J. DIFFERENTIAL GEOMETRY
31 (1990) 47-72

ANALOGS OF LEFSCHETZ THEOREMS FOR LINEAR SYSTEMS WITH ISOLATED SINGULARITIES

B. G. MOISHEZON

In [6] we briefly described the construction of the first examples of (orientation preserving) homeomorphic and not diffeomorphic surfaces of general type. This construction depended on the existence of "big monodromy diffeomorphism groups" for some classes of algebraic surfaces, which can be deduced from results of Ebeling ([3], [4]) on isolated singular points. To relate "local" and "global" in our context we need some analogs of classical Lefschetz theorems on homologies and vanishing cycles.

In the present article we give a detailed construction of homeomorphic and not diffeomorphic surfaces of general type (§4). We also provide the proofs of all necessary facts on homologies and vanishing cycles of complex algebraic varieties which cannot be found in the literature. Our exposition of these facts is such that it can be used for future references. (In future development of Donaldson theory we expect more examples and a better understanding of homeomorphic and not diffeomorphic surfaces of general type.)

This article is actually a result of some very fruitful discussions with R. Friedman to whom I would like to express my gratitude.

All homology groups which we consider have integral coefficients.

1. Vanishing cycles for holomorphic maps

Let $f\colon W \to T$ be a holomorphic map of connected complex manifolds W and T.

Received August 25, 1987 and, in revised form, June 15, 1988. This material is based upon work partially supported by the National Science Foundation under Grant No. DMS-85-03743.

Definition 1. Let

$$S(f) = \{x \in W | df_x \text{ is not surjective}\},$$
$$q(f) = \{x \in S(f) | f|_{S(f)} \text{ is quasifinite at } x, \text{ that is } x \text{ is an isolated point in } (f|_{S(f)})^{-1} f(x)\}.$$

Remark. $S(f)$ is a closed complex subvariety in W, and $q(f)$ is open in $S(f)$. Thus $q(f)$ is a locally closed complex subvariety in W.

Assume that $f\colon W \to T$ is proper and surjective. Let $m = \dim_{\mathbb{C}} W - \dim_{\mathbb{C}} T$.

Definition 2. Let

$$S_b(f) = f(S(f)), \qquad q_b(f) = f(q(f)).$$

$S_b(f)$ is a proper subvariety in T, so that $T - S_b(f)$ is connected and also open dense in T. It is well known that $f|_{W - f^{-1}(S_b(f))}\colon W - f^{-1}(S_b(f)) \to T - S_b(f)$ is a C^∞-bundle. Denote $E_t = f^{-1}(t)$ for all t. For any $t_1, t_2 \in T - S_b(f)$ and any path γ in $T - S_b(f)$ connecting t_1 with t_2 we denote by ψ_γ a diffeomorphism $E_{t_1} \to E_{t_2}$ induced by γ and by $\psi_{\gamma^*} = H_m(\psi_\gamma)$.

Take $v \in q(f)$ and let $s = f(v)$. By definition of $q(f)$ the point v is an isolated singularity of $f^{-1}(s)$, which is a local complete intersection since W and T are nonsingular. Taking small neighborhoods $\tilde{U}_v$ of v in W and U_s of s in T with $f(\tilde{U}_v) = U_s$ we can consider $f|_{\tilde{U}_v}\colon \tilde{U}_v \to U_s$ as a local deformation of $(f^{-1}(s) \cap \tilde{U}_v, v)$. We can embed $f|_{\tilde{U}_v}\colon \tilde{U}_v \to U_s$ in a versal family of deformations of $(f^{-1}(s) \cap \tilde{U}_v, v)$ (see [7]). Denote this versal family by $F_v\colon U_v^\frown \to U_s^\frown$. So we identify U_s with a closed analytic subset in $U_s^\frown$, $\tilde{U}_v$ with $F_v^{-1}(U_s)$ and $f|_{\tilde{U}_v}\colon \tilde{U}_v \to U_s$ with

$$F_v|_{F_v^{-1}(U_s)}\colon F_v^{-1}(U_s) \to U_s.$$

Let $D_v = \{\tau \in U_s^\frown | F_v^{-1}(\tau) \text{ is singular}\}$. It is known that D_v is irreducible, of codimension one, and that for all nonsingular points $\tau \in D_v$, $F_v^{-1}(\tau)$ has only one singular point which is a nondegenerate quadratic singularity. Denote $D'_v = \{\tau \in D_v | \tau \text{ is nonsingular}\}$, and let c_τ, $\tau \in D'_v$, be the singular point of $F_v^{-1}(\tau)$.

For any $\tau_0 \in D'_v$ there exists a small neighborhood d_{τ_0} of τ_0 in $U_s^\frown$ such that $\forall \tau \in d_{\tau_0} - d_{\tau_0} \cap D_v$ on $F_v^{-1}(\tau)$ a (closed) Milnor fiber corresponding to c_{τ_0} is defined (see [7]). Denote such a Milnor fiber by $M(\tau, c_{\tau_0})$. c_{τ_0} is a nondegenerate quadratic singularity, so $H_m(M(\tau, c_{\tau_0}))$ is infinite cyclic. Choosing a generator in $H_m(M(\tau, c_{\tau_0}))$ we get the so-called vanishing cycle in $M(\tau, c_{\tau_0})$. Denote it by $\delta(\tau, c_{\tau_0})$. The class $\delta(\tau, c_{\tau_0})$ can be represented by a smooth m-sphere on $\operatorname{Int}(M(\tau, c_{\tau_0}))$ which we shall denote by $\boldsymbol{\delta}(\tau, c_{\tau_0})$. Taking a loop $\gamma(\tau_0, c_{\tau_0})$ representing the "positive" generator of $\pi_1(d_{\tau_0} - d_{\tau_0} \cap D_v, \tau)$ and considering the corresponding "monodromy" in $F_v\colon U_v^\frown \to U_s^\frown$ we get a diffeomorphism $\Theta(\tau, c_{\tau_0})\colon M(\tau, c_{\tau_0}) \to M(\tau, c_{\tau_0})$ which is identity on $\partial M(\tau, c_{\tau_0})$. Extending this diffeomorphism by identity to $F_v^{-1}(\tau) \to F_v^{-1}(\tau)$ we get $\Theta^\frown(\tau, c_{\tau_0}) \in \operatorname{Diff}(F_v^{-1}(\tau))$ which we shall call a *Dehn twist* of $F_v^{-1}(\tau)$ defined by $\boldsymbol{\delta}(\tau, c_{\tau_0})$. We should remember that $\Theta^\frown(\tau, c_{\tau_0})$ is identity outside a regular neighborhood of $\boldsymbol{\delta}(\tau, c_{\tau_0})$ in $F_v^{-1}(\tau)$, and that $\Theta^\frown(\tau, c_{\tau_0})$ is well defined up to homotopy by c_{τ_0} and τ sufficiently close to τ_0 in $U_s^\frown$.

Considering small closed balls $B_s^\frown$ in $U_s^\frown$ with the center s and $B_v^\frown$ in $U_v^\frown$ with the center v such that $F_v^{-1}(\tau)$ is transversal to $\partial B_v^\frown$ for any $\tau \in B_s^\frown$, we can replace $U_s^\frown$ by $\operatorname{Int}(B_s^\frown)$ and $U_v^\frown$ by $F_v^{-1}(\operatorname{Int} B_s^\frown) \cap \operatorname{Int}(B_v^\frown)$. Thus we will have well-defined boundaries for $U_s^\frown$, $U_v^\frown$, $F_v^{-1}(\tau)$ $\forall \tau \in U_s^\frown$. In particular we will have a trivial C^∞-bundle

$$F_v|_{\cup_{\tau\in U_s^\frown}(\partial(F_v^{-1}(\tau))}\colon \bigcup_{\tau\in U_s^\frown} (\partial(F_v^{-1}(\tau)) \to U_s^\frown$$

and a C^∞-bundle

$$F_v\colon U_v^\frown \cup \partial(F_v^{-1}(U_s^\frown)) - F_v^{-1}(D_v) \to U_s^\frown - D_v$$

consistent with a trivial C^∞-bundle structure for

$$F_v|_{\cup_{\tau\in U_s^\frown}(\partial(F_v^{-1}(\tau))}\colon \bigcup_{\tau\in U_s^\frown} (\partial(F_v^{-1}(\tau)) \to U_s^\frown.$$

For all $\tau \in U_s^\frown - D_v$ we denote $F_v^{-1}(\tau) \cup \partial(F_v^{-1}(\tau))$ (in older notation $F_v^{-1}(\tau) \cap B_v^\frown$) by $M(\tau, v)$, and call it a Milnor fiber corresponding to v.

For all $\tau_1, \tau_2 \in U_s^\frown - D_v$ and any path γ in $U_s^\frown - D_v$ connecting τ_1 with τ_2 we denote by $\psi_\gamma^\frown$ a diffeomorphism induced by γ from $M(\tau_1, v)$ to $M(\tau_2, v)$ which is an identity on the boundary with respect to a trivialization of $F_v|_{\partial(F_v^{-1}(U_s^\frown))}$ chosen above.

Fix $\tau_1 \in U_s^\frown - D_v$. Take any $\tau_0 \in D_v'$ and any simple path γ from τ_0 to τ_1 with $\gamma \cap D_v = \tau_0$. Considering as above a small neighborhood d_{τ_0} of τ_0 in $U_s^\frown$ and $\tau' \in \gamma \cap (d_{\tau_0} - d_{\tau_0} \cap D_v)$ we get a Milnor fiber $M(\tau', c_{\tau_0}) \subset M(\tau', v)$ and a smooth sphere $\boldsymbol{\delta}(\tau', c_{\tau_0})$ representing a vanishing cycle in $M(\tau', c_{\tau_0})$.

Let γ' be part of γ from τ' to τ_1; we can assume that γ' is a simple path. Denote $\boldsymbol{\delta}(\tau_1, c_{\tau_0}; \gamma') = \psi_{\gamma'}^\frown(\boldsymbol{\delta}(\tau', c_{\tau_0}))$. We shall say that $\boldsymbol{\delta}(\tau_1, c_{\tau_0}; \gamma')$ represents a *vanishing cycle* in $M(\tau_1, v)$ which is an element $H_m(M(\tau_1, v))$ denoted by $\delta(\tau_1, \tau_0; \gamma)$. Extending $\Theta^\frown(\tau', c_{\tau_0})$ by identity to a diffeomorphism of $M(\tau', v)$ and denoting it again by $\Theta^\frown(\tau', c_{\tau_0})$ we define a *Dehn twist* of $M(\tau_1, v)$ determined by $\boldsymbol{\delta}(\tau_1, c_{\tau_0}; \gamma')$ as follows:

$$\Theta^\frown(\boldsymbol{\delta}(\tau_1, c_{\tau_0}; \gamma')) = \psi_{\gamma'}^\frown \circ \Theta^\frown(\tau', c_{\tau_0}) \circ (\psi_{\gamma'}^\frown)^{-1}.$$

Denote by $\Lambda^{\frown}(\tau_1, v)$ the set of all vanishing cycles $\{\delta(\tau_1, \tau_0; \gamma)\}$ obtained for all $\tau_0 \in D'_v$ and paths γ from τ_0 to τ_1 as above.

For any compact C^∞-manifold $\mathfrak{X}$ with the boundary $\partial\mathfrak{X}$ we denote by $\mathfrak{M}\mathrm{ap}(\mathfrak{X}, \partial\mathfrak{X})$ the group of homotopy classes of diffeomorphisms of $\mathfrak{X}$ the mapping class group of $\mathfrak{X}$, which is identity on the boundary. When $\partial\mathfrak{X} = \varnothing$ we denote this group by $\mathfrak{M}\mathrm{ap}(\mathfrak{X})$, mapping class group of $\mathfrak{X}$.

Each $\Theta^{\frown}(\boldsymbol{\delta}(\tau', c_{\tau_0}; \gamma'))$ defines uniquely an element in $\mathfrak{M}\mathrm{ap}(M(\tau_1, v), \partial M(\tau_1, v))$ which depends only on the choice of τ_0 and γ. We denote this element by $\Theta^{\frown}(\tau_1, \tau_0; \gamma)$ and call it the *Dehn twist* defined by a smooth sphere $\boldsymbol{\delta}(\tau_1, c_{\tau_0}; \gamma')$ representing a vanishing cycle in $M(\tau_1, v)$.

Denote by $G^{\frown}(\tau_1, v)$ the subgroup of $\mathfrak{M}\mathrm{ap}(M(\tau_1, v), \partial M(\tau_1, v))$ generated by *Dehn twists* $\Theta^{\frown}(\tau_1, \tau_0; \gamma)$ corresponding to all choices of τ_0 and γ as above. $G^{\frown}(\tau_1, v)$ acts naturally on $H_m(M(\tau_1, v))$ and on the set $\Lambda^{\frown}(\tau_1, v)$ of all vanishing cycles in $M(\tau_1, v)$.

From the irreducibility of D_v it follows that $G^{\frown}(\tau_1, v)$ acts transitively on $\Lambda^{\frown}(\tau_1, v)$; actually $\Lambda^{\frown}(\tau_1, v)$ is a $G^{\frown}(\tau_1, v)$-orbit in $H_m(M(\tau_1, v))$ (see [7]).

Now take $\tau_1 \in U_s - S_b(f) = U_s - D_v \cap U_s$. From $M(\tau_1, v) \subset E_{\tau_1}$ and the corresponding $i_m \colon H_m(M(\tau_1, v)) \to H_m(E_{\tau_1})$ we obtain the set $\Lambda(\tau_1, v) = i_m(\Lambda^{\frown}(\tau_1, v))$. Elements of $\Lambda(\tau_1, v)$ we call vanishing cycles induced by v on the neighboring fiber.

Using $M(\tau_1, v) \subset E_{\tau_1}$ we extend by identity each $\Theta^{\frown}(\tau_1, \tau_0; \gamma)$ as above to an element of $\mathfrak{M}\mathrm{ap}(E_{\tau_1})$, which we denote by $\Theta(\tau_1, v; \gamma)$. $\Theta(\tau_1, v; \gamma)$ defines an automorphism of $H_m(E_{\tau_1})$. We denote the last automorphism by $\theta(\tau_1, \tau_0; \gamma)$ and call it the *Picard-Lefschetz transformation* defined by the vanishing cycle $\delta = i_m(\delta(\tau_1, \tau_0; \gamma))$. It is convenient to write θ_δ instead of $\theta(\tau_1, \tau_0; \gamma)$.

It is known that θ_δ is defined by the formula

$$\theta_\delta(z) = z + (-1)^{(m+1)(m+2)/2}(z \cdot \delta)\delta.$$

Now take any $t \in T - S_b(f)$. For any $s \in q_b(f)$, $v \in q(f)$, $f(v) = s$, and any simple path Γ connecting s with t such that $\Gamma \cap S_b(f) = s$ we can choose $\tau_1 \in \Gamma - s$ sufficiently close to s, and denoting by Γ' the part of Γ from τ_1 to t we define the set

$$\Lambda(t, v, \Gamma) = \psi_{\Gamma'*}(\Lambda(\tau_1, v)).$$

Elements of $\Lambda(t, v, \Gamma)$ we call vanishing cycles on E_t induced by v and Γ.

