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WITTEN'S COMPLEX AND INFINITE DIMENSIONAL MORSE THEORY

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Abstract

We investigate the relation between the trajectories of a finite dimensional gradient flow connecting two critical points and the cohomology of the surrounding space. The results are applied to an infinite dimensional problem involving the symplectic action function.

1. Introduction

Let M be a smooth finite dimensional manifold and let $f\colon M \to \mathbb{R}$ be a smooth function. It is the aim of Morse theory to relate the topological type of M and the number and types of critical points of f, i.e. of points $x \in M$ with $df(x) = 0$. For example, if M is compact and all critical points of f are nondegenerate, then there are the well-known Morse inequalities (see e.g. [6]) relating the number of critical points and their Morse indices to the dimension of the graded vector spaces $H^*(M,\mathbb{F})$, where $\mathbb{F}$ is any field and $H^*(M,\mathbb{F})$ is the graded cohomology of M with coefficients in $\mathbb{F}$. (Throughout the paper, we will use Alexander-Spanier cohomology; see [12].) The Morse inequalities are usually stated as a relation between polynomials in $\mathbb{F}[t]$, but can be formulated equivalently as follows: Let us denote by $C_{\mathbb{F}}^*$ the free $\mathbb{F}$-vector space over the set $\mathbf{C}$ of critical points of f. That is, $\mathbf{C}_{\mathbb{F}}^* \simeq (\mathbb{F})^{|C|}$, is identified with a set of generators of $\mathbf{C}_{\mathbb{F}}^*$. Then the Morse inequalities are equivalent to the existence of an $\mathbb{F}$-linear map, called a coboundary operator $\delta_{\mathbb{F}} : \mathbf{C}_{\mathbb{F}}^* \to \mathbf{C}_{\mathbb{F}}^*$ so that $\delta_{\mathbb{F}}\delta_{\mathbb{F}} = 0$ and

$$\ker\delta_{\mathbb{F}} / \operatorname{im}\delta_{\mathbb{F}} = H^*(M,\mathbb{F}). \tag{1.1}$$

The central tool in the proof of the Morse inequalities is the gradient flow of f: If g is a Riemannian metric on M, and $\nabla_g f$ denotes the gradient vector field of f with respect to g, then the solutions of the ordinary differential equation

$$\dot{x}(t) + \nabla_g f(x(t)) = 0, \qquad x(0) = x, \tag{1.2}$$

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for $x \in M$ define homotopies

$$\chi: M \times \mathbb{R} \to M; \qquad \chi(x,t) = x \cdot t := x(t).$$

This is called the gradient flow; the solutions of (1.2) are called flow trajectories. Clearly, the function decreases on the flow trajectories unless $df(x) = 0$, in which case $x \cdot t = x$ for all t. As a result, one can show that for compact M, the topology of the sets

$$M^a = \{x \in M | f(x) \leq a\}$$

for $a \in \mathbb{R}$ changes only at those real numbers a which are values of critical points of f. By analyzing these changes, one obtains the Morse inequalities (see e.g. [6]).

There is a different approach to the Morse inequalities, which gives a geometrical interpretation of $\delta_{\mathbb{R}}$ in terms of flow trajectories "connecting" critical points. (For a precise statement, see Definition 2.1 and the corollary in §2.) This approach depends on certain generic properties of gradient flow discovered by Smale [11]. The existence of δ_{F} was known to Thom [13] and Smale [10]. The connection with the flow trajectories was partly used in Milnor's lectures [7] on the h-cobordism theorem (in fact, Lemma 7.2 and Theorem 7.4 of [7] establish a proof of (1.1) in the special case of "self-indexing" functions). However, it did not receive much attention in general. It was rediscovered by Witten, who in [14] gave a "physicist's proof" of the Morse inequalities in the form of (1.1) by interpreting the critical points of f as "ground states" and the connecting trajectories as "tunneling effects" in a (super-symmetric) quantum dynamical system. Witten's arguments were subsequently extended and made more precise in [5], for example. Because of [14], we call the complex of (1.1) Witten's complex.

In this paper, we are interested in the complex (1.1) for rather different reasons, concerning extensions of Morse theory of infinite dimensional manifolds M. It is well known that the traditional approach to Morse theory via an analysis of the sets M^a gives meaningful results only if the critical points of f (or its negative) have finite Morse index. If both the negative and the positive eigenspace of $D^2 f(x)$ are infinite dimensional, then the homotopy type of M^a does not change when a passes through $f(x)$. On the other hand, it was shown in [3] that for a certain function of the latter type, the symplectic action function on the loop space of a symplectic manifold, the spaces of trajectories connecting critical points behave essentially as in the finite dimensional case. In fact, in this situation one can define a cohomology group based on an operator δ_{F} as described above. This was done in [2] for $\mathbb{F} = \mathbb{Z}_2$. It cannot be directly identified with the cohomology of the Conley index of a compact invariant set in finite dimensional Morse theory. This Conley index was defined in [1] to extend Morse theory to arbitrary flows on locally compact spaces (although we restrict ourselves here to gradient flows on arbitrary finite dimensional manifolds). The definition is given in §2.

