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COMPLEX MANIFOLD GEOGRAPHY IN DIMENSION 2 AND 3

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Introduction

The term geography is used to describe the distribution of Chern numbers of algebraic manifolds of general type. The term was introduced by Persson [73], where he studied the question for algebraic surfaces of general type. Although the Chern numbers are quite rough invariants of manifolds, they are in general easily calculated, and are perhaps the only such invariants for algebraic manifolds of general type. Furthermore, there seem to be in general certain bounds on these Chern numbers, so the natural question arises as to whether there exists an algebraic manifold Y for every given set of numbers fulfilling the bounds, such that Y has precisely those Chern numbers. In the second part of this paper, which is mainly expository, we study these questions for algebraic 3-folds. To put this in its proper perspective, in the first part we review most of the constructions used to get the known results in the surface case. This subject has a history going back at least to about 1950.

About that time Thom created his cobordism theory [86] by which the set of cobordism classes of differentiable manifolds became a ring under the operations of cartesian product and disjoint sum, each class being characterized by its Pontrajagin numbers. The signature theorem, proven in 1953 [24], showed that the Pontrajagin numbers could not be arbitrary: there are integrality conditions which the Pontrajagin numbers of smooth manifolds had to fulfill. A complex analogue was developed by Milnor [61], and the Riemann-Roch theorem, proven in 1954 [24], showed that also the Chern numbers of (almost) complex manifolds had to fulfill certain integrality relations (which were classically known in (complex) dimensions 1 and 2). Milnor was then able to show that these integrality conditions are essentially *all* the conditions which must be fulfilled in the complex cobordism ring. Hirzebruch suggested then that if one restricts

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consideration to *connected* manifolds, the Chern numbers should probably fulfill certain *inequalities* [23, 7(3)].

The examples of algebraic surfaces known then were mainly the ones found lying around in nature—the complete intersections. Complete intersection surfaces do indeed fulfill an inequality: $c_1^2 \leq 2c_2$ (if they are of general type). Zappa conjectured that this inequality might hold for all surfaces of general type. By the signature theorem, $c_1^2 - 2c_2 = \tau$, the signature, so the conjecture took the form: general type surfaces have negative signature. Then in 1955 Hirzebruch proved the famous proportionality principle [22], which implies that a compact, smooth quotient of the 2-ball $\mathbf{B}^2 = \{(x, y) \in \mathbf{C}^2 \mid |x|^2 + |y|^2 < 1\}$, if one exists, is a surface of general type with $c_1^2 = 3c_2$. While such a ball quotient would have disproven the conjecture above, this result suggested another natural candidate for a bound. The existence of such quotients (i.e., the existence of discrete subgroups $\Gamma \subset SU(2, 1)$ acting properly discontinuously and freely on $\mathbf{B}^2$) was soon shown. With this the search for minimal surfaces of general type with positive index began.

It was also known by about 1957 that the signature behaves multiplicatively in fiber bundles, provided the fundamental group of the base acts trivially on the homology of the fiber [9]. Since the signature of any curve is, by definition, zero, this implies that surfaces fibering to curves necessarily have zero signature, provided the fundamental group of the base curve acts trivially. One therefore thought of constructing surfaces fibering onto curves, but where the fundamental group acts nontrivially on the fibers. Examples of general type surfaces with positive index (signature) were constructed in this manner by K. Kodaira [57]. But still for some time to come examples of positive index surfaces were quite scarce.

In the meantime progress had been made in proving the inequality above for general type surfaces. Van de Ven [91], [92] used algebraic-geometric methods to prove $c_1^2 \leq 8c_2$. These methods were improved upon by Bogomolov, who was able to prove $c_1^2 \leq 4c_2$ [79]. Then, in 1977, Miyaoka, improving on Bogomolov's method, proved the inequality $c_1^2 \leq 3c_2$ for all general type surfaces [62]. In the same year, using difficult methods of differential geometry and partial differential equations, Yau solved Calabi's conjecture [92], and as a corollary could prove the famous inequality for the Chern numbers of varieties V of dimension N with K_V ample:

$$(-1)^N c_1^N(V) \leq (-1)^N \frac{2(N+1)}{N} c_1^{N-2} c_2(V),$$

with equality holding if and only if V is a smooth, compact quotient of the N-ball [91]. The solution of Calabi's conjecture implies the existence of a unique Kähler-Einstein metric on V under the assumption K_V ample, and the inequality above had been known earlier for Kähler-Einstein metrics [54]. The most stunning thing of this result is its generality; ball quotients are extreme in all dimensions. Also, the result is stronger than just the converse of Hirzebruch proportionality.

Today it is known that the complete set of inequalities, which the Chern numbers of a minimal surface S of general type fulfill, is [4]:

$$\begin{aligned}
&\text{(i)} \quad c_1^2 > 0, \quad c_2 > 0,\\
&\text{(ii)} \quad c_1^2 + c_2 \equiv 0\ (12),\\
&\text{(iii)} \quad c_1^2 \geq \begin{cases} c_2/5 - 36/5, & c_1^2 \text{ even},\\ c_2/5 - 6, & c_1^2 \text{ odd},\end{cases}\\
&\text{(iv)} \quad c_1^2 \leq 3c_2.
\end{aligned}$$

We already eluded to (ii) and (iv) above; (i) and (iii) are classical. (iii) is Noethers inequality, which was originally formulated in the more conceptual form:

$$\text{(iii)}' \quad c_1^2 \geq 2p_g - 4, \quad p_g = \text{geometric genus of } S.$$

This inequality states roughly "many holomorphic 2-forms drive c_1^2 up", and in this sense can be expected to have an analogue in any dimension. This kind of inequality, however, will not be expressible solely in terms of Chern numbers, as in the case of surfaces.

For surface researchers, the work was clearly cut out: find surfaces "filling in" the area of possible Chern numbers delineated by (i)–(iv) above. Refined methods of construction were needed. One method of construction which had been known for some time, although not yet sufficiently utilized, was ramified coverings of known spaces. The late 70's and early 80's saw a flurry of activity in this direction ([72], [58], [65], [26], [33], [27], [73], [74], [46], [8], [84], [60]). Its systematic use started more or less with the work of Persson which culminated in his famous Compositio paper [73]. The strategy laid out there has been reutilized and refined ever since. This is our point of departure for Part I of this paper.

In all of Part I we concentrate on giving as complete descriptions as feasible of all constructions utilized. §0 is included for the benefit of those not familiar with the theory of algebraic surfaces. §0.3 in particular motivates what follows. In §1 we review Persson's original construction and subsequent generalizations of it. The main result is

Theorem 1. *Let $(x,y) \in \mathbf{Z} \times \mathbf{Z}$ such that $\frac{1}{5}y - \frac{36}{5} \leq x \leq 2y$ and $x \neq 2y - 3k$, where $k \in \{2, 1, 3, \cdots, 15, 19\}$. Then there exists a minimal genus two fibration $S \to C$ with $c_1^2(S) = x$, $c_2(S) = y$.*

This pretty much "fills" the area $\frac{1}{5}c_2 - \frac{36}{5} \leq c_1^2 \leq 2c_2$. In §§1.4 and 1.5 we present some generalizations. The result of §1.4 is on the Picard number of surfaces with fixed (c_1^2, c_2):

Theorem 2. *If X is a Horikawa surface (see §0.3 for definition) with $\chi(X) \equiv 0$ (6), then there is a deformation of X with maximal Picard number.*

This theorem, although in appearance quite different than Theorem 1, is actually proven in a similar manner. The difference lies in the type of singularities one allows the branch locus of a double cover to have. In §1.5 we describe a construction due to Xiao. This construction yields a proof of

Theorem 3. *Let $(x, y) \in \mathbf{Z} \times \mathbf{Z}$ such that x is sufficiently large and*

$$\frac{352}{89}x + 280x^{2/3} < y < \frac{18644}{2129}x - 2555x^{2/3}.$$

Then there exists a simply connected surface X of general type with

$$c_1^2(X) = y, \chi(X) = \frac{c_1^2 + c_2}{12} = x.$$

Although not as precise as Theorem 1 this is stronger in two ways: first, the ratio c_1^2/c_2 is asymptotically $18644/6904 = 2.7$, and second, the constructed surfaces are simply connected.

In §2 we turn to somewhat different construction, due to Miyaoka. The main result here is

Theorem 4. *Every surface $S \subset \mathbf{P}^N$ has a Galois covering $X \to S$ such that X has positive signature (i.e., $c_1^2 > 2c_2$).*

§3 describes the beautiful theory of Picard Modular Surfaces, researched by Holzapfel and his students in Berlin, Hauptstadt der DDR. This yields examples of general type surfaces with c_1^2/c_2 near 3. In §3.1 we describe the logarithmic Chern numbers used for such noncompact quotients.

In §4 we describe most of the constructions known to us which yield examples of (compact or not) ball quotients. These are arranged more or less in chronological ordering. In §4.1 we describe covers of the elliptic modular surfaces, branched along a union of sections of the fibration, constructed by Livné [58] and independently by Inoue [44]. These examples turn up again in relation with the examples constructed in the following three sections. Finally, in §4.5 we describe the generalization of the Yau inequality, due to Miyaoka [63] and R. Kobayashi [52], [53], which accommodates both actions with fixed points and noncompact quotients.

We end the first part with a description of A. Sommese's result [84]. If one considers the quotient c_s^2/c_2 instead of the pair (c_1^2, c_2), then the bounds (i), (iii) and (iv) above can be wrapped up in a single statement:

$$(*) \qquad \text{asymptotically } c_1^2/c_2 \in \left[\tfrac{1}{5}, 3\right].$$

Viewing things this way leads to

Problem. Which $p/q \in [\frac{1}{5}, 3]$ are accumulation points for a series of minimal surfaces of general type?

Sommese considers this problem, and using an elegant construction (by simply taking fiber products) he proves

Theorem 5. *Every $p/q \in [\frac{1}{5}, 3]$ is an accumulation point for a series of minimal surfaces of general type. In fact, for each $p/q \in [\frac{1}{5}, 3]$, there is a minimal surface S of general type with $c_1^2(S)/c_2(S) = p/q$.*

Remark. Of course considering just the quotient c_1^2/c_2 is much weaker than considering pairs (c_1^2, c_2), i.e., given p/q, there are infinitely many possible values for (c_1^2, c_2) with $c_1^2/c_2 = p/q$. Such pairs are, however, bounded in one direction since $c_1^2 > 0$, $c_2 \geq 0$ (condition (i) stated above).

This result pretty much completes the picture of surface geography.

Moving on to dimension 3 there are new difficulties all along the way. First of all, there is no corresponding theory of minimal models in dimension 3. Because of this we have a good theory of geography only if we restrict the class of 3-folds. We discuss these matters in §§6.5 and 7.1. It turns out we get a good theory for 3-folds X with ample canonical bundle, and also for minimal models which are smooth. More generally, we get a 'rational' theory for general minimal models. This point must always be in the back of our heads when doing 3-fold geography.

The next problem in dimension 3 is that we have three numbers, or two ratios to consider. An algebraic 3-fold Y (Y smooth and minimal) of general type determines a point

$$[x_0 : x_1 : x_2] = [c_1^3(Y) : c_1c_2(Y) : c_3(Y)]$$

in the rational projective plane $\mathbf{P}^2(\mathbf{Q})$ with homogeneous coordinates $[x_0 : x_1 : x_2]$. Thus we consider only the two ratios from the start. Of course, just as above, this is a weaker condition than considering triples $[c_1^3, c_1c_2, c_3]$, but this formulation turns out to be *tractable* with current methods. In this set-up, looking for bounds on the Chern numbers of Y is like looking for some curves in $\mathbf{P}^2(\mathbf{Q})$ bounding some (probably convex) finite area $D \subset \mathbf{P}^2(\mathbf{Q})$ of *possible* Chern numbers for algebraic 3-folds Y (with K_Y ample, say). Then comes the question as to whether each $x \in D$ is really the Chern numbers of an actual 3-fold Y.

However the curves bounding D are not known, and therefore it is desirable to "fill in" as much area with known examples, perhaps shedding light as to the whereabouts of the aforementioned curves in $\mathbf{P}^2(\mathbf{Q})$. It is this philosophy we pursue in Part II of this paper.

