{KL}}(M) = E_N^k = E_\infty^k$ and this space, when tensored with $\mathscr{L}$, is naturally isomorphic to the deRham cohomology*:

$$ {}^{\mathscr{V}}H^k_{\mathrm{KL}}(M) \otimes \mathscr{L} \leftrightarrow H^k_{\mathrm{dR}}(M) \otimes \mathscr{L} \quad (\textit{as } \mathscr{L} \textit{ modules.}) \tag{8}$$

In particular it follows from (8) *that*

$$\dim {}^{\mathscr{V}}H^k_{\mathrm{KL}}(M) = \dim H^k_{\mathrm{dR}}(M). \tag{9}$$

As in the Leray spectral sequence the terms E_1 and E_2 can be explicitly identified. Let $H^j_{\mathrm{Hd}}(F_y)$ be the Hodge cohomology of the fiber F_y, i.e., the space of harmonic j-forms with respect to the metric induced by g_∞ (and hence each g_x, $x > 0$). These spaces vary smoothly with y and so define vector bundles, $\mathscr{H}^j_{\mathrm{Hd}}$, over Y. The tensor product with the form bundles gives bundles which we denote by

$$ {}^{\mathscr{V}}L^k = \sum_{j=0}^{k} \mathscr{H}^j_{\mathrm{Hd}} \otimes x^{-(k-j)} \Lambda^{k-j} Y. \tag{10}$$

We shall show below that the sections of these bundles are just the spaces

$$E_1^k = \mathscr{C}^\infty(Y; {}^{\mathscr{V}}L^k). \tag{11}$$

That is, the space E_1^k consists of the fiber-harmonic ($\mathscr{V}$-)k-forms over Y.

Trivializations of the fibration π give identifications of neighboring fibers by diffeomorphisms which are near the identity and so induce the identity mapping on cohomology. The bundles $\mathscr{H}^*_{\mathrm{Hd}}$ therefore have natural locally flat connections, where a section is locally constant if it represents, locally, a fixed cohomology class on the fibers. Exterior differentiation therefore extends from forms on Y to a differential operator, d'_Y, on elements of E_1^k, i.e., sections of ${}^{\mathscr{V}}L^k$, so that $(d'_Y)^2 = 0$. (In general this connection is *not* orthogonal.) The L^2 inner product on fiber-harmonic forms and the metric inner product on the form bundle over Y together give E_1^k an inner product which allows the adjoint δ'_Y of d'_Y, and hence the associated Laplacian

$$\Delta'_Y = d'_Y \delta'_Y + \delta'_Y d'_Y \tag{12}$$

to be defined. Then

$$(13)\qquad \begin{aligned} E_2^k &= \{u \text{ is fiber-harmonic and a harmonic section of } {}^{\mathscr{V}}L^k\} \\ &= \{u \in \mathscr{C}^\infty(Y; {}^{\mathscr{V}}L^k); d'_Y u = \delta'_Y u = 0\}. \end{aligned}$$

The ellipticity of Δ'_Y means that E_2^k is finite dimensional. The determination of the remaining terms in the sequence

$$(14)\quad E_0^k = E_1^k \supset E_2^k \supset E_3^k \supset \cdots \supset E_N^k = E_{N+1}^k = \cdots = E_\infty^k = {}^{\mathscr{V}}H^k_{\mathrm{HL}}(M)$$

is more subtle.

The spaces E_∞^k are fixed by formal power series computations. The main analytic result of this paper is that this ($\mathscr{V}$-)Hodge-Leray cohomology is actually the limit as $x \downarrow 0$ of the Hodge cohomology with respect to the metrics g_x. As already noted the proof depends on the use of $\mathscr{V}$-forms.

(15) Theorem. *The space of smooth harmonic $\mathscr{V}$-forms is the space of $\mathscr{C}^\infty$ sections of a vector bundle over* $[0, \infty)$:

$$(16)\qquad \{u \in \mathscr{C}^\infty(X; {}^{\mathscr{V}}\Lambda^k); \Delta_x u = 0\} = \mathscr{C}^\infty([0,\infty); {}^{\mathscr{V}}\widetilde{\mathscr{H}}^k_{\mathrm{HL}})$$

where the fibers of the ($\mathscr{V}$-)Hodge-Leray bundle, ${}^{\mathscr{V}}\widetilde{\mathscr{H}}^k_{\mathrm{HL}}$, are

$$(17)\qquad \begin{aligned} ({}^{\mathscr{V}}\widetilde{\mathscr{H}}^k_{\mathrm{HL}})_x &\cong H^k_{\mathrm{Hd}}(M_x), \qquad x > 0, \\ ({}^{\mathscr{V}}\widetilde{\mathscr{H}})_0 &= {}^{\mathscr{V}}H^k_{\mathrm{HL}}(M), \end{aligned}$$

i.e. harmonic $\mathscr{V}$-forms in the sense of formal power series are realizable as the Taylor series of harmonic $\mathscr{V}$-forms.

From this it is easy to deduce the results mentioned above for ordinary forms.

(18) Corollary. *The space of smooth harmonic forms is the space of $\mathscr{C}^\infty$ sections of a vector bundle over* $[0, \infty)$:

$$(19)\qquad \{u \in \mathscr{C}^\infty(X; \Lambda^k); \Delta_x u = 0\} = \mathscr{C}^\infty([0,\infty); \widetilde{\mathscr{H}}^k_{\mathrm{HL}}),$$

where the fibers of the Hodge-Leray bundle $\widetilde{\mathscr{H}}^k_{\mathrm{HL}}$ are

$$(20)\qquad (\widetilde{\mathscr{H}}^k_{\mathrm{HL}})_x \cong H^k_{\mathrm{Hd}}(M_x), \qquad x > 0,$$

$$\begin{aligned} (\widetilde{\mathscr{H}}^k_{\mathrm{HL}})_0 = H^k_{\mathrm{HL}}(M) = \{\phi \in \mathscr{C}^\infty(M; \Lambda^k); \\ \exists \tilde{\phi} \in \mathscr{C}^\infty(X; \Lambda^k M) \text{ with } \Delta_x \tilde{\phi} = O(x^\infty)\}, \end{aligned}$$

i.e., harmonic forms in the sense of formal power series at $x = 0$ are realizable as the Taylor series of harmonic forms.

The main step in the proof of Theorem (7) is the proof of (13) which implies in particular that E_2^k is finite dimensional and hence that the sequence (14) stabilizes. The remainder of the proof uses Hodge theory in the sense of formal power series. This is carried out in §5. The corresponding formal part of Corollary (18) is proved in §6. The proof of Theorem (15) is mainly analytic, namely the construction of a parametrix for Δ_x which is uniform down to $x = 0$, to remove the rapidly vanishing error terms which remain after the application of Borel's lemma to the formal power series construction. To construct such a uniform parametrix we follow the general idea of the "microlocalization of boundary-fibration structures" which underlies [7], [8] and [9].

The results obtained here arose from joint work with Boyd Livingston, in the examination of the boundary behavior of harmonic forms for certain degenerate and singular metrics on manifolds with boundary [7]. The computations needed to analyze the L^2 cohomology there bring out the connection between such degenerate boundary problems and the adiabatic limit for a fibration. Apart from our collaborator, we are happy to acknowledge useful conversations concerning this note with Charlie Epstein, Dan Freed, Victor Guillemin and Bob MacPherson.

1. Boundary-fibration structure

We start by carrying out the rescaling of the lift of the form bundle on M to the product $X = [0,\infty)\times M$. This product has an induced fibration, with leaves F and base $[0,\infty)\times Y$. Consider the space $\mathscr{V}$ of $\mathscr{C}^\infty$ vector fields on X which are tangent to the fibers, M, of the product structure and which are as well tangent to the fibers of the fibration, π, above $M_0 = \{x = 0\}$. In local coordinates $y_1, \cdots, y_p, z_1, \cdots, z_s$ in M, where the y's give coordinates in Y, the elements of $\mathscr{V}$ are exactly the vector fields of the form

$$\sum_{j=1}^{p} a_j(x,y,z)x\partial_{y_j} + \sum_{l=1}^{s} b_l(x,y,z)\partial_{z_l}. \tag{21}$$

Since the coefficients in (21) are arbitrary, there is a vector bundle, which we denote ${}^{\mathscr{V}}T_XM = {}^{\mathscr{V}}TM$, of which $\mathscr{V}$ is the full set C^∞ sections:

$$\mathscr{V} = \mathscr{C}^\infty(X;\ {}^{\mathscr{V}}TM).$$

Notice that as a bundle this is isomorphic to the lift, T_XM, to X of the tangent bundle to M but not naturally so. The natural map is a bundle map

$$ {}^{\mathscr{V}}T_XM \to T_XM. \tag{22}$$

This is an isomorphism except over M_0 where it has range equal to the tangent bundle to the fibers. The dual bundle, ${}^{\mathscr{V}}T^*M$, to ${}^{\mathscr{V}}TM$ plays an important role below. The transpose of (22) gives a $\mathscr{C}^\infty$ map

$$\iota_{\mathscr{V}}\colon T^*_X M \to {}^{\mathscr{V}}T^*M, \tag{23}$$

where $T^*_X M$ is the dual to $T_X M$, the bundle of M-valued forms over X. The range of (23) over M_0 is a subbundle which is naturally isomorphic to the bundle of forms on the fibers, since the null space of (23) is just the conormal bundle to the fibers.

In fact the restriction of ${}^{\mathscr{V}}T^*M$ to the boundary, M_0, of X naturally splits

$${}^{\mathscr{V}}T^*_{M_0}M = T^*F \oplus x^{-1}T^*Y, \qquad u_0 = \iota_{\mathscr{V}}(\alpha) + x^{-1}\pi^*\beta.$$

This means that all the "compressed" form bundles, i.e., the exterior powers of ${}^{\mathscr{V}}T^*M$, also split at the boundary

$${}^{\mathscr{V}}\Lambda^k_p = \bigoplus_{j=0}^{k} {}^{\mathscr{V}}\Lambda^{k,j}, \qquad {}^{\mathscr{V}}\Lambda^{k,j} = \Lambda^{k-j}_p(F_y) \otimes x^{-j}\Lambda^j_y Y, \qquad y = \pi(p) \in Y. \tag{24}$$

The family of metrics g_x on M lifts to a nondegenerate inner product, G, on ${}^{\mathscr{V}}T_X M$, and dually on ${}^{\mathscr{V}}T^*_X M$. Moreover, the decomposition over the boundary (24) is orthogonal with respect to this metric. This splitting, together with power series expansion, replaces the more algebraic passage from the filtration of the form bundle to the associated gradation which occurs in the spectral sequence of the fibration (see [4]).