The purpose of this paper is twofold: First, we extend the statement of Witten's complex to cover the Conley index in noncompact manifolds as well as to arbitrary rings $\mathbb{F}$. It turns out that a proof of this extended statement can be given by rather elementary methods of Conley's index theory (see §3). The second purpose is to use this result to calculate the "cohomological" Conley index of [2] in a special limit situation. In view of the "deformation invariance" of the construction of [2], this determines the cohomological Conley index in a rather general situation, and is therefore the final step in the estimates of Lagrangian intersections in [2].

2. The Conley index

If M is not compact, then the Morse inequalities (1.1) generally fail to hold. In this case, a different relation can be recovered for a set of critical points provided that the flow-invariant set of points in M lying on connecting trajectories is compact and isolated.

Here, we call a compact invariant set S isolated if there exists a compact neighborhood U of S whose maximal invariant set

$$S(U) := \{x \in U | \mathbb{R} \cdot x \subset U\}$$

coincides with S. In this case, Conley [1] defined the following invariant of S: For each neighborhood U as above, there exists a compact subset $A \subset U - S$ satisfying the following:

(1) If $x \in U$ and $x \cdot t \notin U$ for some $t > 0$, then there exists $t' \in [0, t]$ such that $x \cdot t' \in A$.

(2) If $x \in A$ and $x \cdot t \notin A$ for $t > 0$, then $x \cdot t \notin U$.

Because of these properties, A is called an exit set for U. A pair (U, A) satisfying these conditions is called an index pair for S. The Conley index $I(S)$ is defined as the pointed homotopy type of the topological quotient X/A with base point $\{A\}$, i.e., as its equivalence class under the relation of pointed homotopy equivalence. It is shown in [1] that it does not depend on the choice of the index pair (U, A) and that it is invariant under continuous deformations of the flow as long as S remains isolated in an appropriate sense. In particular, the relative cohomology

$$I^*(S;\mathbb{F}) = H^*(U,A;\mathbb{F}) \tag{2.1}$$

is well defined for any coefficient ring $\mathbb{F}$. It is the cohomology which takes the place of $H^*(M,\mathbb{F})$ in the Morse inequalities.

Example 1. If M is compact, then an index pair of $S = M$ is given by $(M,\varnothing)$, so that $I(M) = (M \cup \{\varnothing\}, \{\varnothing\})$.

Example 2. One easily shows that any isolated critical point x of f is an isolated invariant set with respect to any gradient flow of f. If x is nondegenerate, i.e. if the linearization

$$A_x = D\nabla_g f(x), \qquad T_xM \to T_xM$$

is an isomorphism, then a special index pair can be constructed as follows: Since A_x is selfadjoint, it splits T_xM into a positive and negative eigenspace $E_x^\pm$. Then, if $\exp_x\colon T_xM \to M$ is the Gauss normal chart and $B_\varepsilon^\pm$ is the ε-ball in $E_x^\pm$, the image under $\exp_x$ of

$$(U,A) = (B_\varepsilon^+ \otimes B_\varepsilon^-, B_\varepsilon^+ \otimes \partial B_\varepsilon^-)$$

is an index pair of x for ε small enough (see [1]). This is also called an isolating block. Hence

$$\begin{aligned} I^*(x;\mathbb{F}) &= H^*(B^+\varepsilon_\otimes B_\varepsilon^-, B_\varepsilon^+ \times \partial B_\varepsilon^-, \mathbb{F}) \\ &= H^*(B_\varepsilon^-, \partial B_\varepsilon^-; \mathbb{F}) \simeq \mathbb{F}. \end{aligned}$$

Conley index itself is a pointed sphere of dimension $\mu(x)$). Note that the choice of a generator of $I^*(x,\mathbb{F})$ corresponds to the choice of an orientation of E_x^-.

Example 3. It is easy to see from the definition that if disjoint sets S_1, S_2 are compact isolated invariant, then so is $S_1 \cup S_2$ and

$$I(S_1 \cup S_2) = I(S_1) \vee I(S_2).$$

Here, $A \vee B$ is the 1-point union of pointed topological spaces. From Examples 2 and 3, we conclude that any finite set C of nondegenerate critical points has a Conley index

$$I^*(C;\mathbb{F}) = C_\mathbb{F}^*. \tag{2.2}$$

The grading of $I^*(C,\mathbb{F})$ corresponds to the grading $C_\mathbb{F}^* = \bigotimes_{\mu \geq 0} C_\mathbb{F}^\mu$ of $C_\mathbb{F}^*$ defined by the Morse index $\mu(x) = \dim E_x^-$. Moreover, particular isomorphisms in (2.2) are specified by a family o of orientations on E_x^-, $x \in C$.