The *known* bounds on the Chern numbers of 3-folds Y, where we now assume K_Y ample, can be summarized as follows:

$$\begin{array}{ll} \text{(i)} & c_1^3(Y) < 0; \quad c_1c_2(Y) < 0, \\ \text{(ii)} & c_1c_2(Y) \equiv 0 \ (24), \\ \text{(iii)} & -c_1^3(Y) \leq \frac{8}{3}(-c_1c_2(Y)). \end{array}$$

Under further assumptions on Y other bounds can be shown to hold. For example, under certain assumptions on the canonical map M. Reid and his student Fletcher have shown $K_Y^3 \geq 2p_g - 4$ to hold, an inequality which corresponds to the Noether inequality for surfaces. This inequality, however, can only be expressed in terms of Chern numbers under restrictive assumptions, i.e. if $g_2 = g_1$ ($g_i = h^0(Y, \Omega_Y^i)$), this becomes $-c_1^3 \geq -c_1c_2/12 - 2$. But notice that nothing is said about the Euler number $c_3(Y)$, and as a substantial difference to the surface case

$$\text{(i)}' \quad c_3(Y) \text{ may be } <, > \text{ or } = 0.$$

Van de Ven suggested the following procedure to get an inequality containing $c_3(Y)$. Suppose $i\colon X \subset \mathbf{P}^N$ is the canonical embedding and is smooth (i.e. $K_X = i^*\mathcal{O}_{\mathbf{P}^N}(1)$). Let

$$\begin{array}{rl} f\colon & X \to G(N+1, 4) \\ & x \mapsto \text{tangent plane to } X \text{ at } x \end{array}$$

be the Gauss mapping. There is the usual bundle sequence on $G(N+1,4)$:

$$0 \to S \to \mathbf{C}^{N+1} \to Q \to 0,$$

where S is the universal bundle (see [18, Chapter I, section on Grassmannians]), which pulls back to an exact sequence on X

$$0 \to f^*S \to f^*\mathbf{C}^{N+1} \to f^*Q \to 0.$$

On the other hand we have the usual exact sequence on $\mathbf{P}^N$:

$$0 \to \mathcal{O}_{\mathbf{P}^N} \to (\mathcal{O}_{\mathbf{P}^N}(1))^{N+1} \to T_{\mathbf{P}^N} \to 0,$$

which gives

$$0 \to \mathcal{O}_{\mathbf{P}^N}(-1) \to \mathcal{O}_{\mathbf{P}^N}^{N+1} \to T_{\mathbf{P}^N}(-1) \to 0,$$

which can be pulled back to X:

$$0 \to i^*\mathcal{O}_{\mathbf{P}^N}(-1) \to i^*\mathcal{O}_{\mathbf{P}^N}^{N+1} \to i^*T_{\mathbf{P}^N}(-1) \to 0.$$

In addition we have the adjunction sequence on X:

$$0 \to T_X \to i^*T_{\mathbf{P}^N} \to N_{\mathbf{P}^N}X \to 0,$$

which gives

$$0 \to T_X(-1) \to i^*T_{\mathbf{P}^N}(-1) \to N_{\mathbf{P}^N}X(-1) \to 0.$$

These three sequences fit together in a diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & \nearrow & f^*S & \longrightarrow & T_X(-1) & \searrow & \\
 & & \nearrow & & \downarrow & & \downarrow & & \searrow \\
0 & \longrightarrow & i^*\mathcal{O}_{\mathbf{P}^N}(-1) & \longrightarrow & i^*(\mathcal{O}_{\mathbf{P}^N}^{N+1}) & \longrightarrow & i^*(T_{\mathbf{P}^N}(-1)) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & f^*Q & & N_{\mathbf{P}^N}X(-1) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

which factorizes like the dotted line yielding the sequence

$$0 \to \mathcal{O}_X(-K_X) \to f^*S \to T_X(-K_X) \to 0.$$

It is well known that $c_3(f^*S) = f^*c_3(S) \leq 0$, so using the formula

$$c(f^*S) = c(\mathcal{O}_X(-K_X))c(T_X(-K_X))$$

for the total Chern class this turns into an *inequality*:

$$c_3(f^*S) = c_3(T_X(-K_X)) - K_X \cdot c_2(T_X(-K_X)) \leq 0,$$

$$c_3 - c_2 \cdot K_X + c_1 \cdot K_X^2 - K_X^3 - K_X \cdot (3K_X^2 + 2 \cdot K_X^2 + c_2) \leq 0,$$

$$c_3 + 2c_1c_2 - 7K_X^3 \leq 0,$$

or

$$\frac{c_3}{c_1c_2} \geq -2 - 7\frac{c_1^3}{c_1c_2}.$$

Similar reasoning applied to the m-canonical map yields an even weaker inequality, much too weak to be useful. But this does give us perhaps an idea of what to expect in general.

We now outline the contents of Part II of this paper. In §6 we describe for each arrangement of planes $\mathcal{L} \subset \mathbf{P}^3$ a corresponding branched covering

$X \to \mathbf{P}^3$, branched along $\mathscr{L}$. X may be singular, with a desingularization Y which is a branched covering of $\hat{\mathbf{P}}^3$, some blow-up of $\mathbf{P}^3$:

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ \hat{\mathbf{P}}^3 & \longrightarrow & \mathbf{P}^3, \end{array}$$

where the horizontal arrows are birational maps and the vertical ones branched coverings. We call Y the *Fermat cover* associated to the arrangement $\mathscr{L}$. (The name stems from the fact that X may be viewed as a singular complete intersection of Fermat hypersurfaces. This is explained in detail in [41].)

We then use specific interesting examples to get an idea where the bounds of D are. In §7.2 we discuss different parts of $\mathbf{P}^2(\mathbf{Q})$ and to organize things, we break the area where D must exist into different zones. We are convinced that one zone which we discuss, the Zone E in the map, belongs to the complement of D. We describe a curve which seems to be a likely candidate for a border of D. This curve looks like a polynomial branch. In §7.3 we give detailed examples of several Fermat covers of general type (and even K_Y ample) which have positive Euler number. In §7.4, which is independent of the rest of the paper, we briefly discuss an interesting phenomenon which has no surface analogue.

§8 contains the only theorems in this paper, which give 3-dimensional analogues (albeit very weak results) of Persson's and Sommese's work. We prove the following:

Theorem. *Let* $[\alpha : \beta : \gamma]$ *be a point (in homogeneous coordinates on* $\mathbf{P}^2(\mathbf{Q})$*) in one of the triangles* $\triangle ABC$ *or* $\triangle CDE$ *with vertices*

$$\begin{aligned} A &= [12 : 11 : 1], \quad & B &= [6 : 5 : 3], \quad & C &= [42 : 33 : 19], \\ C &= [5 : 5 : -2], \quad & D &= [3 : 3 : 2], \quad & E &= [96 : 87 : 55]. \end{aligned}$$

Then there exists an algebraic 3*-fold of general type* Y *with* $[c_1^3(Y) : c_1c_2(Y) : c_3(Y)] = [\alpha : \beta : \gamma]$.

These two triangles are drawn on the map in the appendix. It is seen that this result is but a small start in the right direction. Much work remains to be done.

In §9, which is somewhat independent of §8, we study the relationship between the combinatorial data of an arrangement $\mathscr{L}$ and the ratios of the Chern numbers of the corresponding Fermat covers. It is of independent interest.

Finally, the appendix is the map of 3-fold geography. We have done our best to give a first understanding into this very complex topic by considering as many interesting examples as possible.

I would like to thank A. J. Sommese both for the inspiration of his paper [84], which got this work going, and for his encouragement during my stay at the Max-Planck Institut für Mathematik in Bonn. And of course the idea of Fermat covers was Hirzebruch's, without which none of these interesting examples would be known.[1] It is my pleasure to thank him for his interest in this work as it progressed. It is also my pleasure to thank the referee for an incredibly meticulous job of looking through the text, finding small errors, inconsistencies and suggesting several improvements. Finally, I would like to thank S.-T. Yau for his interest in the original preprint of what is now Part II, and his suggestion of expanding the paper to include what is now Part I.

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[1](Added in proof) Example 7.4.3 in §7 has recently been identified as the Siegel Modular 3-fold of level 4; see the discussion there.

PART I. CONSTRUCTIONS OF SURFACE GEOGRAPHY

0. Families of surfaces of general type

0.1. Basic results. Let X be a compact, complex, analytic manifold of dimension n. For the classification of X two notions are particularly important.

Definition 0.1.1. Let $K_X = \bigwedge^n T^*X$ be the canonical bundle, and

$$\varphi_{mK}\colon X \to W \subset \mathbf{P}^{\dim |mK|}$$

the pluricanonical map. The *Kodaira dimension* of X is

$$\kappa(X) := \begin{cases} \max_m \dim W & \text{if } |mK| \neq \varnothing \text{ for some } m, \\ -\infty & \text{if } |mK| = \varnothing \text{ for all } m. \end{cases}$$

Here $|mK|$ is the linear system of all effective divisors linearly equivalent to mK, and $|mK| = \varnothing$ for all m means there are no effective divisors linearly equivalent to mK for any m. Examples of this are given by projective space.

Definition 0.1.2. The *algebraic dimension* of X is

$$a(X) := \operatorname{tran}_{\mathbf{C}} K(X),$$

where $K(X)$ is the field of rational (meromorphic) functions on X.

It follows immediately from the definitions that

$$\kappa(X) \leq a(X) \leq n = \dim X.$$

If $\kappa(X) = n$, X is said to be of *general type*. If $a(X) = n$, X is called *Moišhezon*. In this case X has the function field of a projective algebraic variety, and Moišhezon X is Kähler if and only if it is projective algebraic. By the inequality above, any X of general type is Moišhezon.

Now let X be a compact, complex analytic surface. There is a strengthening of the above, due to Kodaira and Chow:

Theorem 0.1.3. *If $a(X) = 2$, then X is projective algebraic.*

Therefore, if X is of general type, it is automatically projective algebraic. Surface geography is the study of general type surfaces, so from now on we may assume X to be projective algebraic.

Every algebraic surface with $\kappa(X) \geq 0$ has a unique minimal model [4, p. 79], an old result due to Zariski. (A smooth surface is called minimal if there are no rational curves with self-intersection (-1) (so-called exceptional curves of the first kind) lying on it.) Therefore, in studying surface geography it is sufficient to restrict attention to minimal surfaces.

There are lots of invariants of algebraic surfaces, i.e. Hodge and Betti numbers, $\pi_1(X)$, signature, etc. The basic invariants for surfaces of general type however are just the Chern numbers $c_1^2(X)$ and $c_2(X)$ of the tangent bundle of X. The Chern numbers are not birational invariants, but since we may assume X to be minimal we get a well-defined map

$$\left\{\begin{array}{c}\text{minimal}\\ \text{surfaces } X \text{ of}\\ \text{general type}\end{array}\right\} \to \mathbf{Z} \oplus \mathbf{Z} \qquad (0.1.4)$$

$$X \mapsto (c_1^2(X), c_2(X)).$$

The minimal surfaces of general type can be parametrized in a satisfactory way, a theorem due to Gieseker [17, p. 236]:

Theorem 0.1.5. *There exists a quasi-projective coarse moduli scheme for the minimal surfaces of general type X with fixed Chern numbers c_1^2 and c_2.*

This theorem implies that the inverse image of fixed $(c_1^2, c_2) \in \mathbf{Z} \oplus \mathbf{Z}$ under the map (0.1.4) is a countable number of quasiprojective families.

Actually, just the existence of the Hilbert scheme implies that by desingularization of 2-dimensional irreducible varieties of given degree in a fixed $\mathbf{P}^N$ there can be at most finitely many diffeomorphism types, and so the fact that for $N \geq 5$, the N-canonical map φ_{NK} is a birational morphism onto a normal 2-dimensional subvariety of degree $N^2c_1^2(X)$ in some $\mathbf{P}^N$ (N

depending only on c_1^2 and c_2) implies that for given c_1^2 and c_2 there are only finitely many diffeomorphism types of surfaces with given $(c_1^2, c_2) \in \mathbf{Z} \oplus \mathbf{Z}$. Gieseker's theorem says these are nicely parametrized.

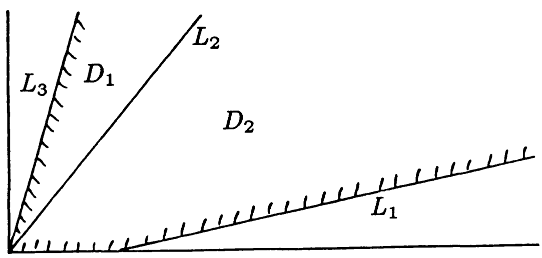
The point of this discussion is that surface geography can (without restricting the generality) be reduced to the following questions:

(a) For which $(x, y) \in \mathbf{Z} \oplus \mathbf{Z}$ is the inverse image of (0.1.4) not empty?

(b) For $(x, y) \in \mathbf{Z} \oplus \mathbf{Z}$ with nonempty inverse image, how many different surfaces (up to deformations) are there, and what are their properties? (A favorite question here: how many are simply connected?)

The *necessary* conditions on $(x, y) \in \mathbf{Z} \oplus \mathbf{Z}$ to be nonempty were stated in the introduction. Using these, there is a region in the plane $\mathbf{Z}^2$ where the possible Chern numbers lie. This is seen in the following diagram.