Starting as an operator on $\mathscr{C}^\infty$ sections of Λ^*M, the exterior differential operator, d, can be lifted to act on sections of $\Lambda^*_X M$ and hence, since the spaces of sections are the same away from M_0, to ${}^{\mathscr{V}}\Lambda^*M$, where we denote it d_M. The action there can be expressed in the following way. Let $\mathrm{Diff}^*_{\mathscr{V}}(X)$ be the ring of differential operators which can be written locally as sums of products of elements of $\mathscr{V}$. Since this is clearly a local $\mathscr{C}^\infty(X)$-module, the definition can be extended to sections of any vector bundle over X; we denote by $\mathrm{Diff}^q_{\mathscr{V}}(X; E, F)$ these "$\mathscr{V}$-differential operators" (of order at most q) from sections of one bundle, E, to another, F. Then,

$$d_M \in \mathrm{Diff}^1_{\mathscr{V}}(X;\ {}^{\mathscr{V}}\Lambda^k,\ {}^{\mathscr{V}}\Lambda^{k+1}) \quad \forall k.$$

This is quite obvious in local coordinates, where it follows from the formula

(25)
$$d_M\left[a(x,y,z)dz^\alpha \wedge \left(\frac{dy}{x}\right)^\beta\right] = \partial_z a \cdot dz \wedge dz^\alpha \wedge \left(\frac{dy}{x}\right)^\beta + (-1)^{|\alpha|}(x\partial_y a)dz^\alpha \wedge \frac{dy}{x} \wedge \left(\frac{dy}{x}\right)^\beta.$$

(26) Lemma. *Acting on the $\mathscr{V}$-form bundles the Laplacian is a $\mathscr{V}$-differential operator*

$$\Delta \in \mathrm{Diff}^2_{\mathscr{V}}(X;\ {}^{\mathscr{V}}\Lambda^k) \quad \forall k. \tag{27}$$

Proof. The metric G gives a positive-definite fiber inner product on the bundles ${}^{\mathscr{V}}\Lambda^k$. Moreover the volume form for g_x is of the form $x^{-p}\nu_x$ for a smooth nonvanishing volume form ν_x on M. Since the adjoint of a $\mathscr{V}$-differential operator with respect to such a volume form and inner product is again $\mathscr{V}$-differential, it follows that

$$\delta \in \mathrm{Diff}^1_{\mathscr{V}}(X;\ {}^{\mathscr{V}}\Lambda^k,\ {}^{\mathscr{V}}\Lambda^{k-1}) \quad \forall k \tag{28}$$

and hence that (27) holds for $\Delta = \delta d + d\delta$.

2. Formal deRham theorem

For any vector bundle, E, over a manifold with boundary, X, the space of Laurent series (at the boundary) with coefficients in E can be defined as follows:

$$\mathscr{L}(X;E) = \{u \in \mathscr{C}^\infty(\mathring{X},E); x^q u \in \mathscr{C}^\infty(X;E) \text{ for some } q \in \mathbf{Z}\}/\dot{\mathscr{C}}^\infty(X;E)$$

where $\dot{\mathscr{C}}^\infty(X;E)$ is the space of $\mathscr{C}^\infty$ sections vanishing to all orders at the boundary. If x is a defining function for the boundary then $\mathscr{L}(X;E)$ is a module over the ring of formal Laurent series

$$\mathscr{L} = \mathscr{L}([0,\infty);\mathbf{R})$$

in the variable x. From the definition of the bundle ${}^{\mathscr{V}}\Lambda^*M$ (see in particular (4)) it follows that for any element $u \in \mathscr{C}^\infty(\mathring{X},\ {}^{\mathscr{V}}\Lambda^*M)$ such that $x^p u \in \mathscr{C}^\infty(X;\ {}^{\mathscr{V}}\Lambda^*M)$, for some $p \in \mathbf{N}$, $x^q \in \mathscr{C}^\infty(X;\Lambda^*M)$, for some (generally larger) $q \in \mathbf{N}$ and conversely. Thus the space of Laurent series sections is the same as for Λ^*M:

$$\mathscr{L}(X;\ {}^{\mathscr{V}}\Lambda^*M) \cong \mathscr{L}(X;\Lambda^*M). \tag{29}$$

Any differential operator $D \in \mathrm{Diff}^*(X;E,F)$ lifts to an operator on formal power series $D\colon \mathscr{L}(X;E) \to \mathscr{L}(X;F)$. Thus we can define the various spaces of formally closed and exact forms

$$\begin{aligned}
\mathscr{C} &= \{u \in \mathscr{L}(X;\Lambda^*M); du = 0\},\\
\mathscr{E} &= \{u \in \mathscr{L}(X;\Lambda^*M); u = dv, v \in \mathscr{L}(X;\Lambda^*M)\},\\
{}^{\mathscr{V}}\mathscr{C} &= \{u \in \mathscr{L}(X;\ {}^{\mathscr{V}}\Lambda^*M); d_M u = 0\},\\
{}^{\mathscr{V}}\mathscr{E} &= \{u \in \mathscr{L}(X;\ {}^{\mathscr{V}}\Lambda^*M); u = d_M v, v \in \mathscr{L}(X;\ {}^{\mathscr{V}}\Lambda^*M)\}.
\end{aligned}$$

Since $[d, x] = 0$ these spaces are $\mathcal{L}$-modules. In fact the isomorphism (29) identifies $\mathcal{C}$ with ${}^{\mathcal{V}}\mathcal{C}$, and also $\mathcal{E}$ with ${}^{\mathcal{V}}\mathcal{E}$, since it identifies d with d_M. From this we conclude:

(30) Lemma. *The formal deRham cohomology is*

$$\mathcal{C}/\mathcal{E} \cong {}^{\mathcal{V}}\mathcal{C}/{}^{\mathcal{V}}\mathcal{E} \cong H^*_{\mathrm{dR}}(M) \otimes \mathcal{L} \cong H^*_{\mathrm{dR}}(M; \mathcal{L}). \tag{31}$$

3. E_1-term

The restriction to the boundary (always in the sense of ${}^{\mathcal{V}}\Lambda^*$) of $d_M a$ depends only on the restriction of a to the boundary, assuming a to be smooth. From (25) we conclude that the resulting operator, called the indicial operator of d_M,

$$\begin{aligned} &I(d_M): \mathcal{C}^\infty(M_0; {}^{\mathcal{V}}\Lambda^k) \to \mathcal{C}^\infty(M_0; {}^{\mathcal{V}}\Lambda^{k+1}), \\ &I(d_M)v = d_M u \restriction M_0 \quad \text{if } u \in \mathcal{C}^\infty(X; {}^{\mathcal{V}}\Lambda^k), u \restriction M_0 = v, \end{aligned} \tag{32}$$

is just

$$I(d_M) = d_F, \tag{33}$$

where d_F is d acting on the fibers with respect to the decomposition (24). From the definition by duality the same is true of the adjoint

$$I(\delta_M) = \delta_F, \quad \text{i.e. } \delta_M \tilde{u}_{|x=0} = \delta_F u \text{ if } u = \tilde{u}_{|x=0}. \tag{34}$$

Hence

$$I(\Delta) = \Delta_F, \qquad \Delta\tilde{u}_{|x=0} = \Delta_F u \text{ if } u = \tilde{u}_{|x=0} \in \mathcal{C}^\infty(M_0; {}^{\mathcal{V}}\Lambda^k). \tag{35}$$

Now, for a fixed partial metric g_∞ consider the space of sections of ${}^{\mathcal{V}}\Lambda^*$ which are fiber-harmonic at the boundary:

$$\{\tilde{u} \in \mathcal{C}^\infty(X; {}^{\mathcal{V}}\Lambda^k); \Delta_F(\tilde{u} \restriction M_0) = 0\}. \tag{36}$$

The boundary values of these sections therefore give the bundle

$${}^{\mathcal{V}}L^k = \oplus_{r=0}^k \mathcal{H}^r_{\mathrm{Hd}}(F) \otimes x^{r-k}\Lambda^{k-r}Y \quad \text{over } Y, \qquad E_0^k = \mathcal{C}^\infty(Y, {}^{\mathcal{V}}L^k).$$

(37) Lemma. *In terms of definition* (5) $E_0^k = E_1^k$ *for all* k *so*

$$E_1^k = \mathcal{C}^\infty(Y, {}^{\mathcal{V}}L^k) \quad \forall k. \tag{38}$$

Proof. Since $E_1^k \subset E_0^k$ we only need to show the reverse. Suppose that $u \in E_0^k$, i.e., there is an extension $\tilde{u} \in \mathcal{C}^\infty(X; {}^{\mathcal{V}}\Lambda^k)$ of u such that $\Delta\tilde{u}_{|x=0} = 0$. As already noted this means that $\Delta_F u = 0$ and hence that $d_M\tilde{u} = x\tilde{v}$

and $\delta_M \tilde{u} = x\tilde{w}$ for some $\tilde{v} \in \mathscr{C}^\infty(X; {}^{\mathscr{V}}\Lambda^{k+1})$ and $\tilde{w} \in \mathscr{C}^\infty(X; {}^{\mathscr{V}}\Lambda^{k-1})$. Thus

(39) $$\Delta\tilde{u} = x\tilde{g}, \qquad \tilde{g} = (\delta_M \tilde{v} + d_M \tilde{w}).$$

Since $g = \tilde{g}_{|x=0} = \delta_F v + d_F w$ it follows that g is in the range of Δ_F, $g = \Delta_F q$. If $\tilde{q}$ is an extension of q, then

(40) $$\Delta(\tilde{u} - x\tilde{q}) = x(\tilde{g} - \Delta\tilde{q}) = x^2\tilde{g}'.$$

This is just the condition that $u \in E_1^k$, proving the lemma.