The coboundary operator δ_F is now defined as follows: For $x, y \in C$, define the space $M_g(x,y)$ of trajectories of the gradient flow connecting x and y. It can be described as the intersection

$$M_g(x,y) = W^u(x) \cap W^s(y) \tag{2.3}$$

of the stable and unstable manifolds at x and y, respectively. We say that the gradient flow is of Morse-Smale type if all intersections in (2.3) are transversal. This can be shown to be the case for a dense set of metrics g on M (see [11]). Then it is easy to verify that

$$\dim M_g(x,y) = \mu(x) - \mu(y).$$

Note that due to the invariance of the flow equation with respect to translations in τ, any nonempty trajectory space has dimension at least one, i.e. for a Morse-Smale flow there do not exist nonconstant trajectories connecting critical points x, y for $\mu(x) \leq \mu(y)$. Now for two critical points x, y with $\mu(x) = \mu(y) + 1$, we define the matrix element $\langle x, \delta y \rangle$ by counting the trajectories between x and y up to translations. There are of course two problems involved. First, we must make sure that the set $\hat{M}(x,y) = M(x,y)/\mathbb{R}$ is finite, and second, we must assign a sign to each of its elements in order to obtain an "invariant" result. To treat the first problem, note that we can identify $\hat{M}(x,y)$ with $M(x,y) \cap M_a$, where $M_a = \{x \in M | f(x) = a\}$ for any (say regular) value a between $f(x)$ and $f(y)$.

Lemma 2.1. *For $\mu(x) = \mu(y) + 1$, $M(x,y) \cap M_a$ is compact for every regular value a.*

Proof. Let u_i be a sequence of trajectories in $M(x,y)$ with $u_i(0) = a$. Clearly, since $u_i(0) \in S$ and S is compact, we have $u_i(0) \to x \in M_a \cap S$ for some subsequence. If this subsequence does not converge in $M(x,y)$, then $x = v(0)$ for some trajectory in S which does not connect x with y. Repeating the process for different regular values a leads to the conclusion that there exist critical points $x = z_0, z_0, \cdots, z_n = y$ trajectories u_i in $M(z_{i-1}, z_i)$ in the closure of $M(x,y)$ in M. But this is impossible for a Morse-Smale flow if $\mu(x) = \mu(y) + 1$, so that $M_a \cap M(x,y)$ is compact. q.e.d.

As to the second problem, it turns out that each of the manifolds $\hat{M}(x,y)$ for arbitrary $\mu(x) - \mu(y)$ actually has a natural normal framing defined by frames on E_x^- and E_y^-. To define it, note that frames on E_x^+ and E_y^- define normal frames $\phi_\pm$ on $W^u(x)$ and $W^s(y)$. On the transverse intersection, they define a normal frame $\phi_+ \oplus \phi_-$ of $M(x,y)$ in TM. To obtain a normal frame of $\hat{M}(x,y)$ in TM, we now only have to add the vector given by the flow direction. Now note that TM is trivial over $M(x,y)$. In fact, a framing of TM over all of $W^u(x)$ is given by a frame on T_xM, that is by an additional frame on E_x^-. It is then easy to see that up to isomorphy, the normal framing of M depends only on the frames of E_x^- and E_y^-. In this paper, we will only make use of the orientation on the normal (and hence on the tangent) bundle of $\hat{M}(x,y)$, which only depends on orientations of E_x^- and E_y^-.

Definition 2.1. *If $o = \{o_x\}$ is a choice of orientation on E_x^- for all critical points x of f, then we define $\langle x, \delta_o y\rangle \in \mathbb{Z}$ as the intersection number of the oriented manifolds $W^u(x) \cap M_a$ and $W^s(y) \cap M_a$ in M_a. Moreover, we define for any coefficient ring $\mathbb{F}$,*

$$C^\mu_{\mathbb{F}} \to C^{\mu+1}_{\mathbb{F}}, \qquad \delta_o y = \sum_{x \in C} x\langle x, \delta y\rangle.$$

Now recall that o also defines the isomorphism (2.2). We are therefore ready to state

Theorem 1. *Let S be a compact invariant set of a Morse-Smale gradient flow on M. Then if S is isolated, we have $\delta_o\delta_o = 0$ and an isomorphism of $\mathbb{F}$-modulus:*

$$I^*(S, \mathbb{F}) \simeq \ker \delta_o / \operatorname{im} \delta_o.$$

If M is compact and $S = M$, then Theorem 1 yields the well-known Morse inequalities.

Corollary. *For any Morse-Smale gradient flow on a compact manifold M, we have*

$$H^*(M, \mathbb{F}) = \ker \delta_o / \operatorname{im} \delta_o.$$

In a more general situation, Fransoza [4] defined a "connection matrix" recovering the cohomology of the index of S (see also [9]). In fact, the existence of the homomorphism δ follows directly from Fransoza's work. While it is known that the connection matrix is related to flow trajectories, it is only in the case of a smooth Morse-Smale flow on a finite dimensional manifold that it can actually be obtained from those. Our methods in proving Theorem 1 do not use Fransoza's work but only the elementary technique of Morse decompositions as introduced in [1].

The fact that the trajectory spaces are framed rather than only oriented suggests the following extension of the above program: If h^* denotes a general cohomology theory, then it should be possible to obtain $h^*(I(S))$ in a way similar to that above through an analysis of trajectory spaces. The chain complex would have to be replaced by $h^*(I(C)) = \bigoplus_{x \in C} h^{*-\mu(x)}$, and the δ-homomorphism would, in contrast to the singular case, depend on trajectory spaces of arbitrary dimension. For example, in the case of stable cohomotopy $h^* = \pi_s^*$, the contribution of every compact component of $\hat{M}(x,y)$ should be given by the element of π_s^* classifying its framed cobordism type. (In fact, the spaces $\hat{M}(x,y)$ undergo framed cobordisms under a change of the metric.) Of course, the higher dimensional components of $\hat{M}(x,y)$ do not have to be compact, but a reasonable modification of the program should lead to a spectral sequence converging to $h^*(I(S))$, as one would expect. This program is only of limited use for finite dimensional Morse theory, but might have applications to infinite dimensional cases.