Using elements $\Theta(\tau_1, v; \gamma) \in \mathfrak{M}\mathrm{ap}(E_{\tau_1})$ defined above, we define elements $\Theta(t, v; \gamma, \Gamma) \in \mathfrak{M}\mathrm{ap}(E_t)$ by

$$\Theta(t, v; \gamma, \Gamma) = \psi_{\Gamma'} \circ \Theta(\tau_1, v; \gamma) \circ (\psi_{\Gamma'})^{-1}.$$

If δ is an element of $\Lambda(\tau_1, v)$ corresponding to $\Theta(\tau_1, v; \gamma)$, and $\delta_t = \psi_{\Gamma'*}(\delta) \in \Lambda(t, v, \Gamma)$, we define the Picard-Lefschetz transformation $\theta_{\delta_t} \in \operatorname{Aut}(H_m(E_t))$ by $\theta_{\delta_t}(z) = z + (-1)^{(m+1)(m+2)/2}(z \cdot \delta_t)\delta_t$. It is clear that θ_{δ_t} is induced by a diffeomorphism of E_t corresponding to $\Theta(t, v; \gamma, \Gamma)$.

Definition 3. Denote by $\Lambda(t)$ the union of $\Lambda(t, v, \Gamma)$ for all $v \in q(f)$ and all Γ in T connecting $f(v)$ with t and such that $\Gamma \cap S_b(f) = f(v)$. We call the elements of $\Lambda(t)$ the *vanishing cycles* in E_t. For any $\delta_t \in \Lambda(t)$ we have a *Picard-Lefschetz transformation* θ_{δ_t} corresponding to δ_t.

Remark. Remember that each θ_{δ_t} is induced by a diffeomorphism of E_t, a "Dehn twist" corresponding to a smooth m-sphere $\boldsymbol{\delta}_t$ representing the class $\delta_t \in H_m(E_t)$.

Denote by $G(t)$ (resp. $G(t, v; \Gamma)$) the subgroup of $\operatorname{Aut}(H_m(E_t))$ generated by all $\theta_{\delta_t} \in \Lambda(t)$ (resp. by all $\delta_t \in \Lambda(t, v; \Gamma)$).

It follows from above that $G(t, v; \Gamma)$ acts transitively on $\Lambda(t, v; \Gamma)$. In particular, each $\Lambda(t, v; \Gamma)$ belongs to a single $G(t)$-orbit in $\Lambda(t)$.

Denote by $\Psi_t \colon \pi_1(T - S_b(f), t) \to \operatorname{Aut}(H_m(E_t))$ the homomorphism defined by the C^∞-bundle structure on $f|_{W - f^{-1}(S_b(f))} \colon W - f^{-1}(S_b(f)) \to T - S_b(f)$ (a "monodromy homomorphism" of $f \colon W \to T$).

Proposition 1. *Assume that $q(f)$ is connected and that* $\operatorname{Im}\Psi_t$ *belongs to $G(t)$. Then $\Lambda(t)$ belongs to only one $G(t)$-orbit in $H_m(E_t)$.*

Proof.

Claim 1. $\forall v \in q(f)$ let $\tilde{U}_v$ and $U_{f(v)}$ be small neighborhoods respectively in W and T considered above. Then $\forall v_0 \in q(f)$ there exists an open neighborhood Q_{v_0} in $q(f)$ such that

(1) $Q_{v_0} \subset f^{-1}(U_{f(v_0)})$;

(2) $\forall v \in Q_{v_0}$ and $t \in (U_{f(v_0)} - U_{f(v_0)} \cap S_b(f)) \cap (U_{f(v)} - U_{f(v)} \cap S_b(f))$, $\Lambda(t, v)$ and $\Lambda(t, v_0)$ belong to one and the same $G(t)$-orbit in $H_m(E_t)$.

Proof. Let $F_{v_0} \colon U_{v_0}^{\frown} \to U_{f(v_0)}^{\frown}$ be a versal family for $(E_{f(v_0)} \cap \tilde{U}_{v_0}, v_0)$ such that $U_{f(v_0)}$ can be identified with a closed analytic subset in $U_{f(v_0)}^{\frown}$, $\tilde{U}_{v_0}$ with $F_{v_0}^{-1}(U_{f(v_0)})$ and $f|_{\tilde{U}_{v_0}} \colon \tilde{U}_{v_0} \to U_{f(v_0)}$ with $F_{v_0}|_{\tilde{U}_{v_0}} \colon F_{v_0}^{-1}(U_{f(v_0)}) \to U_{f(v_0)}$. Denote $\tilde{D}_{v_0} = S(f) \cap \tilde{U}_{v_0}$. Taking $\tilde{U}_{v_0}$ smaller we can assume that $\forall v \in \tilde{D}_{v_0}$ $\exists$ neighborhoods $Y_v^{\frown}$ of v in $U_{v_0}^{\frown}$, $Y_{f(v)}^{\frown}$ of $f(v)$ in $U_{f(v)}^{\frown}$, $\tilde{Y}_v$ of v in $\tilde{U}_v$ and $Y_{f(v)}$ in $U_{f(v)}$ such that $F_v(Y_v^{\frown}) = Y_{f(v)}^{\frown}$ and $F_v|_{Y_v^{\frown}} \colon Y_v^{\frown} \to Y_{f(v)}^{\frown}$ is a versal family for $(E_{f(v)} \cap \tilde{Y}_v, v)$, $Y_{f(v)}$ is a closed analytic subset in $Y_{f(v)}^{\frown}$, $\tilde{Y}_v$ coincides with $(F_v|_{Y_v^{\frown}})^{-1}(Y_v)$, and $F_v|_{\tilde{Y}_v} \colon \tilde{Y}_v \to Y_{f(v)}$ coincides with $f|_{\tilde{Y}_v} \colon \tilde{Y}_v \to Y_{f(v)}$.

Take any $t_1 \in Y_{f(v)} - Y_{f(v)} \cap S_b(f)$. Then from the definition of $\Lambda(t, v)$ it follows that $\Lambda^{\frown}(t_1, v) \subseteq \Lambda^{\frown}(t_1, v_0)$ and $G^{\frown}(t_1, v) \subset G^{\frown}(t_1, v_0)$.

Since $G^\frown(t_1, v)$ (resp. $G^\frown(t_1, v_0)$) acts transitively on $\Lambda^\frown(t_1, v)$ (resp. on $\Lambda^\frown(t_1, v_0)$), we see that $\Lambda^\frown(t_1, v)$ and $\Lambda^\frown(t_1, v_0)$ belong to one and the same $G^\frown(t_1, v_0)$-orbit in $H_m(M(t_1, v_0))$.

The mapping $i_m: H_m(M(t_1, v_0)) \to H_m(E_{t_1})$ induces a homomorphism $G^\frown(t_1, v_0)$ in $G(t_1)$. Thus we see that $\Lambda(t_1, v)$ and $\Lambda(t_1, v_0)$, which are i_m-images of $\Lambda^\frown(t_1, v)$ and $\Lambda^\frown(t_1, v_0)$, belong to one and the same orbit of $G(t_1)$ in $H_m(E_{t_1})$.

Now let $Q_{v_0} = \tilde{U}_{v_0} \cap q(f) = \tilde{U}_{v_0} \cap S(f)$, and recall that $q(f)$ is open in $S(f)$. Denote $U'_{f(v)} = U_{f(v)} - S_b(f) \cap U_{f(v)}$, and take any $v \in Q_{v_0}$ and $t \in U'_{f(v_0)} \cap U'_{f(v)}$. Considering for v the point t_1 as above we can assume also that $t_1 \in U'_{f(v_0)} \cap U'_{f(v)}$. Let γ' be a simple path in $U'_{f(v_0)} \cap U'_{f(v)}$ connecting t_1 with t.

From our definitions it follows that $\Lambda(t, v) = \psi_{\gamma'*}\Lambda(t_1, v)$, $\Lambda(t, v_0) = \psi_{\gamma'*}\Lambda(t_1, v_0)$, and that the isomorphism $\psi_{\gamma'*}$ induces an isomorphism from $G(t_1)$ to $G(t)$. Thus $\Lambda(t, v)$ and $\Lambda(t, v_0)$ belong to one and the same orbit of $G(t)$ in $H_m(E_t)$. q.e.d.

We continue to use the notation $U'_{f(v)} = U_{f(v)} - U_{f(v)} \cap S_b(f)$ for $v \in q(f)$.

Claim 2. Let $v_1, v_2 \in q(f)$, $t_1 \in U'_{f(v_1)}$, $t_2 \in U'_{f(v_2)}$. Then $\exists$ a path in $T - S_b(f)$ connecting t_1 with t_2 such that $\psi_{\gamma*}(\Lambda(t_1, v_1)$ and $\Lambda(t_2, v_2)$ belong to one and the same $G(t_2)$-orbit in $H_m(E_{t_2})$.

Proof. Let $\tilde{\gamma}$ be a path in $q(f)$ connecting v_1 with v_2, and recall that $q(f)$ is connected. Using open neighborhoods Q_v constructed in Claim 1 above we get an open covering $\{Q_v, v \in \tilde{\gamma}\}$ of $\tilde{\gamma}$ in $q(f)$. Take from it a finite open covering $\{Q_{v^{(l)}}, l = 1, \cdots, p\}$ such that $v^{(1)} = v_1$, $v^{(p)} = v_2$ and $\forall l \in \{1, \cdots, p-1\}$, $Q_{v^{(l)}} \cap Q_{v^{(l+1)}} \neq \varnothing$.

In each $Q_{v^{(l)}} \cap Q_{v^{(l+1)}}$, $l \in \{1, \cdots, p-1\}$, choose a point $w^{(l)}$. Clearly

$$f(w^{(l)}) \in U_{f(v^{(l)})} \cap U_{f(v^{(l+1)})} \quad \text{(Claim 1(1))},$$

so

$$U'_{f(w^{(l)})} \cap U'_{f(v^{(l)})} \cap U'_{f(v^{(l+1)})} \neq \varnothing.$$

Take $t^{(l)} \in U'_{f(w^{(l)})} \cap U'_{f(v^{(l)})} \cap U'_{f(v^{(l+1)})}$. By Claim 1 $\Lambda(t^{(l)}, w^{(l)})$ and $\Lambda(t^{(l)}, v^{(l)})$ (resp. $\Lambda(t^{(l)}, w^{(l)})$ and $\Lambda(t^{(l)}, v^{(l+1)})$) belong to one and the same $G(t^{(l)})$-orbit in $H_m(E_{t^{(l)}})$. Thus $\Lambda(t^{(l)}, v^{(l)})$ and $\Lambda(t^{(l)}, v^{(l+1)})$ belong to one and the same $G(t^{(l)})$-orbit in $H_m(E_{t^{(l)}})$. Let $t^{(0)} = t_1$ and $t^{(p)} = t_2$. So we get the sequence $\{t^{(l)}, l = 0, 1, \cdots, p\}$. It is clear that $\forall l \in (0, 1, \cdots, p-1)$ the points $t^{(l)}, t^{(l+1)} \in U'_{f(v^{(l+1)})}$. Let $\gamma(l)$ be a simple path in $U'_{f(v^{(l+1)})}$ connecting $t^{(l)}$ with $t^{(l+1)}$. From the definition of $\Lambda(t, v)$ it follows that

$$\psi_{\gamma(l)*}(\Lambda(t^{(l)}, v^{(l+1)}) = \Lambda(t^{(l+1)}, v^{(l+1)}).$$

Let $v^{(0)} = v_1$. By $\psi_{\gamma(l)^*}(G(t^{(l)})) = G(t^{(l+1)})$ we see that $\psi_{\gamma(l)^*}(\Lambda(t^{(l)}, v^{(l)}))$ and $\Lambda(t^{(l+1)}, v^{(l+1)})$ belong to one and the same $G(t^{(l+1)})$-orbit in $H_m(E_{t^{(l+1)}})$.

Similarly, if $\gamma = \gamma(0) \circ \gamma(1) \circ \cdots \circ \gamma(p-1)$, then $\psi_{\gamma^*}(\Lambda(t^{(0)}, v^{(0)}))$ and $\Lambda(t^{(p)}, v^{(p)})$ belong to one and the same $G(t^{(p)})$-orbit in $H_m(E_{t^{(p)}})$.

Now recall that $v^{(0)} = v_1$, $t^{(0)} = t_1$, $v^{(p)} = v_2$, and $t^{(p)} = t_2$. q.e.d.

Consider now any $t \in T - S_b(f)$ and any $\delta_1, \delta_2 \in \Lambda(t)$. From the definition of $\Lambda(t)$ it follows that for each δ_i $(i = 1, 2)$ there exist $v_i \in q(f)$, t_i in $U'_{f(v_i)}$, path Γ'_i in $T - S_b(f)$ connecting t_i with t and $\delta'_i \in \Lambda(t_i, v_i)$ $(\subset H_m(E_{t_i}))$ such that $\delta_i = \psi_{\Gamma'^*_i}(\delta'_i)$. Using Claim 2 we see that there exists a path γ in $T - S_b(f)$ connecting t_1 with t_2 such that $\psi_{\gamma^*}(\Lambda(t_1, v_1))$ and $\Lambda(t_2, v_2)$ belong to one and the same $G(t_2)$-orbit in $H_m(E_{t_2})$. Let $\Gamma = \Gamma'^{-1}_1 \circ \gamma \circ \Gamma'_2$. From $\Lambda(t, v_i, \Gamma'_i) = \psi_{\Gamma'^*_i}\Lambda(t_i, v_i)$ $(i = 1, 2)$ it follows that $\Psi(\Gamma)(\Lambda(t, v_1, \Gamma'_1))$ and $\Lambda(t, v_2, \Gamma'_2)$ belong to one and the same $G(t)$-orbit in $H_m(E_t)$. By our assumptions $\Psi(\Gamma) \in G(t)$. So $\Lambda(t, v_1, \Gamma'_1)$ and $\Lambda(t, v_2, \Gamma'_2)$, and in particular δ_1 and δ_2 belong to one and the same orbit of $G(t)$ in $H_m(E_t)$. q.e.d.

Assume now that $\dim_{\mathbb{C}} T = 1$. Consider again $v \in q(v)$, $s \in q_b(f)$, $f(v) = s$, small neighborhoods U_s of s in T, $\tilde{U}_v$ of v in W and a versal family $F_v: U_v^{\frown} \to U_s^{\frown}$ such that U_s can be identified with a closed analytic subset in $U_s^{\frown}$, $\tilde{U}_v$ with $F_v^{-1}(U_s)$ and $f|_{\tilde{U}_v}: \tilde{U}_v \to U_s$ with $F_v|_{F_v^{-1}(U_s)}: F_v^{-1}(U_s) \to U_s$.

