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EMBEDDED MINIMAL SURFACES IN MANIFOLDS DIFFEOMORPHIC TO THE THREE-DIMENSIONAL BALL OR SPHERE

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1. Introduction

The celebrated result of Lusternik-Schnirelman asserts that for any Riemannian metric on a 2-sphere S^2, there exist at least three closed geodesics without selfintersections. In the present note, we shall establish an analogue of this theorem for minimal surfaces in three-dimensional manifolds by proving the following two theorems.

Theorem 1. *Let A be a compact body in $\mathbb{R}^3$, homeomorphic to the closed unit ball, with boundary ∂A of class C^4, and of positive mean curvature with respect to the interior normal. Then A contains at least three embedded minimal disks meeting ∂A orthogonally along their boundary, i.e., solving a free boundary problem with boundary ∂A.*

Theorem 2. *Let M be a compact three-dimensional Riemannian manifold, diffeomorphic to the three-sphere S^3. Then M contains at least four embedded minimal two-spheres.*

Let us first discuss the previous results about these problems.

It follows from the work of Sacks and Uhlenbeck [8] that any three-dimensional Riemannian manifold diffeomorphic to S^3 contains a minimal two-sphere. This two-sphere, however, need not be embedded and may have branch points. On the other hand, it follows from the work of Pitts [7] that such a manifold contains a compact embedded minimal surface. [7], however, yields no control over the topological type of this surface. Finally, Simon and Smith [9] succeeded in proving the existence of an embedded minimal S^2 in each manifold diffeomorphic to S^3. White [12] showed that a manifold of positive Ricci curvature, diffeomorphic to S^3, contains at least two embedded minimal two-spheres.

Concerning the free boundary problem addressed in Theorem 1, the first results were due to Smyth [10] and Struwe [11]. Smyth showed that

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for a tetrahedron in $\mathbb{R}^3$, one finds at least three nontrivial embedded minimal disks solving the free boundary problem. Our result does not cover Smyth's, because a tetrahedron does not satisfy the smoothness and positive mean curvature assumption of Theorem 1. Although his arguments are of a rather elementary nature and do not seem to be extendable to more general boundary configurations, we hope to be able to include his result by an approximation argument in the future. Also, it seems that all three of his solutions can be obtained as first order saddle points, whereas for our solutions we also need higher order saddle point constructions. That his solutions should be first order saddle points becomes clear, when one looks at an equilateral tetrahedron in which case all three solutions are congruent and in particular have the same area. If they were higher order saddle points, then there should exist infinitely many solutions of equal area which does not seem to be the case in this example.

Struwe [11] showed the existence of a (possibly branched) minimal immersion of a disk for smooth ∂A, and Grüter and Jost [1] and Jost [3], [4] obtained an embedded minimal disk for a strictly convex boundary, respectively one of nonnegative mean curvature. In [3], Theorem 1 was shown for strictly convex bodies for which the ratio between outer and inner diameter does not exceed $\sqrt{2}$. The method of [3] can also be used to establish Theorem 2 provided that the curvature of M satisfies a suitable pinching condition.

Although the abstract topological setting of the three-disk problem and the four-sphere problem solved in our theorems is the same as in the construction of Lusternik and Schnirelman, several new difficulties occur when one attempts to extend the analysis from geodesics to minimal surfaces. In particular, for saddle point constructions leading to minimal surfaces one gets a much weaker type of convergence than for geodesics. One particular difficulty arises from the fact that the limiting minimal surface may be disconnected or of higher multiplicity so that, for example, the solution obtained from a second order saddle point construction might just be the same as the one from a first order construction, covered twice. Although one can easily exclude this multiple covering in many cases, it seems plausible that it might occur in certain other cases. For this reason, the strategy of our proof is as follows:

If such a multiple covering or disconnectedness occurs in a second or third order saddle point argument, then we construct new classes, in which we perform new saddle point constructions, exploiting the fact that in this case approximating surfaces cannot remain invariant under suitable rotations of the limit surfaces. Some geometric interpretations of our constructions will be given at the end of the paper, and will be displayed there rather than in the introduction here as they cannot be understood without first trying to look at the details of our proof.

All other technical problems were already overcome in our previous paper [3] on which many of our arguments depend.

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2. Proof of Theorem 1

We shall first consider the proof of Theorem 1. The proof of Theorem 2 is very similar, so that after giving the details for Theorem 1, a short sketch at the end will suffice for Theorem 2. In the case of Theorem 2, the existence of the first solution already follows from [9], while the arguments of [3] can be directly generalized to this setting.

Let us start with some remarks about the notation.

$I := [0, 1]$.

$D := \{(x_1, x_2)\colon x_1^2 + x_2^2 \le 1\}$.

$B := \{(x_1, x_2, x_3)\colon x_1^2 + x_2^2 + x_3^2 \le 1\}$.

A is a closed subset of $\mathbb{R}^3$, diffeomorphic to B, with boundary ∂A of class C^4 and with positive mean curvature with respect to the interior normal. (Note that, in contrast, A was open in [3].)

We let $\lambda\colon B \to A$ be a diffeomorphism.

We let $\mathscr{M}$ be the class of all embedded disks of class C^2 in A, meeting ∂A transversally along their boundary.

We shall use the same notation as in [3] except that instead of $\kappa(0,0)$, $\kappa(0,1), \kappa(1,1)$, we write $\kappa_1, \kappa_2, \kappa_3$, etc. All notation concerning varifolds will be taken from [3].

Let us first review the topological constructions of Lusternik and Schnirelman; we shall use the presentation of [6] which offers a slight technical advantage over the presentation of [5] used in [3]. Here, we shall not address the analytic arguments needed to adapt these constructions to minimal surfaces as those can be readily found in [3]. Also, we need not consider the question of regularity of the solutions at the free boundary, as it was already solved in [1] and [2].

1. We let u_1 assign to $t \in I$ the set

$$u_1(t) := \{x_3 = 2t - 1\} \cap B.$$

We let V_1 be the collection of all cycles homologous mod 2 to $\lambda \circ u_1$ in $\mathscr{M}$ mod ∂A.

2. We let u_2 assign to $(t_1, t_2) \in I^2$ the set

$$u_2(t_1, t_2) = \rho^1(t_2)u_1(t_1),$$

where $\rho^i(t)$ is the positive rotation by πt around the x_i-axis. We obtain the identifications

$$u_2(t_1, 1) = -u_2(1 - t_1, 0) \quad \text{for } 0 \leq t_1 \leq 1,$$

where a minus $(-)$ sign denotes a change of orientation. We let V_2 be the class of all cycles homologous mod 2 to $\lambda \circ u_2$ in $\mathscr{M}$ mod ∂A.

In certain cases, one also has to consider so-called subordinated cycles of the following type: Let $l = I \to I^2$ be continuous with $l(\phi) = (t_1(\phi), t_2(\phi))$ and

$$t_1(1) = 1 - t_1(0), \quad t_2(0) = 0, \quad t_2(1) = 1.$$

Then $\lambda \circ u_2(t_1(\phi), t_2(\phi)) \in V_1$, because $l(\phi)$ is homologous mod 2 in $\mathscr{M}$ mod ∂A to $u_1(\phi)$ (and thus also to $u_2(\phi, 0)$).

More generally, each $v \in V_2$ contains such subordinated cycles, and these cycles play a crucial role in the topological arguments of Lusternik and Schnirelman.

3. We let u_3 assign to $(t_1, t_2, t_3) \in I^3$ the set

$$u_3(t_1, t_2, t_3) = \rho^3(t_3)u_2(t_1, t_2),$$

obtaining the identifications

$$u_3(t_1, 1, t_3) = -_3(1 - t_1, 0, t_3), \qquad u_3(t_1, t_2, 1) = -u_3(1 - t_1, 1 - t_2, 0).$$

We let V_3 be the class of all cycles homologous mod 2 to $\lambda \circ u_3$ in $\mathscr{M}$ mod ∂A.