3. Proof of Theorem 1

The Conley index of a compact isolated invariant set S can be related to the Conley indices of isolated invariant subsets by the notion of a "Morse decomposition" of S. To describe this, let us denote for any two subflows S_1, S_2 in S the subflow $S_1 \& S_2$ as the union of S_1, S_2, and of all trajectories "connecting" S_1 and S_2. Then a collection of disjoint compact invariant sets $(M_1, \cdots, M_n)$ is called a Morse decomposition of S, if for all $x \in S$ there exist $1 \leq i \leq j \leq n$ so that $x \in M_i \& M_j$. That is, each point in $S - \bigcup M_i$ lies on a trajectory connecting some M_i with some M_j for $j > i$. In particular, there are no trajectories from M_i to M_i which are not completely contained in M_i, and there are no trajectories at all from M_i to M_j if $j < i$. For example, if the gradient flow of (f, g) on M is of Morse-Smale type, then the sets

$$M_i = \{x \in S | df(x) = 0, \mu(x) = i\} \tag{3.1}$$

define a Morse decomposition. We define an index filtration associated with the Morse decomposition $(M_1, \cdots, M_n)$ as an increasing family $N_0 \subset \cdots \subset N_n$ of compact subsets of M such that (N_i, N_{i-1}) is an index pair for M_i. In the case (3.1), this implies that

$$H^i(N_i, N_{i-1}) = I^i(M_i) = C^i,$$

and $I^p(M_i) = 0$ for $p \neq i$ (see the remarks at the end of the preceding section). Throughout this section, $H^*(X, A)$ denotes cohomology with coefficients in $\mathbb{F}$. A choice o of orientations on E_x^- defines a preferred set of generators in $H^*(N_i, N_{i-1})$. Moreover, an index pair of $M_i \& M_{i+1}$ is given by (N_{i+1}, N_{i-1}). An index pair of $M_i \& M_{i+1} \& M_{i+1}$ is given by (N_{i+2}, N_{i-1}) and so on. In particular, (N_n, N_0) is an index pair of S. Now consider the exact triangle (the long exact cohomology sequence) of the triple (N_{i+1}, N_i, N_{i-1}):

$$\begin{tikzcd} & I^*(M_i \& M_{i+1}) \arrow[dl] & \\ I^*(M_i) \arrow[rr, "\delta_T"] & & I^*(M_j) \arrow[ul] \end{tikzcd},$$

where δ_T is of degree 1.

Lemma 3.1. *With respect to the preferred basis of C^k given by o, we have*

$$\delta_T = \delta_o \colon I^i(M_i) \simeq C^i \to C^{i+1} \simeq I^{i+1}(M_i).$$

Lemma 3.1 implies Theorem 1, since we can apply to the index filtration $(N_0, \cdots, N_n)$ the procedure which in [8, Appendix A] is applied to the cell filtration of a CW-complex. In fact, this procedure depends only on the algebraic fact that $H^*(N_i, N_{i-1})$ is free and has only one nontrivial dimension.

To prove Lemma 3.1, first note that one can treat each pair of points $(x, y) \in M_{i_1} \times M_i$ separately. In fact, by using the index filtration of the Morse decomposition $(M_i - \{y\}, \{y\}, \{x\}, M_{i+1} - \{x\})$ of $M_i \& M_j$, one can reduce Lemma 3.1 to the following statement: For each $x \in M_{i+1}$ and $y \in M_i$, the operator δ_T in the exact triangle

$$\begin{tikzcd} & I^*(x \& y) \arrow[dl] & \\ \mathbb{Z} = I^*(y) \arrow[rr, "\delta_T"] & & I^*(x) = \mathbb{Z} \arrow[ul] \end{tikzcd}$$

is given by multiplication with $k = \langle x, \delta y \rangle$. This is essentially proved in Lemma 7.2 of [7]. To give a proof in terms of the Conley index, we use the freedom of choosing suitable index pairs for x and y: First, choose a regular value a of f in $(f(x), f(y))$, and define $M_0 = \{x \in M | f(x) = a\}$ and $M_\pm = \{x \in M | \pm f(x) \geq \pm a\}$. Since the invariant set S was assumed to be compact, so is the closure of $W^u(x)$ intersected with M_+ and the closure of $W^s(y)$ intersected with M_-. Moreover, let B_x, B_y denote blocks as in Example 2 above around x and y of some (small) radius $\varepsilon > 0$. Then

$$U := B_x \cdot [0, R] \cap M_+ \quad \text{and} \quad V = B_y \cdot [-R, 0] \cap M_-$$

are compact sets, too. Clearly, if $B_x^- = \exp_x(B_\varepsilon^+ \times \partial B_\varepsilon^-)$ is the exit set of B_x, then

$$A := U \cap (M_0 \cup B_x \cdot R)$$

is by this construction an exit set for U, so that (U, A) is an index pair for x. Similarly,