As above denote $D_v = \{\tau \in U_s^{\frown} \,|\, F_v^{-1}(\tau) \text{ is singular}\}$ and D'_v the nonsingular part of D_v. Let $Z = D_v - D'_v$. If $Z \not\ni s$ we say that $f_{(0)} = f|_{\tilde{U}_v}: \tilde{U}_v \to U_s$ is *stable*. It is equivalent to say that v is a nondegenerate quadratic singularity in $f_{(0)}^{-1}(s)$.

Assume $Z \ni s$. Because $U_s^{\frown}$ and U_s are nonsingular, $U_s \cap Z = s$ and $\operatorname{codim}_{U_s^{\frown}} Z \geq 2$, there exists a nonsingular 2-dimensional complex analytic subset $U_s^{(2)}$ in $U_s^{\frown}$ such that $\tilde{U}_v^{(2)} = F_v^{-1}(U_s^{(2)})$ is nonsingular, $U_s^{(2)} \supset U_s$ and $U_s^{(2)} \cap Z = s$. Taking all neighborhoods smaller we can assume that $U_s^{(2)} = U_s \times \Delta$, where Δ is an open disc in $\mathbb{C}^1$, that $\tilde{U}_v^{(2)} = \tilde{U}_v \times \Delta$, $\Delta \ni (0)$, embeddings $U_s \subset U_s^{(2)}$ and $\tilde{U}_v \subset U_v^{(2)}$ coincide with $U_s \times 0 \subset U_s \times \Delta$ and $\tilde{U}_v \subset \tilde{U}_v \times \Delta$, and that

$$\begin{array}{ccc} (\tilde{U}_v \times \Delta =) U_v^{(2)} & & \\ \downarrow {\scriptstyle F_v|_{U_v^{(2)}}} & \searrow^{\text{proj}} & \\ (U_s \times \Delta =) U_s^{(2)} & \xrightarrow[\text{proj}]{} & \Delta \end{array}$$

is commutative.

Let $g = F_v|_{U_v^{(2)}}: U_v^{(2)} \to U_s^{(2)}$ and, $\forall \lambda \in \Delta$, $f_{(\lambda)} = g|_{\tilde{U}_v \times \lambda}: \tilde{U}_v \times \lambda \to U_s \times \lambda$. Let $C_v = D_v \cap U_s^{(2)}$. Since $Z \cap U_s^{(2)} = s$ $(= s \times 0)$, we conclude that

$\forall \lambda \in \Delta - 0$, $f_{(\lambda)}$ has only "stable singularities", that is, on each singular fiber of $f_{(\lambda)}$ corresponding to a point of $C_v \cap (U_s \times \lambda)$ there is only one singularity which is nondegenerate quadratic. We call $g\colon \tilde{U}_v^{(2)} \to U_s^{(2)}$ *a stabilizing family* for $f|_{\tilde{U}_v}\colon \tilde{U}_v \to U_s$ and each $f_{(\lambda)}$, $\lambda \neq 0$, we call a stabilizer of $f|_{\tilde{U}_v}$ $(= f_{(0)})$. For the uniformity of notation in the case when $f|_{\tilde{U}_v}$ is stable, we can take $g = (f|_{\tilde{U}_v}) \times \mathrm{Id}$.

Consider $\tilde{U}_v$ as an open set in $\mathbb{C}^n$ ($n = \dim_{\mathbb{C}} W$) and U_s as an open set in $\mathbb{C}^1$. Let D_r be a small (closed) ball in $\tilde{U}_v$ centered at v and of radius r. Taking a positive $\rho \ll r$ we get a closed disc Δ_ρ centered at s and of radius ρ such that:

(i) $\forall t \in \Delta_\rho - s$, $f_{(0)}^{-1}(t)$ has no singularities in D_r, and $f_{(0)}^{-1}(s)$ has only one singularity v in D_r;

(ii) $\forall t \in \Delta_\rho$, $f_{(0)}^{-1}(t)$ intersects transversally with $S_r^{2n-1} = \partial D_r$.

Fixing a point $u_0 \in \partial\Delta_\rho$ we identify $f_{(0)}^{-1}(u_0) \cap D_r$ with a Milnor fiber $M(u_0, v)$ of $f_{(0)}$ over u_0.

Let $g\colon \tilde{U}_v \times \Delta \to U_s \times \Delta$ be a stabilizing family of $f_{(0)}$ (as above). Taking a positive $\varepsilon \ll \rho$ and replacing Δ by a disk $\Delta_{\varepsilon,0}$ centered at (0) and of radius ε we can see the following:

(iii) $\forall \lambda \in \Delta_{\varepsilon,0}$, $f_{(\lambda)}$ has no critical points in $f_{(\lambda)}^{-1}(\partial\Delta_\rho \times \lambda)$;

(iv) $\forall \lambda \in \Delta_{\varepsilon,0}$ and $\forall t \in \Delta_\rho \times \lambda$, $f_{(\lambda)}^{-1}$ intersects transversally with $S_r^{2n-1} \times \lambda = \partial D_r \times \lambda$.

Denote $N(f_{(\lambda)}, u_0) = f_{(\lambda)}^{-1}(u_0 \times \lambda) \cap (D_r \times \lambda)$, $N(f_{(\lambda)}) = f_{(\lambda)}^{-1}(\Delta_\rho \times \lambda) \cap (D_r \times \lambda)$, $N(g) = g^{-1}(\Delta_\rho \times \Delta_{\varepsilon,0}) \cap (D_r \times \Delta_{\varepsilon,0})$, and by $p\colon N(g) \to \Delta_{\varepsilon,0}$ the natural projection. Clearly $N(f_{(0)}, u_0)$ coincides with the Milnor fiber $M(u_0, v)$ of $f_{(0)}$ over u_0. From (iii)-(iv) it follows that $p\colon N(g) \to \Delta_{\varepsilon,0}$ is a (trivial) C^∞-bundle, so that we can identify (diffeomorphically) $p\colon N(g) \to \Delta_{\varepsilon,0}$ with $N(f_{(0)}) \times \Delta_{\varepsilon,0} \overset{\mathrm{proj}}{\to} \Delta_{\varepsilon,0}$.

We can choose a trivialization of $p\colon N(g) \to \Delta_{\varepsilon,0}$ so that

$$g^{-1}(u_0 \times \Delta_{\varepsilon,0}) \cap (D_r \times \Delta_{\varepsilon,0}) \quad \Big(= \bigcup_{\lambda \in \Delta_{\varepsilon,0}} f_{(\lambda)}^{-1}(u_0 \times \lambda) \cap (D_r \times \lambda)\Big)$$

will be a subproduct of $N(f_{(0)}) \times \Delta_{\varepsilon,0}$. Choosing a trivialization of $p\colon N(g) \to \Delta_{\varepsilon,0}$ we denote corresponding projection $N(g) \to N(f_{(0)})$ by $q\colon N(g) \to N(f_{(0)})$. Denote by $q_\lambda\colon N(f_{(\lambda)}) \to N(f_{(0)})$ the diffeomorphism defined by the chosen trivialization of $p\colon N(g) \to \Delta_{\varepsilon,0}$, $q_\lambda = q|_{N(f_{(\lambda)})}$.

Take $\lambda \in \Delta_{\varepsilon,0} - (0)$ and let $\mathbf{a}$ be a critical value of $f_{(\lambda)}$ in $\Delta_\rho \times \lambda$. Connect $\mathbf{a}$ with $u_0 \times \lambda$ in $\Delta_\rho \times \lambda$ by a simple path $\gamma\colon [0,1] \to \Delta_\rho \times \lambda$ avoiding other critical values of $f_{(\lambda)}$. Let $c(\mathbf{a})$ be the critical point of $f_{(\lambda)}$ in $f_{(\lambda)}^{-1}(\mathbf{a})$.

Because $c(\mathbf{a})$ is a nondegenerate critical point of $f_{(\lambda)}$ ($\lambda \neq 0$), there exists a continuous map $\varphi_\gamma\colon S^{n-1} \times [0,1] \to N(f_{(\lambda)})$ such that $\forall \mu \in [0,1]$, $\varphi_\gamma(S^{n-1} \times \mu) \subset f_{(\lambda)}^{-1}(\gamma(\mu)) \cap N(f_{(\lambda)})$, $\varphi_\gamma(S^{n-1} \times 0) = c(\mathbf{a})$, and $\forall \mu \in [0,1]$, $\mu \neq 0$, $\varphi_\gamma(S^{n-1} \times \mu)$ is a smooth $(n-1)$-sphere in $\operatorname{Int}(f_{(\lambda)}^{-1}(\gamma(\mu)) \cap (D_r \times \lambda))$ which for μ sufficiently close to (0) represents a generator of H_{n-1} (Milnor fiber of $c(\mathbf{a})$ over $\gamma(\mu)$), and $\operatorname{Im}(\varphi_\gamma)$ is a smooth n-disk in $N(f_{(\lambda)})$ with the boundary equal to $\varphi_\gamma(S^{n-1} \times 1) \subset N(f_{(\lambda)}, u_0)$ $(= f_{(\lambda)}^{-1}(\gamma(1)) \cap N(f_{(\lambda)}))$.

Denote $\Delta_{\mathbf{a},\gamma,u_0} = \operatorname{Im}(\varphi_\gamma)$ and call $\Delta_{\mathbf{a},\gamma,u_0}$ a relative Lefschetz cycle in $N(f_{(\lambda)})$ (corresponding to $\mathbf{a}, \gamma, u_0$).

Clearly $\partial\Delta_{\mathbf{a},\gamma,u_0}$ represents a vanishing cycle in $f_{(\lambda)}^{-1}(u_0 \times \lambda)$.

Definition 4. We say that a smooth n-disk $\Delta(u_0)$ is a relative Lefschetz cycle in $(N(f_{(0)}), N(f_{(0)}, u_0))$ if in some $N(f_{(\lambda)})$, $\lambda \neq 0$, there exists a relative Lefschetz cycle $\Delta_{\mathbf{a},\gamma,u_0}$ such that $\Delta(u_0) = q_\lambda(\Delta_{\mathbf{a},\gamma,u_0})$ (or $\Delta(u_0) = q(\Delta_{\mathbf{a},\gamma,u_0})$). Evidently $\partial\Delta(u_0)$ represents a vanishing cycle in $f_{(0)}^{-1}(u_0)$.

Denote

$$N'(f_{(\lambda)}, u_0) = \partial N(f_{(\lambda)}, u_0),$$

$$N'(f_{(\lambda)}) = \bigcup_{t \in \Delta_\rho \times \lambda} (f_{(\lambda)}^{-1}(t) \cap (\partial D_r \times \lambda)),$$

$$N'(g) = \bigcup_{\lambda \in \Delta_{\varepsilon,0}} N'(f_{(\lambda)}) \quad \left(= \bigcup_{(t,\lambda) \in \Delta_\rho \times \Delta_{\varepsilon,0}} (g^{-1}(t,\lambda) \cap (\partial D_r \times \lambda)) \right).$$

From (iv) above it follows that each $N'(f_{(\lambda)}, u_0)$ is a C^∞-manifold, $f_{(\lambda)}|_{N'(f_{(\lambda)})}\colon N'(f_{(\lambda)}) \to \Delta_\rho$ is a C^∞-bundle with the typical fiber $N'(f_{(\lambda)}, u_0)$, and $g|_{N'(g)}\colon N'(g) \to \Delta_\rho \times \Delta_{\varepsilon,0}$ is a C^∞-bundle with the typical fiber $N'(f_{(0)}, u_0)$.

We can choose trivializations of $p\colon N(g) \to \Delta_{\varepsilon,0}$ (see above), of $f_{(0)}|_{N'(f_{(0)})}\colon N'(f_{(0)}) \to \Delta_\rho$ and of $g|_{N'(g)}\colon N'(g) \to \Delta_\rho \times \Delta_{\varepsilon,0}$, so that after identifying $N(g)$ with $N(f_{(0)}) \times \Delta_{\varepsilon,0}$, $N'(f_{(0)})$ with $N'(f_{(0)}, u_0) \times \Delta_\rho$ and $N'(g)$ with $N'(f_{(0)}, u_0) \times \Delta_\rho \times \Delta_{\varepsilon,0}$ we will get that $(N'(f_{(0)}, u_0) \times \Delta_\rho) \times \Delta_{\varepsilon,0}$ will be a subproduct of $N(f_{(0)}) \times \Delta_{\varepsilon,0}$.

After choosing such trivializations we also have for each $\lambda \in \Delta_{\varepsilon,0}$ a trivialization of $f_{(\lambda)}|_{N'(f_{(\lambda)})}\colon N'(f_{(\lambda)}) \to \Delta_\rho \times \lambda$ such that $q_\lambda\colon N(f_{(\lambda)}) \to N(f_{(0)})$ (obtained from $N(g) = N(f_{(0)}) \times \Delta_{\varepsilon,0} \xrightarrow{\text{proj}} N(f_{(0)})$) transforms it to the chosen trivialization of $f_{(0)}|_{N'(f_{(0)})}\colon N'(f_{(0)}) \to \Delta_\rho$.

Proposition 2. *There exists a finite number of relative Lefshetz cycles* $\Delta_1(u_0), \cdots, \Delta_\nu(u_0)$ *in* $N(f_{(0)})$ (*that is, smooth n-discs in* $N(f_{(0)})$ *with* $\Delta_i(u_0) \cap N(f_{(0)}, u_0) = \partial\Delta_i(u_0)$, $i = 1, \cdots, \nu$, *each* $\partial\Delta_i(u_0)$ *represents a vanishing cycle in* $f_{(0)}^{-1}(u_0)$), *such that* $N(f_{(0)})$ *can be retracted to* $N(f_{(0)}, u_0) \cup (\bigcup_{i=1}^{\nu} \Delta_i(u_0))$. *Moreover, this retraction can be chosen so that on* $N'(f_{(0)})$ *it* *coincides with a retraction of* $N'(f_{(0)})$ *to* $N'(f_{(0)}, u_0)$, *corresponding to the chosen trivialization of* $f_{(0)}|_{N'(f_{(0)})}: N'(f_{(0)}) \to \Delta_\rho$ *and a retraction of* Δ_ρ *to* u_0.