(0.1.6)



The region $D = D_1 \cup D_2$ is where the Chern numbers of general type X must lie. These are delineated by three lines:

$$L_1 = \text{line given by Noether-inequality},$$
$$L_2 = \{c_1^2 = 2c_2\}, \qquad (0.1.7)$$
$$L_3 = \{c_1^2 = 3c_2\}.$$

0.2. Hirzebruch proportionality. To understand the importance of the lines L_2 and L_3 in (0.1.6) we state the Hirzebruch proportionality theorem for surfaces. Let U be a hermitian symmetric domain in dimension 2, i.e., either $U = \mathbf{B}^1 \times \mathbf{B}^1$ or $U = \mathbf{B}^2$, where the complex N-ball is defined by

$$\mathbf{B}^N = \left\{ (x_1, \cdots, x_N) \in \mathbf{C}^N \middle| \sum |x_i|^2 < 1 \right\}.$$

Let Γ be a discrete subgroup of $\mathrm{Aut}(U)$ acting properly discontinuously on U with compact quotient $X = \Gamma \backslash U$.

Theorem 0.2.1 [22, *Satz* 3]. *If* $U = \mathbf{B}^1 \times \mathbf{B}^1$ *then* $c_1^2(X) = 2c_2(X)$. *If* $U = \mathbf{B}^2$ *then* $c_1^2(X) = 3c_2(X)$.

From this we see that the lines L_2 and L_3 in (0.1.6) are quite natural boundaries of the "domains" D_1 and D_2. We also point out the following significance: The signature of the intersection form on the free part of $H^2(X, \mathbf{Z})$ for a compact, complex surface X, $\tau(X)$, can be expressed in terms of the Chern numbers $c_1^2(X)$ and $c_2(X)$: $\tau(X) = c_1^2(X) - 2c_2(X)$. So $D_1 = \{(c_1^2(X), c_2(X)) | \tau(X) > 0\}$ and $D_2 = \{(c_1^2(X), c_2(X)) | \tau(X) < 0\}$.

The proportionality principle also holds for noncompact quotients if the right Chern numbers are assigned to the noncompact X. These turn out to be logarithmic Chern numbers of a compactification of X. This notion was introduced and the proportionality theorem was proved by Mumford [69].

Let $\bar{X} = X \cup D$ be a smooth compactification of a noncompact quotient $\Gamma \backslash \mathbf{B}^2 = X$ with D a disjoint union of smooth elliptic curves. The logarithmic Chern numbers of $(\bar{X}, D)$ are denoted $\bar{c}_1^2(\bar{X}, D)$, $\bar{c}_2(\bar{X}, D)$. (See §3.1 for more details on logarithmic Chern numbers.)

Theorem 0.2.2 [69, *Theorem* 3.2 *and Proposition* 3.4.a]. *Suppose* $\bar{X} = X \cup D$ *as above. Then* $\bar{c}_1^2(\bar{X}, D) = 3\bar{c}_2(\bar{X}, D)$.

0.3. Horikawa surfaces. Let $X \to Y$ be a double cover of surfaces with Y smooth and X normal, and let $B \subset Y$ be the branch locus. X is singular only at singular points of B.

Definition 0.3.1. X has *simple singularities* (rational double points, negligible singularities), if B has the following singularities:

$$\begin{aligned} A_n &: \quad x^2 + y^{n+1} = 0 \quad (n \geq 1), \\ D_n &: \quad y(x^2 + y^{n-2}) = 0 \quad (n \geq 4), \\ E_6 &: \quad x^3 + y^4 = 0, \\ E_7 &: \quad x(x^2 + y^3) = 0, \\ E_8 &: \quad x^3 + y^5 = 0. \end{aligned}$$

Let Σ_n be the Hirzebruch surface, i.e., the unique rational ruled surface with section S_n of self-intersection $-n = S_n^2$. Let F be the class of a fiber (for more on Σ_n see [18], [21], [73], or §1.2 below).

Definition 0.3.2. A divisor D on Σ_n homologous to $aS_n + bF$ is called a *divisor of type* (a, b).

We consider surfaces lying on the line L_1 introduced in the last section. These have been thoroughly studied by Horikawa ([39], [40]). He proves:

Theorem 0.3.3. *Let X be a minimal surface of general type with c_1^2 even and $c_1^2 = 2p_g - 4$ $(= \frac{1}{5}c_2 - \frac{36}{5})$. Then the* 1*-canonical map φ_K is a double cover of a surface Y of degree $p_g - 2$ in $\mathbf{P}^{p_g - 1}(\mathbf{C})$. The minimal resolution of Y, Y' is either $\mathbf{P}^2\mathbf{C}$ or a Hirzebruch surface Σ_n. X is the minimal resolution of a double covering $\pi \colon X' \to Y'$, branched along a curve of type* $(6, *)$ *if*

$Y' = \Sigma_n$ or a plane curve of degree 8 or 10 if $Y' = \mathbf{P}^2(\mathbf{C})$, with simple singularities only.

For each fixed c_1^2, there are finitely many possibilities for Y' and a finite number of possibilities for the type of branch curve. Conversely, starting from Y' and a divisor of type $(6, *)$ if $Y' = \Sigma_n$ or a plane curve of degree 8 or 10 if $Y' = \mathbf{P}^2(\mathbf{C})$ with only simple singularities as listed in 0.3.1, the minimal resolution of the double cover $X' \to Y'$ is a minimal surface of general type with $c_1^2 = 2p_g - 4$.

Similar statements are true also for the case of c_1^2 odd, described in [39] and [40] as well as in [4].

Discussion 0.3.4. These theorems give as a dictionary (let B denote the branch curve on Y')

$$\left\{\begin{array}{c} X, c_1^2(X) = 2p_g(X) - 4 \\ = k \\ c_2(X) = 5k + 36 \end{array}\right\} \leftrightarrow \left\{\begin{array}{l} \text{data } Y', \text{ type} \\ \text{of } B\colon (a, b) \end{array}\right\},$$

so that fixing $c_1^2 = k$ on the line L_1 of (0.1.6), the set of all X minimal general type with $c_1^2(X) = k$, $c_2(X) = 5k + 36$ is a finite set parametrized by the data: $\{Y'$, the pair $(6, *)$ or the degree 8 or 10 as the case may be, and the singularities as in 0.3.1 which the branch locus $B \subset Y'$ has$\}$. The correspondence is given by the 1-canonical map, which turns out to be a double cover. The utility of this method is immediately apparent: by varying the data $\{Y'$, type $(6, *)$ or degree, singularity types$\}$ on the right-hand side we may be able to vary the constants (c_1^2, c_2) on the left-hand side of the dictionary.

Examples 0.3.5. Before proceeding further we give two examples of Horikawa surface which are easy to describe.

A. Let $B \subset \mathbf{P}^2$ be a smooth octic curve and $X \xrightarrow{\pi} \mathbf{P}^2$ the corresponding double cover. Then $c_1(X) = \tau^*(3[H] - \frac{8}{2}[H]) = \tau^*(-H)$, so

$$c_1^2(X) = 2 \cdot H^2 = 2.$$

An elementary calculation ($c_2(X) =$ Euler-Poincaré characteristic) shows that

$$c_2(X) = 2\left(3 - \tfrac{1}{2}(-40)\right) = 46,$$

so $p_g(X) = \chi(X, \mathcal{O}_X) + q - 1 = (c_1^2 + c_2)/12 - 1 = 3.$

It follows that $\pi\colon X \to \mathbf{P}^2$ is the 1-canonical map,

$$\begin{aligned} \pi\colon X &\mapsto \mathbf{P}^2 \\ x &\mapsto [\omega_1(x) : \omega_2(x) : \omega_3(x)], \end{aligned}$$

where ω_i is a base of holomorphic 2-forms.

B. Let $Q \subset \mathbf{P}^3$ be a quintic hypersurface. By adjunction

$$c_1(Q) = (4-5)[H]_{|Q},$$

where $[H]$ is the hyperplane class in $\mathbf{P}^3$. It follows that

$$c_1^2(Q) = [-H]_{|Q}^2 = 5,$$

while by adjunction

$$c_2(Q) = c_2(\mathbf{P}^3)_{|Q} - c_1(Q)_{|Q} = 6 \cdot 5 - (-5 \cdot 5) = 55,$$

so $p_g = 4$ and it is readily seen that

$$\begin{gathered} Q \subset \mathbf{P}^3 \\ x \mapsto [\omega_1(x) : \omega_2(x) : \omega_3(x) : \omega_4(x)] \end{gathered}$$

is the 1-canonical map (which may have rational double points).

1. Persson's construction and generalizations

1.1. Double coverings. We start with some general remarks on the singularities of double coverings, which will be applied to construct surfaces with desired Chern numbers. Let $X \xrightarrow{\pi} Y$ be a double cover with branch locus $B \subset Y$. The covering π is necessarily Galois with Galois group $\mathbf{Z}_2$; X has an involution, and the ramification locus $R \subset X$ is its fix point set. Set theoretically we have $\pi^{-1}(B) = R$. X is nonsingular if and only if B is nonsingular; if B has a singularity given locally by $f(x,y) = 0$ at $p \in B$, then X has a singular point with equation $w^2 + f(x,y) = 0$ at $\pi^{-1}(p) \in R$.

Definition 1.1.1. A singular point $q \in R$ of the double cover $X \to Y$ is called a *simple* (inessential) *singularity* if for X' the resolution of singularities

$$\begin{array}{ccc} X' & \xrightarrow{\zeta} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow{\zeta} & Y \end{array}$$

induced by the resolution $Y' \to Y$ of the branch locus $B \subset Y$ we have

$$c_1^2(X') = \{\pi^*(c_1(Y) - \tfrac{1}{2}[B])\}^2, \qquad c_2(X') = \pi^*(c_2(Y) - \tfrac{1}{2}c_1(B)),$$

that is, with no additional corrections to c_1^2 and c_2 induced by the singular point $q \in R \subset X$.

Persson has proven this to be equivalent to the earlier used definition. More precisely:

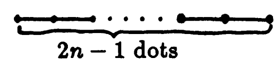
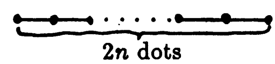
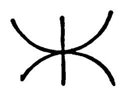
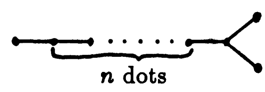
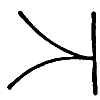
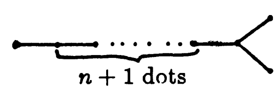
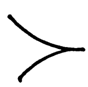
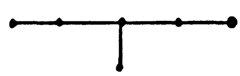
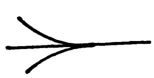
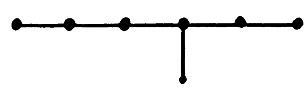
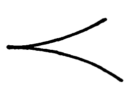
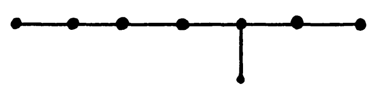
Proposition 1.1.2. *$R \subset X$ has only inessential singularities iff:*

(a) *$p = \pi(q) \in B$ is a double point (or arbitrarily high order) or a triple point, not all three branches of which are tangent*

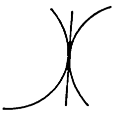
$\Leftrightarrow$

(b) *the corresponding surface singularity is a rational double point.*

The following table gives the translation between (a)$\Leftrightarrow$(b):

singularity of B	equation	name	resolving graph on X
	$x^2 + y^{2n}$	A_{2n-1}	
	$x^2 + y^{2n+1}$	A_{2n}	
	$x^2y + y^{2n+3}$	D_{2n+4}	
	$x^2y + y^{2n+4}$	D_{2n+5}	
	$x^3 + y^4$	E_6	
	$x^3 + xy^3$	E_7	
	$x^3 + y^5$	E_8	

The simplest type of nonsimple (essential) singularity is an infinitely near triple point, which corresponds to a singularity of B of the following kind:



$x^3 + y^3$ $\tilde{E}_6$ simple elliptic $o - 1$ elliptic curve.

Proposition 1.1.3. *Let $\bar{X} \to X$ be the minimal resolution of singularities of a branched cover $X \to Y$ with one infinitely near triple point. The Chern numbers of $\bar{X}$ are easily related to those of X (where $c_1(X) = \pi^*(c_1(Y) - \frac{1}{2}[B])$, $c_2(X) = \pi^*(c_2(Y) - \frac{1}{2}c_1(B)))$* [73, *Proposition* 1.11]*:*

$$c_1^2(\bar{X}) = c_1^2(X) - 1, \qquad c_2(\bar{X}) = c_2(X) - 11.$$

Thus introducing infinitely near triple points into the branch locus gives us a tool for changing the *ratio* c_1^2/c_2. (The fact that c_2 changes by 11 is due to the fact that the smallest possible change for $c_1^2 + c_2$ is 12.)

1.2. Ruled surfaces. The surface Y to be used in the upcoming constructions will be a ruled surface. We begin by reviewing the necessary facts. (See also [18], [21], [73].)