$$B = \overline{\partial V - (M_0 \cup B_y \cdot (-R))}$$

is an exit set for V, so that (V, B) is an index pair for y. (It is instructive to visualize this and the following construction in the case where all values between $f(x)$ and $f(x)$ are regular. Then one can choose R large enough so that $A = U \cap M_0$ and $B = \partial V - M_0$.) Now an index filtration N_0, N_1, N_2 of $(\{x\}.\{y\})$ is given by

$$N_0 = B \cup \overline{A - V}, \quad N_1 = V \cup A, \quad N_2 = U \cup V.$$

Note that we have excision isomorphisms $H^*(N_1, N_0) \to H^*(V, B)$ and $H^*(N_2, N_1) \to H^*(U, A)$. Hence we have to prove that the δ-homomorphism of the exact triangle

$$\begin{array}{c} I^*(U \cup V, B \cup \overline{A - V}) \\ \downarrow \quad \uparrow \\ H^*(V, B) \simeq H^*(V \cup A, B \cup \overline{A - V}) \xrightarrow{\delta} H^*(V \cup U, V \cup A) \simeq H^*(U, A) \end{array}$$

maps the generator e of $H^*(V, B)$ into the k-fold of the generator f of $H^*(U, A)$. The δ-homomorphism of a triple factors through

$$H^\mu(V \cup A, B \cup \overline{A - V})^{j^*} \to H^\mu(V \cup A) \to H^{\mu+1}(V \cup U, V \cup A) = \mathbb{Z}.$$

Note that (U, A) is a thickening of a cell $(D^{\mu+1}, S^\mu)$. Now if $[A] \in H_n(V \cup A)$ denotes the corresponding n-cycle, then $j^*e[A] = k$ by the definition of the intersection number. Moreover, the δ-homomorphism of the pair $(V \in U,\ V \in A)$ satisfies $\delta\alpha = \alpha[A]f$. This completes the proof of Theorem 1.

4. An application to symplectic geometry

Let P be a smooth manifold of even dimension $2n$ which carries a symplectic form ω. Let L and L' be two Lagrangian submanifolds of P, i.e., L and L' are of dimension n and ω vanishes on TL and TL'. In [2] and [3], we studied the Morse theory for the symplectic action function on the space $\Omega(L, L')$ of smooth paths in P connecting L and L'. More precisely, the symplectic action a is defined on the universal covering $\tilde{\Omega}$ of Ω by the condition

$$da(z)\xi = \int_0^1 \omega(\dot{z}(t), \xi(t))\, dt.$$

The critical points of a are the constant paths $z(t) = x$, where x by definition of Ω must be an intersection of L with L'. This variational problem can therefore be used to estimate the number of such intersections. To give a rough outline of the method, let J_t, $t \in [0, 1]$, be any smooth family of smooth almost complex structures on P satisfying the Kähler condition with respect to ω. This means that the bilinear forms

$$g_t = \omega(J_t \cdot, \cdot) \tag{4.1}$$

are symmetric and positive for each t. Such a family J_t will be called an almost Kähler structure. Then we consider for each pair (x_+, x_-) of intersections of L and L' the set

(4.2)

$$\begin{aligned} \mathbf{M}_J(x_-, x_+) = \{u\colon \mathbb{R} \times [0,1] \to P \mid\, & (1)\ u(\mathbb{R} \times \{0\}) \subset L; \\ & (2)\ u(\mathbb{R} \times \{1\}) \subset L'; \\ & (3)\ \lim_{\tau \to \pm\infty} u(\tau, t) = x_\pm \quad \text{for all } t \in [0,1]; \\ & (4)\ \overline{\partial}_J u = 0\}. \end{aligned}$$

Here, $\overline{\partial}_J$ is the Cauchy-Riemann operator

$$\overline{\partial}_J u(\tau, t) = \frac{\partial u(\tau, t)}{\partial \tau} + J_t(u(\tau, t)) \frac{\partial u(\tau, t)}{\partial t}.$$

Although $\mathbb{P} \times [0,1]$ is not compact, the image of each $u \in \mathbf{M}_J(x, y)$ turns out to be precompact due to the asymptotic conditions. If J is independent of t, then u can be considered as a holomorphic disc in P with two "corners". Here, however, we interpret the vanishing of $\overline{\partial}_J u$ as the trajectory equation for the "flow" generated by the "vector field" $g(z)(t) = J_t\, dz(t)/dt$ on the space $\Omega(L, L')$, which is in fact the gradient of the action functional with respect to an L^2-inner product on $T\Omega(L, L')$. Of course this "gradient flow" is not well defined as a family of maps from Ω into itself. However, the spaces (4.2) of "trajectories" connecting two zeros of g share many properties of trajectory spaces in finite dimensional Morse theory. For example, $\mathbf{M}_J(x, y)$ "generically" is a smooth manifold, as in the case for trajectory spaces for finite dimensional gradient flows. We will discuss this property later on. Also, the compactness properties of $\mathbf{M}_J(x, y)$ are similar to those of trajectory spaces of finite dimensional manifolds. For a detailed study of the Cauchy-Riemann flow, see [3].