4. E_2-term

As just noted, if $\tilde{u}_{|x=0} \in E_1^k$ then $d_M\tilde{u}$ vanishes at $x = 0$. Thus the boundary value $x^{-1}d_M\tilde{u} \restriction M_0$ makes sense. Let Π_1 be projection onto the fiber-harmonic part of a form on the fibers. Then consider

(41) $$d'_Y u = \Pi_1[x^{-1}d_M\tilde{u}], \quad \tilde{u} \in E_1^k, \quad \tilde{u} = u \restriction M_0.$$

If $\tilde{u} \restriction M_0 = 0$, so $\tilde{u} = xu', u'$ smooth, then $d_M\tilde{u} = xd_M u'$ and from (33) $du' \restriction M_0 = d_F(u' \restriction M_0)$ is exact, so is annihilated by Π_1. This means that the operator

(42) $$d'_Y : \mathscr{C}^\infty(Y; {}^{\mathscr{V}}L^k) \to \mathscr{C}^\infty(Y; {}^{\mathscr{V}}L^{k+1})$$

is well defined.

(43) Lemma. *For the operator defined by* (42) *on* $\mathscr{C}^\infty(Y; {}^{\mathscr{V}}L^k)$, $d_Y'^2 = 0$.

Proof. By Lemma (37) if $u \in \mathscr{C}^\infty(Y; {}^{\mathscr{V}}L^k)$, i.e., $u \in \mathscr{C}^\infty(M; {}^{\mathscr{V}}\Lambda^k)$, and $\Delta_F u = 0$, then u can be extended to $\tilde{u} \in \mathscr{C}^\infty(X; {}^{\mathscr{V}}\Lambda^k)$ with $d_M\tilde{u} = x\tilde{v}$. So, setting $v = \tilde{v}_{|x=0}$, $d'_Y u = \Pi_1 v$. Now $0 = d_M^2\tilde{u} = xd_M\tilde{v}$ so $d_M\tilde{v} = 0$. This implies that $v = \Pi_1 v + d_F w$ for some $w \in \mathscr{C}^\infty(M; {}^{\mathscr{V}}\Lambda^{k-1})$. Then $\tilde{v} - d_M\tilde{w}$ extends $v - d_F w$ if $\tilde{w}$ extends w, and hence

$$d_Y'^2 u = \Pi_1[x^{-1}(d_M(\tilde{v} - d_M\tilde{w}))_{|x=0}] = 0.$$

By analogy with (41) we set

(44) $$\delta'_Y u = \Pi_1(x^{-1}\delta\tilde{u}_{|x=0}) \quad \text{if } u \in E_1^k,\ \tilde{u} \in \mathscr{C}^\infty(X; {}^{\mathscr{V}}\Lambda^k) \text{ and } u = \tilde{u}_{|x=0}.$$

Here δ is the family of adjoints δ_x of d with respect to the metrics g_x. By (28) δ maps $\mathscr{C}^\infty(X; {}^{\mathscr{V}}\Lambda^k)$ into $\mathscr{C}^\infty(X; {}^{\mathscr{V}}\Lambda^{k-1})$. Moreover it is easily seen that δ'_Y is well defined as an operator

(45) $$\delta'_Y : \mathscr{C}^\infty(Y, {}^{\mathscr{V}}L^k) \to \mathscr{C}^\infty(Y, {}^{\mathscr{V}}L^{k-1}) \quad \forall k$$

as in the discussion surrounding (42).

(46) Lemma. *The adjoint of d'_Y with respect to the fiber metric induced by g_∞ and h on the ${}^{\mathscr{V}}L^k$ and the volume form of h is precisely δ'_Y so that the associated Laplacian is*

$$\Delta'_Y = \delta'_Y d'_Y + d'_Y \delta'_Y. \tag{47}$$

Proof. Notice that the inner product on E_1^k is just the sum over the decomposition (3) of the tensor products of the inner products induced by g_∞ on ${}^{\mathscr{V}}L^{j,0}$ and by h on $x^{-(k-j)}\Lambda^{k-j}Y$ integrated over the fibers of π. Thus if u_i, $i = 1, 2$, are elements of E_1^k and $\tilde{u}_i$ are $\mathscr{C}^\infty$ sections of ${}^{\mathscr{V}}\Lambda^*$ with boundary values u_i, then, always evaluating at $x = 0$,

$$\langle u_1, u_2\rangle = x^p \langle \widetilde{u_1}, \widetilde{u_2}\rangle_G, \qquad p = \dim Y.$$

From this it follows that

$$\langle d'_Y u_1, u_2\rangle = \langle x^{p-1} d_M \widetilde{u_1}, \widetilde{u_2}\rangle_G = \langle \widetilde{u_1}, x^{p-1}\delta_M \widetilde{u_2}\rangle_G = \langle u_1, \delta'_Y u_2\rangle.$$

That is, the adjoint of d'_Y is given by $\delta'_Y u = \Pi_1[x^{-1}\delta_M u]$. This proves

(48) Lemma. *The differential operator d'_Y from sections of ${}^{\mathscr{V}}L^k$ to sections of ${}^{\mathscr{V}}L^{k+1}$ over Y commutes with the decomposition* (10) *and is such that $x d'_Y$ defines a map from sections of $\mathscr{H}^j_{\mathrm{Hd}}$ to sections of $\mathscr{H}_{\mathrm{Hd}} \otimes \Lambda^1$ over Y which is just the natural flat connection on the bundles of the fiber-harmonic forms.*

Proof. We have already noted that d'_Y only depends on the choice of the metric g_∞ restricted to the fibers, which enters through the definition of the Hodge cohomology on the fibers. If the Hodge theorem is used to identify the harmonic forms with the cohomology then we need to show that d'_Y does not depend on the metric at all. If α is a fiber-harmonic form for one metric then the form α' representing the same section of the cohomology bundle is of the form $\alpha + d_F\alpha''$ with the property $\Pi'_1\alpha = \alpha'$. Lifting α'' to a $\mathscr{C}^\infty$ form γ note that the exact term $d\gamma$ is annihilated at $x = 0$ by Π'_1, the projection for the second metric. This shows the independence of the choice of metric.

Since d'_Y is defined locally in the base, a product metric can be used to compute it. This shows directly that d'_Y is flat as a connection on the cohomology bundles. Hence the lemma is proved.

From the definition in (5)

$$\begin{aligned} E_2^k = \{v \in \mathscr{C}^\infty(M_0; {}^{\mathscr{V}}\Lambda^k); v \in \mathscr{C}^\infty(Y, {}^{\mathscr{V}}L^k) \\ \exists \tilde{v} \in \mathscr{C}^\infty(X; {}^{\mathscr{V}}\Lambda^k) \text{ with } \Delta\tilde{v} = x^3\tilde{w} \text{ and } \tilde{v}_{|x=0} = v\}. \end{aligned}$$

(49) Lemma. *The space $E_2^k \in \mathscr{C}^\infty(M_0; {}^{\mathscr{V}}\Lambda^k)$ consists precisely of those elements $v \in E_1^k$ which, when viewed as $\mathscr{C}^\infty$ sections of ${}^{\mathscr{V}}L^k$ over Y, are* *harmonic in the sense that*

$$\Delta'_Y v = 0. \tag{50}$$

Proof. Essentially by definition the space $E_1^k = E_0^k$ is characterized as the boundary values of harmonic ${}^{\mathscr{V}}\Lambda^k$-forms modulo first (or second) order errors. Now if $d_M\tilde{u} = x\tilde{v}_1$ and $\delta_M\tilde{u} = x\tilde{v}_2$, then

$$\Delta_x u = x(\delta_M\tilde{v}_1 + d_M\tilde{v}_2). \tag{51}$$

Denote the boundary values of the $\tilde{v}_i$ by v_i. As shown in the proof of Lemma (37), there exists $w \in \mathscr{C}^\infty(X; {}^{\mathscr{V}}\Lambda^k)$ such that

$$\begin{aligned} &[\tilde{v}_1 - d_M\delta_M\tilde{w}]_{|x=0} = \Pi_1 v_1 \in E_1^{k+1}, \\ &[\tilde{v}_2 - \delta_M d_M\tilde{w}]_{|x=0} = \Pi_1 v_2 \in E_1^{k-1}. \end{aligned} \tag{52}$$

Now, from (51),

$$\Delta(\tilde{u} - x\tilde{w}) = x[\delta_M(\tilde{v}_1 - d_M\tilde{w}) + d_M(\tilde{v}_2 - \delta_M\tilde{w})]. \tag{53}$$

Using (52) this can be written

$$\Delta(\tilde{u} - x\tilde{w}) = x^2\tilde{v}, \qquad \tilde{v} = [x^{-1}\delta_M(\tilde{v}_1 - d_M\tilde{w}) + x^{-1}d_M(\tilde{v}_2 - \delta_M\tilde{w})],$$

From (41) and (44)

$$\begin{aligned} \delta'_Y[\Pi_1 v_1] &= x^{-1}\delta_M(\tilde{v}_1\delta_M(\tilde{v}_1 - d_M\tilde{w})_{|x=0}, \\ d'_Y[\Pi_1 v_2] &= x^{-1}d_M(\tilde{v}_2 - d_M\tilde{w})_{|x=0}. \end{aligned}$$

Combining this with (44) we conclude that the boundary value, v, of $\tilde{v}$ is just

$$v = \delta'_Y\Pi_1 v_1 + d'_Y\Pi_1 v_2 = (\delta'_Y d'_Y + d'_Y\delta'_Y)u = \Delta'_Y u. \tag{54}$$

This shows that if $\Delta'_Y u = 0$ then $u \in E_2^k$. The converse follows similarly, proving the lemma.

Certainly then, as the null space of an elliptic operator, E_2^k is finite dimensional.

Choosing an extension map

$$\mathscr{C}^\infty(M_0; {}^{\mathscr{V}}\Lambda^*) \to \mathscr{C}^\infty(X; {}^{\mathscr{V}}\Lambda^*) \to \mathscr{L}(X; {}^{\mathscr{V}}\Lambda^*)$$

we can identify $\mathscr{L} \cdot E_2^k$ with a subspace of $\mathscr{L}(X; {}^{\mathscr{V}}\Lambda^k)$. This subspace is finite dimensional over $\mathscr{L}$ so has a basis $e_1, \cdots, e_L$. The inner product G gives a bilinear form

$$\mathscr{L}(X; {}^{\mathscr{V}}\Lambda^*) \times \mathscr{L}(X; {}^{\mathscr{V}}\Lambda^*) \to \mathscr{L}.$$

Using this to orthonormalize the basis we obtain a projection operator onto a complement to $\mathscr{L} \cdot E_2^k$:

$$\Pi_2^{\perp}: \mathscr{L}(X; {}^{\mathscr{V}}\Lambda^*) \to \mathscr{L}(X; {}^{\mathscr{V}}\Lambda^*), \qquad \Pi_2^{\perp}(f) = f - \sum_{j=1}^{L} \langle f, e_j \rangle^G e_j.$$

(55) Lemma. *The operator* $P_2 = \Pi_2^{\perp} \Delta_x \Pi_2^{\perp}$ *is an isomorphism on* $\Pi_2^{\perp} \mathscr{L}(X; {}^{\mathscr{V}}\Lambda^*)$.