Proof. Take $\lambda \in \Delta_{\varepsilon,0} - 0$ and let $a_1, \cdots, a_\nu$ be all the critical values of $f_{(\lambda)}$ in $\Delta_\rho \times \lambda$. Choose a system $\gamma_1, \cdots, \gamma_\nu$ of simple paths in $\Delta_\rho \times \lambda$ connecting $a_1, \cdots, a_\nu$ with $u_0 \times \lambda$ and meeting only at $u_0 \times \lambda$. For $i = 1, \cdots, \nu$ let $\Delta_i'(u_0) = \Delta_{a_i, \gamma_i, u_0}$ (relative Lefshetz cycle in $N(f_{(\lambda)})$ corresponding to a_i, γ_i, u_0). Then there exists a retraction of $N(f_{(\lambda)})$ to $N(f_{(\lambda)}, u_0) \cup (\bigcup_{i=1}^{\nu} \Delta_i'(u_0))$ which on $N'(f_{(\lambda)})$ coincides with a retraction of $N'(f_{(\lambda)})$ to $N'(f_{(\lambda)}, u_0)$, corresponding to the chosen trivialization of $f_{(\lambda)}|_{N'(f_{(\lambda)})}: N'(f_{(\lambda)}) \to \Delta_\rho \times \lambda$ and a retraction of Δ_ρ to u_0.

Applying the diffeomorphism $q_\lambda: N(f_{(\lambda)}) \to N(f_{(0)})$ and observing that $q_\lambda(N(f_{(\lambda)} \times u_0)) = N(f_{(0)}, u_0)$, and each $q_\lambda(\Delta_i'(u_0))$ is a relative Lefschetz cycle in $(N(f_{(0)}), N(f_{(0)}, u_0))$, we obtain from the retraction above a retraction of $N(f_{(0)})$ satisfying all the conditions of the proposition. q.e.d.

Going back to $f: W \to T$ ($\dim_{\mathbf{C}} T = 1$) let us assume that there is a closed submanifold Y in W such that $f(Y) = T$, $f|_Y: Y \to T$ is a C^∞-bundle and $Y \cap S(f) = \varnothing$.

Take $t \in T - S_b(f)$ and connect s ($\in q_b(f)$) to t by a simple path Γ in T with $\Gamma \cap S_b(f) = s$.

Considering again $N(f^{(0)})$ and $N(f^{(0)}, u_0)$ ($= M(u_0, v)$, the Milnor fiber corresponding to v and u_0), assume that $u_0 = \Gamma \cap \partial\Delta_\rho$ and denote by Γ' the part of Γ from u_0 to t.

$f|_{f^{-1}(\Gamma')}: f^{-1}(\Gamma') \to \Gamma'$ is a C^∞-bundle, and $f|_{f^{-1}(\Gamma') \cap Y}: f^{-1}(\Gamma') \cap Y \to \Gamma'$ is a C^∞-subbundle of it. Take a trivialization of $f^{-1}(\Gamma') \to \Gamma'$ such that it will give also a trivialization of $f^{-1}(\Gamma') \cap Y \to \Gamma'$. Let $\Pi: f^{-1}(\Gamma') \to E_t$ ($= f^{-1}(t)$) be the corresponding projection. Then $\Pi(f^{-1}(\Gamma') \cap Y) = Y \cap E_t$.

Take $N(f^{(0)})$ small enough so that $N(f^{(0)}) \cap Y = \varnothing$. Let $\Delta(u_0)$ be a relative Lefshetz cycle in $(N(f^{(0)}), N(f^{(0)}, u_0))$. Then $\Delta(u_0) \cap f^{-1}(u_0) = \partial\Delta(u_0)$ is a smooth $(n-1)$-sphere representing a vanishing cycle in $E_{u_0} = f^{-1}(u_0)$. Denote

$$\Delta(u_0, \Gamma) = \Delta(u_0) \cup (\Pi^{-1}(\Pi(\partial\Delta(u_0)))).$$

Because $\Delta(u_0)$ is a smooth n-disk, $\Delta(u_0) \cap \Pi^{-1}(\Pi(\partial\Delta(u_0))) = \partial\Delta(u_0)$, and $\Pi^{-1}(\Pi(\partial\Delta(u_0)))$ is diffeomorphic to $S^{n-1} \times [0, 1]$, we see that $\Delta(u_0, \Gamma)$ is an n-disk in W. Clearly $\Delta(u_0, \Gamma) \cap Y = \varnothing$, $\Delta(u_0, \Gamma) \cap E_t = \partial\Delta(u_0, \Gamma)$ and $\partial\Delta(u_0, \Gamma)$ is a smooth $(n-1)$-sphere representing a vanishing cycle in E_t.

Definition 5. We call any $\Delta(u_0, \Gamma)$ obtained as above a relative Lefschetz cycle in (W, E_t) or $(W - Y, E_t - E_t \cap Y)$.

Proposition 3. *Let $f\colon W \to T$ and $Y \subset W$ be as above. Assume that T is topologically an open 2-disc, and that $S(f) = q(f)$, that is, the fibers of f have only isolated singularities. Let $t_0 \in T - S_b(f)$ and $E_{t_0} = f^{-1}(t_0)$. Then there exists a finite number of relative Lefschetz cycles $\Delta_{(1)}, \cdots, \Delta_{(\mu)}$ in $(W - Y, E_{t_0} - E_{t_0} \cap Y)$ (that is, smooth n-disks in $W - Y$ with $\Delta_{(i)} \cap E_{t_0} = \partial\Delta_{(i)}$ (u_0), $i = 1, \cdots, \mu$, each $\partial\Delta_{(i)}$ represents a vanishing cycle in E_{t_0}) such that W can be retracted to $E_{t_0} \cup (\bigcup_{i=1}^{\mu} \Delta_{(i)})$. Moreover, this retraction can be chosen so that on Y it coincides with a "trivial retraction" of Y to $Y \cap E_{t_0}$ $(= (f|_Y)^{-1}(t_0))$ corresponding to a trivialization of $f|_Y\colon Y \to T$ and a retraction of T (topologically a 2-disc) to t_0. In particular, $W - Y$ can be retracted to $(E_{t_0} - E_{t_0} \cap Y) \cup (\bigcup_{i=1}^{\mu} \Delta_{(i)})$.*

Proof. Let $a_1, \cdots, a_\mu$ be all the critical values of f in T, for each $i = 1, \cdots, \mu$ let d_i be a small disc centered at a_i, and let a_i' be a point on ∂d_i. Let Γ_i', $i = 1, \cdots, \mu$, be simple paths in $\overline{T - \bigcup_{i=1}^{\nu} d_i}$ connecting the a_i''s with t_0 and meeting only at t_0. First we retract W to $f^{-1}(\bigcup_{i=1}^{\mu}(\Gamma_i' \cup d_i))$ so that on Y we use a retraction corresponding to a trivialization of $f|_Y\colon Y \to T$ and a retraction of T to $\bigcup_{i=1}^{\nu}(\Gamma_i' \cup d_i)$.

Next, using Proposition 2 we retract each $f^{-1}(d_i)$, $i = 1, \cdots, \mu$, to $f^{-1}(a_i') \cup (\bigcup_{j=1}^{\nu_i} \Delta_j(a_i'))$ respecting "trivial retraction" on Y, where $\Delta_j(a_i')$ are some relative Lefschetz cycles in $(f^{-1}(d_i), f^{-1}(a_i'))$. Thus, we get a retraction of $f^{-1}(\bigcup_{i=1}^{\mu}(\Gamma_i' \cup d_i))$ to $f^{-1}(\bigcup_{i=1}^{\mu}\Gamma_i') \cup (\bigcup_{i=1}^{\mu}\bigcup_{j=1}^{\nu_i} \Delta_j(a_i'))$, respecting a "trivial retraction" on Y.

To finish, use a trivialization of $f^{-1}(\bigcup_{i=1}^{\mu}\Gamma_i') \to \bigcup_{i=1}^{\mu}\Gamma_i'$ respecting the chosen trivialization of $f|_Y\colon Y \to T$, a retraction of $f^{-1}(\bigcup_{i=1}^{\mu}\Gamma_i')$ to E_{t_0} corresponding to this trivialization and a retraction of $\bigcup_{i=1}^{\mu}\Gamma_i'$ to t_0, and the definition of relative Lefschetz cycles in $(W - Y, E_{t_0} - Y \cap E_{t_0})$.

2. Linear systems of "Lefschetz type"

Let X be an n-dimensional compact complex manifold, $\mathfrak{D}$ a base point free linear system on X, and $\mathbb{C}P^N$ the parameter space for $\mathfrak{D}$. Denote, $\forall t \in \mathbb{C}P^N$, by E_t the divisor in $\mathfrak{D}$ corresponding to t, and the graph of $\mathfrak{D}$ by $W = \{(x, t) \in X \times \mathbb{C}P^N | x \in E_t\}$.

Let $f\colon W \to \mathbb{P}^N$, $p\colon W \to X$ be the natural projections. From the fact that $\mathfrak{D}$ is base point free, it easily follows that W is nonsingular. In fact, $p\colon W \to X$ is a holomorphic $\mathbb{C}P^{N-1}$-bundle over X.

Denote $S(D) = S(f)$, $S_b(D) = S_b(f)$, $q(D) = q(f)$ and $q_b(D) = q_b(f)$. $\forall t \in \mathbb{C}P^N$ we naturally identify E_t with $f^{-1}(t)$. As in §1 (see Definition 3) we define for any E_t, $t \in \mathbb{C}P^N - S_b(D)$, the set $\Lambda(t) \in H_{n-1}(E_t)$, the elements of which we call *vanishing cycles* in E_t. We shall write $\Lambda_D(t)$ instead of $\Lambda(t)$. We also define the subgroup $G_D(t) \subset \mathrm{Aut}(H_{n-1}(E_t))$ generated by Picard-Lefschetz transformations θ_δ, $\delta \in \Lambda_D(t)$: $\theta_\delta(z) = z + (-1)^{n(n+1)/2}(z,\delta)\delta$. It follows from §1 that each $\delta \in \Lambda_D(t)$ can be represented by a smooth $(n-1)$-sphere $\boldsymbol{\delta}$ in E_t, and that each θ_δ is induced by an orientation preserving diffeomorphism of E_t, a "Dehn twist" defined by $\boldsymbol{\delta}$.

Proposition 4. *Assume that* $f(S(D) - q(D))$ *has codimension* ≥ 2 *in* $\mathbb{C}P^N$. *As in §1, we denote by*

$$\Psi_t\colon \pi_1(P^N - S_b(D), t) \to \mathrm{Aut}(H_{n-1}(E_t))$$

the natural homomorphism corresponding to the C^∞*-bundle* $f|_{W-f^{-1}(S_b(d))}$: $W - f^{-1}(S_b(D)) \to P^N - S_b(D)$. *Then*

$$\mathrm{Im}\,\Psi_t \subseteq G_D(t).$$

Proof. Take a generic line L in $\mathbb{C}P^N$. Since $\mathrm{codim}_{P^N} f(S(D)-q(D)) \geq 2$, we have $L \cap f(S(D)-q(D)) = \varnothing$. That means that any singular element E_s of the pencil $\mathfrak{D}_L = \{E_t, t \in L\}$ has only isolated singularities.

Using local versal families of isolated singularities (as in §1) it is easy to show that any local monodromy automorphism of $H_{n-1}(E_{t'})$, where $E_{t'}$ is in a neighborhood of E_s with only isolated singularities, is a product of Picard-Lefschetz transformations (and their inverses) corresponding to some vanishing cycles, that is an element of $G_D(t')$.

We get that $\forall$ loop Γ in $L - L\cap S_b(f)$ starting at some $t_0 \in L - L\cap S_b(f)$ the corresponding $\Psi(\Gamma) \in G_D(t_0)$. Because $\pi_1(L - L\cap S_b(f), t_0) \to \pi_1(\mathbb{C}P^N - S_b(f), t_0)$ is surjective for a generic line L in $\mathbb{C}P^N$, we see that

$$\mathrm{Im}\,\Psi_{t_0} \subseteq G_D(t_0).$$

Proposition 5. *Let* $\mathfrak{D}$ *be a base point free linear system on* X *with* $q(\mathfrak{D})$ *connected and* $\mathrm{codim}_{\mathbb{C}P^N} f(S(D) - q(D)) \geq 2$. *Then for* $t \in \mathbb{C}P^N - S_b(D)$ *the set* $\Lambda_D(t)$ *(vanishing cycles in* E_t*) belongs to only one* $G_D(t)$*-orbit in* $H_{n-1}(E_t)$.

Proof. The proof follows immediately from Propositions 1 and 4.

Let $[D]$ be the complex line bundle on X and let V be the linear subspace in $H^0(X, \mathcal{O}_X[D])$ which defines $\mathfrak{D}$. For any subvariety $Y \subseteq X$ we define the linear system $\mathfrak{D}|_Y$, the restriction of $\mathfrak{D}$ to Y, by the line bundle $[D]|_Y$ and the linear subspace in $H^0(Y, \mathcal{O}_Y[D]|_Y)$ which is the image of V under the restriction homomorphism:

$$H^0(X, \mathcal{O}_X[D]) \to H^0(Y, \mathcal{O}_Y[D]|_Y).$$

Definition 6. We say that a linear system $\mathfrak{D}$ on X is of *Lefschetz type* if the following conditions are satisfied:

(a) $\mathfrak{D}$ is infinite and base point free;

(b) $\operatorname{codim}(f(S(D) - q(\mathfrak{D})))$ in $\mathbb{C}P^N$ (the parameter space of $\mathfrak{D}$) is ≥ 2;

(c) in the case $\dim_{\mathbb{C}} X \geq 2$, for a generic $E_t \in \mathfrak{D}$ the linear system $\mathfrak{D}|_{E_t}$ on E_t is of Lefschetz type.

Let $\mathfrak{D}$ be a linear system of Lefschetz type on X. Because $\mathfrak{D}$ is base point free, any generic $E_t \in \mathfrak{D}$ is nonsingular (Bertini's theorem). Moreover, if L is a generic line in $\mathbb{C}P^N$, and $\mathfrak{D}_L$ is the pencil in $\mathfrak{D}$ parametrized by L, then it follows from (b) above that any singular element $E_s \in \mathfrak{D}_L$ has only isolated singular points. In the case $\dim_{\mathbb{C}} X \geq 2$, taking a generic $E_t \in D$ we get from (c) and (a) above that $\mathfrak{D}|_{E_t}$ is infinite and a base point free linear system on E_t. Thus two generic E_t, $E_{t'} \in \mathfrak{D}$ intersect transversally at nonempty $E_t \cap E_{t'}$ by applying Bertini's theorem to $\mathfrak{D}|_{E_t}$. In particular, for a generic pencil $\mathfrak{D}_L$ in $\mathfrak{D}$ the base point set B_L is nonempty and nonsingular, and $\forall t_1, t_2 \in L$ the corresponding E_{t_1}, E_{t_2} intersect transversally at $B_L = E_{t_1} \cap E_{t_2}$.

Andreotti and Frankel proofs of the First and Second Lefschetz theorems can be used almost without changes to prove similar theorems for linear systems of Lefschetz type.