In this paper, we approach the relation between the Cauchy-Riemann flow and finite dimensional Morse theory from a different angle. Let L be any smooth manifold, and denote by $P = T^*L$ the cotangent bundle of L with the canonical symplectic structure. Then L is a Lagrangian submanifold of P, and so is the graph of any closed 1-form on L. In particular, any smooth function $f\colon L \to \mathbb{R}$ determines a Lagrangian submanifold $L' = \{(x, df(x)) | x \in l\}$ in P. Clearly, L and L' intersect precisely at the critical points of f. Our aim is to compare the gradient flow of f with respect to some metric g on L with the Cauchy-Riemann flow in $\Omega(L, L')$ with respect to a suitable family J_t of almost complex structures.

Theorem 2. *Let L be a smooth manifold with metric g. Then there exists a constant $\varepsilon > 0$ such that if $f \colon L \to \mathbb{R}$ is a smooth function satisfying*

$$|f(x)| + |\nabla f(x)| + |\nabla\nabla f(x)| < \varepsilon \tag{4.3}$$

for all $x \in L$, then there exists a family $J = \{J_t\}_{t \in [0,1]}$ as in (4.1) *such that the assignment*

$$\mathbf{M}_J(x, y) = C^\infty(\mathbb{R}, L), \qquad u \to u_0(\tau) = u(\tau, 0)$$

is a bijection onto the set of bounded trajectories of the gradient flow connecting two critical points x and y of f.

In [2], we defined an algebraic invariant for the Cauchy-Riemann flow, which depends solely on the space of bounded trajectories and the linearized flow operator

$$E_u = D\overline{\partial}(u) = \nabla_1 + J_u \nabla_2 + Z_u$$

along them. Here Z is a zero order operator, which depends on the connection ∇. If ∇ is the Levi-Civita connection, then the L^2-adjoint of E_u is

$$E_u^+ = -\nabla_1 + J_u \nabla_2 + Z_u.$$

E_u and e_u^+ act on the space of smooth sections ξ of the bundle u^*TP satisfying the boundary conditions $\xi(\tau, 0) \in TL$ and $\xi(\tau, 1) \in TL'$. We denote by $\ker E_u$ the space of bounded solutions of $E_u \xi = 0$ and by $\operatorname{cok} E_u$ the space of bounded solutions of $E_u^+ \xi = 0$. If $u \in \mathbf{M}(x, y)$ and x and y are nondegenerate critical points, then these spaces are related to the stable and unstable manifolds $W^u(x)$, $W^s(y)$ as follows.

Proposition 1. *Let x, y be two nondegenerate critical points and let $u \in \mathbf{M}_J(x, y)$ be as in Theorem* 1. *Define $p = u(0, 0)$. Then we have an isomorphism*

$$\ker E_v \to T_p W^u(x) \cap T_p W^s(y), \qquad \xi \to \xi(0, 0).$$

The same map also defines an isomorphism between $\operatorname{cok} E_v$ *and a complement of $T_p W^u(x) + T_p W^s(y)$ in TL.*

We now call $u \in \mathbf{M}_J(x, y)$ regular if $\operatorname{cok} E_u = 0$. It can be shown (see [3, Theorem 3]) that if x and y are transverse intersections of L and L' and if u is regular, then $\mathbf{M}_J(x, y)$ is a smooth manifold near u, whose tangent space is isomorphic to $\ker E_u$.

Definition 1. Assume the following:

(1) The spaces $\mathbf{M}_J(x, y)$ are regular.

(2) Whenever $\dim \mathbf{M}_J(x,y) = 1$, then $\mathbf{M}_J(x,y)/\mathbb{R}$ is finite.

(3) The $\mathbb{Z}_2$-numbers

$$(4.4)\qquad \langle x, \delta y\rangle = \begin{cases} \#(\mathbf{M}_J(x,y)/\mathbb{R}) \bmod 2 & \text{if } \dim \mathbf{M}_J(x,y) = 1, \\ 0 & \text{otherwise} \end{cases}$$

satisfy

$$\sum_{y\in L\cap L'} \langle x, \delta y\rangle \langle y, \delta z\rangle = 0.$$

We then define the index cohomology of L and L' with respect to L as

$$I^*(L, L', J) = \ker \delta / \operatorname{im} \delta,$$

where δ is the operator on the free $\mathbb{Z}_2$-vector space over the set $L \cap L'$ defined by the matrix elements $\langle x, \delta y\rangle$ of (4.4).

Here, $\mathbf{M}_J(x,y)/\mathbb{R}$ is the quotient by the translational symmetry of $\mathbf{M}_J(x,y)$ in the τ-variable, which can be shown to be a smooth manifold. For appropriate J, conditions (1)–(3) can be shown to hold in the special case where P is compact, L' is an "exact deformation" of L, and $\pi_2(P,L) = 0$. Now Theorems 1 and 2 together yield