Proof. By Lemma (49) the leading term of any element of $\Pi_2^{\perp} \mathscr{L}(X; {}^{\mathscr{V}}\Lambda^*)$ in the null space of P_2 must be zero. Thus P_2 is injective.

Surjectivity follows by an inductive argument. If $f \in \Pi_2^{\perp} \mathscr{L}(X; {}^{\mathscr{V}}\Lambda^k)$ then the leading term $x^l f_l$, $l \in \mathbf{Z}$, must have coefficient $f_l \perp E_2^k$. Thus we can choose $u_l \in \mathscr{C}^{\infty}(M_0; {}^{\mathscr{V}}\Lambda^k)$ so that $f_l' = \Delta_F u_l - f_l \in \mathscr{C}^{\infty}(Y, {}^{\mathscr{V}}L^k)$, i.e., is fiber-harmonic. If u' has leading term u_l, then

$$f' = \Delta_x u' - f \text{ has leading term } x^l (f_l').$$

In fact $f_l' = \Delta_Y' u_{l-2}$ for some u_{l-2} which is also a section of ${}^{\mathscr{V}}L^k$. By Lemma (49) we can choose u'' with leading term $x^{l-2} u_{l-2}$ so that

$$\Delta_x u'' - f' \text{ has leading term } x^l (\Delta_Y' u_{l-2} - f_l') = 0.$$

Thus $\Delta_x(u' - u'') - f = O(x^{l+1})$. It follows that $\Pi_2^{\perp} \Delta_x (u' - u'') - f = O(x^{l+1})$. Let $v = \Pi_2^{\perp}(u' - u'')$. Since $v' = u'' - v = \Pi_2(u' - u'')$ and $u' - u''$ has leading term x^{l-2}, from the definition of Π_2 it follows that $\Delta_x v' = O(x^{l+1})$ is three orders smaller than $u' - u''$. Thus $\Pi_2^{\perp} \Delta_x v - f = O(x^{l+1})$ too. Proceeding inductively in the Laurent series we can construct u such that $\Pi_2^{\perp} \Delta_x \Pi_2^{\perp} u = f$.

5. Formal Hodge cohomology

We are now in a position to prove a version of the Hodge decomposition in terms of Laurent series at $x = 0$.

(56) Proposition. *Let* $\mathscr{N} \in \mathscr{L}(X; {}^{\mathscr{V}}\Lambda^*)$ *be the null space of* Δ_x *in Laurent series. Then* $\mathscr{N}$ *is finite dimensional as an* $\mathscr{L}$*-module,* Δ_x *is an isomorphism on its orthocomplement*

$$\begin{aligned} \mathscr{N}^{\perp} &= \{u \in \mathscr{L}(X; {}^{\mathscr{V}}\Lambda^*); \langle u, v \rangle^G \sim 0 \ \forall v \in \mathscr{N}\} \\ &= d_M(\mathscr{L}(X; {}^{\mathscr{V}}\Lambda^*)) \oplus \delta_M(\mathscr{L}(X; {}^{\mathscr{V}}\Lambda^*)), \end{aligned} \tag{57}$$

and the inclusion $N \hookrightarrow {}^{\mathscr{V}}\mathscr{C}$ *in the space of formally closed* ${}^{\mathscr{V}}\Lambda^*$*-forms projects to an isomorphism*

$$ {}^{\mathscr{V}}H_{\mathrm{HL}} \otimes \mathscr{L} = \mathscr{N} \cong H^*_{\mathrm{dR}}(M; \mathscr{L}). \tag{58}$$

Proof. Certainly $\mathcal{N}$ is a finite dimensional $\mathcal{L}$-module, since the projection Π_2 is injective on it. Let B be the inverse of $\Pi_2^{\perp}\Delta_x\Pi_2^{\perp}$ from Lemma (55). The projection of the matrix equation

$$\begin{pmatrix} \Pi_2\Delta_x\Pi_2 & \Pi_2\Delta_x\Pi_2^{\perp} \\ \Pi_2^{\perp}\Delta_x\Pi_2 & \Pi_2^{\perp}\Delta_x\Pi_2^{\perp} \end{pmatrix} \begin{pmatrix} u' \\ u'' \end{pmatrix} = \begin{pmatrix} f' \\ f'' \end{pmatrix} \tag{59}$$

onto the image of Π_2 is then

$$Qv = [\Pi_2\Delta_x\Pi_2 - \Pi_2\Delta_x\Pi_2^{\perp}B\Pi_2^{\perp}\Delta_x\Pi_2]v = g = f' - \Pi_2\Delta_x\Pi_2^{\perp}Bf''. \tag{60}$$

That is, if (u', u'') satisfies (59) then $v = u'$ satisfies (60), and conversely if v satisfies (60) then

$$\begin{pmatrix} u' \\ u'' \end{pmatrix} = \begin{pmatrix} v \\ Bf'' - B\Pi_2^{\perp}\Delta_x v \end{pmatrix}$$

satisfies (59). The null space of this self-adjoint operator, Q, is $\Pi_2\mathcal{N}$, and by standard arguments it is an isomorphism on the orthocomplement in $\Pi_2\mathcal{L}(X; {}^{\mathcal{V}}\Lambda^*)$. This shows that Δ_x is an isomorphism on $\mathcal{N}^{\perp}$.

The remainder of the proposition follows directly. The decomposition (57) as usual is a consequence of the commutation properties $[\Delta_x, d_M] = [\Delta_x, \delta_M] = 0$. The identification, (58), of the Hodge cohomology with the deRham cohomology follows from this and (31).

One immediate consequence of Proposition (56) is the solvability in formal power series of

$$\Delta_x u = f. \tag{61}$$

Set

$$N(k) = \min\{N; E_N^k = E_\infty^k\}. \tag{62}$$

(63) Corollary. *If $f \in \mathcal{C}^\infty(X; {}^{\mathcal{V}}\Lambda^k)$ and $\langle f, v\rangle = O(x^\infty)$ for all $v \in \mathcal{C}^\infty(X; \Lambda^k)$ with $\Delta_x v = O(x^\infty)$, then there exists $u \in x^{-N(k)}\mathcal{C}^\infty(X; {}^{\mathcal{V}}\Lambda^k)$ satisfying* (61) *in the sense of Laurent series at $x = 0$.*

Proof. Since Δ_x is an isomorphism on $\mathcal{N}^{\perp}$, the only point to check is the order of singularity of the solution. If $u \in x^{-P}\mathcal{C}^\infty(X; {}^{\mathcal{V}}\Lambda^k)$ then

$$\Delta_x(x^P u) = O(x^P) \Rightarrow u_0 = (x^P u)_{|x=0} \in E_P^k.$$

Thus, if $P > N(k)$ then $\lim\langle u, v\rangle = \langle u_0, v_0\rangle = 0$ for all $v \in E_P^k = E_\infty^k$. This implies $u_0 = 0$, so $P \le N(k)$.

6. Hodge-Leray cohomology

We proceed to prove the formal part of Corollary (18), and more particularly

(64) Lemma. *The space* $H^k_{\mathrm{HL}}(M)$ *defined by* (20) *consists of* $\mathscr{C}^\infty$ *closed forms on* M *representing the deRham cohomology, i.e.,*

$$H^k_{\mathrm{HL}}(M) \ni \phi \mapsto [\phi] \in H^k_{\mathrm{dR}}(M) \quad \textit{is an isomorphism.} \tag{65}$$

Proof. Certainly $H^k_{\mathrm{HL}}(M)$ consists of closed forms since the smooth extension $\tilde{\phi}$ is closed to infinite order at $x = 0$. Thus the map (65) is well defined.

Consider the space $\Phi \subset \mathscr{C}^\infty(X; \Lambda^k)/d\mathscr{C}^\infty(X; \Lambda^k)$ of formal power series of $\mathscr{C}^\infty$ forms which are formally harmonic, $\tilde{\phi} \in \Phi \Leftrightarrow \Delta_x \tilde{\phi} = O(x^\infty)$. Then $H^k_{\mathrm{HL}}(M)$ is by definition just the set of boundary values of Φ. Similarly let $\Phi \subset \mathscr{L}(X; \Lambda^k)$ be the space of Laurent series of harmonic forms. Clearly

$$\mathscr{L}(X, \Lambda^k) \equiv \mathscr{L}(X; {}^{\mathscr{V}}\Lambda^k) \Leftrightarrow \Phi' \equiv \mathscr{N} \tag{66}$$

is just the space of formally harmonic $\mathscr{V}$-forms. Suppose that ψ is a closed $\mathscr{C}^\infty$ k-form. Let $\tilde{\psi}$ be its trivial extension to X, independent of x. Let

$$\tilde{\psi} = \psi_d + \psi_\delta + \psi_0 \tag{67}$$

be the decomposition of the Taylor series of $\tilde{\psi}$ as a $\mathscr{V}$-form, from (57). Certainly $d_M \tilde{\psi} = 0$ so $\psi_\delta = 0$. Writing out (67) as an equation in $\mathscr{L}(X, \Lambda^*)$ rather than $\mathscr{L}(X, {}^{\mathscr{V}}\Lambda^*)$ and equating coefficients of x show that $\psi = \phi + d\eta$ for some $\mathscr{C}^\infty$ form η. Thus the map (65) is surjective. The injectivity of (65) is similar.

(68) Corollary. *If* $N(k)$, *defined by* (62), *is the length of the sequence* (14), *then* $N(k)$ *is at most the number of summands in* (10).