A. Rational ruled surfaces Σ_n (Hirzebruch surface). A (geometrically) ruled surface is a surface Y with a projection morphism $\pi: Y \to C$ onto a curve C, such that $\pi^{-1}(x)$ is a rational curve for all $x \in C$, and no fiber has self-intersection (-1). Y is called *rationally ruled* if $C = \mathbf{P}^1$ is a projective line. Rationally ruled surfaces are characterized by a natural number n, such that there is a unique section S_n of $\Sigma_n \to \mathbf{P}^1$ with $S_n^2 = -n$ (the infinite section). The corresponding surface is called the Hirzebruch surface Σ_n. If F denotes the homology class of a fiber, then S_n and F generate the divisor group modulo linear equivalence. The zero section of $\Sigma_n \to \mathbf{P}^1$ (which is not unique) has class $S_0 = S_n + n \cdot F$, $S_0^2 = S_n^2 + 2nS_n \cdot F + n^2F^2 = -n + 2n = n$. A divisor $D \subset Y$ which has class $D = aS_n + bF$ is called of type (a, b).

Σ_N is a two-fold cover of Σ_{2N} branched over S_0 and S_{2N} on Σ_{2N}. To describe this neatly we change notation: S_0^N, S_∞^N are the curves denoted S_0 and S_N above on Σ_N. Then we have

$$\pi: \Sigma_N \to \Sigma_{2N},$$
$$(S_\infty^N)^2 = \left(\pi^*\left(\tfrac{1}{2}S_\infty^{2N}\right)\right)^2 = 2\tfrac{1}{4}(S_\infty^{2N})^2 = -N,$$
$$(S_0^N)^2 = \pi^*\left(\left(\tfrac{1}{2}S_0^{2N}\right)\right)^2 = 2\left(\tfrac{1}{4}(S_0^{2N})\right) = N.$$

The factor $\frac{1}{2}$ is because $\pi^*(S_0^{2N}) = 2S_0^N$, so $(\pi^*(S_0^{2N}))^2 = 2 \cdot (S_0^{2N})^2 = 4(S_0^N)^2$.

B. Irrational ruled surfaces. The notation Σ_n^{q+1} will be used for an irrational ruled surface $\Sigma_n^{q+1} \to C$, where C is a curve with genus q, and a section S of $\Sigma_n^{q+1} \to C$ has $S^2 = n$. We say a curve $\Delta \subset \Sigma_n^{q+1}$ with

$\Delta \sim aS + bF$ has class (a, b) (note the difference of the type on a rational ruled surface).

1.3. Theorems on geography.

1.3.1. Strategy. Persson's results are proven in two steps.

Step 1. Construct double covers $X \to Y$, $Y = \Sigma_N^q$, with branch locus $D = (2a, 2b)$ a smooth divisor on Y. We get a family of surfaces $X(q, N, a, b)$ depending on the constants q, N, a and b, such that the Chern numbers $c_1^2(X(q, N, a, b))$, $c_2(X(q, N, a, b))$ are sufficiently spread out in the $\mathbf{Z}^2$-plane.

Step 2. Introduce infinitely near triple points to D, while leaving it in the same class, to "fill in" the gaps left in Step 1.

Theorem 1.3.2 [73, *Theorem* 2]. *Let* $(x, y) \in \mathbf{Z}^2$ *be in the region* D_2 *of* (0.1.6), $\frac{1}{5}y - \frac{36}{5} \leq x \leq 2y$, *and* $x \neq 2y - 3k$, $k \in \{2, 1, 3, \cdots, 13, 15, 19\}$. *Then there exists a minimal surface* X *of general type, such that*

$$c_1^2(X) = x, \qquad c_2(X) = y.$$

X *may be taken to be a genus two fibration.*

Proof Idea. We outline the proof utilizing Steps 1 and 2 as above.

Step 1. $X_{q,b}^N$ is defined to be the double cover of Σ_N^q, branched along a smooth, irreducible $D \subset X$ with $D \approx 6S + 2bF$. (Because $a = 3$, we are a priori constructing genus two fibrations.) $X_{q,b}^N$ has Chern numbers

$$c_1^2(X_{q,b}^N) = 8(q-1) + 4b + 6N,$$
$$c_2(X_{q,b}^N) = 4(q-1) + 20b + 30N.$$

Here, $c_1^2 = 2c_2 - 36b - 54N = 2c_2 - 18(2b + 3N)$ so any $(x, y) \in \mathbf{Z}^2$, $y = 2x - 18p$, $y \geq \frac{1}{5}x - \frac{36}{5}$, is realized as the Chern numbers of the genus two fibration $X_{q,b}^N$, $p = 2b + 3N$. This completes Step 1.

Step 2. Suppose we introduce k infinitely near triple points into the branch locus D. Then [73, Lemma 4.1.5]:

$$c_1^2 = 2c_2 - 18p + 7k,$$

that is, each singularity raises c_1^2 by 7. Of course, the question as to whether such singular D exists is far from trivial. But to finish the proof of Theorem 1.3.2, the following result suffices:

Lemma 1.3.3 [73, 4.1.6]. *For sufficiently large* b ($2b + 3N \geq 21$ *suffices*) *and any* k, $1 \leq k \leq 5$, *there exists a* D *of class* $(6, 2b)$ *with exactly* k *infinitely close triple points and no other essential singularities.*

Proof of 1.3.3. Let $C \to \mathbf{P}^1$ be a double covering branched at $2(q+1)$ points, so C is a curve of genus q. Consider the following diagram:

$$\begin{array}{ccc} \Sigma^q_{2N} & \xrightarrow{\pi} & \Sigma_N \\ \downarrow & & \downarrow \\ C & \longrightarrow & \mathbf{P}^1 \end{array}$$

where Σ_N is the Hirzebruch surface (1.2), and $\Sigma^q_{2N} \to \Sigma_N$ is the fiber product. Let $\tilde{S}_0 = \pi^*(S_0)$ and $\tilde{S}_\infty = \pi^*(S_\infty)$ be the pullbacks of the zero and infinite section,[2] respectively. Since $\tilde{S}_\infty \sim \tilde{S}_0 - 2NF$, the union $\tilde{S}_\infty \cup \tilde{S}_0 \sim 2\tilde{S}_0 - 2NF$ is even, and there is a double cover (for the precise statement see §4.1.2)

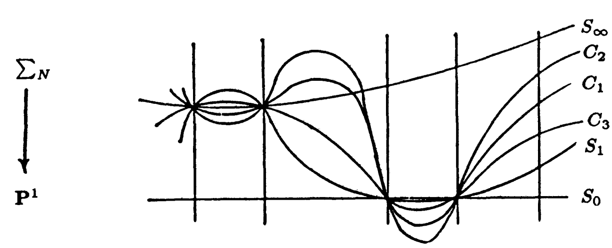
$$\Sigma^q_N \xrightarrow{\pi'} \Sigma^q_{2N}$$

branched at $\tilde{S}_0$ and $\tilde{S}_\infty$. We can now construct the divisor D on Σ^q_N as $\pi'^*\pi^*(B)$ for an appropriate $B \subset \Sigma_N$. This is done as follows: choose an irreducible class S_1 on Σ_N of type $(1, c)$ intersecting S_0 and S_∞ transversely. S_1 intersects S_0 in $c+N$ distinct points, and it intersects S_∞ and N points, no two of which are in the same fiber of $\Sigma_N \to \mathbf{P}^1$. Let S_2 be a (reducible) class of type $(1, c)$, such that

(i) if $k \le c$, S_2 is the union {a section S not passing through any of the intersection points $S_1 \cap S_0$}$\cup${k fibers through intersections $S_1 \cap S_\infty$}$\cup${$c-k$ generic fibers}.

(ii) if $k \ge c$, S_2 is the union {S_∞} $\cup$ {$k-c$ fibers through any of the intersection points $S_1 \cap S_0$} $\cup$ {$2c+N-k$ generic fibers}.

S_1 and S_2 generate a pencil (one-dimensional linear system) of divisors of type $(1, c)$ with exactly k basepoints (every divisor in the pencil passes through these points) on the union of S_0 and S_∞. Choose three distinct generic curves C_1, C_2 and C_3 in this pencil; the union $B = C_1 \cup C_2 \cup C_3$ has exactly k ordinary triple points, i.e.,



[2]Henceforth, S_∞ will denote the curve S^N_∞ (infinite section, $S^2_\infty = -N$), and S_0 the curve S^N_0 (with $S^2_0 = N$) of 1.2.A above.

lifting C_1, C_2 and C_3 to Σ_{2N}^q, we get three curves which have $2k$ ordinary triple points, all of which lie on the branch locus of

$$\Sigma_N^q \xrightarrow{\pi'} \Sigma_{2N}^q.$$

A local calculation shows that $\pi'^*\pi^*(B) = D$ is now a union of three curves which have $2k$ infinitely near triple points (an ordinary triple point on the branch locus becomes an infinitely near one on the double cover, compare [73, 1.2.2]), and D has class $(6, 2c)$.

This can be further modified to yield divisors D of class $(6, 2b)$ with an odd number of infinitely near triple points [73, p. 40 bottom].

Thus we get examples of surfaces with $c_1^2 = 2c_2 - k$, with the exception of $k = 2, 6, 10, 14, 1, \cdots, 15, 19, 23, 27$ or 31. The cases $k = 6, 10, 14, 23, 27$ or 31 are constructed by ad-hoc methods in [73]. This completes the proof of 1.3.2.

Simple connectivity. As mentioned above, after answering the question above it is natural to ask which properties (other than given Chern invariants) the constructed surfaces have. In particular, one would like to know if, for given $(x, y) \in \mathbf{Z}^2$ there exists a *simply connected* minimal surface of general type X with $c_1^2(X) = x$, $c_2(X) = y$. In this respect, Persson states the following conjecture, due to Bogomolov [73, p. 6]:

Conjecture 1.3.4. If X is simply connected, then $c_1^2 \leq 2c_2$.

As we will see below, this conjecture is false, as was first proven by Moišhezon, who showed that one of Miyaoka's examples below (§2) with $c_1^2 > 2c_2$ is simply connected.

Persson's main result about simply connected surfaces is:

Theorem 1.3.5 [73, *Theorem* 3, *p.* 45]. *Let* $(x, y) \in \mathbf{N}^2$ *with*

$$y - 24 \leq 2x, \qquad x \leq 2y - 2\frac{9}{3\sqrt{12}}(x+y)^{2/3}.$$

Then there exists a simply connected minimal surface of general type X, *with* $c_1^2(X) = x$, $c_2(X) = y$.

Before going into the construction used we first state the basic result which is used to show simple connectivity of the surfaces, a result well known to topologists.

Lemma 1.3.6 [73, *Lemma* 3.20]. *Let* X *be a real* 4*-dimensional manifold with a fibration* $\pi\colon X \to B$ *onto a real two-dimensional simply connected manifold, with path connected fibers. Assume that there are no multiple fibers, and that there exists at least one simply connected fiber. Then* X *is simply connected.*

Proof Idea of 1.3.5. Once again the proof follows the two steps described above.

Step 1. We construct a family of surfaces $X(a,c,d)$ as follows: Let $D_1 \subset \mathbf{P}^1 \times \mathbf{P}^1$ and $D_2 \subset \mathbf{P}^1 \times \mathbf{P}^1$ be two divisors of types $(2a, 2)$ and $(2c, 2d)$, respectively, let

$$\pi_1 \colon Y \to \mathbf{P}^1 \times \mathbf{P}^1$$

be the double cover branched along D_1, and let $\pi \colon X(a,c,d) \to Y$ be the double cover branched along $\pi_1^*(D_2)$. The Chern numbers are given by [73, Proposition 1.28]:

$$(1.3.7) \qquad \begin{aligned} c_1^2 &= 8((a+c)-2)(d-1), \\ c_2 &= 4((a+c)-2)(d-1) + 12(cd+a). \end{aligned}$$

Without loss of generality we may assume D_2 to contain fibers, so $X(a,c,d)$ will be simply connected by 1.3.6.

Before proceeding to Step 2, we briefly explain how to determine the set of $(x,y) \in \mathbf{Z} \oplus \mathbf{Z}$ which can be obtained by means of (1.3.7). Fix d; then (1.3.7) for c_1^2 and c_2 becomes *linear* in a and c, yielding a linear map

$$\begin{aligned} \varphi_d \colon \mathbf{Z} \times \mathbf{Z} &\to \mathbf{Z} \times \mathbf{Z} \\ (a,c) &\mapsto (c_1^2, c_2). \end{aligned}$$

The (a,c)-lattice is mapped to a sublattice of the (c_1^2, c_2)-lattice. In fact, [73, p. 45] the image of φ_d is a sublattice of co-area $8(d-1)^2$, and (x,y) in the (c_1^2, c_2)-lattice is in the image of the first quadrant if and only if [73, 4.2.2]

(1.3.8)

$$2 \cdot \frac{d-1}{4d-1} \cdot y - \frac{24(d-1)(d+1)}{(4d-1)} \leq x \leq 2\frac{d-1}{d+2} \cdot y - \frac{24(d-1)(d+1)}{d+2}.$$

Step 2. Now add infinitely near triple points to $\pi_1^*(D_2)$.