Theorem 3. *Let $L \subset T^*L$ be the zero section and let L' be the graph of df, where f is a Morse function on L satisfying* (4.3). *Then there exists a family $J = \{J_t\}$ as in* (4.1) *such that*

$$I(L, L'; J) = H^*(L, \mathbb{Z}_2).$$

5. Proof of Theorem 2 and Proposition 1

Let us start with defining the family J_t of almost complex structures on T^*L. Every metric g on L defines an almost complex structure J_g on T^*L which satisfies (4.1) with respect to the canonical symplectic structure on T^*L and which for every $(x,\xi) \in T^*L$ maps the vertical tangent vectors to horizontal tangent vectors with respect to the Levi-Civita connection of g. In particular, for $(x,0) \in L \subset T^*L$, it assigns to each $\phi \in T_xL \subset T(T^*L)$ the cotangent vector $J\phi = g(\phi,\cdot) \in t^*L \subset T(T^*L)$. Now let $\phi_t T^*L \to T^*L$ denote the exact deformation induced by the Hamiltonian $H(p) = f(\pi p)$, where $\pi\colon T^*L \to L$ is the projection. Define

$$J_t = (\phi_t)_* J (\phi_t)_*^{-1}.$$

Clearly, $\phi_1(L) = L' \subset T^*L$ is the graph of df. Then for each smooth map $x\mathbb{R} \to L$, we define

$$(\Gamma x)(\tau, t) = \bar{x}(\tau, t) = \phi_t(x(\tau)).$$

Lemma 5.1. *If* $x'(\tau) + \operatorname{grad}_g f(x(\tau)) = 0$ *for all* $\tau \in \mathbf{R}$, *then* $\overline{\partial x} = 0$.

Proof. We have

$$\begin{aligned}\frac{\partial \bar{x}(\tau,t)}{\partial \tau} &= \frac{\partial \phi_t(x(\tau))}{\partial \tau} = D\phi_t(x(\tau))\frac{dx(\tau)}{d\tau} \\ &= D\phi_t(x(\tau))\operatorname{grad}_g f(x(\tau)).\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{\partial \bar{x}(\tau,t)}{\partial t} &= \frac{\partial \phi_t(x(\tau))}{\partial t} = X_H(\phi_t(x(\tau))) = D\phi_t(x(\tau))X_H(x(\tau)) \\ &= D\phi_t(x(\tau))J(x(\tau))\operatorname{grad}_g f(x(\tau)) \\ &= J_t(\bar{x}(\tau,t))D\phi_t(x(\tau)).\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial \bar{x}(\tau,t)}{\partial \tau} + J_t\frac{\partial \bar{x}(\tau,t)}{\partial t} &= -\, D\phi_t(x(\tau))\operatorname{grad}_g f(x(\tau)) \\ &\quad - J_t^2 D\phi_t(x(\tau))\operatorname{grad}_g f(x(\tau)) \\ &= 0. \quad \text{q.e.d.}\end{aligned}$$

It remains to show that the map Γ is surjective for f small enough in $C^2(L,\mathbf{R})$. We therefore invert the above construction: For any path u in $\Omega(L,L')$, we define the path

$$\bar{u}\colon \mathbf{R}\times[0,1]\to T^*L, \qquad \bar{u}(\tau,t) = \phi_{-t}u(\tau,t)$$

in $\Omega(L,L)$. Clearly, u is in the image of the map Γ if and only if for each $\tau \in \mathbf{R}$,

$$\bar{u}(\tau)(t) = \bar{u}(\tau,t)$$

is constant. This however does not follow from the Cauchy-Riemann equations. In fact, if $\overline{\partial}_0\bar{u} = \partial\bar{u}/\partial\tau + J_0\partial\bar{u}/\partial t$, then we have

$$\begin{aligned}(5.1)\quad \overline{\partial}\tilde{u}(\tau,t) &:= \frac{\overline{\partial u}(\tau,t)}{\partial\tau} + J_0(u(\tau,t))\frac{\partial\bar{u}(\tau,t)}{\partial t} \\ &= D\phi_{-t}(u(\tau,t))\frac{\partial u(\tau,t)}{\partial\tau} \\ &\quad + J_0(\bar{u}(\tau,t))\left[D\phi_{-t}(u(\tau,t))\frac{\partial u(\tau,t)}{\partial t} - \dot{\phi}_{-t}(u(\tau,t))\right].\end{aligned}$$

Here,

$$\dot{\phi}_t(z) = \frac{d\phi_t(z)}{dt} = X_H(\phi_t(z)) = -(J_0H')(\phi_t(z))$$

is the Hamiltonian vector field. Moreover, by the definition of J_t and for $\overline{\partial}u = 0$,1 we have

$$\overline{\partial u}(\tau,t) + H'(\bar{u}(\tau,t)) = 0.$$

Let us now write $\bar{u}(\tau,t)=(x(\tau,t),y(\tau,t))$, where $x(\tau,t)\in L$ and $y(\tau,t)\in T^*_{x(\tau,t)}L$. Then (5.1) decomposes into its projection onto L and

$$\frac{\partial x(\tau,t)}{\partial\tau}+\nabla_t y(\tau,t)+f'(x(\tau,t))=0,$$

and its g_0-orthogonal projection onto the fibers

$$\nabla_\tau y(\tau,t)-\frac{\partial x(\tau,t)}{\partial t}=0. \tag{5.2}$$

Clearly, if we can show that y vanishes identically, then x is constant in t by (5.2) and satisfies $x'(\tau)+\operatorname{grad}_g f(x(\tau))=0$. Moreover, by the above construction, we then have $u=\bar{x}$. To prove that $y=0$, define

$$\gamma(\tau)=\tfrac{1}{2}\langle y(\tau),y(\tau)\rangle,$$

where the inner product $\langle\ ,\ \rangle$ between sections of $(\bar{u}(\tau))^*TP$ is defined as

$$\langle\xi,\zeta\rangle\int g(\xi(t),\zeta(t))\,dt,$$

and g is the standard extension of the Riemann metric on L to the cotangent bundle. The following lemma proves that γ and therefore y vanish identically.