7. The parametrix for Δ_x

The transition from the formal power series result, Proposition (56), to the existence of a smooth basis, as in Theorem (15), is accomplished by the construction of a parametrix which is uniform as $x \downarrow 0$. From the smoothness of the error term in the parametrix one can extract a basis for ${}^{\mathscr{V}}\mathscr{H}^k_{\mathrm{HL}}$ smooth down to $x = 0$. The construction itself is quite typical of the (more intricate) methods used in [9] and [7]. Thus we proceed to construct the Schwartz kernel of the parametrix directly, but on a blown up version of the product manifold. The construction proceeds in three steps. The first is a symbolic construction equivalent to the construction of a parametrix on a compact manifold, but uniform up to the boundary. This is followed by a global construction on a Euclidean bundle over the fibers which distinguishes the fiber-harmonic forms at the boundary. Finally there is a formal power series step essentially reducing to Corollary (63).

First we describe the blown up product on which the construction takes place.

8. Stretched product

Since x is a parameter throughout, we consider the partial product

$$Z = [0, \infty) \times M \times M$$

with its fibrations and two Lie algebras $\mathscr{V}_L$ and $\mathscr{V}_R$, being respectively $\mathscr{V}$ on the second and third (left and right) factors. Inside Z consider the submanifold Q:

$$Q = \{(0, p, p'); \pi_L(p) = \pi_R(p')\},$$

where π_L and π_R are the left and right fibrations. This is simply the "fiber diagonal" over the boundary. We carry out the construction of the parametrix on the manifold with corners which is obtained by blowing up Z along the submanifold Q:

$$Z_Q = SNQ \sqcup [Z \backslash Q].$$

As a set it is given by replacing Q by its inward-pointing unit spherical bundle. It comes equipped with the "blow-down" map

$$\pi_Q \colon Z_Q \to Z, \tag{69}$$

which is the identity away from the "front face" SNQ, which we will denote $\mathrm{ff}(Z_Q)$. The blown up space Z_Q has a unique $\mathscr{C}^\infty$ structure such that π_Q is $\mathscr{C}^\infty$, is a diffeomorphism from $Z_Q \backslash \mathrm{ff}(Z_Q)$ to $Z \backslash Q$ and has rank $\dim Y + 2\dim F + 1$ at $\mathrm{ff}(Z_Q)$ (see for example [10]).

The manifold Z_Q is just the natural domain for polar coordinates around Q. The front face $\mathrm{ff}(Z_Q)$ is fibered by hemispheres $\pi_Q^{-1}(q)$, $q \in Q$. In fact the projection of Q down to the right factor of M in Z shows that $\mathrm{ff}(Z_Q)$ fibers over M with fibers $F \times \mathbf{S}^p_+$, $p = \dim Y$:

$$\begin{array}{ccc} F \times \mathbf{S}^p_+ & \longrightarrow & \mathrm{ff}(Z_Q) \\ & & \downarrow \\ & & M \end{array} \tag{70}$$

The hemispheres $\mathbf{S}^p_+$ are parametrized nonsingularly by the component ω of polar coordinates around Q induced by coordinates on M:

$$\begin{aligned} Z_Q &\supset [0, \infty) \times \mathbf{S}^p \times \mathbf{R}^p \times F \times F \\ &\ni (R = (x^2 + |y - y'|^2)^{1/2},\ \omega = (x, y - y')/R, y, z, z') \\ &\mapsto (x, y, z, y', z') \in Z. \end{aligned} \tag{71}$$

More usefully for computations one can introduce projective coordinates

$$x, u = \frac{y - y'}{x}, y, z, z' \tag{72}$$

valid except at the boundary of the front face. Since u takes values in $\mathbf{R}^p$, the interior of each fiber of the front face has a Euclidean structure, which is independent of the choice of coordinates.

The significance of Z_Q is mainly related to the following simple result.

(73) Lemma. *The closure, $\Delta_{\mathscr{V}}$, in Z_Q of the submanifold $\{(x,p,p);\ x > 0,\ p \in M\}$ of $Z \backslash Q$ is an embedded submanifold which meets the boundary of Z_Q only in the interior of $\mathrm{ff}(Z_Q)$ and does so transversally. The Lie algebras $\mathscr{V}_L$ and $\mathscr{V}_R$ lift to algebras of $\mathscr{C}^\infty$ vector fields on Z_Q which are tangent to all boundary faces, each is transversal to $\Delta_{\mathscr{V}}$, tangent to the spherical fibers of $\mathrm{ff}(Z_Q)$ and restricts to them to the translation invariant vector fields spanning the Euclidean structure.*

Proof. These statements are all local. Moreover they are clearly invariant, so can be checked in any local coordinates x, y, z, y', z', where y, y' are local coordinates in the base. Since Q projects onto the diagonal in $Y \times Y$, the y-coordinates can be taken the same in the two factors. Then Q is just $y = y', x = 0$. In the projective coordinates (72) $\Delta_{\mathscr{V}}$ is given by $u = 0, z = z'$ so is obviously an embedded submanifold transversal to the local boundary component $x = 0$.

The transversality of the lifted Lie algebras also follows from this computation since

$$\partial_{z_l}, x\partial_{y_j} = \partial_{u_j} + x\partial_{y_j} \tag{74}$$

span $\mathscr{V}_L$ and restrict to span the translation-invariant vector fields. This proves the lemma.

The bundles ${}^{\mathscr{V}}\Lambda^k M$, defined as they are from ${}^{\mathscr{V}}TM$, lift to the left factor of $Z = X \times M$ and then up to Z_Q where they embed naturally in Λ^k over the interior of $\mathrm{ff}(Z_Q)$.

(75) Lemma. *The metrics g_x lift to Z_Q from the left factor of M to a $\mathscr{C}^\infty$ 2-cotensor near $\mathrm{ff}(Z_Q)$ which restricts to a product metric on $F \times \mathbf{R}^p$ with second factor Euclidean. The Laplacian lifts to a differential operator with $\mathscr{C}^\infty$ coefficients on Z_Q which is transversally elliptic, with respect to $\Delta_{\mathscr{V}}$, and tangent to the fibers $F \times \mathbf{R}^p$ of the interior of $\mathrm{ff}(Z_Q)$. Its restriction to a fiber is $\Delta_F + \Delta_E$ with Δ_E Euclidean.*

Proof. These statements are all coordinate invariant so can be checked in the projective coordinates (72). The Laplacian is an elliptic combination of the vector fields (74) with $\mathscr{C}^\infty$ coefficients on X.

9. Symbolic construction

We wish to construct a right parametrix for Δ_x, i.e., an operator G_x on compressed k-forms on M such that

$$\Delta_x G = \mathrm{Id} - F, \tag{76}$$

where F is to be made as "small" as possible, in successive steps. Initially then G_1', the first part of G, is a distribution on Z, or rather a distributional section of the homomorphism bundle with density factor

$$G_1' \in \mathscr{C}^{-\infty}(Z; \mathrm{Hom}({}^{\mathscr{V}}\Lambda^k, {}^{\mathscr{V}}\Lambda^k) \otimes \Omega_R).$$

Here Ω_R is the density bundle lifted from the right factor of M, allowing the formula

$$G_1' u(x, y) = \int_M G_1' \cdot u = \int_M G_1 \cdot u \, dg_x, \tag{77}$$

to be interpreted correctly, where the Riemannian density dg_x has been used to trivialize the density bundle. Thus G_1 is a distributional section of the homomorphism bundle. It should be noted that

$$dg_x = x^{-r}\mu, \qquad \mu \text{ is } \mathscr{C}^\infty \text{ and nonvanishing}, \tag{78}$$

as a density on M; so this normalization introduces factors of x.

The successive kernels will be extendible distributions on Z, a manifold with boundary. Since Z_Q is obtained by blowing up a submanifold of the boundary of Z the space of extendible distributions on Z_Q is canonically isomorphic to the space of extendible distributions on Z (see for example [9]). Thus we can just as well consider

$$G_1 \in \mathscr{C}^{-\infty}(Z_Q; \mathrm{Hom}({}^{\mathscr{V}}\Lambda^k, {}^{\mathscr{V}}\Lambda^k)), \tag{79}$$

where the lifts of the bundles over Z (whence they have arrived from M) are not distinguished from the bundles on Z. Thus we shall lift the equation (76) from Z to Z_Q.

Consider the kernel of the identity operator. As a distribution on Z it is a nonvanishing Dirac delta section over the diagonal. With the density factor removed by the Riemannian density it takes the form in local coordinates

$$\mathrm{Id} = x^p h \delta(y - y')\delta(z - z') \quad \text{on } Z,$$

where h is an isomorphism on the bundle. Lifted to Z_Q the factors of x just compensate for the homogeneity of the delta function

$$\mathrm{Id} = h\delta(z-z')\delta(u) \quad \text{on } Z_Q. \tag{80}$$

Lemma (75) shows that the lifted operator is $\mathscr{C}^\infty$ and transversally elliptic (i.e., noncharacteristic with respect) to the lifted diagonal $\Delta_{\mathscr{V}}$. Since this is the carrier of the singularity of the kernel of the identity, in (80), for the first step in the construction of a parametrix, we can use standard elliptic theory.

(81) Lemma. *The kernel G_1 in* (79) *can be chosen as an* (*elliptic*) *conormal distribution with respect to the submanifold $\Delta_{\mathscr{V}}$* :

$$G_1 \in I^{-2}(Z_Q, \Delta_{\mathscr{V}}; \mathrm{Hom}({}^{\mathscr{V}}\Lambda^k, {}^{\mathscr{V}}\Lambda^k)), \tag{82}$$

which vanishes in a neighborhood of $\partial Z_Q \setminus \mathrm{ff}(Z_Q)$, the part of the boundary other than the front face, and is such that

$$\Delta_x G_1 - \mathrm{Id} = F_1 \in \mathscr{C}^\infty(Z_Q, \mathrm{Hom}({}^{\mathscr{V}}\Lambda^k, {}^{\mathscr{V}}\Lambda^k)). \tag{83}$$

Proof. This is a standard symbolic argument. Since G_1 will be chosen $\mathscr{C}^\infty$ away from the lifted diagonal, $\Delta_{\mathscr{V}}$, and this submanifold only meets the boundary in the front face, by Lemma (73), the extra vanishing condition is trivially obtained.

10. Laplacian on Euclidean space

The main step in the iterative solution of the model problem, carried out below, is the inversion of the standard flat Laplacian on Euclidean space. The only subtlety is that we need to solve

$$\Delta_E v = f \quad \text{on } \mathbf{R}^p, \tag{84}$$

where f may have an asymptotic expansion at infinity, and we need to control the asymptotic behavior of the solution at infinity. We shall introduce a suitable space on which this can be done.