Lemma 1.3.9 [73, 4.2.3]. *On $\pi_1^*(D_2) \subset Y$ we can introduce exactly k infinitely near triple points (and no other essential singularities) for $0 \leq k \leq 8(d-1)^2$, if*

$$c \geq 9(d-1) + 2, \qquad 3a \geq 8(d-1)^2.$$

This is done in the same way as in 1.3.3, utilizing the composition

$$X \xrightarrow{\pi} Y \xrightarrow{\pi_1} \mathbf{P}^1 \times \mathbf{P}^1$$

of double covers. Find $D \subset \mathbf{P}^1 \times \mathbf{P}^1$ with k triple points, linearly equivalent to D_2, with all triple points lying on the branch locus of π_1. The inequalities of 1.3.9 are sufficient to construct such a D. Take D to consist of three disjoint curves of class (c^1, d^1) with k triple points, where $2c-2 \leq 3c^1+1 \leq 2c$ and $2d-2 \leq 3d^1 \leq 2d$, plus a sufficient number of horizontal and vertical fibers (at least one vertical fiber) to give it the right class. We have to show that the branch locus of π_1 can be chosen to pass through the k triple points of D, then $\pi_1^*(D)$ will have k infinitely near triple points. Under the assumptions of the lemma such a branch divisor $S \subset \mathbf{P}^1 \times \mathbf{P}^1$ of class $(2a, 2)$ can be found passing through the k triple points by [73, 4.2.9].

Now Y fibers over $\mathbf{P}^1$ with fiber a curve C which is a double cover of $\mathbf{P}^1$ branched over $2a$ points, and $\pi_1^*(D)$ contains fibral components. It follows that $X(a,b,d)_k$ just constructed fibers over $\mathbf{P}^1$ with at least one simply connected fiber, so by 1.3.6, it is simply connected.

What remains is to determine which (x,y) in the (c_1^2, c_2) lattice are covered by this construction. Studying (1.3.7) and (1.3.8) yields the inequalities stated in 1.3.5 [73, p. 47].

1.4. Theorems on the Picard number. In the next two sections we present generalizations of Persson's original work, constructions aimed at yielding different results. In this section we discuss the construction of Persson [74] designed to study the Picard number of double coverings. In the final section we discuss a construction due to G. Xiao, described in [8], which gives many examples of simply connected minimal surfaces of general type with positive index, thus yielding counterexamples to the conjecture 1.3.4. Both of these generalizations use the same strategy as in §1.3, but concentrate on different types of singularities of the branch locus.

The Picard number. The Picard number ρ of an algebraic variety is a subtle arithmetic invariant, defined to be the $\mathbf{Q}$-rank of the Neron-Severi group NS = {divisors modulo numerical equivalence} $\otimes \mathbf{Q}$. By the Hodge decomposition for an algebraic surface S

$$H^2(S,\mathbf{C}) = H^{2,0}(S) \oplus H^{1,1}(S) \oplus H^{0,2}(S),$$
$$H^2(S,\mathbf{C}) \cap H^{1,1}(S) \simeq \mathrm{NS}(S),$$

so if $p_g(S) = 0$ then $H^2(S,\mathbf{C}) = H^{1,1}(S)$ and $\rho = b_2$, the second Betti number, and is a topological invariant. But when $p_g(S) > 0$, ρ is not constant in deformation families, due to the existence of transcendental cycles. This suggests:

Problem 1.4.1. Given a family of surfaces, find members of the family with maximal Picard number $\rho = h^{1,1}$.

There is in general no solution to 1.4.1; easy counterexamples are the ball quotients constructed by Livné [58] and discussed in §4.1 below. They have $\rho < h^{1,1}$, while ball quotients are rigid [7], i.e., have no deformations at all.

The following two theorems are proved in [74].

Theorem 1.4.2. *Let X be a Horikawa surface (i.e., $c_1^2 = 2p_g - 4$). If $c_1^2 + c_2 \not\equiv 0$ (72), then X has a deformation with maximal Picard number.*

Theorem 1.4.3. *For every even integer $m = 2n$ there exists a plane curve of degree m such that the (minimal resolution) of the double cover has maximal Picard number.*

This gives a solution to the problem for these two (very special) families of surfaces. The idea of proof is now slightly different than in §1.3. Whereas there we disregarded simple singularities and concentrated on infinitely near triple points (driving the ratio c_1^2/c_2 up), we now wish to fix c_1^2 and c_2, and studying simple singularities yields results on ρ. In fact

1.4.4. A χ_n singularity ($\chi = A$, D or E) of the branch curve yields a rational double point upstairs, whose resolution contributes exactly n (independent) cycles to the Neron-Severi group of the covering.

Let C be a curve on Y with singularities of type A, D or E as above, and let $\sigma(C) = \sum m_i$, where the singularities of C are of types χ_{m_i}. Assume $E_1, \cdots, E_n$ is a collection of numerically independent rational curves, *disjoint* from the branch locus, such that the restriction of the intersection form on their span is negative definite. Then [74, Proposition 1.5]: $\rho(\tilde{X}) \geq \rho(Y) + \sigma(C) + n$, where $\tilde{X}$ is the minimal resolution of singularities of the double cover $X \to Y$, branched along C. The following then makes sense:

Definition 1.4.5. A curve $C \subset Y$ with only χ_m-singularities is called *maximizing* $\Leftrightarrow \sigma(C) = h^{1,1}(\tilde{X}) - \rho(Y) - n$.

If C is maximizing, then the desingularization $\tilde{X}$ of the double cover X has maximal Picard number.

We set $Y = \mathbf{P}^2$ ($n = 0$); Theorem 1.4.3 above is then an immediate consequence of

Lemma 1.4.6. *For every even integer $m = 2n$ there is a maximizing curve $C_m \subset \mathbf{P}^2$ with degree m.*

Proof. We use the following construction, which is a universal one for constructing hypersurfaces in $\mathbf{P}^n$ with special kinds (and lots) of singularities. Let $[x_0 : \cdots : x_n]$ be a set of homogeneous coordinates on $\mathbf{P}^n$, and let

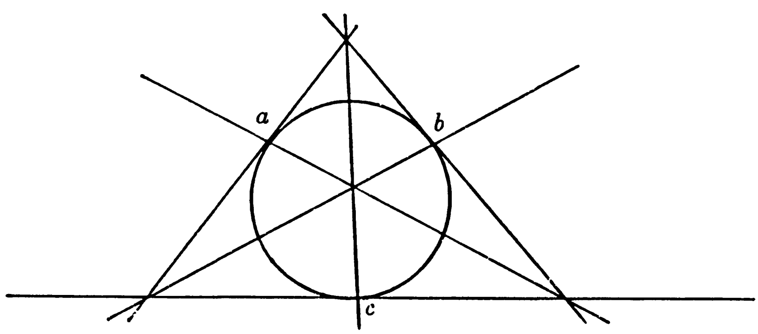
$$\varphi_d : \mathbf{P}^n \to \mathbf{P}^n$$
$$[x_0 : \cdots : x_n] \to [x_n^d : \cdots : x_n^d] = [Z_0 : \cdots : Z_n]$$

be the canonical nth power map. A hyperplane H in $\mathbf{P}^n$ given by $a_0Z_0 + \cdots + a_nZ_n = 0$ is covered under φ_d by a Fermat hypersurface of degree d:

$$a_0x_0^d + \cdots + a_nx_n^d = 0.$$

Since H meets the coordinate axi transversely, $\varphi_d^{-1}(H)$ will be smooth, tangent (to order d) at the inverse images of the intersections of H and the coordinate axis.

Now consider the following arrangements of lines in $\mathbf{P}^2$:



Let a, b, c, denote the 2-fold points of this arrangement, and let C be the conic tangent to the coordinate axis at a, b and c, as pictured. Let

$$F_d = \varphi_d^{-1}(D) = \{(x_0^d + x_1^d + x_2^d)^2 - 4(x_0^dx_1^d + x_1^dx_2^d + x_0^dx_2^d) = 0\}.$$

F_d has $3d$ A_{d-1} singularities, arranged d by d on the coordinate axis, which are the inverse images of a, b, and c on C. (If C is given locally near a by $y - x^2 = 0$, then on F_d it has the form $y^d - x^2 = 0$.) Now add two coordinate planes:

$$C_d := x_0 \cdot x_1F_d = x_0 \cdot x_1\{(x_0^d + x_1^d + x_2^d)^2 - 4(x_0^dx_1^d + x_1^dx_2^d + x_2^dx_0^d)\}.$$

Claim. *C_d has*
(i) 1 A_1,
(ii) $2d$ D_{d+2},
(iii) d A_{d-1}, *singularities, i.e.,*

$$\sigma(C_d) = 1 + 2d(d+2) + d(d-1) = 3d(d+1) + 1.$$

Proof. The A_{d-1} singularity of F_d is a D_{d+2} of C_d if it lies on $x_0 = 0$ or $x_1 = 0$, while it remains an A_{d-1} if it lies on $x_2 = 0$. The A_1 singularity is the point $x_0 = x_1 = 0$.

To show that C_d is maximizing, we must calculate $h^{1,1}(\bar{X}_d)$, where $\bar{X}_d \to X_d$ is the desingularization of the double cover of $\mathbf{P}^2$ branched along C_d. Since X_d only has rational double points, by 1.1.1, the invariants c_1^2 and c_2 of X_d will be the same as for a double cover branched along a smooth curve of degree $2n = 2(d+1)$. $h^{1,1}(\bar{X}_d)$ can be calculated from these, and by the method used in example 0.3.5.A, one gets

$$\begin{aligned} c_1^2(\bar{X}_d) &= 2(n-3)^2, \\ c_2(\bar{X}_d) &= 4n^2 - 6n + 6, \\ h^{1,1}(\bar{X}_d) &= 3n(n+1) + 2. \end{aligned}$$

It follows that $\sigma(C_d) = 3n(n+1) + 1 = h^{1,1}(\bar{X}_d) - \rho(\mathbf{P}^2)$, i.e., C_d is maximizing.

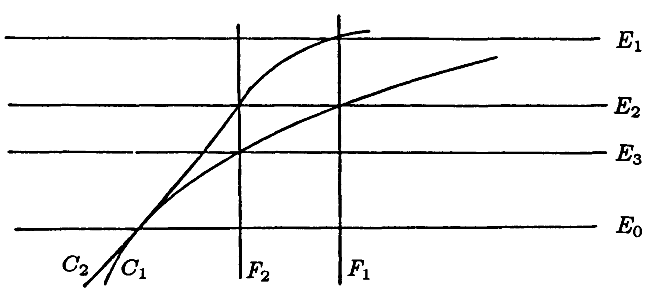
We now turn to the result on Horikawa surfaces, 1.4.2. First note that for a Horikawa surface, since $c_1^2 = 2p_g - 4$ and $c_2 = 2 - 4q + 2p_g + h^{1,1}$, Noether's formula $12\chi = c_1^2 + c_2$ yields

$$\begin{aligned} 12\chi &= 2p_g - 4 + 2 - 4q + 2p_g + h^{1,1} \\ &= (4p_g - 4q + 4) - 6 + h^{1,1}, \end{aligned}$$

or $h^{1,1}(X) = 8\chi + 6$. ($\chi(S) = p_g(S) - q(S) + 1$ is the arithmetic genus.) The method of construction differs with the modality of χ. As an example, we describe the construction for $\chi \equiv 1$ (2).

Theorem 1.4.7. *If X is a Horikawa surface with $\chi(X) \equiv 1$ (2), there is a deformation of X with maximal Picard number.*

Sketch of Proof. On $\mathbf{P}^1 \times \mathbf{P}^1$ consider the following configuration of curves:



where E_i is of type $(1,0)$, F_i is of type $(0,1)$ and C_i is of type $(1,1)$. Set $C = C_1 \cup C_2 \cup E_0 \cup E_1 \cup E_2 \cup E_3$. C has a D_6 and 6 A_1 singularities. Take the fiber product

$$\begin{array}{ccc} \mathbf{P}^1 \times \mathbf{P}^1 & \longrightarrow & \mathbf{P}^1 \times \mathbf{P}^1 \\ \downarrow & & \downarrow \\ \varphi_\kappa \colon E_0 & \longrightarrow & E_0 \end{array}$$

where $\varphi_\kappa \colon E_0 \to E_0$, $\varphi_\kappa(x_0 : x_1) = (x_0^\kappa : x_1^\kappa)$ is the natural κth power map on the projective line E_0. $C_\kappa := \varphi_\kappa^{-1}(C)$ is a curve of type $(6, 2\kappa)$ on $\mathbf{P}^1 \times \mathbf{P}^1$ with κ D_6 singularities, 4 $A_{2\kappa-1}$ and 2κ A_1-singularities. In other words, $\sigma(C_\kappa) = 6\kappa + 4(2\kappa - 1) + 2\kappa = 16\kappa - 4$ while $h^{1,1}(\tilde{X}) = 8(2\kappa - 1) + 6 = 16\kappa - 2$, where $\tilde{X} \to X$ is the desingularization of the double cover $X \to \mathbf{P}^1 \times \mathbf{P}^1$ branched along C_κ. It follows that C_κ is maximizing.