Proposition 6. *Let $\mathfrak{D}$ be a linear system of Lefschetz type on X, E_∞ be a nonsingular (generic) element of $\mathfrak{D}$, $X' = X - E_\infty$, E_0 be another generic element in $\mathfrak{D}$, and $E'_0 = E_0 - E_0 \cap E_\infty$. Then:*

(1) *there exist smooth n-discs $\Delta_{(1)}, \cdots, \Delta_{(\mu)}$ in X', such that $\forall i = 1, \cdots, \nu$, $\Delta_{(i)} \cap E'_0 = \partial\Delta_{(i)}$, $\partial\Delta_{(i)}$ represents a vanishing cycle in E_0 ($\Delta_{(i)}$ is transversal to E'_0 at $\Delta_{(i)} \cap E'_0$), and a retraction of X' to $E'_0 \cup (\bigcup_{i=1}^{\mu} \Delta_{(i)})$,*

(2) *X' is homotopically equivalent to an n-dimensional complex.*

Proof. (1) Let $\mathfrak{D}_L$ be the pencil in $\mathfrak{D}$ containing E_0 and E_∞ (L the parameter line of $\mathfrak{D}_L$), $B = E_0 \cap E_\infty$ be the base point set of $\mathfrak{D}$, $W_L \subset X \times L$ be the graph of $\mathfrak{D}_L$ in $X \times L$, and $f_L \colon W_L \to L$, $\varphi \colon W_L \to X$ be the natural projections. It is easy to see that φ is the blowing-up of X with center B, so that $\varphi|_{W_L - \varphi^{-1}(B)} \colon W_L - \varphi^{-1}(B) \to X - B$ is an isomorphism. Let (0), $(\infty) \in L$ be the points corresponding to $E_{(0)}$ and $E_{(\infty)}$. To prove (1) we apply Proposition 3 (§1) with $T = L - (\infty)$, $t_0 = (0)$, $W = W_L - f_L^{-1}((\infty))$, $Y = \varphi^{-1}(B) - \varphi^{-1}(B) \cap f_L^{-1}((\infty))$, and $f = f_L|_{W_L - f_L^{-1}((\infty))}$.

(2) To prove (2) we apply induction on the dimension of X and observe that $E'_0 = E_0 - E_0 \cap E_\infty = E_0 - B$, where B is a generic element of a system of Lefschetz type $\mathfrak{D}|_{E_0}$ on E_0. q.e.d.

Let $\mathfrak{D}$ be a linear system of Lefschetz type on X, and E a nonsingular element of $\mathfrak{D}$.

Definition 7. A relative Lefschetz cycle in (X,E) is a smooth n-disc Δ on X ($n = \dim_{\mathbf{C}} X$) with the following properties:

(a) $\Delta \cap E = \partial\Delta$ and $\partial\Delta$ represents a vanishing cycle in E;

(b) Δ is transversal to E at each point of $\Delta \cap E$.

We also say that an element $\delta \in H_{n-1}(E)$ is a *good vanishing cycle in E* if $\exists$ a relative Lefshetz cycle Δ in (X,E) such that $\partial\Delta$ represents δ.

Denote by i_r the canonical homomorphism

$$H_r(E) \to H_r(X).$$

Proposition 7 (*1st Lefschetz Theorem*; *cf.* [1]). *$\forall r \leq n-2$, i_r is an isomorphism and i_{n-1} is an epimorphism.*

Proof (cf. [1]). We use Proposition 6 with $E_\infty = E$ and $X' = X - E$. Then it follows from Proposition 6(2) that $H^r(X') = 0$ $\forall r \geq n+1$. Thus by the Lefschetz Duality Theorem

$$H_r(X,E) = 0 \quad \forall r \leq n-1.$$

Now use homology exact sequence for (X,E).

Proposition 8. (1) (*2nd Lefschetz Theorem*; *cf.* [2]) $\mathrm{Ker}(i_{n-1})$ *is generated by good vanishing cycles in E.*

(2) *$H_n(X,E)$ is generated by relative Lefschetz cycles in (X,E).*

(3) *Any element $z \in H_n(X)$ can be represented by a cycle of the following form:*

$$\sum_{i=1}^{\mu} m_i \Delta_{(i)} - \gamma,$$

where $m_i \in \mathbb{Z}$, all $\Delta_{(i)}$ are relative Lefschetz cycles in (X,E), and γ is an n-chain in E with $\partial\gamma = \sum_{i=1}^{\mu} m_i \partial\Delta_{(i)}$, γ representing a relation between vanishing cycles $\delta_i = \partial\Delta_{(i)}$, $i = 1, \cdots, \mu$.

Proof. (1) Using Proposition 6 we can repeat word by word the Andreotti-Frankel proof of the 2nd Lefschetz Theorem (see [2, §6, Theorem 3]; our Proposition 6 replaces Theorem 1 of [2].)

(2) Consider

$$H_n(X) \underset{j_n}{\to} H_n(X,E) \underset{\partial_n}{\to} H_{n-1}(E) \underset{i_{n-1}}{\to} H_{n-1}(X).$$

Take any $a \in H_n(X,E)$, and let $b = \partial_n(a)$. Since $b \in \mathrm{Ker}\, i_{n-1}$, it follows from part (1) of the proposition that we can write $b = \sum_{i=1}^{\nu} m_i \delta_i$, where $m_i \in \mathbb{Z}$, and each δ_i is a good vanishing cycle in E. Thus there exist relative Lefschetz cycles $\Delta_{(i)}$, $i = 1, \cdots, \nu$, in (X,E) such that each $\delta_i = \partial\Delta_{(i)}$.

Let $a_1 = a - \sum_{i=1}^{\nu} m_i \Delta_{(i)}$. Then $\partial a_1 = \partial a - \sum_{i=1}^{\nu} m_i \delta_i = b - \sum_{i=1}^{\nu} m_i \delta_i = 0$, and so there exists $z \in H_n(X)$ such that $a_1 = j_n(z)$.

Let $\{E_t, t \in \mathbb{C}P^1\}$ be a generic pencil in $\mathfrak{D}$ containing E. Denote $E_0 = E$, and let E_∞ be another nonsingular element of $\{E_t, t \in \mathbb{C}P^1\}$, and $B = E_t \cap E$. Let $\Phi\colon \tilde{X} \to X$ be the blow-up of X with center B, $\tilde{B} = \Phi^{-1}(B)$, and $\tilde{E}_t$ be the proper transform of E_t in $\tilde{X}$. Because $\Phi|_{\tilde{E}_t}\colon \tilde{E}_t \to E_t$ is an isomorphism, we identify E_0 with $\tilde{E}_0$ and E_∞ with $\tilde{E}_\infty$. Let $B_\infty = \tilde{E}_\infty \cap \tilde{B}$, let

$$H_n(\tilde{X} - \tilde{E}_\infty) \xrightarrow{l} H_n(\tilde{X}) \xrightarrow{\sigma} H_{n-2}(\tilde{E}_\infty),$$

$$H_n(\tilde{B}) \xrightarrow{q} H_{n-2}(B_\infty)$$

be parts of Thom-Ghysin sequences for $(\tilde{X}, \tilde{E}_\infty)$ and $(\tilde{B}, B_\infty)$ respectively. Let $\varphi_n = H_n(\Phi)$. It follows from Theorem 2 of [2] that φ_n is surjective. Thus for z above $\exists \tilde{z} \in H_n(\tilde{X})$ with $z = \varphi_n(\tilde{z})$. Consider the following commutative diagram of canonical homomorphisms:

$$\begin{array}{ccccc}
 & & H_n(\tilde{B}) & \xrightarrow{q} & H_{n-2}(B_\infty) \\
 & & \downarrow \tilde{\imath}_n & & \downarrow \mu_{n-2} \\
H_n(\tilde{X} - \tilde{E}_\infty) & \xrightarrow[l]{} & H_n(\tilde{X}) & \xrightarrow[\sigma]{} & H_{n-2}(\tilde{E}_\infty) \\
\varphi_n l \downarrow & \searrow \tilde{\jmath}_n & & & \\
H_n(X) & & H_n(\tilde{X} - \tilde{E}_\infty, \tilde{E}_0) & & \\
j_n \downarrow & \swarrow \rho\ (=\Phi_*) & & & \\
H_n(X, E) & & & &
\end{array}$$

It is easy to see that q and μ_{n-2} are surjective by using $\tilde{B} = B \times \mathbb{C}P^1$ and Proposition 7 for (E_∞, B_∞), so there exists $y \in H_n(\tilde{B})$ such that $\sigma(\tilde{z}) = \mu_{n-2} q(y)$. Since $\sigma \tilde{\imath}_n(y) = \mu_{n-2} q(y)$, we get $\sigma(\tilde{z}) = \sigma \tilde{\imath}_n(y)$. Let $\tilde{Z}_1 = \tilde{z} - \tilde{\imath}_n(y)$. Then $\sigma(\tilde{Z}_1) = 0$ and so $\exists \tilde{w} \in H_n(\tilde{X} - E_\infty)$ such that $\tilde{Z}_1 = l(\tilde{w})$. Thus $j_n \varphi_n(\tilde{Z}_1) = j_n \varphi_n l(\tilde{w}) = \rho \tilde{\jmath}_n(\tilde{w})$.

It follows from Proposition 6 that $H_n(\tilde{X} - \tilde{E}_\infty, \tilde{E}_0)$ is generated by relative Lefschetz cycles, say $\tilde{\Delta}_1, \cdots, \tilde{\Delta}_p$, in $(\tilde{X} - \tilde{E}_\infty - \tilde{B}, \tilde{E}_0 - \tilde{E}_0 \cap \tilde{B})$. Clearly each $\overline{\Delta}_j = \Phi(\tilde{\Delta}_j)$, $j = 1, \cdots, p$, is a relative Lefschetz cycle in (X, E).

Since $\tilde{\jmath}_n(\tilde{w}) \in H_n(\tilde{X} - \tilde{E}_\infty, \tilde{E}_0)$ we can write $\tilde{\jmath}_n(\tilde{w}) = \sum_{j=1}^{p} n_j \tilde{\Delta}_j$, with all $n_j \in \mathbb{Z}$. Thus $\rho \tilde{\jmath}_n(\tilde{w}) = \sum_{j=1}^{p} n_j \overline{\Delta}_j$, and $j_n \varphi_n(\tilde{Z}_1) = \sum_{j=1}^{p} n_j \overline{\Delta}_j$. Because $\tilde{z} = \tilde{Z}_1 + \tilde{\imath}_n(y)$, we have

$$\begin{aligned} a_1 &= j_n(z) = j_n \varphi_n(\tilde{z}) = j_n \varphi_n(\tilde{Z}_1) + j_n \varphi_n \tilde{\imath}_n(y) \\ &= \sum_{j=1}^{p} n_j \overline{\Delta}_j + j_n \varphi_n \tilde{\imath}_n(y), \qquad y \in H_n(\tilde{B}). \end{aligned}$$

Claim. $\forall y \in H_n(\tilde{B})$, $j_n \varphi_n \tilde{\imath}_n(y) = 0$.

Proof. Consider the commutative diagram:

$$\begin{array}{ccccc} & & H_n(\tilde{B}) & & \\ & \swarrow^{\varphi_n'} & \downarrow \tilde{i}_n & & \\ H_n(B) & & H_n(\tilde{X}) & & \\ \downarrow \overline{\mu}_n & & \downarrow \varphi_n & & \\ H_n(E) & \xrightarrow[i_n]{} & H_n(X) & \xrightarrow[j_n]{} & H_n(X,E) \end{array}$$

We see that $j_n\varphi_n\tilde{i}_n(y) = j_n i_n \overline{\mu}_n \varphi_n'(y) = 0$ since $j_n i_n = 0$. q.e.d.

Now we continue to prove Part (2) of Proposition 8. From the above we get $a_1 = \sum_{j=1}^{p} n_j\overline{\Delta}_j$ and $a = a_1 + \sum_{i=1}^{\nu} m_i\Delta_{(i)} = \sum_{j=1}^{p} n_j\overline{\Delta}_j + \sum_{i=1}^{\nu} m_i\Delta_{(i)}$ where all $n_j, m_i \in \mathbb{Z}$ and all $\overline{\Delta}_j, \Delta_{(i)}$ are relative Lefschetz cycles in (X, E).

(3) Part (3) follows immediately from (2) and the exact sequence:

$$H_n(E) \underset{i_n}{\to} H_n(X) \underset{j_n}{\to} H_n(X,E) \underset{\partial_n}{\to} H_{n-1}(E).$$

3. Sequences of finite cyclic coverings

Proposition 9. *Let X be an n-dimensional compact complex manifold, S an $(n-1)$-dimensional complex submanifold of X, and $f\colon Y \to X$ a finite cyclic covering of X ramified at S.*[1] *Let $\mathfrak{D}$ be a linear system of Lefschetz type on X (Definition 6) such that $\mathfrak{D}|_S$ is a linear system of Lefschetz type on S. Then $f^*\mathfrak{D}$ is a linear system of Lefschetz type on Y.*

Proof. Use induction on $\dim_{\mathbb{C}} X$. It is clear that $f^*\mathfrak{D}$ is infinite and base point free. Let $\mathfrak{D}_L$ be a generic pencil in $\mathfrak{D}$, and L the parameter line of $\mathfrak{D}_L$. Let $s \in L$ and $c \in f^*E_s \cap f^{-1}(S)$. We can assume that there exist complex analytic coordinates $y_1, \cdots, y_n$ in a neighborhood of c on Y, and $x_1, \cdots, x_n$ in a neighborhood of $f(c)$ in X such that locally f is given by $x_i = y_i$, $i = 1, \cdots, n-1$, $x_n = y_n^m$ $(m > 1)$, $x_n = 0$ is a local equation of S at $f(c)$. Let $F(x_1, \cdots, x_n) = 0$ be a local equation of E_s at $f(c)$. Then f^*E_s in a neighborhood of c has local equation $F(y_1, \cdots, y_{n-1}, y_n^m) = 0$, and c is singular on f^*E_s iff

$$\frac{\partial F}{\partial y_i}(c) = 0, \quad i = 1, \cdots, n-1, \qquad m \cdot y_n(c)^{m-1}\frac{\partial F}{\partial y_n}(c) = 0.$$

Since $y_n(c) = 0$, this system of equations is equivalent to

$$\frac{\partial F}{\partial y_i}(c) = 0, \quad i = 1, \cdots, n-1; \quad \text{or} \quad \frac{\partial F}{\partial x_i}(f(c)) = 0, \quad i = 1, \cdots, n-1.$$

[1]We consider only totally ramified cyclic coverings (e.g. $f^{-1}(S) \simeq S$).

This shows that a point $a \in f^*E_s$ is singular on f^*E_s iff $f(a)$ is a singular point of E_s or of $E_s|_S$.

Because $\mathfrak{D}_L|_S$ is a generic pencil in $\mathfrak{D}|_S$, and $\mathfrak{D}$ and $\mathfrak{D}|_S$ are of Lefschetz type, we see that singular points of E_s and $E_s|_S$ $(s \in L)$ must be isolated, that is, f^*E_s has only a finite number of singular points. Thus $f^*\mathfrak{D}$ satisfies condition (b) in Definition 6 the definition of Lefschetz type linear systems.