Again, we can construct subordinated cycles. Namely, let $l_1 \colon I^2 \to I^3$ be continuous with

$$l_1(\rho_1, \rho_2) = (t_1(\rho_1, \rho_2), t_2(\rho_1, \rho_2), t_3(\rho_1, \rho_2))$$

and

$$\begin{aligned} t_1(1, \rho_2) &= 1 - t_1(0, \rho_2), & t_1(\rho_1, 1) &= 1 - t_1(\rho_1, 0), \\ t_2(1, \rho_2) &= 1 = t_3(\rho_1, 1), & t_2(0, \rho_2) &= 0 = t_3(\rho_1, 0), \\ t_2(\rho_1, 1) &= 1 - t_2(\rho_1, 0), & t_3(1, \rho_2) &= t_3(0, \rho_2). \end{aligned}$$

Then $\lambda \circ u_3(t_1(\rho_1, \rho_2), t_2(\rho_1, \rho_2), t_3(\rho_1, \rho_2)) \in V_2$, because $l_1(\rho_1, \rho_2)$ is homologous mod 2 in $\mathscr{M}$ mod ∂A to $u_2(\rho_1, \rho_2)$.

Again, each $v \in V_3$ contains such subordinated cycles.

The three homology classes V_1, V_2, V_3 describe the topology of $\mathscr{M} \bmod \partial A$. Rotation around the remaining axis does not give rise to a new cycle anymore; namely put

$$u_4(t_1, t_2, t_3, t_4) = \rho^2(t_4) u_3(t_1, t_2, t_3).$$

Then one has the boundary identifications

$$u_4(t_1, t_2, t_3, 1) = -u_4(1 - t_1, t_2, 1 - t_3, 0),$$

and gets nothing new. u_4, however, is useful for the geometric intuition in case 3.3 below.

In the case of Theorem 2, one likewise considers the class $\mathscr{M}$ of embedded two-spheres of class C^2 in M. In this case, $\mathscr{M} \bmod M$ has four nontrivial cycles mod 2, constructed in a similar way as above. These cycles can be most easily described if one represents S^3 as the unit sphere in $\mathbb{R}^4$. A fourth order homology class in the space of embedded two-spheres $\mathscr{M}$ ($\bmod S^3$) is created by rotating a third order class (constructed as in 3. above) about the x_4-axis by an angle of 180 degrees.

Remark. In [3, p. 406f.], in the definition of the corresponding classes $V(0,0), V(0,1), V(1,1)$, it was in addition required that each element can be parametrized in such a way on the unit disk D that the parametrization depends smoothly on $t \in J$, where J is the appropriate domain of definition (the corresponding map was denoted by k in [3]). It turns out that on one hand checking this condition for a given class leads to a nontrivial global topological problem, while on the other hand this requirement is not necessary for the considerations of [3] (and the present paper). Namely, one only needs parametrizations depending smoothly on a parameter locally, and this is automatically satisfied for a smooth class. This is an advantage of the nonparametric approach which we have adopted over the classical Lusternik-Schnirelman approach to the theorem of the three-closed geodesics where this global requirement leads to a difficult technical problem.

We now carry out subsequent saddle point constructions.

1. We put

$$\kappa_1 := \inf_{v \in V_1} \sup_{t \in I} |v(t)|.$$

By [3], one obtains a minimaxing varifold of the form

$$W_1 = \sum_{j=1}^{N_1} n_{1,j} \mathbf{v}(\Sigma_{1,j}),$$

where $N_1, n_{1,j}$ are positive integers, $\Sigma_{1,j}$ are pairwise disjoint embedded minimal disks meeting ∂A orthogonally along their boundary, and

$$\kappa_1 = \sum_{j=1}^{N_1} n_{1,j}|\Sigma_{1,j}|.$$

2. We put

$$\kappa_2 := \inf_{v\in V_2} \sup_{t\in J} |v(t)|,$$

J being a parameter domain for v. By [3], one obtains a minimaxing varifold of the form

$$W_2 = \sum_{j=1}^{N_2} n_{2,j}\mathbf{v}(\Sigma_{2,j})$$

with the same properties as W_1, and

$$\kappa_2 = \sum_{j=1}^{N_2} n_{2,j}|\Sigma_{2,j}|.$$

We consider several cases:

2.1. We find a surface $\Sigma_{2,j}$ different from all $\Sigma_{1,i}$, $i = 1, \cdots, N_1$.

2.2. $\kappa_2 = \kappa_1$. In this case, by the constructions of [3],we find infinitely many solutions as follows. We consider subordinated cycles $l(\phi)$ as described above. We look at $v \circ l(\phi)$, for $v \in V_2$, and, as noted above, $v \circ l \in V_1$. Hence, for every such v and l,

$$\sup_{\phi\in I} |v \circ l(\phi)| \geq \kappa_1,$$

because κ_1 was the minimaxing value for V_1.

If one could find some $v \in V_2$ with $\sup_{t\in J} |v(t)| = \kappa_2$ and with all $v(t)$ with $|v(t)| = \kappa_2$ consisting of minimal surfaces, then for every such l actually

$$\sup_{\phi\in I} |v \circ l(\phi)| = \kappa_1 = \kappa_2,$$

and since the boundary identifications required for u_2 make I^2 into a Möbius-strip, one concludes (cf. [6, p. 349] or [3, p. 423]) that there exists a 1-dimensional cycle of solutions, i.e., there exist infinitely many solutions.

In general, it is not clear whether one can find some $v \in V_2$ with $\sup_{t\in J} |v(t)| = \kappa_2$, and this leads to certain technical complications. These complications, however, are handled by the argument of [3, p. 422f], and one obtains a 1-dimensional cycle of solutions.

We also remark that, because of

$$\sup_{\phi \in I} |v \circ l(\phi| \geq \kappa_1$$

as $v \circ l \in V_1$, case 1 of [3, p. 421] cannot occur. Precisely the same reasoning applies to all considerations below whenever we work with subordinated cycles.

2.3. $\{\Sigma_{2,1}, \cdots, \Sigma_{2,N_2}\} \subset \{\Sigma_{1,1}, \cdots, \Sigma_{1,N_1}\}$, but $\kappa_2 > \kappa_1$.

Case 2.3 will be considered below. In case 2.1 or 2.2, we proceed to

3. We put

$$\kappa_3 := \inf_{v \in V_3} \sup_{t \in J} |v(t)|,$$

J being a parameter domain for v. By [3] again, one obtains a minimaxing varifold of the form

$$W_3 = \sum_{j=1}^{N_3} n_{3,j} \mathbf{v}(\Sigma_{3,j})$$

with the same properties as W_1, and

$$\kappa_3 = \sum n_{3,j} |\Sigma_{3,j}|.$$

We consider several cases:

3.1. We find a surface $\Sigma_{3,j}$ different from all $\Sigma_{1,i}, \Sigma_{2,i}$.

3.2. $\kappa_3 = \kappa_2$. In this case again, for $v \in V_3$ and l_1 as above, $v \circ l_1 \in V_2$, hence

$$\sup_{t \in J} |v \circ l_1(t)| \geq \kappa_2,$$

and in the same way as in 2.2 (cf. [3, p. 424f.] for the detailed argument) we obtain a one-parameter family of solutions.

3.3. $\{\Sigma_{3,1}, \cdots, \Sigma_{3,N_3}\} \subset \{\Sigma_{1,1}, \cdots, \Sigma_{2,N_2}\}$.

In case 3.1 or 3.2 the theorem is proved. Case 3.3 will be considered below.