The two other cases to be considered are $\chi \not\equiv 0$ (3) and $\chi \equiv 3$ (4) [74, Propositions 4.6.–4.7].

1.5. Xiao's generalization. We now discuss a generalization of Persson's Theorem 1.3.5 above, a construction method due to G. Xiao and described in [8] by Zhijie Chen. The main result is Theorem 1.5.16, which yields examples of minimal general type surfaces of positive index which are simply connected. This construction uses several types of essential singularities. We begin with a brief discussion of these.

Essential singularities. Let $X \to Y$ be a double cover with branch locus $B \subset Y$. Let $p_1, \cdots, p_s$ be the set of singular points of B. There is a canonical resolution of singularities (which need not equal the minimal resolution) induced by resolving the branch locus B in Y, $\tilde{Y} \to Y$, taking the fiber product

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \tilde{Y} & \longrightarrow & Y \end{array}$$

and then normalizing. Let d_i =multiplicity of $P_i \subset B$ and let $m_i = [d_i/2]$ be the greatest integer not exceeding $d_i/2$. Formulas for the Chern numbers of $\tilde{X}$ are easily calculated to be

$$c_1^2(\tilde{X}) = 2\left(c_1(Y) - \tfrac{1}{2}[B]\right)^2 - 2\sum(m_i - 1)^2,$$
$$c_2(\tilde{X}) = 2\left(c_2(Y) - \tfrac{1}{2}c_1(B)\right) - 2\sum(m_i - 1)(2m_i + 1),$$

where $[B]$ is the fundamental class of $B \subset Y$.

Definition 1.5.1. The *specialization vector* of the singularity at P_i is

$$(a,b)=\left(2\sum(m_i-1)^2, 2\sum(m_i-1)(2m_i+1)\right).$$

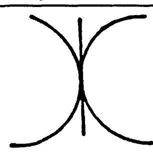
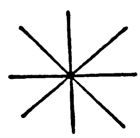
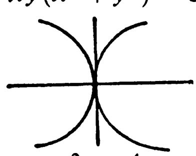
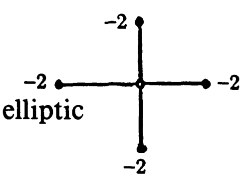
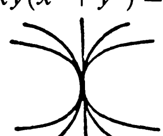
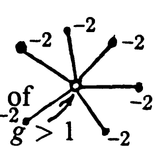
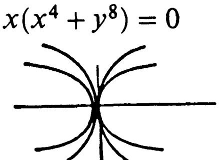
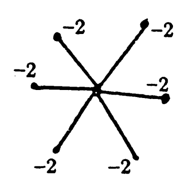
Thus,

$$c_1^2(\bar{X})=c_1^2(X)-a, \qquad c_2(\bar{X})=c_1^2(X)-b,$$

where symbolically, $c_1^2(X) = 2(c_1(Y)-\frac{1}{2}[B])^2$ and $c_2(X) = 2(c_2(Y)-\frac{1}{2}c_1(B))$.

We now list the essential singularities used in Xiao's construction.

1.5.2. List.

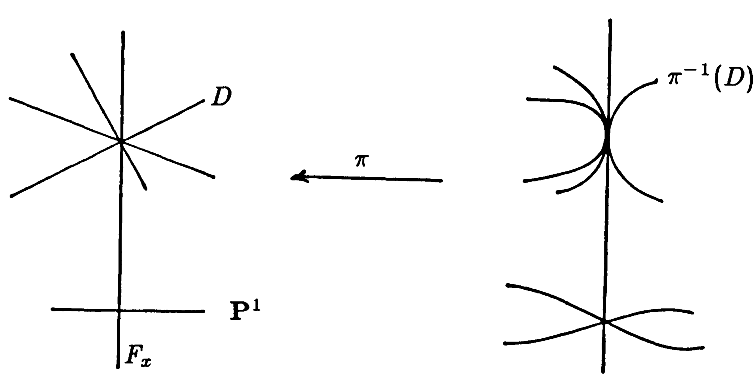
Type	Name	curve singularities	resolving divisor	specialization vector
(a)	infinitely near triple point	 $x(x^2+y^4)=0$	$\circ -1$ elliptic curve	$(1,11)$
(b)	ordinary 4-fold point	 $xy(x^2+y^2)=0$	$\circ -2$ elliptic curve	$(2,10)$
(c)	infinitely near 4-fold point	 $xy(x^2+y^4)=0$	 elliptic	$(2,10)$
(d)	infinitely near 5-fold point	 $x(x^4+y^8)=0$	curve of genus $g>1$ 	$(9,39)$
(e)	infinitely near 6-fold point	 $xy(x^4+y^8)=0$		$(10,38)$

1.5.3. Xiao's construction. Let $G \subset \mathrm{Aut}(\mathbf{P}^1)$ be a finite group acting on $\mathbf{P}^1$. This defines a divisor on $\mathbf{P}^1 \times \mathbf{P}^1$, $D := \{(x, \gamma(x)),\ \gamma \in G\}$. D is the sum of $|G|$ smooth curves of type $(1, 1)$. The fixed points of G on $\mathbf{P}^1$ fall into three conjugacy classes, with the order of the isotropy group of G at a fixed point in the ith conjugacy class $= a_i$ as given in the following table:

		$\|G\|$	a_1	a_2	a_3
Dihedral group	D_{2k} $(k \geq 2)$	$2k$	2	2	k
Tetrahedral group	T_{12}	12	2	3	3
Octahedral group	O_{24}	24	2	3	4
Icosahedral group	I_{60}	24	2	3	5

Therefore, if $d_i = |G|/a_i$, then D has d_1^2 points where a_1 components meet, d_2^2 points where a_2 components meet, and d_3^2 points where a_3 components meet. Viewing $\mathbf{P}^1 \times \mathbf{P}^1$ as a quadric hypersurface Q in $\mathbf{P}^3$, the configuration D of curves may be viewed as the intersection of Q with an arrangement of the appropriate numbers of planes in $\mathbf{P}^3$. For example, D for $G = O_{24}$ is the intersection of Q with $A_1^3(24)$, the arrangement defined by the group G_{576} acting on $\mathbf{P}^3$, the symmetry group of the regular 24-cell (in $\mathbf{R}^4$). (The intersection being "generic" in the sense that Q does not meet any of the singular points of $A_1^3(24)$, only singular lines.)

1.5.4. Introducing singularities. As in Step 2 of strategy 1.3.1, but now more subtle, we add singularities to D, take their lifts under a double cover and use it as the branch locus of a double cover. The idea is as follows: if $p \in D$ is a k-fold point of D, lying on the fiber F_x, $x \in \mathbf{P}^1$, then on a double cover p becomes infinitely near:



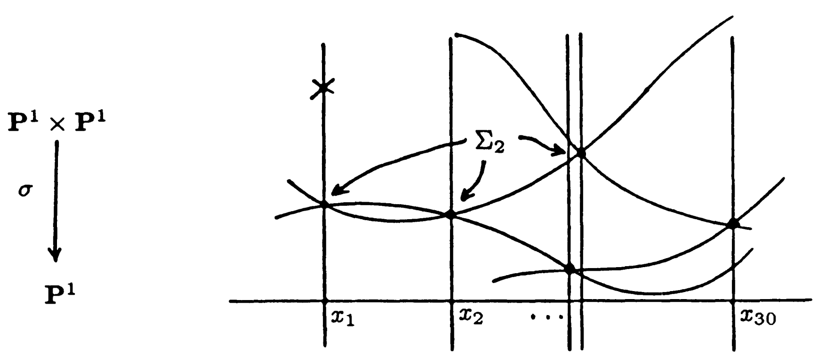
We now describe the construction of a simply connected surface of general type $X(k,n,s,t,r_1,r_2,r_3,r_4)$, or X for short, for integers k, n, s, t, r_1, r_2, r_3, r_4. X is defined by the following sequence of coverings where the individual maps are discussed below:

$$\begin{array}{ccccccccccccc} X & \xrightarrow{\phi} & S & \xrightarrow{\pi} & S_k & \xrightarrow{\pi_k} & S_{k-1} & \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_1} & S_0 & \xrightarrow{\pi_0} & \mathbf{P}^1 \times \mathbf{P}^1 \\ & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \sigma \\ & & & & \mathbf{P}^1 & \xrightarrow{p_k} & \mathbf{P}^1 & \xrightarrow{p_{k-1}} \cdots \xrightarrow{p_1} & \mathbf{P}^1 & \xrightarrow{p_0} & \mathbf{P}^1 \end{array}$$

1.5.5. The fiber square (defines dependency on n).

$$\begin{array}{ccc} \mathbf{P}^1 \times \mathbf{P}^1 = S_0 & \xrightarrow{\pi_0} & \mathbf{P}^1 \times \mathbf{P}^1 \\ \downarrow & & \downarrow \sigma \\ \mathbf{P}^1 & \xrightarrow{p_0} & \mathbf{P}^1 \end{array}$$

Let $D \subset \mathbf{P}^1 \times \mathbf{P}^1$ be the configuration discussed above for the group I_{60} of order 60, and $\Sigma_2 \subset D$ the set of double points. We convert these 900 double points into $900n$ triple points. $p_0\colon \mathbf{P}^1 \to \mathbf{P}^1$ is an n-fold cover branched at two generic points (i.e., $p_1, p_2 \notin \sigma(\Sigma_2)$), and let $\pi_0\colon S_0 \to \mathbf{P}^1 \times \mathbf{P}^1$ be the fiber product as in the diagram above. Consider the divisor consisting of D and one fiber (of σ) through each of the points of $\Sigma_2 \subset D$:



Set $(x_1, \cdots, x_{30}) = \sigma(\Sigma_2)_{\text{red}} \subset \mathbf{P}^1$. Therefore, the divisor

$$S_0 \supset D_0 := \pi_0^* \left(D + \sum_{i=1}^{30} \sigma^*(x_i) \right)$$

is a curve of type $(90n, 60)$ ($= (90nS, 60F)$, $S = \sigma^*(x)$, $F = \sigma_2^*(p)$, $p \in \mathbf{P}^1$, generic (where $\sigma_2\colon \mathbf{P}^1 \times \mathbf{P}^1$ is the other projection) which has $900n$ triple points lying over those in the figure above, as well as $400n$ triple points lying over the 400 triple points of D, and $144n$ 5-fold points lying over those of D.

1.5.6. The fiber squares (for fixed $k \in \mathbf{N}$).

$$\begin{array}{ccc} S_j & \xrightarrow{\pi_j} & S_{j-1} \\ \downarrow & & \downarrow \\ \mathbf{P}^1 & \xrightarrow{p_j} & \mathbf{P}^1 \end{array} \qquad (j = 1, \cdots, k).$$

As above, set $\{x_1, \cdots, x_{30}\} = \sigma(\Sigma_2)_{\text{red}} \subset \mathbf{P}^1$, $\Sigma_3 \in D$ is the set of triple points (400), $\Sigma_5 \subset D$ is the set of 5-fold points (144), and $\{y_1, \cdots, y_{20}\} = \sigma(\Sigma_3)_{\text{red}}$, $\{z_1, \cdots, z_{12}\} = \sigma(\Sigma_5)_{\text{red}}$. $p_j\colon \mathbf{P}^1 \to \mathbf{P}^1$ is a double cover branched at $b_1^{(j)}$, $b_2^{(j)}$, where $b_1^{(j)}$ and $b_2^{(j)}$ are two randomly chosen points in

$$p_{j-1}^* p_{j-2}^* \cdots p_0^*(x_1, \cdots, x_{30}, y_1, \cdots, y_{20}, z_1, \cdots, z_{12}) \\ - \text{ inverse images of } \{b_\mu^{(l)}(l) < j\}$$

$\pi_j\colon S_j \to S_{j-1}$ is the fiber product (double covers branched along fibers over $b_1^{(j)}$ and $b_2^{(j)}$). This construction converts

(1.5.7)

(a) 30 triple point	to	30 infinitely near triple pts. if $b_i^{(j)} \in p_{j-1}^* \circ \cdots \circ p_0^*(x_1, \cdots, x_{30})$,
(b) 20 triple points	to	20 infinitely near triple pts. if $b_i^{(j)} \in p_{j-1}^* \circ \cdots \circ p_0^*(y_1, \cdots, y_{20})$,
(c) 12 5-fold points	to	12 infinitely near t-fold pts. if $b_i^{(j)} \in p_{j-1}^* \circ \cdots \circ p_0^*(z_1, \cdots, z_{12})$.