Lemma 5.2. *If f satisfies* (4.3) *for ε small enough, then for all $\tau\in\mathbb{R}$,*

$$\frac{\partial^2\gamma(\tau)}{\partial\tau^2}\geq\frac{\varepsilon^2}{2}\gamma(\tau).$$

Proof. We have

$$\frac{\partial^2\gamma(\tau)}{\partial\tau^2}=\|(\nabla_\tau y)(\tau)\|_2^2+\langle y(\tau),(\nabla_\tau^2 y)(\tau)\rangle.$$

Now since the Levi-Civita connection is torsion free, we have

$$\begin{aligned}\nabla_\tau^2 y&=\nabla_\tau\frac{\partial x(\tau,t)}{\partial t}=\nabla_t\frac{\partial x(\tau,t)}{\partial\tau}\\&=-\nabla_t\{\nabla_t y(\tau,t)+f'(x(\tau,t))\}\\&=-\nabla_t^2 y(\tau,t)-f''(x(\tau,t))\frac{\partial x(\tau,t)}{\partial t}\\&=-\nabla_t^2 y(\tau,t)-f''(x(\tau,t))\nabla_\tau y(\tau,t).\end{aligned}$$

Since $y(\tau,0)=y(\tau,1)=0$, we can perform an integration by parts to obtain

$$\frac{\partial^2\gamma(\tau)}{\partial\tau^2}=\|\nabla_\tau y(\tau)\|_2^2+\|\nabla_t y(\tau)\|_2^2-\langle y(\tau),f''(x(\tau))\nabla_\tau y\rangle. \tag{5.3}$$

Using again the fact that $y(\tau)$ vanishes on the boundary of the interval $[0, 1]$, we conclude that there exists a constant γ independent of $u = (x, y)$ and of h such that

$$\|y\|_2 \leq C\|\nabla_t y\|_t.$$

We can therefore choose f small enough in $C^2(L, \mathbb{R})$ such that

$$\begin{aligned}\langle y(\tau), f''^{\circ}(x(t))\nabla_\tau y(\tau)\rangle &\leq C \max_{x\in L}|f''(x)| \cdot \|\nabla_t y(\tau)\|_2\|\nabla_\tau y(\tau)\|_2 \\ &\leq \tfrac{1}{4}\|\nabla_t y(\tau)\|_2\|\nabla_\tau y(\tau)\|_2.\end{aligned}$$

Then since

$$\frac{a^2}{2} + \frac{b^2}{2} - \frac{1}{4}a \cdot b = \left(\frac{a}{2} - \frac{b}{2}\right)^2 \geq 0,$$

(5.3) yields the estimate

$$\frac{\partial^2 \gamma(\tau)}{\partial \tau^2} \geq \frac{1}{2}\|\nabla_t^2 y(\tau)\|n_2^2 \geq \frac{1}{2C^2}\|y(\tau)\|_2^2 \geq \frac{1}{C^2}\gamma(\tau).$$

This completes the proof of Lemma 5.2, and hence of Theorem 2. In order to prove Proposition 1, we proceed similarly.

References

[1] C. C. Conley, *Isolated invariant sets and the Morse index*, CBMS Regional Conf. Ser. in Math., Vol. 38, Amer. Math. Soc., Providence, RI, 1978.

[2] A. Floer, *Morse theory for Lagrangian intersections*, to appear.

[3] ——, *The unregularized gradient flow of the symplectic action*, to appear.

[4] R. Fransoza, *Index filtrations and connection matrices for partially ordered Morse decompositions*, preprint.

[5] B. Helffer & J. Sjöstrand, *Puits multiples en mechanique semiclassique* IV; *etude du complexe de Witten*, Comm. Partial Differential Equations **10** (1985) 245–340.

[6] J. Milnor, *Morse theory*, Ann. of Math. Studies, No. 51, Princeton University Press, Princeton, NJ, 1963.

[7] ——, *Lectures on the H-cobordism theorem*, Math. Notes, Princeton University Press, Princeton, NJ, 1965.

[8] J. Milnor & J. Stasheff, *Characteristic classes*, Ann. of Math. Studies, No. 76, Princeton University Press, Princeton, NJ, 1974.

[9] R. Moeckel, *Morse decompositions and connection matrices*, preprint, University of Minnesota–Twin Cities.

[10] S. Smale, *Morse inequalities for dynamical systems*, Bull. Amer. Math. Soc. **66** (1960), 43–49.

[11] ——, *On gradient dynamical systems*, Ann. of Math. (2) **74** (1961), 199–206.

[12] E. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.

[13] R. Thom, *Sur une partition en cellules associée à une fonction sur une variété*, C.R. Acad. Sci. Paris **228** (1949), 973–975.

[14] E. Witten, *Supersymmetry and Morse theory*, J. Differential Geometry **17** (1982), 661–692.

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