We return to case 2.3. We let $\varepsilon_n \to 0$, and look at cycles $v_n : I^2 \to \mathcal{M}$ contained in V_2, with

$$v_n(\tfrac{1}{2}, \tfrac{1}{2}) \subset N(W_2, \varepsilon_n), \tag{1}$$

where N denotes a neighborhood with respect to the **F**-distance function for varifolds. We then consider all smooth three-parameter families of diffeomorphisms $\psi_n(t_1, t_2, t_3) : A \to A$, $0 \leq t_1, t_2 \leq 1$, $0 \leq t_3 \leq 2$, with

$$\psi_n(\tfrac{1}{2}, \tfrac{1}{2}, t_3)_\#(W_2) = W_2, \quad \psi_n(\tfrac{1}{2}, \tfrac{1}{2}, t_3)_\#(N(W_2, \varepsilon_n)) \subset N(W_2, \varepsilon_n) \tag{2}$$

for all t_3,

$$\psi_n(\tfrac{1}{2},\tfrac{1}{2},t+s) = \psi_n(\tfrac{1}{2},\tfrac{1}{2},t)\circ\psi_n(\tfrac{1}{2},\tfrac{1}{2},s) \quad \text{for all } t,s, \tag{3}$$

$$\psi_n(t_1,t_2,1) \neq \mathrm{id}, \quad \psi_n(t_1,t_2,2) = \mathrm{id} \quad \text{for all } t_1,t_2. \tag{4}$$

As $\sum n_{2,j} > 1$ in the present case, by (1), (2), and (4) we obtain

$$\psi_n(\tfrac{1}{2},\tfrac{1}{2},1)(v_n(\tfrac{1}{2},\tfrac{1}{2})) \neq v_n(\tfrac{1}{2},\tfrac{1}{2}). \tag{5}$$

We look at families of the form

$$w_n(t_1,t_2,t_3) = \psi_n(t_1,t_2,t_3)(v_n(t_1,t_2)), \qquad v_n \in V_2.$$

From (1) and (2) we obtain

$$w_n(\tfrac{1}{2},\tfrac{1}{2},t_3) \subset N(W_2,\varepsilon_n) \quad \text{for all } t_3. \tag{6}$$

On the other hand, because of (5), $w_n(\frac{1}{2},\frac{1}{2},t_3)$ is nonconstant. More generally, we can use any $u_n \in V_2$ homologous mod 2 to v_n in $\mathscr{M} \bmod \partial A$ and satisfying (1) for some parameter value.

We thus obtain classes $V_{2.3}^n$ of three-parameter families in $\mathscr{M}$. (Instead of restricting the parameter domain to the product of a two-dimensional domain and an interval, one can also admit more general three-dimensional families. Conditions (2) and (3) above then have to be satisfied, along a suitable line. For the sequel, however, the present class is already enough, except in case 2.3.2 below, where one needs this generalization for 3.4.2.)

We put

$$\kappa_{2.3} := \lim_{n\to\infty} \inf_{w_n \in V_{2.3}^n} \sup_{t\in J} |w_n(t)|,$$

J being a parameter domain for w_n. Similarly as before, we obtain a varifold $W_{2.3}$ with $\kappa_{2.3} = \|W_{2.3}\|$, the support of $W_{2.3}$ again consisting of disjoint embedded minimal disks with integer multiplicity.

We consider several cases:

2.3.1. $\kappa_{2.3} = \kappa_2$, $W_{2.3} = W_2$.

2.3.2. Not all components of $W_{2.3}$ are contained in the collection of surfaces $\Sigma_{1,1},\cdots,\Sigma_{2,N_2}$.

2.3.3. $\kappa_{2.3} > \kappa_2$, but $W_{2.3}$ has no new components. In this case, we repeat the above procedure with an additional parameter:

There again exist cycles $v_n\colon I^2\times[0,2]\to\mathscr{M}$ (we assume only for simplicity that the parameter domain is $I^2\times[0,2]$; other parameter domains are possible and can be handled in the same way, cf. 2.3) contained in $V_{2.3}^n$ with, say,

$$v_n(\tfrac{1}{2},\tfrac{1}{4},\tfrac{1}{2}) \subset N(W_{2.3},\delta_n),$$

with $\delta_n \to 0$. We then look at smooth four-parameter families of diffeomorphisms $\psi_n(t_1, t_2, t_3, t_4)\colon A \to A$, $0 \le t_1, t_2 \le 1$, $0 \le t_3, t_4 \le 2$, with the properties

$$\psi_n(\tfrac{1}{2}, \tfrac{1}{2}, t_3, t_4)_\#(W_2) = W_2,$$

$$\psi_n(\tfrac{1}{2}, \tfrac{1}{2}, t_3, t_4)_\#(N(W_2, \varepsilon_n)) \subset N(W_2, \varepsilon_n) \quad \text{for all } t_3, t_4,$$
$$\psi_n(\tfrac{1}{2}, \tfrac{1}{2}, t+s, t_4) = \psi_n(\tfrac{1}{2}, \tfrac{1}{2}, t, t_4) \cdot \psi_n(\tfrac{1}{2}, \tfrac{1}{2}, s, t_4) \quad \text{for all } s, t, t_4,$$
$$\psi_n(t_1, t_2, 1, 0) \ne \text{id}, \quad \psi_n(t_1, t_2, 2, 0) = \text{id} \quad \text{for all } t_1, t_2.$$

There exists a line $(t_1(\phi), t_2(\phi), t_3(\phi), t_4(\phi)) \subset I^2 \times [0, 2]^2$ with

$$t_1(0) = \tfrac{1}{2}, \quad t_2(0) = \tfrac{1}{4}, \quad t_3(0) = \tfrac{1}{2}, \quad t_4(0) = 0,$$
$$t_4(1) = \tfrac{1}{2}, \quad t_2(1) = \tfrac{3}{4}, \quad t_3(1) = \tfrac{1}{2}, \quad t_4(1) = 1,$$
$$t_i(2) = t_i(0), \qquad i = 1, \cdots, 4;$$
$$\psi_n(t_1(\phi), t_2(\phi), t_3(\phi), t_4(\phi))_\#(W_{2.3}) = W_{2.3},$$
$$\psi_n(t_1(\phi), t_2(\phi), t_3(\phi), t_4(\phi))_\#(N(W_{2.3}, \delta_n)) \subset N(W_{2.3}, \delta_n)$$

for all ϕ; we also require the semigroup property of these diffeomorphisms with respect to ϕ, and finally

$$\psi_n(t_1, t_2, t_3, 1) \ne id, \quad \psi_n(t_1, t_2, t_3, 2) = \text{id} \quad \text{for all } t_1, t_2, t_3.$$

We form the classes $V^n_{2.3.3}$ as before and put

$$\kappa_{2.3.3} := \lim_{n \to \infty} \inf_{w_n \in V^n_{2.3.3}} \sup_{t \in J} |w_n(t)|.$$

We again obtain a minimaxing varifold $W_{2.3.3}$ and have to consider three cases, the last one corresponding to $\kappa_{2.3.3} > \kappa_{2.3}$, but without obtaining any new minimal disk. We keep on repeating this process with additional parameter values until after at most a finite number of steps (dependent on the geometry of A) one of the first two cases (getting the same critical value as in the previous step or obtaining a new minimal disk) occurs. We can then perform the same constructions below as in case 2.3.1 or 2.3.2 respectively, but with some additional parameters.

The claim that after a finite number of steps one of the first two cases has to occur can be seen by the following explicit (and therefore very elementary) construction of suitable comparison paths. The construction presented here should also shed some light on the difference between the one-dimensional case of closed geodesics and the two-dimensional case of minimal surfaces, the essential difference stemming from the fact that in the latter case one can always add arbitrarily long necks to connect pieces of a surface without significantly increasing area, provided these necks are sufficiently thin. Of course, no such construction is possible in the one-dimensional case.

We shall exhibit two constructions below, the second one is more general and covers all cases that have to be considered, whereas the first one, while of more limited scope, is geometrically more instructive.

Let us set up a little bit of notation first.