To check (c) of this definition for $f^*\mathfrak{D}$, we must take a generic $E_1 \in \mathfrak{D}$ and show that $f^*\mathfrak{D}|_{f^*E_1}$ is of Lefschetz type on f^*E_1. Since $\mathfrak{D}|_S$ is of Lefschetz type, $E_1|_S$ is nonsingular, that is, E_1 is transversal to S. Thus $S_1 = E_1 \cap S$ and f^*E_1 are nonsingular, and $f_1 = f|_{f^*E_1}: f^*E_1 \to E_1$ is a finite cyclic covering of E_1 ramified at S_1. Let $\mathfrak{D}_1 = \mathfrak{D}|_{E_1}$. To use the induction where E_1 will replace X, we have to check only that $\mathfrak{D}_1|_{S_1}$ is of Lefschetz type on S_1, or $\mathfrak{D}|_{S\cap E_1}$ is of Lefschetz type on $S \cap E_1$. But $\mathfrak{D}|_{S\cap E_1} = (\mathfrak{D}|_S)|_{E_1\cap S}$. Since $\mathfrak{D}|_S$ is of Lefschetz type on S and $E_1 \cap S$ is a generic element of $\mathfrak{D}|_S$, we have that $(\mathfrak{D}|_S)_{E_1\cap S}$ is of Lefschetz type on $E_1 \cap S$.

Proposition 10. *Let $f: Y \to X$ be a finite cyclic covering of n-dimensional compact complex manifolds ramified at a complex submanifold S of X. Assume that S is an element of a linear system of Lefschetz type on X. $\forall k$, denote by $\varphi_k: H_k(Y) \to H_k(X)$ the canonical homology homomorphism corresponding to f. Then $\forall k = 1, \cdots, n-1$, φ_k is an isomorphism, and φ_n is an epimorphism.*

Proof. Let $\tilde{S} = f^{-1}(S)$, $X' = X - S$, $Y' = Y - \tilde{S}$ and $f' = f|_{Y'}: Y' \to X'$. From Proposition 6(2) it follows that X' is homotopically equivalent to an n-complex. Since $f': Y' \to X'$ is a nonramified covering, Y' is homotopically equivalent to an n-complex. In particular $H^k(Y - \tilde{S}) = 0$ $\forall k \geq n+1$. By the Lefschetz Duality Theorem $H_k(Y, \tilde{S}) = 0$ $\forall k \leq n-1$, and for the same reasons, $H_k(X,S) = 0$ $\forall k \leq n-1$.

Claim 1. The canonical homomorphism

$$\psi_n: H_n(Y, \tilde{S}) \to H_n(X,S)$$

is an epimorphism.

Proof. We can find a (closed) tubular neighborhood TS of S in X such that $f^{-1}(TS)$ is a tubular neighborhood of $\tilde{S}$ in Y. Denote $T\tilde{S} = f^{-1}(TS)$, and by $\psi_{n,T}$ the canonical homomorphism $H_n(Y, T\tilde{S}) \to H_n(X, TS)$; let $\tau: H_n(X, TS) \to H_n(X,S)$ be the canonical isomorphism.

Clearly it is enough to show that $\psi_{n,T}$ is an epimorphism.

It follows from Proposition 8(2) that $H_n(X,S)$ is generated by relative Lefschetz cycles, say $\Delta_{(1)}, \cdots, \Delta_{(\nu)}$, in (X,S). Since each $\Delta_{(i)}$ is transversal to S, we can assume that each $\Delta'_{(i)} = \Delta_{(i)} - \Delta_{(i)} \cap \mathrm{Int}(TS)$ is a smooth n-disc in $X - S$ with $\partial\Delta'_{(i)} \subset \partial TS$. Due to the fact that $f|_{Y-\tilde{S}}\colon Y - \tilde{S} \to X - S$ is a nonramified covering we can lift each $\Delta'_{(i)}$ to a smooth n-disc $\tilde{\Delta}_{(i)}$ in $Y - \tilde{S}$ with $\partial\tilde{\Delta}_{(i)} \subset \partial T\tilde{S}$. Because $f|_{\tilde{\Delta}_{(i)}}\colon \tilde{\Delta}_{(i)} \to \Delta'_{(i)}$, $i = 1, \cdots, \nu$, is a homeomorphism, we see that each $\tilde{\Delta}_{(i)}$ represents an element of $H_n(Y, T\tilde{S})$ with $\psi_{n,T}(\tilde{\Delta}_{(i)}) = \Delta'_{(i)}$ in $H_n(X, TS)$. Thus $\tau\psi_{n,T}(\tilde{\Delta}_{(i)}) = \Delta_{(i)}$ in $H_n(X,S)$. Because $\Delta_{(1)}, \cdots, \Delta_{(\nu)}$ were generators of $H_n(X,S)$, $\psi_{n,T}$ is an epimorphism. q.e.d.

Clearly $f|_{\tilde{S}}\colon \tilde{S} \to S$ is an isomorphism. Consider the following commutative diagrams:

$((*)_\alpha)$

$$\begin{array}{ccccccccccc}
H_n(\tilde{S}) & \xrightarrow{\tilde{i}_n} & H_n(Y) & \xrightarrow{\tilde{j}_n} & H_n(Y,\tilde{S}) & \xrightarrow{\tilde{\partial}_n} & H_{n-1}(\tilde{S}) & \xrightarrow{\tilde{i}_{n-1}} & H_{n-1}(Y) & \xrightarrow{\tilde{j}_{n-1}} & H_{n-1}(Y,\tilde{S}) \\
\downarrow \lambda_n & & \downarrow \varphi_n & & \downarrow \psi_n & & \downarrow \lambda_{n-1} & & \downarrow \varphi_{n-1} & & \downarrow \psi_{n-1} \\
H_n(S) & \xrightarrow[i_n]{} & H_n(X) & \xrightarrow[j_n]{} & H_n(X,S) & \xrightarrow[\partial_n]{} & H_{n-1}(S) & \xrightarrow[i_{n-1}]{} & H_{n-1}(X) & \xrightarrow[j_{n-1}]{} & H_{n-1}(X,S)
\end{array}$$

$$((*)_\beta) \qquad \begin{array}{ccccccc}
H_k(Y,\tilde{S}) & \xrightarrow{\tilde{\partial}_k} & H_{k-1}(\tilde{S}) & \xrightarrow{\tilde{i}_{k-1}} & H_{k-1}(Y) & \xrightarrow{\tilde{j}_{k-1}} & H_{k-1}(Y,\tilde{S}) \\
\downarrow \psi_k & & \downarrow \lambda_{k-1} & & \downarrow \varphi_{k-1} & & \downarrow \psi_{k-1} \\
H_k(X,S) & \xrightarrow[\partial_k]{} & H_{k-1}(S) & \xrightarrow[i_{k-1}]{} & H_{k-1}(X) & \xrightarrow[j_{k-1}]{} & H_{k-1}(X,S)
\end{array}$$

Claim 2. φ_{k-1} is an isomorphism for $k \leq n - 1$.

Proof. Taking $k \leq n - 1$ in $(*)_\beta$, and using $H_k(X,S) = H_{k-1}(X,S) = H_k(Y,\tilde{S}) = H_{k-1}(Y,\tilde{S}) = 0$ (for $k \leq n - 1$) we get that i_{k-1} and $\tilde{i}_{k-1}$ are isomorphisms in $(*)_\beta$ ("1st Lefschetz Theorem"). Since λ_{k-1} is an isomorphism, we conclude that φ_{k-1} is an isomorphism for $k \leq n - 1$.

Claim 3. φ_{n-1} is an isomorphism.

Proof. Consider $(*)_\alpha$. Because i_{n-1} is surjective ($H_{n-1}(X,S) = 0$), φ_{n-1} is an epimorphism, so we have to prove that $\operatorname{Ker}\varphi_{n-1} = 0$. Take any $a \in \operatorname{Ker}\varphi_{n-1}$. Since $H_{n-1}(Y,\tilde{S}) = 0$, $\tilde{i}_{n-1}$ is surjective. Thus $\exists b \in H_{n-1}(\tilde{S})$ with $\tilde{i}_{n-1}(b) = a$, and $0 = \varphi_{n-1}\tilde{i}_{n-1}(b) = i_{n-1}\lambda_{n-1}(b)$. Furthermore, $\exists c \in H_n(X,S)$ with $\partial_n(c) = \lambda_{n-1}(b)$. Since ψ_n is surjective from Claim 1 above, $\exists c_1 \in H_n(Y,\tilde{S})$ with $\psi_n(c_1) = c$ and $\partial_n\psi_n(c_1) = \lambda_{n-1}(b)$, or $\lambda_{n-1}\tilde{\partial}_n(c_1) = \lambda_{n-1}(b)$. But λ_{n-1} is an isomorphism, so $\tilde{\partial}_n(c_1) = b$. Hence

$$a = \tilde{i}_{n-1}(b) = \tilde{i}_{n-1}\tilde{\partial}_n(c_1) = 0.$$

Claim 4. φ_n is an epimorphism.

Proof. Take any $a \in H_n(X)$. Because ψ_n is surjective from Claim 1, $\exists b \in H_n(Y,\tilde{S})$ with $\psi_n(b) = j_n(a)$. Since $\lambda_{n-1}\tilde{\partial}_n(b) = \partial_n\psi_n(b) = \partial_n j_n(a) = 0$ and λ_{n-1} is an isomorphism, we get $\tilde{\partial}_n(b) = 0$. Thus $\exists c \in H_n(Y)$ such that $b = \tilde{j}_n(c)$, and $j_n(a - \varphi_n(c)) = \psi_n(b) - \psi_n\tilde{j}_n(c) = \psi_n(b) - \psi_n(b) = 0$. Further, $\exists c_1 \in H_n(S)$ with $a - \varphi_n(c) = i_n(c_1)$. Since λ_n is an isomorphism, $\exists c_2 \in H_n(\tilde{S})$ with $\lambda_n(c_2) = c_1$. Now we have $a - \varphi_n(c) = i_n\lambda_n(c_2) = \varphi_n\tilde{i}_n(c_2)$ and $a = \varphi_n(c + \tilde{i}_n(c_2))$. Hence φ_n is an epimorphism.

Corollary of Proposition 10. *Let z be any primitive element in $H^n(X, \mathbb{Z})$. Then f^*z is a primitive element in $H^n(Y, \mathbb{Z})$.*

Proof. Since z is primitive, $\exists$ an element $a \in H_n(X)$ with $z(a) = 1$. By Proposition 10, $\varphi_n: H_n(Y) \to H_n(X)$ is surjective. Thus $\exists \tilde{a} \in H_n(\tilde{Y})$ with $f_*(\tilde{a}) = \varphi_n(\tilde{a}) = a$, and

$$f^*z(\tilde{a}) = z(f_*\tilde{a}) = z(a) = 1.$$

So f^*z is primitive.

Let X be an n-dimensional nonsingular projective algebraic variety, $X \subset \mathbb{C}P^N$, and $\mathfrak{D}$ be the linear system of hyperplane sections of X. Assume that X does not belong to any hyperplane of $\mathbb{C}P^N$. Let $S_1, \cdots, S_k$ be subvarieties of codimension one on X such that $\forall j_1, \cdots, j_l \in \{1, 2, \cdots, k\}$, $S_{j_1} \cap \cdots \cap S_{j_l}$ is nonsingular and $\bigcup_{j=1}^k S_j$ has only normal crossing singularities.

Let $m_1, \cdots, m_k$ be positive integers such that S_j is divisible by m_j in $\operatorname{Pic} X$ for any $j = 1, \cdots, k$. Define inductively finite morphisms $\alpha_j: X_j \to X$, $j = 0, \cdots, k$, as follows: $X_0 = X$, $\alpha_0 = \mathrm{Id}$, and if $\alpha_{j-1}: X_{j-1} \to X$ is defined, let $\beta_j: X_j \to X_{j-1}$ be the cyclic covering of degree m_j ramified at $\alpha_{j-1}^{-1}(S_j)$, $\alpha_j = \alpha_{j-1} \circ \beta_j$.

Denote $\mathfrak{D}_j = \alpha_j^*\mathfrak{D}$, inverse images of divisors of $\mathfrak{D}$, and by $\mathfrak{D}_{x,\operatorname{sing} x}$, $\forall x \in X$, the set of elements of $\mathfrak{D}$ which pass through x and are singular at x. $\mathfrak{D}_{x,\operatorname{sing} x}$ is a linear subsystem of $\mathfrak{D}$.

Proposition 11. *Assume that $\forall x \in X$ the linear system $\mathfrak{D}_{x,\operatorname{sing} x}$ is infinite and base point free in $X - x$. Let $\tilde{E}_t$ be any nonsingular element in $\mathfrak{D}_k$, $\Lambda_{D_k}(t) \subset H_{n-1}(\tilde{E}_t)$ be the set of vanishing cycles in $\tilde{E}_t$, and $G_{D_k}(t)$ be the subgroup of $\operatorname{Aut}(H_{n-1}(\tilde{E}_t))$ generated by all Picard-Lefschetz transformations $\{\theta_\delta, \delta \in \Lambda_{D_k}(t)\}$. Then $\Lambda_{D_k}(t)$ belongs to only one $G_{D_k}(t)$-orbit in $H_{n-1}(\tilde{E}_t)$.*

Proof. Denote by $T = \mathbb{C}P^{N*}$ the parameter space of $\mathfrak{D}$. Let $E_t \in \mathfrak{D}$ be the element of $\mathfrak{D}$ corresponding to t $\forall t \in T$. As in §2 denote the graph of $\mathfrak{D}$ by $W = \{(x, t) \in X \times T | x \in E_t\}$. Let $f: W \to T$ and $p: W \to X$ be the natural projections. It is easy to see that W is nonsingular, and actually $p: W \to X$ is a holomorphic $\mathbb{C}P^{N-1}$-bundle over X.

Set $W_k = W \times_X X_k$, denote the canonical projections by $g_k: W_k \to W$ and $p_k: W_k \to X_k$, and set $f_k = f \circ g_k: W_k \to T$. It is clear that $W_k = \{(\tilde{a}, t) \in X_k \times T | \tilde{a} \in \tilde{E}_t\}$. We claim that $q(\mathfrak{D}_k) = q(f_k)$ is connected.[2] To see this we have to use the following two claims.