1.5.8. The double cover (introducing dependency on t, r_4).

$$\begin{array}{ccc} S & \xrightarrow{\pi} & S_k \\ & & \downarrow{\scriptstyle\varphi} \\ & & \mathbf{P}^1 \end{array}$$

Let $D_k \subset S_k$, $D_k = \pi_k^* \circ \cdots \circ \pi_1^*(D_0)$, and let

$$\left.\begin{array}{l} D = p_k^* \circ \cdots \circ p_1^*(\sigma(\Sigma_2))_{\text{red}} \\ T = p_k^* \circ \cdots \circ p_1^*(\sigma(\Sigma_3))_{\text{red}} \\ Q = p_k^* \circ \cdots \circ p_1^*(\sigma(\Sigma_5))_{\text{red}} \end{array}\right\} \quad \text{divisors on } \mathbf{P}^1.$$

We will now subtract off of $D+T+Q$ the points used as branch loci for the p_j, to get essential singularities of the desired kind. Set

$$R=\sum_{\mu=1,2}(p_k^*\circ\cdots\circ p_1^*(b_\mu^{(1)})+p_k^*\circ\cdots\circ p_2^*(b_\mu^{(2)})+\cdots+p_k^*(b_\mu^{(k)})),$$

and subtract off the inverse images of the $b_i^{(j)}$:

$$\bar{D}=D-(D\cap R),\quad \bar{T}=T-(T\cap R),\quad \bar{Q}=Q-(Q\cap R).$$

Set $2q_1=\deg(D\cap R)$, $2q_2=\deg(T\cap R)$ and $2q_3=\deg(Q\cap R)$. Then it is easy to see that

$$\begin{aligned}&\deg\bar{D}=2^k\cdot 30-2q_1, &&\deg\bar{T}=2^k\cdot 20-2q_2,\\ &\deg\bar{Q}=2^k\cdot 12-2q_3, &&q_1+q_2+q_3=2(2^k-1).\end{aligned}\tag{1.5.9}$$

$\pi\colon S\to S_k$ will be the double cover branched over the divisor

$$B=\sigma_0+\sigma_1+p^*(\theta),$$

where σ_0, σ_1 are sections transversal to p (type $(1,0)$), and $\theta\subset\mathbf{P}^1$ is the divisor defined as follows: $\bar{D}'=\{r_4 \text{ distinct points}\}\subset\mathbf{P}^1$,

$$\theta=\theta(t,r_4)=\bar{D}-\bar{D}'+\bar{T}+\bar{Q}+\xi,$$

where $\deg\xi=2t+r_4$, ξ generic. Therefore $\deg\theta=2^k\cdot 58+2t+4$ and $B\subset\mathbf{P}^1\times\mathbf{P}^1=S_k$ is of type $(2,2^k\cdot 58+2t+4)$. We let $\pi\colon S\to S_n$ denote the minimal resolution of singularities, and set

$$\mathscr{D}=\pi^*(D_k)\quad \text{type}(90n,2^k\cdot 60).$$

Since we have subtracted off the fibers of $\varphi\colon S_k\to\mathbf{P}^1$ lying over the $b_\mu^{(j)}$ in B, the infinitely near points constructed by 1.5.7 above do not lie on the branch locus, and so are doubly covered as they are. On the other hand, the other (not infinitely near; transversal) triple points and 5-fold points lie on the branch locus and are converted to infinitely near triple points and infinitely near 5-fold points, respectively. The result is summarized below.

(1.5.10)

on D_k	$\overset{\pi}{\longleftarrow}$	on $\mathscr{D}$
transversal {triple point, 5-fold point}	1-1	infinitely near {triple point, 5-fold point}
infinitely near {triple point, 5-fold point}	2-1	2 infinitely near {triple points, 5-fold points}

1.5.11. The double cover ϕ (introducing dependence on s, r_1, r_2, r_3).

$$\begin{array}{ccccc} X & \xrightarrow{\phi} & S & \xrightarrow{\pi} & S_k \\ & & & & \downarrow{\scriptstyle p} \\ & & & & \mathbf{P}^1 \end{array}$$

We form the divisor B_ϕ as follows:

$$B_\phi = \mathscr{D} + \pi^*(p^*(R_D + R_T + R_Q + \bar{D}' + \xi')),$$

where $\bar{D}' = \{r_4 \text{ distinct points}\} \subset \bar{D}$ as above, ξ' is reduced generic points, and

$$\begin{aligned} \deg \xi' &= 2s - r_1 - r_2 - r_3 - r_4, \\ R_D &= \{r_1 \text{ distinct points}\} \subset D, \\ R_T &= \{r_2 \text{ distinct points}\} \subset T, \\ R_Q &= \{r_3 \text{ distinct points}\} \subset Q. \end{aligned}$$

B_ϕ has the class of $\pi^*(90nS_0+(2^k\cdot 60+25)F)$. We define X to be the double cover branched along B_ϕ. X will be singular where B_ϕ has its singularities. From (1.5.7), (1.5.9) and (1.5.10), the kinds and numbers of singularities of B_ϕ can be determined. They are listed in the following table.

1.5.12. Singularities of B_ϕ.

Singularity type	location (on $\pi^{-1}(\cdots)$)	number of singularities
(a)	$\bar{D}', D-\bar{D}-R_D, \bar{T}$, $T-\bar{T}-R_T$	$2^k(1300n-30)$ $-(30n-1)(r_4+2r_1)-40nr_2$
(b)	$\bar{D}-\bar{D}'$	$2\cdot(30n-1)r_4$
(c)	R_D, R_T	$2(30n-1)r_1+40nr_2$
(d)	$\bar{Q}, Q-R_Q$	$2^k\cdot 144n-24nr_3$
(e)	R_Q	$24nr_3$

Now applying the standard formula for the Chern numbers of the canonical resolution of singularities, then minimalizing, one gets

Theorem 1.5.13. *$X(k,n,s,t,r_1,r_2,r_3,r_4)$ is a minimal surface of general type, and*

$$\begin{aligned} c_1^2(X) &= 2^k(18644n-442)+8(45n-1)(s+t)-3(30n-1)r_4 \\ &\quad -2(30n-1)r_1-40nr_2-24nr_3, \\ c_2(X) &= 2^k\cdot 6904n+8\left(90n-\tfrac{1}{2}\right)s+2(90n+4)t-9(30n-1)r_4 \\ &\quad +2(30n-1)r_1+40nr_2+24nr_3+24. \end{aligned}$$

Lemma 1.5.14. *The surface X constructed above is simply connected.*

Proof. Since B_ϕ contains at least one component of the type $\pi^*(p^*(pt))$, which is a projective line, Lemma 1.3.6 applies.

Example 1.5.15. Setting $s = 1$, $r_i = 0$, $t = 0$ we get a surface $X = X(k, n, 1)$ with

$$c_1^2 = 2^k(18644n - 442) + 8(45n - 1), \quad c_2 = 2^k \cdot 6904n + 4(180n - 1),$$

so $c_1^2 - 2c_2 = 2^k(4836n - 442) - 8(135n) > 0$ for $n \geq 2$, $k \geq 2$. This is a relatively simple example of a simply connected surface with $c_1^2 > 2c_2$. Of course, to just construct one example this complicated construction would not have been necessary. But using the parameters k, n, s, t, r_1, r_2, r_3 and r_4, the following theorem can be proved [8, Theorem 1].

Theorem 1.5.16. *Let x, y be integers with*

$$\frac{352}{89}x + 280x^{2/3} < y < \frac{18644}{2129}x - 2555x^{2/3}$$

and x sufficiently large. Then there exists a simply connected surface X of general type with

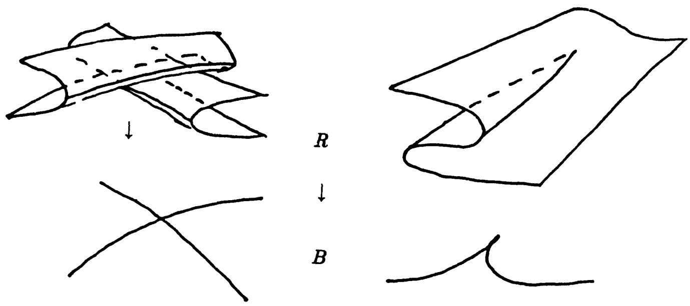
$$c_1^2(X) = y, \qquad \chi(X) = x \quad (\chi = \textit{arithmetic genus}).$$

2. Miyaoka's construction

We review a quite different kind of construction, due to Miyaoka [61], which also produces lots of examples of surfaces of general type with $c_1^2 > 2c_2$. In fact, he shows that for a given projective surface S, there exists a finite ramified Galois covering X of S with $c_1^2 > 2c_2$.

2.1. Projections. Let S be an algebraic surface, and $S \subset \mathbf{P}^N$ a projective embedding (which we assume is smooth, for simplicity). Let $p\colon S \to \mathbf{P}^2$ be a generic projection onto a projective plane. If the degree of the embedding of S is d, then p is a finite map (branched cover) of degree d. The covering $p\colon S \to \mathbf{P}^2$ in general is, however, *not* Galois. The branching behavior of p is easy to determine.

Proposition 2.1.1 [61]. *The ramification locus $R \subset S$ is a smooth, irreducible, reduced divisor on S. p maps R birationally onto $B \subset \mathbf{P}^2$, where B is the branch divisor. The singular locus of B is a disjoint union $B_{\text{sing}} = \Gamma \cup \Delta$, where $\Gamma =$ ordinary double points of B and $\Delta = (2,3)$ cusps.*



Remark 2.1.2. The given properties hold for a generic projection, not for any given projection. The properties of R follow from Bertini's theorem [21, III, 10.9 and Example 11.3], since R moves in a sufficiently large linear system of divisors.

2.2. Chern numbers of the Galoisization. The covering $X \to S$ is constructed as the "Galoisization" of the projection $p\colon S \to \mathbf{P}^2$, as follows. Since $p\colon S \to \mathbf{P}^2$ is a d-fold cover, we have a natural representation $\pi_1(\mathbf{P}^2 - B) \to S_d$, where S_d is the symmetric group. We can form the associated Galois cover with $|S_d| = d!$ sheets over $\mathbf{P}^2 - B$. There is a unique way to compactify (Fox completion [15]) such that $\pi\colon X \to \mathbf{P}^2$ ramifies to degree 2 along B, to degree $2^2 = 4$ at $p \in \Gamma$, and to degree $2 \cdot 3 = 6$ at the cusps. (The factor 3 is since the double cover of the cusp has a quotient singularity of the form $\mathbf{C}/\mathbf{Z}_3$ over the cusp point.)

To calculate X's Chern invariants, we start with the following observation. We have on S

$$p^*(c_1(\mathbf{P}^2)) - c_1(S) = R$$

[16, 3.2.10] so that $\deg R$, which is $H \cdot R$ (H hyperplane class in $\mathbf{P}^N$), by this equation equals

$$\begin{aligned} \deg R &= H \cdot \pi^* c_1(\mathbf{P}^2) - c_1(S) \cdot H \\ &= 3d + m \qquad (m = H \cdot K_S) \\ &= \deg B. \end{aligned}$$

Note that R may be viewed as the desingularization of B, so the Plücker formula gives us [18, p. 280]

$$c_1(R) = -(3d+m)^2 + 3(3d+m) + 2\sharp\Gamma + 2\sharp\Delta. \tag{2.2.1}$$

By the description of the covering above, the Euler number $c_2(X)$ is

$$c_2(X) = d!(e(\mathbf{P}^2 - B) + \tfrac{1}{2}e(B - \Gamma - \Delta) + \tfrac{1}{4}\sharp\Gamma + \tfrac{1}{6}\sharp\Delta),$$

$$\begin{aligned} c_2(X) &= d!\,(3 - \tfrac{1}{2}e(B) - \tfrac{1}{4}\sharp\Gamma - \tfrac{1}{3}\sharp\Delta) \qquad (e(B) = c_1(R) - \sharp\Gamma) \\ &= (d!/2)\left(6 - (-(3d+m)^2 + 3(3d+m) + \sharp\Gamma + 2\sharp\Delta) - \tfrac{1}{2}\sharp\Gamma - \tfrac{2}{3}\sharp\Delta\right) \\ &= (d!/2)\left(6 + (3d+m)^2 - 3(3d+m) - \tfrac{3}{2}\sharp\Gamma - \tfrac{8}{3}\sharp\Delta\right). \end{aligned}$$

An easy computation expressing $c_1^2(S)$ and $c_2(S)$ in terms of d, m, $\sharp\Delta$ and $\sharp\Gamma$ yields

$$\begin{aligned} \sharp\Gamma &= \tfrac{1}{2}(3d+m)^2 - 42d - 30m + \text{const}, \\ \sharp\Delta &= 12d + 3m + \text{const}, \end{aligned} \tag{2.2.2}$$

where "const" is an expression involving $c_1^2(S)$ and $c_2(S)$ and therefore independent of the embedding. This yields the formula

$$c_2(X) = (d!/2)\left[\tfrac{1}{4}(3d+m)^2 - \tfrac{1}{2}(19d+9m) + \text{const}\right]. \tag{2.2.3}$$

On the other hand, since $c_1(X) = \pi^*(c_1(\mathbf{P}^2) - \frac{1}{2}[B])$, the number

$$c_1^2(X) = d!\left[9 - 3(3d+m) + \tfrac{1}{4}(3d+m)^2\right] \tag{2.2.4}$$

does not depend on the number of singularities, only on the degree, of the branch curve B.