For the path u_1 described above we choose parametrizations on the unit disk D in such a way that

$$u_1(t)(x_1, x_2) = (\sqrt{1-t^2}x_1, \sqrt{1-t^2}x_2, t) \in B,$$

and for abbreviation, we put, for $-1 \le \alpha \le 1$,

$$\{x_1 \ge \alpha\} := \{(x_1, x_2) \in D \colon x_1 \ge \alpha\}.$$

We then look at the path $u(t_1, t_2)$:

$$\begin{aligned}
u(t_1,t_2) &= u_1(t_2)\{x_1 \ge 1 - 2t_1/t_2\} \cup -u_1(t_2)\{x_1 \ge 1 - 2t_1/t_2\}, \\
&\qquad\qquad 0 \le t_1 \le t_2,\ 0 < t_2 \le 1/2, \\
u(t_1,t_2) &= u_1(t_1) \cup \textstyle\bigcup_{t=2t_2-t_1}^{t_1} u_1(t)\{x = -1\} \cup -u_1(2t_2 - t_1), \\
&\qquad\qquad t_2 \le t_1 \le 2t_2,\ 0 < t_2 \le 1/2, \\
u(t_1,t_2) &= u_1(t_1) \cup \textstyle\bigcup_{t=-2t_2+t_1}^{t_1} u_1(t)\{x = -1\}, \qquad 2t_2 \le t_1 \le 1,\ 0 < t_2 \le \frac{1}{2}, \\
u(t_1, 1/2) &= u_1(1/2)\{x \ge 1 - 4t_1\} \cup -u_1(1/2)\{x \ge -4t_1\}, \qquad 0 \le t_1 \le 1/2, \\
u(t_1, 1/2) &= u_1(t_1) \cup \textstyle\bigcup_{t=1-t_1}^{t_2} u_1(t)\{x = -1\} \cup -u_1(1-t_1), \qquad 1/2 \le t_1 \le 1, \\
u(t_1,t_2) &= u_1(t_2)\{x_1 \ge 1 - 2t_1/1 - t_2\} \\
&\qquad \cup -u_1(t_2)\{x_1 \ge 1 - 2t_1/1 - t_2\},\ 0 \le t_1 \le 1 - t_2, 1/2 \le t_2 < 1, \\
u(t_1,t_2) &= -u_1(1-t_1) \cup \textstyle\bigcup_{t=1-t_1}^{2t_2+t_1-1} u_1(t)\{x = -1\} \cup u_1(2t_2 + t_1 - 1), \\
&\qquad\qquad 1 - t_2 \le t_1 \le 2(1-t_2),\ 1/2 \le t_2 < 1, \\
u(t_1, 2_1) &= -u_1(1-t_1) \cup \textstyle\bigcup_{t=1-t_1}^{3-2t_2-t_1} u_1(t)\{x = -1\}, \\
&\qquad\qquad 2(1-t_2) \le t_1 \le 1,\ \frac{1}{2} \le t_2 < 1.
\end{aligned}$$

Let us first look at the case where the support of W_1 has only one component, denoted by Σ_1. We then choose a diffeomorphism $\lambda \colon B \to A$ with $\lambda(u_1(\frac{1}{2})) = \Sigma_1$, and also put

$$v_1(t) := \lambda(u_1(t)), \qquad v(t_1, t_2) := \lambda(u(t_1, t_2)).$$

Then $v_1 \in V_1$, and v is the limit of paths in V_2.

We point out that we do not require that

$$\sup_{t \in I} |v_1(t)| = |\Sigma_1|.$$

If one assumes, however, that ∂A is strictly convex, then one can construct a path v_1 satisfying this requirement (cf. [3, Lemma 5.3]). In this case, one thus also sees directly that

$$\sup_{(t_1,t_2)} |v(t_1,t_2)| = 2|\Sigma_1| = 2\kappa_1,$$

implying $\kappa_2 \leq 2\kappa_1$. Actually, in the strictly convex case, by a slight modification of the above path v (or by an estimate of Pitts-Rubinstein), one can even see that $\kappa_1 < 2\kappa_1$ so that in this case the existence of a second solution follows directly from the arguments of [3]. Here, however, we do not want to elaborate on this point, and mention it only because it should be helpful for the geometric intuition.

Even in the general case, one can find a sequence of paths $v_1^n \in V_1$ with

$$\lim_{n\to\infty} v_1^n(\tfrac{1}{2}) = n_1\Sigma_1, \qquad n_1|\Sigma_1| = \kappa_1,$$

where n_1 is the multiplicity of the saddle point solution Σ_1. Constructing paths v^n from v_1^n in the same way as v was constructed from v^1, one sees that in any case $\kappa_2 \leq 2\kappa_1$. Of course, no such relation holds in the one-dimensional case, and we shall see this difference between the one- and the two-dimensional cases again later in the discussion of the ellipsoid.

Moreover, by putting

$$v(t_1,t_2,t_3) = \lambda(\rho^3(t_3)u(t_1,t_2)),$$

we note

$$\sup |v(t_1,t_2,t_3)| = \sup |v(t_1,t_2)|,$$

since, apart from connecting lines of vanishing area, every surface $v(t_1,t_2,t_3)$ is contained in the union of two surfaces of the family $v_1(t_1)$. Therefore, we conclude that if $\kappa_2 = 2\kappa_1$, only case 2.3.1 can occur.

It remains, however, to discuss the case $\kappa_1 < \kappa_2 < 2\kappa_1$, where 2.3 nevertheless may occur because the varifold W_1 has as its support a minimal surface Σ_1 covered with multiplicity $n_1 \geq 2$ or more than one minimal surface. Of course, in order to prove the theorem, we may assume for simplicity that the number of surfaces is at most 2; denote them by $\Sigma_{1,1}, \Sigma_{1,2}$. This latter assumption implies that there exist diffeomorphisms ψ_n as above (satisfying (2)–(5)) with the additional property $\psi_n(\Sigma_1) = \Sigma_1$, respectively $\psi_n(\Sigma_{1,j}) = \Sigma_{1,j}$, $j = 1, 2$.

We want to point out that the construction to follow also applies to the previously discussed case $\kappa_2 = 2\kappa_1$.

First of all, in the case of 2.3.3, we choose a path $t_i(\phi)$, $i = 1, \cdots, 4$, from

$$t_1 = \alpha_1, \quad t_2 = \beta_1, \quad t_3 = \tfrac{1}{2}, \quad t_4 = 0 \quad (\text{actually here } \alpha_1 = \tfrac{1}{2}, \alpha_2 = \tfrac{1}{4})$$

to

$$t_1 = 1 - \alpha_1, \quad t_2 = 1 - \beta_1, \quad t_3 = \tfrac{1}{2}, \quad t_4 = 1,$$

which is disjoint from $(t_1 = \frac{1}{2}, t_2 = \frac{1}{2})$, $\phi \in [0, 2]$. Also the map $\phi \to t_4(\phi)$ should be injective; without loss of generality we may assume $t_4(\phi) = \phi$.

We take $v = v_n \in V_2$ as above and put, dropping the index n,

$$w(t_1, t_2, t_3) := \psi(t_1, t_2, t_3)(v(t_1, t_2)).$$

We then set

$$\psi'(t_1, t_2, t_3, t_4) := \psi(t_1, t_2, t_3 + t_4),$$
$$w'(t_1, t_2, t_3, t_4) := \psi'(t_1, t_2, t_3, t_4)(v(t_1, t_2)).$$

We now have to modify w' in such a way that along the path $t_i(\phi)$, $i = 1, \cdots, 4$, Σ_1 or $\Sigma_{1,1}$ and $\Sigma_{1,2}$, respectively, are covered with prescribed multiplicity. We choose a path $y(\phi, \theta)$, $0 \le \theta \le 1$, for each $\phi \in [0, 2]$, in $\mathscr{M} \bmod \partial A$, or more precisely in the closure of this space, connecting $w'(t_1(\phi), \cdots, t_4(\phi))$, corresponding to $\theta = 0$, to a configuration corresponding to $\theta = 1$ consisting of a point on the boundary of each surface Σ_1, $\Sigma_{1,1}$ or $\Sigma_{1,2}$ which has to be covered with even multiplicity or the surface itself if it has to be covered with odd multiplicity. In case both $\Sigma_{1,1}$ and $\Sigma_{1,2}$ are covered, we also need a connecting line in ∂A in the configuration for $\theta = 1$. We also choose the ϕ-dependence in such a way that the $\theta = 1$ configuration performs a full rotation under $\psi'(t_1(\phi), \cdots, t_4(\phi))$, as ϕ traverses from 0 to 2 (analogous to (3) and (4)).