Claim 1. Let Y be any nonsingular algebraic variety in $\mathbb{C}P^n$. Denote by $\mathscr{H}_t$, $\forall t \in T = \mathbb{C}P^{N*}$, the hyperplane in $\mathbb{C}P^{N*}$ corresponding to t, and by $\mathscr{T}_{Y,a}$, $\forall a \in Y$, the $\dim Y$-dimensional linear subspace in $\mathbb{C}P^N$ which is

[2]The simplified proof of this statement was suggested by the referee.

tangent to Y at a. Let us say that $\mathscr{H}_t$ is tangent to Y at $a \in Y$ if $\mathscr{H}_t \supset \mathscr{T}_{Y,a}$. Denote $\tau_Y = \{(a, t) \in Y \times T | \mathscr{H}_t$ is tangent to Y at $a\}$, and by $p_Y : \tau_Y \to Y$ the natural projection. Then $p_Y : \tau_Y \to Y$ is a holomorphic projective bundle over Y.

Claim 2. (Using the notation of the proposition). Denote by I the set consisting of X and all $S_{j_1} \cap \cdots \cap S_{j_e}$. Then

$$S(f_k) = \bigcup_{C \in I} g_k^{-1}(\tau_C) \quad (\text{each } \tau_C \subset W).$$

The proofs of these two claims are standard. In the proof of Claim 2 we have to use that $\forall \tilde{a} \in X_k$ the map $\alpha_k : X_k \to X$ can be defined locally by: $x_i \to x_i^{a_i}$, $a_i \geq 1$, integers, $i = 1, \cdots, n$.

For any $C \in I$ (see Claim 2) denote $q_C = q(f_k) \cap g_k^{-1}(\tau_C)$. Because $q(f_k)$ is open in $S(f_k)$, each q_C is open in $g_k^{-1}(\tau_C)$. From Claim 2 it follows that $q(f_k) = \bigcup_{C \in I} q_C$. Since $p|_{\tau_C} : \tau_C \to C$ is a projective bundle, we get that $g_k^{-1}(\tau_C) \xrightarrow{p_k} \alpha_k^{-1}(C)$ is a projective bundle and in particular $\forall \tilde{a} \in \alpha_k^{-1}(C)$, $p_k^{-1}(\tilde{a}) \cap g_k^{-1}(\tau_C)$ is a complex projective space. So if $p_k^{-1}(\tilde{a}) \cap q_C \neq \varnothing$, then $p_k^{-1}(\tilde{a}) \cap q_C$ is a nonempty Zariski open subset in $p_k^{-1}(\tilde{a}) \cap g_k^{-1}(\tau_C)$ and thus irreducible and connected. In particular, when $p_k^{-1}(\tilde{a}) \cap q_C \neq \varnothing$, there is only one connected component of q_C intersecting $p_k^{-1}(\tilde{a})$. Denote this component by $q_{C,\tilde{a}}$.

Clearly all connected components of q_C (for $q_C \neq \varnothing$) can be represented as $q_{C,\tilde{b}}$ for some $\tilde{b} \in X_k$. From the assumptions of our proposition it easily follows that $q_X \neq \varnothing$. Thus q_X is connected; it is a nonempty Zariski open subset in the projective bundle $g_k^{-1}(\tau_X)$ over X_k. So to show that $q(f_k)$ is connected, we have to prove only that for each $q_{C,\tilde{b}}$ the intersection $q_{C,\tilde{b}} \cap q_X \neq \varnothing$.

Take any $q_{C,\tilde{b}}$ and let $b = \alpha_k(\tilde{b})$. Because $D_{b,\mathrm{sing}\, b}$ is infinite and base point free on $X - b$, there exists $E_{t_0} \in \mathfrak{D}$ such that $E_{t_0} \ni b$, and b is the only singularity of E_{t_0}. Then $\tilde{E}_{t_0}$ has only isolated singularities, and $\tilde{b}$ is one of them. Thus we get $(\tilde{b}, t_0) \in q(\tilde{f})$, $(b, t_0) \in \tau_X$, and so $(\tilde{b}, t_0) \in q_X$. Since $\mathscr{H}_{t_0}$ must be tangent to X at b, it is also tangent to C at b, that is, $(b, t_0) \in \tau_C$. So $(\tilde{b}, t_0) \in g_k^{-1}(\tau_C) \cap q(\tilde{f}) = q_C$ which implies $(\tilde{b}, t_0) \in q_{C,\tilde{b}}$. Hence $q_{C,\tilde{b}} \cap q_X \neq \varnothing$.

So we have proved that $q(f_k) = q(\mathfrak{D}_k)$ is connected.

By Proposition 9, $\mathfrak{D}_k$ is a linear system of Lefschetz type in X_k. Thus $\operatorname{codim}_T f_k(S(\mathfrak{D}_k) - q(\mathfrak{D}_k)) \geq 2$. Because $q(\mathfrak{D}_k)$ is connected, it follows from Proposition 5 that $\forall t \in T - S_b(\mathfrak{D}_k)$, $\Lambda_{\mathfrak{D}_k}(t)$ belongs to only one $G_{\mathfrak{D}_k}(t)$-orbit in $H_{n-1}(\tilde{E}_t)$. q.e.d.

Assume now that $\dim_{\mathbb{C}} X = 3$, $b_2(X) = 1$, and that for a nonsingular $\tilde{E}_t \in \mathfrak{D}_k$ and the canonical class $K_{\tilde{E}_t}$, $K^2_{\tilde{E}_t} \neq 0$. Let $i\colon H_2(\tilde{E}_t) \to H_2(X_k)$ be the canonical homomorphism for $\tilde{E}_t \hookrightarrow X_k$, and $u\colon H_4(X_k) \to H_2(\tilde{E}_t)$ be the "intersection homomorphism". From the adjunction formula it follows that $\exists e \in H_4(X_k)$ such that $u(e) = K_{\tilde{E}_t}$.

For any vanishing cycle $\delta \in H_2(\tilde{E}_t)$ we have $(K_{\tilde{E}_t} \cdot \delta)_{\tilde{E}_t} = (e \cdot i(\delta))_{X_k} = 0$. Denote by $(K_{\tilde{E}_t})$ the maximal rank-one submodule in $H_2(\tilde{E}_t)$ containing $K_{\tilde{E}_t}$, and by $\operatorname{Aut}(H_2(\tilde{E}_t), (K_{\tilde{E}_t}))$ the group of all intersection preserving automorphisms α of $H_2(\tilde{E}_t)$ such that $\forall z \in (K_{\tilde{E}_t})$, $\alpha(Z) = Z$. From $(K_{\tilde{E}_t} \cdot \delta) = 0$ $\forall$ vanishing cycles $\delta \in H_2(\tilde{E}_t)$ it follows that $\forall \theta \in G_{\mathfrak{D}_k}(t)$ and $\forall z \in (K_{E_t})$ $\theta(z) = z$. Thus $G_{\mathfrak{D}_k}(t) \subset \operatorname{Aut}(H_2(\tilde{E}_t), (K_{\tilde{E}_t}))$.

Denote by V_t the submodule in $H_2(\tilde{E}_t)/(\mathrm{Tor})$ generated by all elements of $\Lambda_{\mathfrak{D}_k}(t)$ (all vanishing cycles in $H_2(\tilde{E}_t)$). Let $\mathscr{O}_{V_t}$ be the group of all automorphisms of V_t preserving the intersection pairing. Evidently there is a natural homomorphism of $G_{\mathfrak{D}_k}(t)$ in $\mathscr{O}_{V_t}$.

From Proposition 11 and $b_2(X) = 1$ it follows that $b_2(X_k) = 1$ and thus $b_4(X_k) = 1$. $K^2_{\tilde{E}_t} \neq 0$ implies now that e generates $H_4(X_k) \otimes \mathbb{Q}$.

Let $z \in H_2(\tilde{E}_t)$ be such that $(z \cdot K_{\tilde{E}_t})_{\tilde{E}_t} = 0$. Thus $0 = (z \cdot u(e))_{\tilde{E}_t} = (i(z) \cdot e)$ and $i(z) = 0$ in $H_2(X) \otimes \mathbb{Q}$. Using also $K^2_{\tilde{E}_t} \neq 0$ we get the orthogonal decomposition:

$$H_2(\tilde{E}_t) \otimes \mathbb{Q} = (\operatorname{Ker} i \otimes \mathbb{Q}) \oplus ((K_{\tilde{E}_t}) \otimes \mathbb{Q}).$$

From this decomposition and the Proposition 8(1) it follows that if the image of $G_{\mathfrak{D}_k}(t)$ in $\mathscr{O}_{V_t}$ has a finite index, then $G_{\mathfrak{D}_k}(t)$ is a subgroup of finite index in $\operatorname{Aut}(H_2(\tilde{E}_t), (K_{\tilde{E}_t}))$.

On the other hand, by results of Ebeling (see [3], [4]), $G_{\mathfrak{D}_k}(t)$ will have a finite index in $\mathscr{O}_V(\tilde{E}_t)$ when the following conditions are satisfied:

(1) $\mathfrak{D}$ has an element E_s with only isolated singularities, one of which has in its versal local deformation the so-called $\mathscr{U}_{12}$ (Arnold) singularity (it is given by $z^3 + y^3 + x^4 = 0$ (see [3]).

(2) $\Lambda_{D_k}(t)$ belongs to only one $G_{D_k}(t)$-orbit in $H_{n-1}(\tilde{E}_t)$.

Proposition 12. *Assume as in Proposition* 11 *that* $\forall x \in X$, $D_{x,\mathrm{sing}\,x}$ *is infinite and base point free in* $X - x$. *Assume also* (*as above*) *that* $\dim_{\mathbb{C}} X = 3$, $b_2(X) = 1$, $K^2_{\tilde{E}_t} \neq 0$ *and that there exists* E_s *in* $\mathfrak{D}$ *such that all singularities of* E_s *are isolated and one of them has in its versal local deformation the so-called* $\mathscr{U}_{12}$ *singularity* (*locally of the form* $z^3 + y^3 + x^4 = 0$). *Then* $G_{D_k}(t)$ *is a subgroup of finite index in* $\operatorname{Aut}(H_2(\tilde{E}_t), (K_{\tilde{E}_t}))$.

Proof. The proof follows from remarks above (Ebeling's results) combined with Proposition 11.

Proposition-Example 13. *Let $X = \mathbb{C}P^3$, $\mathscr{D}$ be the linear system of all quadrics in $\mathbb{C}P^3$, and $S_1, \cdots, S_k$ be k nonsingular surfaces in $\mathbb{C}P^3$, such that each $S_{j_1} \cap \cdots \cap S_{j_l}$ is nonsingular and $\bigcup_{j=1}^k S_j$ has only normal crossing singularities. Let $\{m_1, \cdots, m_k\}$ be positive integers such that m_j divides $\deg S_j$, $j = 1, \cdots, k$. Define inductively $\alpha_j \colon X_j \to X$, $j = 0, \cdots, k$, as follows: $X_0 = X$, $\alpha_0 = \mathrm{Id}$, and if $\alpha_{j-1} \colon X_{j-1} \to X$ is defined, let $\beta_j \colon X_j \to X_{j-1}$ be the degree m_j cyclic covering of X_{j-1} ramified at $\alpha_{j-1}^{-1}(S_j)$, $\alpha_j = \alpha_{j-1} \circ \beta_j$. Let $\mathfrak{D}_k = \alpha_k^* \mathfrak{D}$, and let $\tilde{E}_t$ be a generic element in $\mathfrak{D}_k$. Assume that $\exists j_0 \in \{1, \cdots, k\}$ such that $m_{j_0} \geq 3$ and $\deg S_{j_0} \geq 4$. Then $G_{D_k}(t)$ is a subgroup of finite index in $\mathrm{Aut}(H_2(\tilde{E}_t), (K_{\tilde{E}_t}))$.*

Proof. We shall use Proposition 12. Take any $x \in X$. Considering all quadric cones in $X = \mathbb{C}P^3$ with the vertex at x, we see that $D_{x,\mathrm{sing}\, x}$ is infinite and base point free in $X - x$. Without loss of generality we can assume that $j_0 = 1$.

Let $(x_0 \colon x_1 \colon x_2 \colon x_3)$ be homogeneous coordinates in $X = \mathbb{C}P^3$, $A = x_1^4 + x_2^3 x_0$, Q, F, G be some generic homogeneous polynomials in $(x_0 \colon x_1 \colon x_2 \colon x_3)$ with the following properties:

(a) $\deg Q = 2$, $\deg F = m_1 - 4$, $\deg G = m_1 - 2$;

(b) $\{x_1 = x_2 = 0\} \cap \{Q = 0\} \cap \{G = 0\} = \varnothing$;

(c) $\{A = 0\} \cap \{F = 0\} \cap \{Q = 0\} \cap \{G = 0\} = \varnothing$;

(d) $\{x_1 = x_2 = 0\} \cap \{Q = 0\} \cap \{F = 0\} = \varnothing$.

Using Bertini's Theorem we can see that for generic F, Q, G as above and a generic λ, the surface of degree m_1 given by

$$S_{1,\lambda} = \{A \cdot F + \lambda Q \cdot G = 0\}$$

in $\mathbb{C}P^3$ is nonsingular.

Denote by $E_s = \{Q = 0\}$ and let $\beta_1 \colon X_1 \to X$ be the finite cyclic covering of degree m_1 of $X = \mathbb{C}P^3$ ramified at $S_{1,\lambda}$. Let $\overline{S}_{1,\lambda} = \{S_{1,\lambda} \cap E_s\}$. Then $\overline{S}_{1,\lambda}$ is a curve on E_s which has two singular points, say a_1, a_2, of type $x^4 + y^3 = 0$ (they are the points of $E_s \cap \{x_1 = x_2 = 0\}$). $\overline{S}_{1,\lambda}$ is the branch curve of the cyclic covering $\beta_1|_{\beta_1^{-1}(E_s)} \colon \beta_1^{-1}(E_s) \to E_s$.

Let $E_{s,1} = \beta_1^{-1}(E_s)$, $b_i = \beta_1^{-1}(a_i)$, $i = 1, 2$. Clearly at each b_i, $i = 1, 2$, $E_{s,1}$ has the singularity locally given by $z^{m_1} = y^3 + x^4$. Since we assume that $m_1 \geq 3$, this singularity has in its local versal family the singularity $z^{m_1} + \varepsilon z^3 = y^3 + x^4$ ($|\varepsilon| \ll 1$) equivalent to $z^3 = y^3 + x^4$, that is type $\mathscr{U}_{12}$ singularity.

Now choose generic surfaces $S_{2,\lambda}, \cdots, S_{k,\lambda}$ in $\mathbb{C}P^3$, $\deg S_{j,\lambda} = m_j$, such that each $S_{j_1,\lambda} \cap \cdots \cap S_{j_l,\lambda}$, $j_1, \cdots, j_l \in \{1, \cdots, k\}$, is nonsingular and that $\bigcup_{j=1}^k S_{j,\lambda}$ has only normal crossings. Deforming given $S_1, \cdots, S_k$ to

$S_{1,\lambda}, \cdots, S_{k,\lambda}$ we see that it is enough to prove the proposition in the case $S_j = S_{j,\lambda}$. Since in this case all conditions of Proposition 12 are satisfied, $G_{D_k}(t)$ is of finite index in $\mathrm{Aut}(H_2(\tilde{E}_t), (K_{\tilde{E}_t}))$.