Supposing throughout that $p \geq 2$ set

$$\widehat{\mathscr{T}_l} = \{\hat{f} \in \mathscr{C}^\infty(\mathbf{R}^p \setminus 0); \exists q \in \mathbf{N} \text{ such that } \hat{f}(\eta) = |\eta|^{-2q} g(\eta) \text{ with } g \in \mathscr{S}(\mathbf{R}^p), |\hat{f}(\eta)| \leq C|\eta|^{-l}\}, \quad l \in \mathbf{Z}, \tag{85}$$

$\mathscr{S}(\mathbf{R}^p)$ being Schwartz's space of test functions of rapid decrease. We shall define a regularization map

$$e: \widehat{\mathscr{T}_l} \mapsto \mathscr{S}'(\mathbf{R}^p);\ e(\hat{f})_{|\eta \neq 0} = \hat{f}. \tag{86}$$

There is a reasonably natural choice, once the norm is fixed, given by the introduction of polar coordinates. Consider the "Laurent" series expansion of $\hat{f}$ written in polar coordinates

$$\begin{aligned} \hat{f}(r\theta) &= \sum_{j>p-1} r^{-j}\beta_j(\theta) + r^{-p+1}f'(r,\theta), \\ \beta_j &\in \mathscr{C}^\infty(\mathbf{S}^{p-1}),\ f' \in \mathscr{C}^\infty([0,\infty)\times\mathbf{S}^{p-1}). \end{aligned} \tag{87}$$

Then, recalling that the Lebesgue measure in polar coordinates is $d\eta = r^{p-1}dr\,d\theta$, set

$$\begin{aligned} e(\hat{f})(\phi) = &\sum_{j>p-1} \left\langle r_+^{p-1-j}, \int_{\mathbf{S}^{p-1}} \beta_j(\theta)\phi(r\theta)\,d\theta \right\rangle \\ &+ \int_0^\infty \int_{\mathbf{S}^{p-1}} f'(r,\theta)\phi(r\theta)\,dr\,d\theta, \quad \forall\phi\in\mathscr{S}(\mathbf{R}^p). \end{aligned} \tag{88}$$

Here, the first pairing is of tempered distributions on the line, with r_+^{-k} the regular part at $z=-k$ of the meromorphic extension of

$$r_+^z = \begin{cases} \exp(z\log r), & r>0 \\ 0, & r\le 0 \end{cases} \quad (\Re z > -1). \tag{89}$$

Certainly definition (88) has the desired restriction property (86) and is independent of l since only the expansion (87) is involved. It follows easily that for any symbol p

$$e(p\hat{f}) = pe(\hat{f}) \quad \forall\hat{f}\in\widehat{\mathscr{T}_l},\ \forall p\in S^\infty(\mathbf{R}^p). \tag{90}$$

Clearly if p vanishes at the origin, the "order" decreases:

$$p\in S^\infty(\mathbf{R}^p), \qquad D^\alpha p(0)=0 \quad \forall|\alpha|\le r \Rightarrow p\colon\widehat{\mathscr{T}_l}\to\widehat{\mathscr{T}_{l-r}}. \tag{91}$$

Of course the selection of a regularization of r^{-k} means that some ambiguity is involved in the definition. This appears in the argument below in the form of the noninvariance under general linear transformations.

(92) Lemma. *For any i, j and l*

$$e(\eta_i\partial_{\eta_j}\hat{f}) - \eta_i\partial_{\eta_j}e(\hat{f}) = \sum_{|\alpha|\le l-p+2} c_\alpha D^\alpha\delta(\eta) \quad \textit{if } \hat{f}\in\widehat{\mathscr{T}_l}. \tag{93}$$

Proof. The noncommutativity arises from the fact that r_+^{-k} is not homogeneous. Rather it is quasihomogeneous of degree $-k$ in the sense that

$$r\partial_r r_+^{-k} = -kr_+^{-k} + c_k\partial_r^{k-1}\delta(r). \tag{94}$$

From this (93) follows easily.

Now using the regularization (88) and Fourier transform we define

$$(95)\qquad \mathcal{T}_l \subset \mathcal{S}'(\mathbf{R}^p), \quad g \in \mathcal{T}_l \Leftrightarrow \hat{g} = e(\hat{f}), \quad \hat{f} \in \widehat{\mathcal{T}}_l, \quad \mathcal{T}_l \leftrightarrow \widehat{\mathcal{T}}_l.$$

The point of this construction is simply that the map

$$(96)\qquad \Delta_E : \mathcal{T}_l \leftrightarrow \mathcal{T}_{l-2} \quad \forall l \in \mathbf{Z}$$

is always an isomorphism. Indeed the inverse is given by $\hat{f} \mapsto |\eta|^{-2}\hat{f}$.

To make use of this isomorphism we need some information on the asymptotic behavior of the elements of $\mathcal{T}_l$. This is easily deduced form formula (88).

(97) Lemma. *If $p \geq 2$ then any $f \in \mathcal{T}_l$ is a symbol on $\mathbf{R}^p$ with complete asymptotic expansion at infinity of the form*

$$(98)\qquad \begin{aligned} f(u) &\sim \sum_{j\geq 0} |u|^{l-p-j} f_j(\theta) + \log|u| \cdot p(u), \\ p(u) &= \sum_{0\leq|\alpha|\leq l-p} c_\alpha u^\alpha, \quad \eta = r\theta. \end{aligned}$$

Notice that as a consequence of the properties of $\widehat{\mathcal{T}}_l$ we have

$$(99)\qquad \partial_{u_j} : \mathcal{T}_l \to \mathcal{T}_{l-1}, \qquad u_i\partial_{u_j} : \mathcal{T}_l \to \mathcal{T}_l + \mathcal{P}_{l-p},$$

where $\mathcal{P}_s$ is the space of polynomials of degree at most s.

The case $p = 1$ is even more elementary since no lower order terms and no logarithmic terms arise. Thus set

$$(100)\qquad \mathcal{T}_l = \left\{ g \in \mathcal{C}^\infty(\mathbf{R}); g(u) \sim \sum_{j\leq l-1} (\pm 1)^{j+1} a_j |u|^j \text{ as } u \to \pm\infty \right\}, \qquad (p = 1).$$

Then (96) and (99) hold, the latter without any polynomial error terms.

11. Model problem

It is important to note that the error term F_1 in (83) is *not* a smoothing operator on M, although it is an operator on M, which is smoothing for $x > 0$ and depends smoothly on x down to $x = 0$. To obtain a smoothing (and hence compact) error, we need to choose a second kernel G_2 on Z_Q, which will remove the Taylor series of F_1 at the front face. Let ρ be a defining function for the nonfront face of Z_Q. The choice of G_2 will be (initially) of the form

$$(101)\qquad \begin{aligned} G_2 \in\ & \rho^{p-2}\mathcal{C}^\infty(Z_Q; \operatorname{Hom}({}^{\mathcal{V}}\Lambda^k, {}^{\mathcal{V}}\Lambda^k)) \\ &+ \log\rho \cdot x^{p-2}\mathcal{C}^\infty(Z; \operatorname{Hom}({}^{\mathcal{V}}\Lambda^k, {}^{\mathcal{V}}\Lambda^k)) \end{aligned}$$

with the logarithmic term absent if $p = 1$. The fact that the logarithmic term has as coefficient a $\mathscr{C}^\infty$ section on the original manifold Z is important because of its effect on the regularity properties of the operator G_2. The splitting (101) is not unique, but the space formed by the sum is well defined.

(102) Lemma. *If $p > 1$ then a kernel G_2 of the form (101) can be chosen so that*

$$\Delta_x G_2 - F_1 = F_1' + \log\rho \cdot F_2'', \tag{103}$$

where $F_2', F_2'' \in x^{p-2}\mathscr{C}^\infty(Z; \mathrm{Hom}({}^{\mathscr{V}}\Lambda^k))$, and $F_2'' \cong 0$ in Taylor series at Q. If $p = 1$ then a kernel $G_2 \in \rho^{-1}\mathscr{C}^\infty(Z_G; \mathrm{Hom}({}^{\mathscr{V}}\Lambda^k))$ can be chosen so that

$$\Delta_x G_2 - G_1 = F_2' \in x^{-1}\mathscr{C}^\infty(Z; \mathrm{Hom}({}^{\mathscr{V}}\Lambda^k)). \tag{104}$$

Proof. Initially we consider the case $k = 1$, and the action on functions, and assume $p > 1$. In terms of the polar coordinates (71) we can take the defining function for the nonfront face of Z_Q to be $\rho = \omega_0$. We shall use the two symbols interchangeably. The desired kernel will be constructed in the form

$$\begin{gathered} G_2 \sim \sum_{i=0}^{\infty} x^i e_i(\omega, z, z') + \log\rho \cdot e', \\ e_i \in \rho^{p-2-i}\mathscr{C}^\infty(\mathbf{S}^p \times F \times F), \quad e' \in \mathscr{C}^\infty(Z), \end{gathered} \tag{105}$$

by solving recursively for the e_i. The logarithmic terms will arise naturally in this recursion.

Notice that $x = \rho R$ where R is a defining function for the front face, so the sum in (105) does represent the Taylor series at the front face of a $\mathscr{C}^\infty$ function. Writing the Taylor series as in (105), in terms of x, introduces extra singularities in the e_i, but has the advantage that a corresponding series for $\Delta_x G_2$ is easily obtained because $[\Delta_x, x] = 0$:

$$\Delta_x G_2 \sim \sum_{i=0}^{\infty} x^i \Delta_x e_i(\omega, z, z') + \Delta_x[\log\rho \cdot e'], \tag{106}$$

although now the coefficients may depend on all variables. Consider the form of the $\Delta_x e_i$ in the projective coordinates (71). Notice that the function $1/\rho = 1/\omega_0$ is of precisely linear growth in $|u|$; this follows from

$$\omega_0 \cdot u = \frac{x}{R} \cdot \frac{y - y'}{x} = \omega'.$$

Expanding the coefficients of the operator expressed in terms of the vector fields (74) gives:

$$(107)\quad \begin{aligned} \Delta_x e_i(u,y,z,z') &= B_0(y,z,D_z,D_u)e_i + xB_1(y,z,D_z,D_u,D_y)e_i \\ &\quad + x^2 B_{(2)}(x,y,z,D_z,D_u,D_y)e_i, \\ B_{(2)} &\sim \sum_{j\geq 2} x^j B_j(y,x,D_z,D_u,D_y). \end{aligned}$$

As an operator in u, this has constant coefficients.