In the next step, we extend $y(\phi, \theta)$ to $1 \le \theta \le 2$ so that $y(\phi, 2)$ covers each of the surfaces $\Sigma_1, \Sigma_{1,1}, \Sigma_{1,2}$ with the required multiplicity $n_1, n_{1,1}$ or $n_{1,2}$, respectively, and

$$\sup_{1 \le \theta \le 2} |y(\phi, \theta)| \le n_1 |\Sigma_1| \quad \text{or} \quad n_{1,1}|\Sigma_{1,1}| + n_{1,2}|\Sigma_{1,2}| \text{ respectively.}$$

This latter requirement is trivial to achieve, as a multiply covered surface can be connected to a point on the boundary or the once covered surface, depending on whether the covering is even or odd, by letting each pair of copies of the surface in the covering annihilate itself along a folding line. The prototype of this process is the family

$$u_1(\tfrac{1}{2})\{x_1 \ge 1 - 2\mu\} \cup u_1(\tfrac{1}{2})\{x_1 \ge 1 - 2\mu\} \quad \text{for a parameter } \mu, \ 0 \le \mu \le 1.$$

For the final step, we choose some suitable $\rho > 0$ so that for each ϕ we obtain a domain $B'(\phi)$ by deleting the ball $B((t_1(\phi), t_2(\phi), t_3(\phi)), \rho)$ from the parameter domain for t_1, t_2, t_3, and put

$$B' := \bigcup_{\phi=0}^{2} B'(\phi) \times \{t_4(\phi)\}.$$

We also denote the original parameter domain for $t_1, \cdots, t_4$ by B_0. We choose a surjective map $\mu : B' \to B_0$ which is the identity on the t_4 component (remembering $t_4(\phi) = \phi$) and mapping $\partial B((t_1(\phi), t_2(\phi), t_3(\phi)), \rho)$ onto $(t_1(\phi), t_2(\phi), t_3(\phi))$ and being injective otherwise.

We then obtain $w_0(t_1, t_2, t_3, t_4)$ as $w' \circ \mu$ on B' and $y(\phi, \frac{2}{\rho}(r - \phi))$ on each $B((t_1(\phi), t_2(\phi), t_3(\phi)), \rho)$, where r, $0 \le r \le \rho$, denotes the distance from the center in polar coordinates (r, ω) on each such ball, and the construction is independent of the angular coordinate ω.

We now come to the essential point of the construction: The construction can be repeated with additional rotation parameters $t_5, t_6, \cdots$ and controlled area as follows. On B', w_0 is mapped into itself under rotations so that area here does not increase anymore. Moreover, the area on the path $y(\phi, \theta)$, $0 \le \theta \le 1$, is independent of the multiplicity of the limiting surface, and this property is not altered by further rotations. Finally, the area of $y(\phi, \theta)$, $1 \le \theta \le 2$, is bounded by the area of the limiting surface.

Thus, if case 2.3.3.3 should occur, i.e., when minimaxing over the class $V_{2.3.3}$ we increase the critical value without obtaining a new solution, we introduce a new rotation parameter t_5 with the same properties as before, and choose a path $t_1(\phi_5), \cdots, t_5(\phi_5)$, $0 \le \phi_5 \le 2$, disjoint from $(t_1 = \frac{1}{2}, t_2 = \frac{1}{2})$ and $(t_1(\phi), \cdots, t_4(\phi))$, and a path $y_5(\phi_5, \theta)$, $0 \le \theta \le 2$, in the same way as before.

If necessary, we repeat the construction with a new parameter t_6, and so on, introducing paths $t_1(\phi_\nu), \cdots, t_\nu(\phi_\nu)$, $0 \le \phi_\nu \le 2$, and $y_\nu(\phi_\nu, \theta)$, $0 \le \theta \le 2$. We can do this by our construction in such a way that the supremum of area increases with ν only for $y_\nu(\phi_\nu, \theta)$, $1 \le \theta \le 2$; here, however, it is bounded by the required multiplicity of the limiting surface. Applying a rotation ψ to this family does not yield any new surfaces, and therefore also the maximal area is not increased. Therefore, after finitely many steps, we obtain a family whose area maximum is not increased by further rotations $\psi(t_1, \cdots, t_\nu)$, and we are in the first case.

We now treat case 2.3.1. For simplicity of notation, we shall suppress the index n in the sequel, as its role was already exhibited in 2.3.

Ultimately, we want to consider four-parameter families of elements with the following properties ($0 \le t_1, t_2, t_4 \le 1$, $t_3 \in \mathbb{R}$):

$$v(t_1, 1, 0, 0) = -v(1 - t_1, 0, 0, 0), \tag{7}$$

$$v(t_1, t_2, t_3, 0) = w(t_1, t_2, t_3), \qquad w \text{ as in 2.3;} \tag{8}$$

in particular

(9) $$w(\tfrac{1}{2}, \tfrac{1}{2}, t_3) \subset N(W_2, \varepsilon),$$

(10) $$w(t_1, t_2, 2) = w(t_1, t_2, 0),$$

(11) $$v(t_1, t_2, t_3 + 1, t_4) = -v(1 - t_1, 1 - t_2, t_3, 1 - t_4).$$

As before, we form the class V_4 of all such four-dimensional families. An example of an element of V_4 is indicated in the sketches shown; they are to be understood as follows: A is mapped diffeomorphically onto the unit ball B, the minimal disk obtained in the first step is the disk in the x_1, x_2 plane, we assume that everything is symmetric with respect to the x_1, x_3 plane, and we represent only the intersection of our family with this plane; t_3 then acts by rotations about the x_3 axis. The left column is supposed to depict a class with the same properties as $\lambda \circ u(t_1, t_2)$ defined above. The right column is obtained from the left one by rotation about the x_3-axis by 180 degrees and exchanging t_1 and $1 - t_1$ as well as t_2 and $1 - t_2$. The middle column suggests a path connecting the left and the right ones via the parameter t_4.

2.4. This gives us two additional parameters, namely t_4 and t_3, for which we can perform a saddle point construction. The idea is to minimax first over classes of the form $v(t_1, t_2, 0, t_4)$. At this point, we have assigned an image to t_3 only for $t_4 = 0$ and for $t_4 = 1$; for $t_4 = 0$, this has been carried out in 2.3, and for $t_4 = 1$ this follows because we assume

$$v(t_1, t_2, 0, 1) = -v(1 - t_1, 1 - t_2, 1, 0).$$

We note that $v(t_1, t_2, 0, t_4)$ is not a cycle as it is not closed, and we therefore prescribe boundary conditions for $t_4 = 0$ and $t_4 = 1$.

Again, we can form a class $V_{2.4}$ of families homologous to the one just discussed under the given boundary condition. Minimaxing in $V_{2.4}$, we obtain a critical value $\kappa_{2.4}$ and a varifold

$$W_{2.4} = \sum_{j=1}^{N_{2.4}} n_{2.4,j} \mathbf{v}(\Sigma_{2.4,j})$$

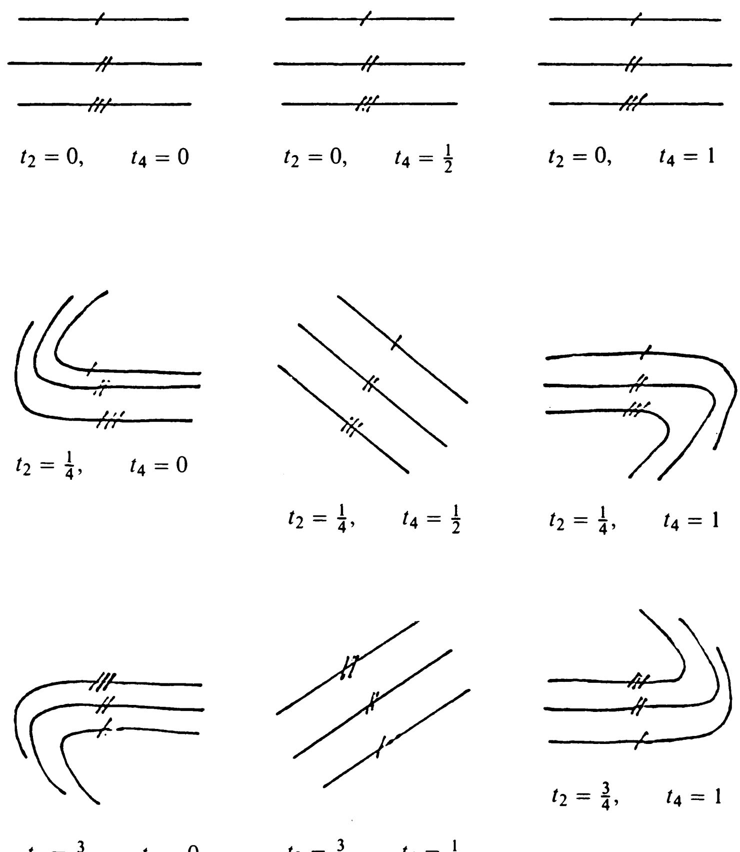
with the same properties as before.