4. Examples of (orientation preserving) homeomorphic and not diffeomorphic simply-connected algebraic surfaces of general type

Using a new invariant of S. Donaldson the following theorem was recently proved by R. Friedman and J. Morgan (see [6, §3]).

Theorem (*R. Friedman-J. Morgan*). *Let S_1 and S_2 be two simply-connected algebraic surfaces of general type. Suppose:*

(i) *orientation preserving diffeomorphisms of S_j $(j = 1, 2)$ induce a subgroup of finite index in* $\mathrm{Aut}(H_2(S_j), (K_{S_j}))$;

(ii) *$p_g(S_1)$ and $p_g(S_2)$ are even*;

(iii) *$K_{S_j} = n_j k_j$, where $n_j \in \mathbb{Z}^+$, $k_j \in H^2(S_j, \mathbb{Z})$ is a primitive cohomology class, and $n_1 \neq n_2$.*

Then S_1 and S_2 are not orientation preserving diffeomorphic.

We shall apply this theorem to some abelian Galois coverings of $\mathbb{C}P^1 \times \mathbb{C}P^1$ with branch loci having only normal crossing singularities. More explicitly let $Y_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$, $l_1 = \mathrm{pt} \times \mathbb{C}P^1$, $l_2 = \mathbb{C}P^1 \times \mathrm{pt}$, and $C = l_1 + l_2$. Let $\{x_1, x_2, \cdots\}$ be a sequence of positive integers. Define finite coverings $g_k(x_1, \cdots, x_k) = g_k : Y(x_1, \cdots, x_k) \to Y_0$ as follows:

Let $g_0 = \mathrm{Id}$. Assume that g_{k-1} is already defined. Let $f_k : Y(x_1, \cdots, x_k) \to Y(x_1, \cdots, x_{k-1})$ be a triple cyclic covering of $Y(x_1, \cdots, x_{k-1})$ (for $k = 1$ of Y_0) with nonsingular branch locus B_k linearly equivalent to $g_{k-1}^*(3x_k C)$. Let $g_k = g_{k-1} \cdot f_k$.

Lemma. *Let $Y_k = Y(x_1, \cdots, x_k)$, and $C_k = g_k^* C$. Then*

(1) $K_{Y_k} = 2(\sum_{i=1}^k x_i - 1)C_k$;

(2) $c_1^2(Y_k) = 8 \cdot 3^k((\sum_{i=1}^k x_i) - 1)^2$;

(3) *the index* $\tau(Y_k) = -16 \cdot 3^{k-1}(\sum_{i=1}^k x_i^2)$;

(4) $\chi(Y_k) = 1 + p_g(Y_k) = 3^k(\sum_{i=1}^k x_i - 1)^2 + 2 \cdot 3^{k-1}(\sum_{i=1}^k x_i^2)$.

Proof. Use induction on k. The lemma is true for $Y_0 = Y(\varnothing)$ by taking "all x_i"=0. Assume it is true for Y_{k-1} $(k \geq 1)$.

Since $K_{Y_k} = f_k^* K_{Y_{k-1}} + \frac{2}{3} f_k^* B_k$ and by the inductive assumption $K_{Y_{k-1}} = 2(\sum_{i=1}^{k-1} x_i - 1)C_{k-1}$, we have

$$K_{Y_k} = 2(\sum_{i=1}^{k-1} x_i - 1)C_k + 2x_k C_k = 2(\sum_{i=1}^k x_i - 1)C_k.$$

Using $C_k^2 = 2 \cdot 3^k$ we get

$$c_1^2(Y_k) = K_{Y_k}^2 = 8 \cdot 3^k(\sum_{i=1}^k x_i - 1)^2.$$

This proves (1) and (2) of the lemma. From $K_{Y_k} = f_k^* K_{Y_{k-1}} + \frac{2}{3} f_k^* B_k$ it follows also that $c_1^2(Y_k) = 3c_1^2(Y_{k-1}) + 4K_{Y_{k-1}} \cdot B_k + \frac{4}{3}B_k^2$. Using $c_2(Y_k) = 3c_2(Y_{k-1}) - 2c_1(B_k)$ and $-2c_1(B_k) = 2(K_{Y_{k-1}} + B_k) \cdot B_k$ we get

$$\begin{aligned}\tau(Y_k) &= \tfrac{1}{3}(c_1^2(Y_k) - 2c_2(Y_k)) \\ &= \tfrac{1}{3}[3c_1^2(Y_{k-1}) + 4K_{Y_{k-1}} \cdot B_k + \tfrac{4}{3}B_k^2 - 6c_2(Y_{k-1}) - 4K_{Y_{k-1}} \cdot B_k - 4B_k^2] \\ &= \tfrac{1}{3}[3(c_1^2(Y_{k-1}) - 2c_2(Y_{k-1})) - \tfrac{8}{3}B_k^2] = 3\tau(Y_{k-1}) - \tfrac{8}{9}B_k^2.\end{aligned}$$

By inductive assumption $\tau(Y_{k-1}) = -16 \cdot 3^{k-2}(\sum_{i=1}^{k-1} x_i^2)$. Because $B_k = g_{k-1}^*(3x_k C)$, we have $B_k^2 = 3^{k-1} \cdot 9 \cdot x_k^2 \cdot 2$ and $-\frac{8}{9}B_k^2 = -16 \cdot 3^{k-1} x_k^2$. Thus

$$\begin{aligned}\tau(Y_k) &= -16 \cdot 3^{k-1}(\textstyle\sum_{i=1}^{k-1} x_i^2) - 16 \cdot 3^{k-1} x_k^2 \\ &= -16 \cdot 3^{k-1} \cdot \textstyle\sum_{i=1}^{k} x_i^2.\end{aligned}$$

This proves (3) of the lemma. Since

$$\tfrac{1}{8}(c_1^2 - \tau) = \tfrac{1}{8}[c_1^2 - \tfrac{1}{3}(c_1^2 - 2c_2)] = \tfrac{1}{24}(3c_1^2 - c_1^2 + 2c_2) = \frac{1}{12}(c_1^2 + c_2) = \chi,$$

we have

$$\begin{aligned}\chi(Y_k) &= \tfrac{1}{8}[c_1^2(Y_k) - \tau(Y_k)] \\ &= \tfrac{1}{8}[8 \cdot 3^k(\textstyle\sum_{i=1}^{k} x_i - 1)^2 + 16 \cdot 3^{k-1} \textstyle\sum_{i=1}^{k} x_i^2] \\ &= 3^k \cdot (\textstyle\sum_{i=1}^{k} x_i - 1)^2 + 2 \cdot 3^{k-1} \textstyle\sum_{i=1}^{k} x_i^2.\end{aligned}$$

It is easy to see that each Y_k is simply connected. So $\chi(Y_k) = 1 + p_g(Y_k)$.

Remark 1. From (1) of the lemma it follows that each $Y(x_1, \cdots, x_k)$ is an even 4-manifold, and from (2) and (3) we can deduce that if $\{x_1, \cdots, x_k\}$ and $\{y_1, \cdots, y_l\}$ are two sequences of positive integers, then the corresponding $Y(x_1, \cdots, x_k)$ and $Y(y_1, \cdots, y_l)$ are (orientation preserving) homeomorphic iff

$$3^k(\textstyle\sum_{i=1}^{k} x_i - 1)^2 = 3^l(\textstyle\sum_{i=1}^{l} y_j - 1)^2,$$

$$3^{k-1} \textstyle\sum_{i=1}^{k} x_i^2 = 3^{l-1} \textstyle\sum_{i=1}^{l} y_j^2,$$

by using $\pi_1(Y(x_1, \cdots, x_k)) = \pi_1(Y(y_1, \cdots, y_l)) = 0$ and the results of M. Freedman [5].

We can reformulate the conditions above as follows: $Y(x_1, \cdots, x_k)$ and $Y(y_1, \cdots, y_l)$ are (orientation preserving) homeomorphic iff

(a) $k \equiv l \pmod 2$;

(b) $\sum_{i=1}^{k} x_i = 3^{(l-k)/2}(\sum_{j=1}^{l} y_j) - 3^{(l-k)/2} + 1$; and

(c) $\sum_{i=1}^{k} x_i^2 = 3^{l-k} \sum_{j=1}^{l} y_j^2$.

Remark 2. From (4) of the lemma it follows that $p_g(Y(x_1,\cdots,x_k))$ is even iff $\sum_{i=1}^k x_i$ is even.

Remark 3. Taking in Proposition-Example 13 (see §3 above) all m_j, $j=1,\cdots,k$, to be equal to three, identifying $Y_0=\mathbb{C}P^1\times\mathbb{C}P^1$ with a generic element E of the linear system $\mathfrak{D}$ of all quadrics in $X=\mathbb{C}P^3$, and choosing appropriate hypersurfaces $S_1,\cdots,S_k$ in $\mathbb{C}P^3$ (see Proposition-Example 13) with $S_j\cap E\in|3x_jC|$, $C\in|l_1+l_2|\subset E=Y_0$, we obtain finite coverings $\alpha_j\colon X_j\to X$ such that $Y(x_1,\cdots,x_k)=\alpha_k^{-1}(E)$, and $g_k=g_k(x_1,\cdots,x_k)\colon Y(x_1,\cdots,x_k)\to Y_0$ coincides with $\alpha_k|_{\alpha_k^{-1}(E)}\colon\alpha_k^{-1}(E)\to E$. Assume $\exists i_0\in(1,\cdots,k)$ such that $x_{i_0}\geq 2$. Then $\deg S_{i_0}\geq 6$ and we get from the Proposition-Example 13 that orientation preserving diffeomorphisms of $Y(x_1,\cdots,x_k)$ $(=\alpha_k^{-1}(E))$ induce a subgroup of finite index in $\operatorname{Aut}(H_2(Y(x_1,\cdots,x_k)),(K_{Y(x_1,\cdots,x_k)}))$.

Remark 4. From the Corollary of Proposition 10 (see §3) it follows that g_k^*C belongs to a primitive cohomology class in $H^2(Y(x_1,\cdots,x_k),Z)$.

Proposition 14. *Let m be any nonnegative integer, and $\{l_1,\cdots,l_m\}$ any sequence of m positive integers with $\sum_{j=1}^m l_j$ even (for $m=0$ we take the empty set). Let $y_1=y_2=y_3=y_4=1$, $y_5=6$ and $\forall j=1,\cdots,m$, $y_{5+j}=l_j$; $x_1=2$, $x_2=10$, $x_3=16$ and $\forall j=1,\cdots,m$, $x_{3+j}=3l_j$. Take $k=m+3$, $l=m+5$. Then the corresponding $Y(x_1,\cdots,x_k)$, $Y(y_1,\cdots,y_l)$ are (orientation preserving) homeomorphic and not diffeomorphic simply connected (minimal) algebraic surfaces of general type.*

Proof. Let us first check that $Y(x_1,\cdots,x_k)$ and $Y(y_1,\cdots,y_l)$ are (orientation preserving) homeomorphic. Use Remark 1 above. Let $M=\sum_{j=1}^m l_j$ and $N=\sum_{j=1}^m l_j^2$. We see that in our case $l-k=2$ and

$$\sum_{i=1}^k x_i=2+10+16+3\sum_{j=1}^m l_j=28+3M;$$

$$\sum_{j=1}^l y_j=1+1+1+1+6+\sum_{j=1}^m l_j=10+M.$$

Thus condition (b) in Remark 1 is equivalent here to $28+3M=30+3M-3+1$ which is true. We also have:

$$\sum_{i=1}^k x_i^2=4+100+256+9\sum_{j=1}^m l_j^2=360+9N,$$

$$\sum_{j=1}^l y_j^2=1+1+1+1+36+\sum_{j=1}^m l_j^2=40+N.$$

Condition (c) of Remark 1 is equivalent here to $360+9N=9(40+n)$ which is true. Thus our $Y(x_1,\cdots,x_)$ and $Y(y_1,\cdots,y_l)$ are (orientation preserving) homeomorphic.

Now let $S_1=Y(x_1,\cdots,x_k)$, $S_2=Y(y_1,\cdots,y_l)$. To prove that S_1 and S_2 are not diffeomorphic we use the theorem of R. Friedman and J. Morgan mentioned above. We have to check that S_1 and S_2 satisfy conditions (i), (ii), (iii) of that theorem. From Remark 3 above it follows that S_1 and S_2 satisfy condition (i).

Because $M = \sum_{j=1}^{m} l_j$ is even, $\sum_{i=1}^{k} x_i = 28 + 3M$ and $\sum_{j=1}^{l} y_j = 10 + M$ are even. Thus by Remark 2 above $p_g(S_1)$ and $p_g(S_2)$ are even numbers. So condition (ii) is satisfied.

To check (iii), let $K_{S_j} = n_j P_j$, where $n_j \in \mathbb{Z}^+$, and $P_j \in H^2(S_j, \mathbb{Z})$ is a primitive cohomology class. Denote $h_1 = g_k(x_1, \cdots, x_k)\colon S_1 \to Y_0$, and $h_2 = g_k(y_1, \cdots, y_l)\colon S_2 \to Y_0$. Using Remark 4 above we see that $h_j^* C$ is primitive in $H^2(S_j, \mathbb{Z})$, $j = 1, 2$. Thus by (1) of the lemma above we get that $n_1 = 2((\sum_{i=1}^{k} x_i) - 1) = 2(28 + 3M - 1)$ and $n_2 = 2(10 + M - 1)$, that is $n_1 = 3n_2$. So $n_1 \neq n_2$.

References

[1] A. Andreotti & T. Frankel, *The Lefschetz Theorem on hyperplane sections*, Ann. of Math. **69** (1969) 713–717.

[2] ——, *The second Lefschetz Theorem on hyperplane sections*, Global Analysis: Papers in Honor of K. Kodaira, Princeton University Press, Princeton, 1969, 1–20.

[3] A. Beauville, *Le groupe de monodromie des familles universel d'hypersurfaces et d'intersections completes*, Complex Analysis and Algebraic Geometry (H. Gráuert, ed.), Lecture Notes in Math., Vol. 1194, Springer, Berlin, 1986, 8–18.

[4] W. Ebeling, *An arithmetic characterization of the symmetric monodromy groups of singularities*, Invent. Math. **77** (1984) 85–99.

[5] M. Freedman, *The topology of four-dimensional manifolds*, J. Differental Geometry **17** (1982) 357–454.

[6] R. Friedman, B. Moishezon & J. W. Morgan, *On the C^∞ invariance of the canonical classes of certain algebraic surfaces*, Bull. Amer. Math. Soc. (N.S.) **17** (1987), 283–286.

[7] E. Looijenga, *Isolated singular points on complete intersections*, Cambridge University Press, Cambridge, 1984.

COLUMBIA UNIVERSITY