From Lemma (75) it follows that B_0 is simply the Laplacian on $F \times \mathbf{R}^p$ with its product metric, as a fiber of the interior of the front face of Z_Q. To invert the operator we treat the harmonic and nonharmonic parts on F separately:

$$B_0 = B_{00} + B_{0\perp}, \begin{cases} B_{00}\colon \mathscr{H}' \to \mathscr{H}', \ \mathscr{H}' = \{e \in \mathscr{C}^\infty(\mathrm{ff}(\mathring{Z}_Q); {}^{\mathscr{V}}\Lambda^*); \Delta_F e = 0\}, \\ B_{0\perp}\colon \mathscr{H}'_\perp \to \mathscr{H}'_\perp, \\ \mathscr{H}'_\perp = \{e \in \mathscr{C}^\infty(\mathrm{ff}(\mathring{Z}_Q); {}^{\mathscr{V}}\Lambda^*); e = \delta_F e' + d_F e''\}. \end{cases}$$

In fact

$$B_{00} = \Delta_E, \qquad B_{0\perp} = \Delta_F + \Delta_E$$

where Δ_E is a Euclidean Laplacian acting on the fibers. Although a constant coefficient operator on each fiber, $\mathbf{R}^p$, the coefficients of this Euclidean Laplacian Δ_E may well depend on the base variables y, since this is just the y-dependence of the metric h.

To apply the discussion of the Laplacian above it is therefore natural to make a linear change of variables in u to reduce the Euclidean metric to standard form. This is easily accomplished by replacing u by

$$(108)\qquad U = A(y)u, \qquad \det A(y) \neq 0.$$

The main adverse effect of this transformation is the change

$$(109)\qquad \partial_y \mapsto \partial_y + \frac{\partial A}{\partial y} A^{-1} U \partial_U$$

which adds a linear vector field in U to ∂_y. In terms of the variable U we can consider the space $\mathscr{T}_l$ of (95), with $\mathscr{C}^\infty$ dependence on the parameters y and z.

The second of the pieces of B_0 is the more easily inverted. Let $\mathscr{T}_l^{\perp}$ be the part of $\mathscr{T}_l$ which is orthogonal to the harmonic functions (i.e., constants) on the fiber F. Since the spectrum of Δ_F is strictly positive (and discrete)

$$(110)\qquad B_{0\perp} \text{ is an isomorphism on } \mathscr{T}_l \quad \forall l \in \mathbf{Z}.$$

To prove (110) take the Fourier transform in U. If η is the variable dual to U, then the inverse of $B_{0\perp}$ becomes $(\Delta_F + |\eta|^2)^{-1}$. This preserves smoothness in $z \in F$ and the parameters y. Acting on $\mathscr{T}_l^{\perp}$ it preserves the orthogonality to harmonic functions, and since it is a multiplication operator in η, it preserves the form (85) of $\mathscr{T}_l$ (see (90)). Thus (110) holds.

The invertibility properties of B_{00} have already been discussed. In particular we shall use (96). Let $\mathscr{T}_l^{0}$ denote the part of $\mathscr{T}_l$ which is harmonic (i.e., constant) on the fibers. Then

$$\Delta_E : \mathscr{T}_l^{0} \leftrightarrow \mathscr{T}_{l-2}^{0} \quad \forall l. \tag{111}$$

To use these invertibility results we need to consider the behavior of the higher order terms in (107). First consider B_1. This arises from the second term in the Taylor series of Δ_x on X together with the cross terms arising from (74), i.e., the fact that $x\partial_y$ lifts to $\partial_u + x\partial_y$, not ∂_u. By (51) the first part has values in the $\mathscr{H}_{\perp}'$. On the other hand the second part must have a factor of ∂_u, and it therefore reduces the order at infinity. Recalling that the change of variable (108) has been made and using (109) we see that

$$B_1 : \mathscr{T}_l^{\perp} \to \mathscr{T}_l + \mathscr{P}_{l-p}, \quad B_1 : \mathscr{T}_l^{0} \to \mathscr{T}_{l-1}^{0} + \mathscr{T}_l^{\perp} + \mathscr{P}_{l-p} \quad \forall l \in \mathbf{Z}. \tag{112}$$

Here of course the polynomials (in u or equivalently U) depend smoothly on y and z. The higher order terms have no special properties so

$$B_i : \mathscr{T}_l \to \mathscr{T}_l + \mathscr{P}_{l-p} \quad \forall i \geq 2. \tag{113}$$

This allows us to solve (103) inductively with

$$e_i \in \mathscr{T}_{i+2}^{0} + \mathscr{T}_i^{\perp} \quad \forall i \in \mathbf{N}. \tag{114}$$

This is accomplished by writing the formal power series version of (103), using (105), (106) and (107), as

$$B_0 e_i = -\sum_{j=0}^{i-1} B_{i-j} e_j + F_{1,i}$$

where the B_i and F_{1i} are the Taylor series of B and F_1 in x. The initial step is

$$B_0 e_0 = F_{1,0} \in \mathscr{T}_0 \Rightarrow e_0 \in \mathscr{T}_2^{0} + \mathscr{T}_0^{\perp}.$$

Then the inductive assumption (114) means that

$$-B_1 e_{i-1} - \sum_{j=0}^{i-2} B_{i-j} e_j \in \mathscr{T}_i + \mathscr{P}_{i-p}.$$

Applying (112) and (113) gives the inductive step.

Thus we have constructed a formal power series as in (105). Notice that $u = (y - y')/x$ and this means that the series for e' is a Taylor series at Q on Z. Using Borel's lemma to sum this to a $\mathscr{C}^\infty$ function on Z and the other series as a $\mathscr{C}^\infty$ function on Z_Q gives G_2 as in (101) satisfying the conditions of Lemma (102).

The case $p = 1$ is similar but simpler, without logarithmic terms. Extension to the case of positive form dimension involves only notational changes, so Lemma (102) is proved.

12. $\mathscr{C}^\infty$ regularity of the parametrix

We shall slightly modify G_2 before examining its regularity as an operator. Writing (101) in the form $G_2 = g_2' + \log\rho \cdot g_2''$ we notice that (106) implies that $\Delta_x g_2'' \cong 0$ in the sense of Taylor series at Q. Thus consider

$$(115)\qquad \begin{gathered} \tilde{G}_2 = G_2 - \log x \cdot g_2'' = \tilde{g}_2' + \log r \cdot \tilde{g}_2'', \\ \tilde{g}_2' \in \rho^{p-2}\mathscr{C}^\infty(Z_Q, {}^{\mathscr{V}}\Lambda^k), \quad \tilde{g}_2'' \in x^{p-2}\mathscr{C}^\infty(Z, {}^{\mathscr{V}}\Lambda^k), \end{gathered}$$

since $x = R\rho$. Then $\tilde{G} = G_1 - \tilde{G}_2$ is a true parametrix for Δ_x:

$$(116)\qquad \Delta_x\tilde{G} = \mathrm{Id} + F_2, \qquad F_2 \in x^{p-2}\mathscr{C}^\infty(Z, {}^{\mathscr{V}}\Lambda^k).$$

One should however recall that a factor of x^{-p} has been absorbed into the measure in (77) and (78). Thus as an operator the error term is a map

$$F_2 \colon \mathscr{C}^\infty(X; {}^{\mathscr{V}}\Lambda^k) \to x^{-2}\mathscr{C}^\infty(X; {}^{\mathscr{V}}\Lambda^k).$$

We next consider the regularity properties of the operator with kernel $\tilde{G}$. Away from $x = 0$ it is a pseudodifferential operator of order -2. The important point is therefore the uniformity up to $x = 0$.

(117) Lemma. *The parametrix constructed above, $\tilde{G} = G_1 - \tilde{G}_2$, satisfies*

$$G \colon \mathscr{C}^\infty(X; {}^{\mathscr{V}}\Lambda^k) \to x^{-2}\mathscr{C}^\infty(X; {}^{\mathscr{V}}\Lambda^k).$$

Proof. To show the regularity of Gu we use an extreme form of a standard method, namely showing tangential and normal regularity separately. The tangential part is in the form of conormal regularity:

$$(118)\qquad G \colon \mathscr{C}^\infty(X; {}^{\mathscr{V}}\Lambda^k) \to \mathscr{A}(X; {}^{\mathscr{V}}\Lambda^k).$$

Here the conormal space is defined by

$$\mathscr{A}(X; {}^{\mathscr{V}}\Lambda^k) = \{u \in \mathscr{C}^{-\infty}(X; {}^{\mathscr{V}}\Lambda^k); \\ \text{for some } s = s(u), \mathscr{V}_b(X)^r u \in H^s(X; {}^{\mathscr{V}}\Lambda^k)\ \forall r \in \mathbf{N}\}.$$

The Lie algebra, $\mathcal{V}_b(X)$, consisting of all $\mathcal{C}^\infty$ vector fields tangent to the boundary of X, acts through some connection, and $H^s(X)$ is the standard Sobolev space.

The main step in proving (118) is to note that there is a lifting map

$$\ell_L : \mathcal{V}_b(X) \to \mathcal{V}_b(Z_Q), \quad \text{with } \ell_L(V) \text{ tangent to } \Delta_{\mathcal{V}}.$$

Suppose that $V \in \mathcal{V}_b(X)$ has support in $[0,\infty) \times \pi^{-1}(\Omega)$ where $\Omega \subset Y$ is a coordinate patch over which π is trivial, $\pi^{-1}(\Omega) = \Omega \times F$. Thus V can be written

$$V = a(x,y,z)x\partial_x + \sum_{i=0}^{p} a_j(x,y,z)\partial_{y_j} + V'$$

with V' tangent to the fibers F. Let $\phi \in \mathcal{C}_c^\infty([0,\infty) \times \Omega)$ be identically equal to 1 on the support of the coefficients, and consider

$$a(x,y,z)x\partial_x + \sum_{i=0}^{n} a_j(x,y,z)[\partial_y + \phi(x,y')\partial_{y'_j}] + V'. \tag{119}$$

This vector field on Z projects to V and is tangent to the diagonal Δ. Since it is also tangent to Q, lifting to Z_Q gives $\ell_L(V)$, locally, as desired. A partition of unity gives a global lifting.