We distinguish three cases:

2.4.1. We obtain a new minimal disk.

2.4.2. $\kappa_{2.4} = \kappa_{2.3}$ $(= \kappa_2)$. In this case, we obtain again infinitely many solutions. Namely, although $v(t_1, t_2, 0, t_4)$ does not represent a cycle, as it is not closed, it can be made into a cycle by composing $v(t_1, t_2, 0, 1)$ with $v(t_1, t_2, t_3, 1)$, $0 \le t_3 \le 1$. This does not increase the critical value,

/ corresponds to $t_1 = \frac{1}{4}$, // to $t_1 = \frac{1}{2}$, /// to $t_1 = \frac{3}{4}$



as the minimax value for the paths $v(t_1, t_2, t_3, 1)$ is $\kappa_{2.3}$. Because of the boundary identifications (11), the argument of [3] then applies to yield the conclusion.

Thus, altogether one traverses from $v(t_1, t_2, 0, 0)$ to $v(t_1, t_2, 1, 1) = -v(1 - t_1, 1 - t_2, 0, 0)$. We then construct a subordinate cycle from the class V_2 as follows. We define

$$l_2\colon [0,1] \times [0,2] \to I^2 \times [0,1] \times I$$

in two steps, first for $0 \le \rho_1 \le 1$, $0 \le \rho_2 \le 1$. We put

$$l_2(\rho_1, \rho_2) = (t_1(\rho_1, \rho_2), t_2(\rho_1, \rho_2), 0, t_4(\rho_1, \rho_2))$$

with

$$\begin{aligned}
&t_1(1,\rho_2) = 1 - t_1(0,\rho_2), &&t_1(\rho_1,1) = 1 - t_1(\rho_1,0),\\
&t_2(1,\rho_2) = 1 = t_4(\rho_1,1), &&t_2(0,\rho_2) = 0 = t_4(\rho_1,0),\\
&t_2(\rho_1,1) = 1 - t_2(\rho_1,0), &&t_4(1,\rho_2) = t_4(0,\rho_2),\\
&(t_1(\rho_1,0), t_2(\rho_1,0)) \neq (\tfrac{1}{2},\tfrac{1}{2}),
\end{aligned}$$

and consequently also $(t_1(\rho_1,1), t_2(\rho_1,1)) \neq (\frac{1}{2},\frac{1}{2})$. Finally, for $0 \le \rho_1 \le 1$, $1 \le \rho_2 \le 2$, we put

$$l_2(\rho_1,\rho_2) = (t_1(\rho_1,1), t_2(\rho_1,1), \rho_2 - 1, t_4(\rho_1,1)).$$

We also denote the above 3-dimensional cycle traversing from $v(t_1,t_2,0,0)$ to $v(t_1,t_2,1,1) = -v(1-t_1,1-t_2,0,0)$ by w. Then $w \circ l_2 \in V_2$ because of the above boundary identifications. Furthermore, we note that $l_2(\rho_1,\rho_2)$, for $0 \le \rho_1 \le 1$ and $1 \le \rho_2 \le 2$, is disjoint from the constrained line $(\frac{1}{2},\frac{1}{2},t_3,1)$.

Thus

$$\sup_t |w \circ l_2(t)| \ge \kappa_{2.3} = \kappa_2.$$

Pointing out again that the boundary conditions for the above cycle w are the same as for cycles in V_3, we conclude in the same way as in 3.2 that we obtain a 1-parameter family of solutions.

2.4.3. $\{\Sigma_{2.4,1},\cdots,\Sigma_{2.4,N_{2.4}}\} \subset \{\Sigma_{1,1},\cdots,\Sigma_{1,N_1}\}$, but $\kappa_{2.4} > \kappa_{2.3}$. We then repeat the above process with additional parameters until we either obtain a new solution or the critical value does not increase anymore. Again, the process has to terminate after a finite number of steps.

This can be seen by the following elementary construction of comparison surfaces. We put $v_i = \lambda \circ u_i$, $i = 1,2,3$, defined in the beginning, and $v = \lambda \circ u$, defined in 2.3.3. We also choose λ in such a way that $v_1(\frac{1}{2})$ is our minimal surface Σ_1.

We first connect $v_2(t_1,\frac{1}{2})$ to $v_1(t_1)$ or $v(t_1,\frac{1}{2})$ (defined in 2.3.3) depending on whether we want to cover a limiting surface Σ_1 with odd or even multiplicity ν, and similarly in case the limiting surface has more than one component. We denote the corresponding family by $z_\nu(t_1,\theta)$; thus $\theta = 0$ corresponds to $v_2(t_1,\frac{1}{2})$, and $\theta = 1$ to $v_1(t_1)$ or $v(t_1,\frac{1}{2})$. In the same way as in 2.3.3, we then use $1 \le \theta \le 2$ to connect $z_\nu(t_1,1)$ to a family $z_\nu(t_1,2)$ for which $z_\nu(\frac{1}{2},2)$ is Σ_1 covered with multiplicity ν, in such a way that

$$\sup_{\substack{0\le t_1\le 1\\ 1\le\theta\le 2}} |z_\nu(t_1,\theta)| = \nu|\Sigma_1|.$$

We also point out that $\sup_{0\le t_1\le 1, 0\le\theta\le 1} |z_\nu(t_1,\theta)|$ is independent of ν.

Now, in case 2.4.3, we want to exhibit a family

$$v(t_1, t_2, t_3, t_4, t_5, t_6), \qquad 0 \le t_1, t_2, t_4, t_6 \le 1,\ 0 \le t_3, t_5 \le 2,$$

with

$$v(t_1, t_2, t_3, t_4, t_5 + 1, t_6) = -v(1 - t_1, 1 - t_2, t_3, 1 - t_4, t_5, 1 - t_6),$$

for which $v(\frac{1}{2}, \frac{1}{2}, t_3, \frac{1}{2}, t_5, 0)$ is $3\Sigma_1$, i.e., Σ_1 is covered with multiplicity 3. We first choose a family $v'(\tau_1, \tau_2, \tau_3, \tau_4)$, $0 \le \tau_1, \tau_2, , \tau_4 \le 1$, $0 \le \tau_3 \le 2$, as depicted in our diagram. This means more precisely that

$$v'(\tau_1, \tau_2, \tau_3, \tfrac{1}{2}) = v_3(\tau_1, \tau_2, \tau_3)$$

(again $v_3 = \lambda \circ u_3$, u_3 is defined in 3 above). We then choose a surjective map

$$\mu\colon I^2 \to I^2, \quad (t_2, t_4) \to (\tau_2(t_2, t_4), \tau_4(t_2, t_4))$$

with the property that for some suitable $\rho > 0$, $I^2 \backslash B((\frac{1}{2}, \frac{1}{2}), \rho)$ is mapped surjectively onto I^2, that μ is the identity on $I^2 \backslash B((\frac{1}{2}, \frac{1}{2}), 2\rho)$ and that $\partial B((\frac{1}{2}, \frac{1}{2}), \rho)$ is mapped onto $(\frac{1}{2}, \frac{1}{2})$.