Consider the space $\mathcal{G}$ of all kernels of the form of a sum (82) and (101). Applying V (through some connection) and integrating by parts gives

$$VGu = (\ell_L(V)G)u + \sum_j G(W'u), \tag{120}$$

where W' is the transpose of the projection of $\ell_L(V)$ onto the right factor of M in Z. Since $\ell_L(V)$ is tangent to the boundary of Z_Q, the kernel $\ell_L(V)G \in \mathcal{G}$, so has the same singularity type as G. Hence iterating (120) it suffices to note that any operator with kernel in $\mathcal{G}G$ is bounded from $\mathcal{C}^\infty(X; {}^{\mathcal{V}}\Lambda^k)$ into some fixed Sobolev space. Since the elements of $\mathcal{G}$ have some fixed regularity, this is certainly so, proving (118).

To show the normal regularity we shall investigate the analyticity properties of the product

$$\frac{x^t}{\Gamma(t)} x^2 G\mu dx dg_x \in \dot{\mathcal{C}}^{-\infty}(Z, \mathrm{Hom}({}^{\mathcal{V}}\Lambda^k)). \tag{121}$$

Here μ is a nonvanishing smooth density on the left factor of M, and dg_x is the Riemannian density from the right factor. Certainly this is well defined for $\Re(t) \gg 0$. We shall show that, as a supported distributional density on Z, it is in fact entire in t. To see this we lift to Z_Q. This is unambiguous in $\Re(t) \gg 0$. Notice that the lift is of the form

$$\frac{R^t}{\Gamma(t)}\rho^t G R^2 \rho^{2-p}\nu \in \dot{\mathscr{C}}^{-\infty}(Z_Q, \mathrm{Hom}(^{\mathscr{V}}\Lambda^k)\otimes\Omega). \tag{122}$$

Here ν is a smooth, nonvanishing section of the standard density bundle over Z_Q. The singular factors in (122) arise from the introduction of polar coordinates:

$$x^2[\mu\, dx\, dg_x] = x^{2-p}\mu\, dx\mu' = x^{2-p}R^p h dR d\omega dy dz dz' = R^2\rho^{2-p}\nu. \tag{123}$$

Here $h \neq 0$.

Now G_1 contributes an entire term to this, since it is $\mathscr{C}^\infty$ up to $R = 0$ (the front face) and on its support $\rho \neq 0$. Thus we have only to consider (122) with G replaced by $\tilde{G}_2$ (see (115)). Since $\rho^{2-p}\tilde{g}_2'$ is $\mathscr{C}^\infty$, we conclude that

$$\frac{R^t}{\Gamma(t)}\frac{\rho^t}{\Gamma(t)}R^2\rho^{2-p}\tilde{g}_2'\nu$$

is entire. The presence of the factor $\log R$ in the second term in (115) introduces the possibility of simple poles at the negative integers, i.e.,

$$\frac{R^t}{\Gamma(t)}\frac{\rho^t}{\Gamma(t)}\log R\cdot R^2\rho^{2-p}\tilde{g}_2'' \text{ has at most simple poles at } t \in -\mathbf{N}.$$

Combining these two statements we see that the distribution (122) is meromorphic in $\mathbf{C}$ with at most double poles at $-\mathbf{N}$. Consider the form of the residues. Since $\rho^t/\Gamma(t)$ is entire and R^t is entire away from the front face, where $\rho^{2-p}\tilde{G}_2$ is also smooth, the support of the residues must be contained in the front face. Moreover all these residues must be smooth in z, z', y as distributions in (R, ω). Projecting down to Z from Z_Q (dual to lifting of smooth functions) gives the distribution (121), which is therefore at worst meromorphic with residues supported in Q. To remove the poles we need only subtract a distribution of the form

$$L = (\log x)^2 f_2(x, y, z, z', y - y') + (\log x) f_1(x, y, z, z', y - y'),$$

where f_1 and f_2 are $\mathscr{C}^\infty$ in x and are chosen to have the correct Taylor series at $x = 0$ to reproduce the residues. Now applying Δ_x we find

$$\Delta_x\tilde{G}_2 = \Delta_x(\tilde{G}_2 - L) + \Delta_x L \in \mathscr{C}^\infty(Z; \mathrm{Hom}(^{\mathscr{V}}\Lambda^k)).$$

Since $x^t x^2(\tilde{G}_2 - L)\mu\, dx/\Gamma(t)$ is entire (by construction), the same is true of $x^t x^2\Delta_x(\tilde{G} - L)\mu\, dx/\Gamma(t)$. This implies that

$$\Delta_x f_i \cong 0 \quad \text{for } i = 1, 2 \tag{124}$$

in Taylor series at $x = 0$. However, from the support and regularity conditions on the kernels noted above, the operators defined by the f_i are of the form

$$\sum_{j=0}^{\infty} x^j P_j(z, z', y, D_y).$$

That is, they are differential operators in y with coefficients which are smoothing operators in z and smooth in y. Then (124) implies that the ranges of these two formal power series operators lie in the null space of Δ_x (in the sense of formal power series). Since the latter space has finite dimensional coefficients, and a differential operator with finite dimensional range must vanish, the f_i vanish as Taylor series in x. By definition this means that (121) is entire. This is the "normal regularity" of G.

Finally then observe that (remembering the density factors)

$$f \in \mathscr{C}^\infty(X; {}^{\mathscr{V}}\Lambda^k) \Rightarrow x^2 G(f) \in \mathscr{A}(X; {}^{\mathscr{V}}\Lambda^k) \cap \mathscr{A}'(X; {}^{\mathscr{V}}\Lambda^k).$$

Here the dual space is characterized [9] by the condition that for any element $u, x^t u/\Gamma(t)$ is entire. In [8] or [9] (see also [6, Volume 3, Chapter 18]) it is shown that on any manifold with boundary

$$\mathscr{A}(X) \cap \mathscr{A}'(X) = \mathscr{C}^\infty(X).$$

This completes the proof of Lemma (117).

13. Proof of Theorem (15) and Corollary (18)

To complete the proof of Theorem (15) we only need to improve the parametrix constructed above to a precise generalized inverse, i.e., inverse modulo a finite rank remainder. By finding an appropriate formal power series we can add to $\tilde{G}$ a term $G_3 \in x^{-1}\mathscr{C}^\infty(Z, \mathrm{Hom}({}^{\mathscr{V}}\Lambda^k))$, to remove all but an error of rank $b_k = \dim H^k(M)$ so that $G = \tilde{G} + G_3$ satisfies

$$\begin{gathered} \Delta_x(G) = \mathrm{Id} + \sum_{i,j=1}^{b_k} a_{ij} h_i(x,y) h_j'(x,y') + F', \\ F' \in \mathscr{C}^\infty(Z, \mathrm{Hom}({}^{\mathscr{V}}\Lambda^k)), \quad F' \cong 0 \text{ at } x = 0. \end{gathered} \tag{125}$$

To see this first note that the leading coefficient of x^{p-2} in F_2 arises directly from the coefficient of x^{p-2} in $\tilde{G} - \tilde{G}_2$, at least away from Q where this is $\mathscr{C}^\infty$. Thus it must be in the range of Δ_F. In particular we can remove it by adding to $\tilde{G}$ a term with leading power x^{p-2} and $\mathscr{C}^\infty$ coefficient. Thus we can assume that $F_2 \in x^{p-1}\mathscr{C}^\infty(Z, \mathrm{Hom}({}^{\mathscr{V}}\Lambda^k))$. Using Lemma (37) the coefficient of x^{p-1} in F_2 can be similarly removed, without making $\tilde{G}$ more singular, since it must also be in the range of Δ_F. Thus we can assume that $F_2 \in x^p\mathscr{C}^\infty(Z, \mathrm{Hom}(^{\mathscr{V}}\Lambda^k))$. Now, solving in formal power series we can remove any terms orthogonal to the null space of Δ_x, at the expense of adding to $\tilde{G}$ a term in $x^{p-N}\mathscr{C}^\infty(Z, \mathrm{Hom}(^{\mathscr{V}}\Lambda^k))$ where N is given in Corollary (68), and is bounded by $p+1$. Thus the new term G_3 is such that xG_3 is $\mathscr{C}^\infty$ and (125) holds.

For x small the operator $\mathrm{Id} + F'$ can be inverted, giving an operator of the same type. Since composition of G with an operator such as F' gives another operator with $\mathscr{C}^\infty$ kernel vanishing to all orders at $x = 0$ the extra term G_3 in (125) can be chosen so that $F' = 0$ identically near $x = 0$. Thus

$$(126)\quad \Delta_x(G) = \mathrm{Id} + \sum_{i,j=1}^{b_k} a_{ij}(x)h_i(x,y)h_j'(x,y'), \qquad h_i, h_i' \in \mathscr{C}^\infty(X; {}^{\mathscr{V}}\Lambda^k),$$

and hence the rank of the remainder term is precisely the dimension of the cohomology.

Since the operator Δ_x is self-adjoint and has range of codimension b_k (when $x > 0$), the $\mathscr{C}^\infty$ sections h_i' must lie in its null space. These provide the basis of the bundle ${}^{\mathscr{V}}\tilde{\mathscr{H}}^k_{\mathrm{HL}}$, considered as a subbundle of $\mathscr{C}^\infty(X, {}^{\mathscr{V}}\Lambda^k)$. Thus the proof of Theorem (15) is complete.

Corollary (18) also follows immediately. That $\tilde{\mathscr{H}}^k_{\mathrm{HL}}$ extends to be $\mathscr{C}^\infty$ as a subspace of the usual form bundle $\mathscr{C}^\infty(X; \Lambda^k)$ follows from the formal power series discussion (see §6).

Various solvability properties of Δ_x also follow from (126).

(127) Corollary. *If f is a $\mathscr{C}^\infty$ form on M depending smoothly on $x \in [0, \infty)$, then there exists a smooth form $u \in \mathscr{C}^\infty(X; \Lambda^k M)$ such that*

$$\Delta_x[x^{-p-1}u] - f \in \mathscr{C}^\infty(X; \Lambda^k M) \textit{ is } g_x\textit{-harmonic.}$$

One can also give L^2-continuity results for the projection onto the harmonic part by noting the (straightforward) L^2-boundedness properties of the parametrix, and hence generalized inverse.

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