We then define $v(t_1, t_2, t_3, t_4)$ by

$$v(t_1, t_2, t_3, t_4) := v'(t_1, \tau_2(t_2, t_4), t_3, \tau_4(t_2, t_4))$$

for $0 \le t_1 \le 1$, $t_2, t_4 \in I^2 \backslash B((\frac{1}{2}, \frac{1}{2}), \rho)$, $0 \le t_3 \le 2$, and choosing polar coordinates (r, ω) on $B((\frac{1}{2}, \frac{1}{2}), \rho)$ where r is the radial coordinate,

$$v(t_1, t_2, t_3, t_4) := \psi(t_3) z_3(t_1, 2 - \tfrac{2}{\rho} r(t_2, t_4)),$$

where $\psi(t_3) := \lambda(\rho^3(t_3))$. Then

$$v(t_1, t_2, t_3, t_4, t_5) := \psi(t_5) v(t_1, t_2, t_3, t_4)$$

for $0 \le t_1, t_2, t_4 \le 1$, $0 \le t_3, t_5 \le 2$.

Finally we construct

$$v'(t_1, t_2, t_3, t_4, t_5, t_6), \qquad 0 \le t_6 \le 1,$$

from $v(t_1, t_2, t_3, t_4, t_5)$ in such a way that

$$v'(t_1, t_2, t_3, t_4, t_5, \tfrac{1}{5}) = v_3(t_1, t_2, t_3 + t_5),$$
$$v'(t_1, t_2, t_3, t_4, t_5, 0) = v(t_1, t_2, t_3, t_4, t_5),$$

and satisfying the above boundary identifications. In order to construct this family, we use as before $z_3(t_1, \theta)$ in order to connect $v'(t_1, t_2, t_3, t_4, t_5, 0)$ with $v_3(t_1, t_2, t_3 + t_5)$.

If necessary, we repeat this construction with additional parameters t_7, t_8, by first constructing a path $v(t_1, t_2, t_3, t_4, t_5, t_6)$ in the same way as above, with

$$v(t_1, \tfrac{1}{2}, t_3, \tfrac{1}{2}, t_5, \tfrac{1}{2}) = \psi(t_3 + t_5) z_4(t_1, 2).$$

If we iterate this construction, we obtain families where on one part of the parameter domain $\{(t_1, t_2, \cdots, t_{2\nu})\}$, the maximal area is independent of ν, whereas on the other part it is bounded by $\nu|\Sigma_1|$. This implies that for sufficiently large ν, the maximal area is $\nu|\Sigma_1|$, and this in turn implies that the process terminates at or before this ν. Namely, we can alway introduce additional parameters $t_{2\nu+1}, t_{2\nu+2}$ and construct a family satisfying all required boundary conditions and constraints, but using only surfaces of the previous family and thus not increasing the maximal area.

This construction can be adapted to the cases where Σ_1 is covered with multiplicity $n_1 \geq 2$ already in the first step, or where the solution of the first step has two components, in the same way as the second construction in 2.3.3 worked for these cases.

In conclusion, let us point out again the idea of this construction, as this construction while completely elementary is somewhat technical. Namely, if we choose λ in such a way that $\lambda \circ u_1(\frac{1}{2}) = \Sigma_1$, then the class $v_3 = \lambda \circ u_3$ and its further rotations under $\psi(t_3) = \lambda(\rho^3(t_3))$ satisfy all boundary conditions required for $V_{2.4}$ and its subsequent rotations and iterations, but not the constraints imposed for $t_1 = \frac{1}{2}, t_2 = \frac{1}{2}, t_4 = 0$, etc. These constraints, however, can be satisfied in the following way: First, one modifies the family so that for the appropriate parameter values one obtains Σ_1 covered with multiplicity 1 or 2, depending on whether the required multiplicity ν is odd or even. This can be achieved with a maximal area bounded independently of ν, and this is not affected by further rotations either. Secondly, one connects this family to one where Σ_1 is covered with multiplicity ν for the appropriate parameter values. The principle here is that a disk covered twice can be connected to a point, by letting the two copies cancel each other. This step requires a maximal area of $\nu|\Sigma_1|$. As for the other steps, the maximal area is bounded independently of ν, and for sufficiently large ν the maximal area of the family is $\nu|\Sigma_1|$. This implies that the iteration process terminates after finitely many steps, because we can extend the family to the additional parameter values $t_{2\nu-1}, t_{2\nu}$ by only using surfaces already contained in the family $v(t_1, \cdots, t_{2\nu-2})$ so that we do not increase the area anymore.

In case 2.3.2, 2.4.1 or 2.4.2, we proceed to

3.4. We minimax over elements of V_4 with $0 \leq t_3 \leq 1$, obtaining a critical value $\kappa_{3.4}$ as the area of a varifold

$$W_{3.4} = \sum_{j=1}^{N_{3.4}} n_{3.4,j}\mathbf{v}(\Sigma_{3.4,j}).$$

We distinguish between the cases:

3.4.1. We obtain a new minimal disk.

3.4.2. $\kappa_{3.4} = \kappa_{2.4}$ (respectively $= \kappa_{2.3}$ in case 2.3.2). In this case, because of the boundary identifications (11), we can again conclude with the help of [3] that we get infinitely many solutions. For example, a subordinate class contained in $V_{2.4}$ is given by $t_i(\rho_1, \rho_2, \rho_3)$, $i = 1, \cdots 4$, $0 \leq \rho_1, \rho_2, \rho_3 \leq 1$, with the following parameter identifications:

$$\begin{aligned}
&t_4(\rho_1, 0, \rho_3) = 0, \quad t_4(\rho_1, 1, \rho_3) = 1, && t_3(\rho_1, \rho_2, 0) = 0, \quad t_3(\rho_1, \rho_2, 1) = 1,\\
&t_3(\rho_1, 1, \rho_3) = t_3(\rho_1, 0, \rho_3), && t_3(1, \rho_2, \rho_3) = t_3(0, \rho_2, \rho_3),\\
&t_4(\rho_1, \rho_2, 1) = 1 - t_4(\rho_1, \rho_2, 0), && t_4(1, \rho_2, \rho_3) = t_4(0, \rho_2, \rho_3),\\
&t_1(1, \rho_2, \rho_3) = 1 - t_1(0, \rho_2, \rho_3), && t_1(\rho_1, 1, \rho_3) = 1 - t_1(\rho_1, 0, \rho_3),\\
&t_1(\rho_1, \rho_2, 1) = 1 - t_1(\rho_1, \rho_2, 0), && t_2(1, \rho_2, \rho_3) = 1, \quad t_2(0, \rho_2, \rho_3) = 0,\\
&t_2(\rho_1, 1, \rho_3) = 1 - t_2(\rho_1, 0, \rho_3), && t_2(\rho_1, \rho_2, 1) = 1 - t_2(\rho_1, \rho_2, 0).
\end{aligned}$$

For a subordinate class in $V_{2.3}$, the roles of t_3 and t_4 have to be reversed.

3.4.3. All surfaces $\Sigma_{3.4,j}$ are contained in the set of solutions from previous steps. In this case, we again repeat the above construction with additional parameters, at most a finite number of times, until one of the other two cases occurs.

It remains to treat case 3.3. We neglect the index n and assume that we have a cycle in V_3 with, say,

$$\mathbf{v}(v(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})) \in N(W_3, \varepsilon).$$

We then look at smooth families $\psi(t_1, t_2, t_3, t_4)\colon A \to A$ of diffeomorphisms, with

$$\begin{aligned}
&\psi(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, t_4)_\#(N(W_3, \varepsilon)) \subset N(W_3, \varepsilon),\\
&\psi(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, t_4)_\#(W_3) = W_3 \quad \text{for all } t_4,\\
&\psi(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, t+s) = \psi(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, t) \circ \psi(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, s) \quad \text{for all } s, t,\\
&\psi(t_1, t_2, t_3, 1) \neq \text{id}, \quad \psi(t_1, t_2, t_3, 2) = \text{id} \quad \text{for all } t_1, t_2, t_3.
\end{aligned}$$

We then construct five-parameter families $v(t_1, t_2, t_3, t_4, t_5)$ in $\mathscr{M}$ with

$$\begin{aligned}
&v(t_1, 1, 0, 0, 0) = -v(1 - t_1, 0, 0, 0, 0),\\
&v(t_1, t_2, 1, 0, 0) = -v(1 - t_1, 1 - t_2, 0, 0, 0),\\
&v(t_1, t_2, t_3, t_4, 0) = \psi(t_1, t_2, t_3, t_4)(v(t_1, t_2, t_3, 0, 0)),\\
&v(t_1, t_2, t_3, t_4 + 1, t_5) = -v(1 - t_1, t_2, 1 - t_3, t_4, 1 - t_5).
\end{aligned}$$

Again

$$\mathbf{v}(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, t_4, 0)) \subset N(W_3, \varepsilon),$$

but $v(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, t_4, 0) \neq \text{const}$.

By a similar reasoning as above, we obtain a new solution by minimaxing over the five-dimensional homology class defined by the preceding construction.

3. Proof of Theorem 2

This proof proceeds by arguments similar to the proof of Theorem 1. Let us for example look at the case where in the second step we obtain the surface Σ_1 of the first step covered with higher multiplicity. We let $\tau \in \mathrm{SO}(3)$, considered as a group of transformations of S^2. As in 2.3, but dropping the index n for simplicity of notation, we denote the critical varifold of the second step by W_2 and assume, for some $v \in V_2$,

$$v(\tfrac{1}{2}, \tfrac{1}{2}) \subset N(W_{2,\varepsilon}),$$

and consider families $\psi(t_1, t_2, \tau) \colon M \to M$ of diffeomorphisms, $0 \leq t_1, t_2 \leq 1$, $\tau \in S^2$, satisfying

$$\begin{aligned}
&\psi(\tfrac{1}{2}, \tfrac{1}{2}, \tau)(W_2) = W_2,\\
&\psi(\tfrac{1}{2}, \tfrac{1}{2}, \tau)(N(W_2, \varepsilon)) \subset N(W_2, \varepsilon) \quad \text{for all } \tau,\\
&\psi(\tfrac{1}{2}, \tfrac{1}{2}, \tau_1 \circ \tau_2) = \psi(\tfrac{1}{2}, \tfrac{1}{2}, \tau_1) \circ \psi(\tfrac{1}{2}, \tfrac{1}{2}, \tau_2) \quad \text{for all } \tau_1, \tau_2,\\
&\psi(\tfrac{1}{2}, \tfrac{1}{2}, \tau) \neq \text{id} \quad \text{for } \tau \neq id.
\end{aligned}$$

We put

$$w(t_1, t_2, \tau) := \psi(t_1, t_2, \tau)(v(t_1, t_2)), \qquad v \in V_2 \text{ as above.}$$

Let us assume for the moment that we are in case 2.3.1 as in the proof of Theorem 1.

We first choose an element $\tau' \in \mathrm{SO}(3)$ of order 2 and introduce a parameter t_5, $0 \leq t_5 \leq 1$, in order to minimax over classes $v(t_1, t_2, \mathrm{id}, t_5)$ with

$$v(t_1, t_2, \mathrm{id}, 1) = -v(1 - t_1, 1 - t_2, \tau', 0),$$

and of course also

$$v(t_1, 1, \mathrm{id}, 0) = -v(1 - t_1, 1, \mathrm{id}, 0),$$
$$v(t_1, t_2, \tau, 0) = w(t_1, t_2, \tau), \quad \text{as above.}$$

Assuming that we obtain a new solution, we then choose two orthogonal circle subgroups of SO(3), parametrized by $t_3 \in [0, 2]$ and $t_4 \in [0, 2]$, with $t_3 = 1$ and $t_4 = 1$ both corresponding to τ' and of course $t_3 = 2$ and $t_4 = 2$ corresponding to id.

This gives us five-dimensional classes $v(t_1, t_2, (t_3, t_4), t_5)$ with identifications

$$v(t_1, t_2, (t_3 + 1, 0), t_5) = -v(1 - t_1, 1 - t_2, (t_3, 0), 1 - t_5),$$
$$v(t_1, t_2, (t_3, t_4 + 1), t_5) = -v(1 - t_1, 1 - t_2, (1 - t_3, 0), 1 - t_5),$$

where the parameter t_3 is extended to the whole real axis by the requirement

$$v(t_1, t_2, (t_3 + 2, t_4), t_5) = v(t_1, t_2, (t_3, t_4), t_5).$$

With these two additional parameters t_3 and t_4, we can then search for two additional solutions.

With such constructions, the proof proceeds in the same manner as the previous one. There is only one point that requires some attention. Namely, when we carried out the iterations in the previous proof, we restricted attention to the case where the solution of the first step has at most two components. The reason for this was that in this case we could find a family of rotations under which every component was individually invariant. If in the present situation, the solution of the first step has three components, however, we can still find a one-parameter family of rotations (instead of a two-parameter family as above) under which each component remains invariant. We can use this family to generate two more solutions as in the proof of Theorem 1 (instead of three generated above) which is more than sufficient to prove the theorem in this case as well.

Remarks. (1) With a similar reasoning, one also establishes

Theorem 3. *Let A be a compact body in some three-dimensional Riemannian manifold, diffeomorphic to the unit ball, with a smooth boundary of positive mean curvature with respect to the interior normal. Suppose that the interior of A contains no embedded minimal two-spheres. Then there exist again at least three embedded minimal disks in A with a free boundary on ∂A.*

(2) White [12] showed that the number 4 of Theorem 2 is optimal by considering ellipsoids with the lengths of the principal axes different from,

but close to, each other. A similar argument shows that 3 is optimal for Theorem 3.

On the other hand, if the ellipsoid becomes very elongated, then there actually exist more solutions. This can be shown with our methods; let us consider the situation of Theorem 1. We thus have an ellipsoid in $\mathbb{R}^3$ with principal axes of different length, where the third axis is much longer than the other two. The first minimal disk just sits in the plane determined by the first two axes; let its area be κ_1. For the second step, one can explicitly construct an element of V_2 with maximal area less than $2\kappa_1$. (In a manner similar to the construction of the cycle $\lambda \circ (t_1, t_2)$ in 2.3 above.) One therefore obtains a second minimal disk which is similar in shape to two parallel copies of the first disk connected by a neck.

If one then rotates the whole 2-dimensional cycle about the third axis, the maximal area does not increase significantly. One then connects each surface in this cycle to a surface obtained from this cycle by a rotation of 180 degrees, with the same parameter identifications as above. Minimaxing produces a third solution as in 2.4. Finally rotating the whole three-dimensional family about the third axis again yields a four-parameter family with suitable parameter identifications, and minimaxing produces another solution as in 3.4. Of course, in this case the third and fourth solutions may just be the planar ones determined by the first and third or the second and third axes respectively. On the other hand, if the ellipsoid is very elongated, then minimaxing solutions will still have smaller area than these planar ones, and one can repeat the procedure. Of course, in any case this argument then also applies to bodies of similar shape as an ellipsoid where, however, one might not know a priori any solutions.

(3) There are various ways to interpret our constructions geometrically. The previous discussion of the ellipsoid suggests a kind of symmetry breaking in the image. Again let the third axis be the longest one. If it is not too much longer than the other ones, then the second solution is planar and invariant under rotation by 180 degrees about this axis. If the length of the third axis increases, then at a certain point the index of this planar minimal surface becomes larger than two, and the second solution is nonplanar and not invariant anymore under this rotation. We acknowledge the helpful contributions by J. Pitts on this example.

It is also helpful for the intuition to consider the cycle $\lambda \circ u(t_1, t_2) \in V_2$ constructed in 2.3 which illuminates the essential difference between the one-dimensional problem of closed geodesics and the two-dimensional problem of minimal surfaces, because in the latter case one can connect pieces that are far apart by necks of arbitrarily small area.

Finally, in case multiplicity occurs in the second or third step, then we actually can interpret this as a degenerate one- (resp. two- in the case of Theorem 2) family of solutions. For example, a minimal surface Σ of multiplicity two occurs as two infinitesimally close surfaces connected at a single point on ∂A or in M, respectively, and the family is obtained by rotating this point on $\Sigma \cap \partial A$ or on Σ, respectively. A S^1- or S^2-family of solutions, however, should contribute topologically in a different way than an isolated solution to the configuration of all solutions, and exploiting this topological difference actually is at the heart of our constructions.

We also remark that the solutions we finally obtain after discarding possibly many higher multiplicity critical points through iterations may be of rather high index; here, however, we do not want to elaborate this point.

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