From (2.2.3) and (2.2.4) it follows that, for d sufficiently large, $c_1^2(X) > 2c_2(X)$. This proves

2.2.5. *For every projective algebraic surface S, there exist a Galois cover X of S such that X has positive signature.*

Remark 2.2.6. Moišhezon has studied the fundamental groups of the surfaces X, and discovered that some surfaces are simply connected. These were actually the first known examples of simply connected surfaces with positive signature.

3. Picard modular surfaces

As opposed to branched coverings another standard method of constructing examples of complex manifolds is by taking quotients of known manifolds (in particular Stein spaces) under sufficiently nice group actions. Quotients of hermitian symmetric domains by discrete subgroups of their automorphism groups which act properly discontinuously are a prime object of study in this respect.

In dimension 2 there are only two kinds of hermitian symmetric domains: a product of 1-disks, $\mathbf{B}^1 \times \mathbf{B}^1$, and the complex 2-ball, $\mathbf{B}^2 = \{(x_1, x_2) \in \mathbf{C}^2 | \sum |x_i|^2 < 1\}$. The corresponding compact duals are $\mathbf{P}^1 \times \mathbf{P}^1$ and $\mathbf{P}^2$, respectively. If $\Gamma_X \subset \mathrm{Aut}(\mathbf{B}^1 \times \mathbf{B}^1)$ and $\Gamma_Y \subset \mathrm{Aut}(\mathbf{B}^2)$ are discrete subgroups which act *freely* (i.e., have no fixed points, neither in the interior nor on the boundary of the domain), then, by a theorem of Cartan, the quotients

$$X = \mathbf{B}^1 \times \mathbf{B}^1/\Gamma_X, \qquad Y = \mathbf{B}^2/\Gamma_Y$$

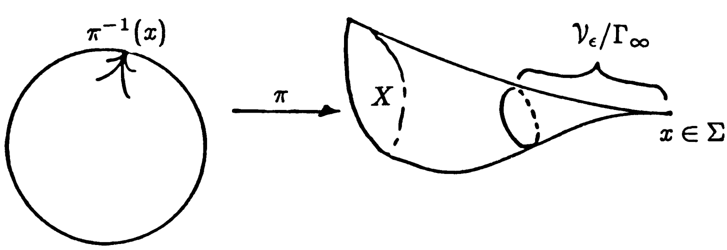
are compact; in fact algebraic manifolds and Hirzebruch proportionality ([22] and §0.2) tell us

$$\begin{aligned} c_1^2(X) &= 2c_2(X), & c_1^2(Y) &= 3c_2(Y), \\ c_2(X) &= \mathrm{vol}(X), & c_2(Y) &= \mathrm{vol}(Y). \end{aligned} \tag{3.0.1}$$

More interesting things may occur if we consider actions by groups Γ which have only parabolic fixed points, that is, fixed points on $\partial(D)$, where D is one of the domains $\mathbf{B}^1 \times \mathbf{B}^1$ or $\mathbf{B}^2$. As compared with the covering constructions above, where we looked for families of divisors with certain properties to act as branching locus, the job now is to find families of discrete subgroups and to study the invariants of the family of quotients. Beautiful examples of this are the Hilbert modular surfaces corresponding to the Hilbert modular groups $SL(\mathcal{O}_d)$, d a prime, $d \equiv 3$ (4), $\mathcal{O}_d$ = ring of integers in $\mathbf{Q}(\sqrt{d})$, which were intensively studied by Hirzebruch and his coworkers in the 1970's. They are noncompact quotients of $\mathbf{B}^1 \times \mathbf{B}^1$ by an irreducible group $\Gamma \subset \mathrm{Aut}(\mathbf{B}^1 \times \mathbf{B}^1)$ ([25], [29]-[32]). In fact, it turns out that any irreducible $\Gamma \subset \mathrm{Aut}(\mathbf{B}^1 \times \mathbf{B}^1)$ (with noncompact quotient) is commensurable to one of the Hilbert modular groups [25, 5.8] so that these yield essentially "most" examples of quotients for the domain $\mathbf{B}^1 \times \mathbf{B}^1$. However from the point of view of surface geography, they are not of interest to us since they all have $c_1^2 < 2c_2$ [25, 3.9, and p. 238].

The analogues in $\mathrm{Aut}(\mathbf{B}^2)$ of the Hilbert modular groups are the Picard modular groups and the quotients of $\mathbf{B}^2$ by them, intensively studied in the 1980's by Holzapfel and coworkers ([35]–[38], [12]–[14]). The Picard modular group is $\Gamma^{(d)} = SU((2,1), \mathcal{O}_{\mathbf{Q}(\sqrt{-d})})$, where d is a square free number. Proper compactifications turn out to be surfaces of general type with ratio c_1^2/c_2 arbitrarily close to 3. In the first section we discuss the compactification of arithmetic ball quotients, and in the following section Mumford's proportionality theorem is used to calculate the Chern numbers. Standard reference here are the original articles listed above, in particular [35].

3.1. Compactification. Let $\Gamma \subset \mathrm{Aut}(\mathbf{B}^2)$ be a discrete arithmetic group acting properly discontinuously. By the fundamental work of Baily-Borel [3], $X = \Gamma\backslash\mathbf{B}^2$ can be embedded in a projective space as an everywhere dense open subset of X^*, where X^* is a normal projective variety. Let $\Sigma = X^* - X$. Σ is called the set of cusps of X^*. The points of Σ are covered by the parabolic fixed points of Γ on $\partial\mathbf{B}^2$. The picture is as follows.



Since the question of compactification is a local one, it is sufficient to study a cusp $y \in \pi^{-1}(X)$ in $\partial\mathbf{B}^2$. To insure that the compactification will be by a smooth elliptic curve, we will assume from now on that Γ is a *neat* subgroup.

Definition 3.1.1. $\Gamma \subset \mathrm{Aut}(\mathbf{B}^2)$ is *neat*, iff for every $\gamma \in \Gamma$, the subgroup (in $\mathbf{C}^*$) generated by its eigenvalues has no torsion.

Fixing a cusp $X_0 = (1,0) \in \mathbf{B}^2$ and an embedding $\mathbf{C}^2 \to \mathbf{P}^2$ given by $(X,Y) \mapsto [X = z_1/z_0, Y = z_2/z_0, 1]$, we apply the Cayley transform

$$\tau\colon (X,Y,1) \mapsto \frac{1}{\sqrt{2}}(X+1, -\sqrt{2}\cdot iY, i(X-1))$$

which transforms $\mathbf{B}^2 = \{(X,Y) \mid |X|^2 + |Y|^2 < 1\}$ into the Siegel domain of type II: $\mathscr{V} = \left\{(u,v) \mid \mathrm{Im}\, u > \frac{1}{2}|v|^2\right\}$. The point $\kappa_0 = (1,0,1) \in \mathbf{B}^2$ is mapped to $\infty = (1,0,0)$. The map is biholomorphic and $\mathrm{Aut}(\mathbf{B}^2)$ acts on $\mathscr{V}$. The full stationary group $P_\infty \subset \mathrm{Aut}(\mathbf{B}^2)$ of $\infty \in \mathscr{V}$ is a parabolic subgroup with unipotent radical U_∞. P_∞ splits:

$$P_\infty = A_\infty \cdot M_\infty \cdot U_\infty,$$

$$A_\infty = \left\{ \begin{pmatrix} r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^{-1} \end{pmatrix}; \ r \in \mathbf{R}_+ \right\}, \quad M_\infty = \left\{ \begin{pmatrix} \theta & 0 & 0 \\ 0 & \theta^{-2} & 0 \\ 0 & 0 & \theta \end{pmatrix}; \ \theta \in U(1) \right\},$$

$$U_\infty = \left\{ \begin{pmatrix} 1 & ia & \frac{1}{2}|a|^2 + r \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} = [a,r]; \ a \in \mathbf{C}, r \in \mathbf{R} \right\}.$$

Since we assume Γ to be neat, only the factor U_∞ will be important for us. Let $\kappa \in \partial\mathbf{B}^2$ be a cusp. Then $\Gamma_\kappa = \{\text{isotropy group of } \gamma\}$ is conjugate to a subgroup $\Gamma_\infty \subset U_\infty$, which is a lattice.

The compactification is described very nicely in [35, pp. 214–215], so we just sketch the argument. Let U_X be a neighborhood of $\Gamma\backslash\mathbf{B}^2$ at the cusp. Then U_X is the total space of a punctured disk bundle over an elliptic curve, and the compactification is by adding to zero section. To see this, let $U_\kappa \subset \pi^{-1}(U_X)$ be a neighborhood of the cusp $\kappa \in \partial\mathbf{B}^2$ covering the neighborhood U_X. Then its transform on $\mathscr{V}$ is a set of the form

$$\mathscr{V}_\varepsilon = \{(u, v) | 2\,\mathrm{Im}\, u - |v|^2 > \varepsilon\}.$$

The action of Γ_∞ on $\mathscr{V}$ can be extended to $\mathbf{C} \times \mathbf{C}$.

We "factorize" an element $\gamma = [a_\gamma, r_\gamma] \in \Gamma_\infty \subset U_\infty$ into its real (r_γ) and complex (a_γ) parts:

$$\begin{aligned} \Delta_\infty &= \{r_\gamma \in \mathbf{R} | [a_\gamma, r_\gamma] \in \Gamma_\infty\} \quad \text{for some } a_\gamma, \\ \Lambda_\infty &= \{a_\gamma \in \mathbf{C} | [a_\gamma, r_\gamma] \in \Gamma_\infty\} \quad \text{for some } r_\gamma. \end{aligned}$$

This defines a diagram:

$$\begin{array}{ccccc} & & \mathbf{C} \times \mathbf{C} & \longrightarrow & F_\infty = \mathbf{C} \times \mathbf{C}/\Lambda_\infty \\ & & \uparrow & & \downarrow \\ \mathbf{C} \times \mathbf{C} & \longrightarrow & \mathbf{C}^* \times \mathbf{C} & \longrightarrow & \bar{\mathscr{V}}/\Gamma_\infty \\ \uparrow & & \uparrow & & \uparrow \\ \mathscr{V} & \longrightarrow & \mathscr{V}/\Delta_\infty & \longrightarrow & \mathscr{V}/\Gamma_\infty \end{array} \tag{3.1.2}$$

F_∞ is a line bundle over the elliptic curve $\Gamma_\infty = 0 \times \mathbf{C}/\Lambda_\infty$, and $\bar{\mathscr{V}}/\Gamma_\infty$ is its compactification. Open sets of the form $\bar{\mathscr{V}}_\varepsilon/\Lambda_\infty$ can be used to desingularize the cusps on the Baily-Borel compactification.

Lemma 3.1.3. *The compactifying torus T_κ has self-intersection number*

$$(T_\kappa)^2 = -2|\Lambda_\kappa|/q_\kappa,$$

where Λ_κ is the area of the lattice $\Lambda_\kappa \subset \mathbf{C}$, q_κ is a generator of Δ_κ, and the subscript κ means the transform of subscript ∞ under τ^{-1}.

3.2. Calculation of Chern numbers. As discussed above, for a smooth, *compact* ball quotient $X = \Gamma\backslash\mathbf{B}^2$ the proportionality $c_1^2 = 3c_2$ holds. Mumford has proven a corresponding proportionality [64] for noncompact quotients $X = \Gamma\backslash\mathbf{B}^2$ which can be compactified $X \subset \bar{X}$ by a disjoint union of elliptic curves. In the noncompact case we must replace the Chern numbers $c_1^2(\bar{X})$, $c_2(\bar{X})$ by the *logarithmic* Chern numbers $\bar{c}_1^2(\bar{X},D)$, $\bar{c}_2(\bar{X},D)$ of X along $D = \bar{X} - X$ (see [69], [43]). Then Mumford's theorem (0.2.2) is as follows.

Theorem 3.2.1. *Let $X \subset \bar{X}$ be the compactification of a ball quotient (by an arithmetic group) such that $D = \bar{X} - X$ is a disjoint union of smooth elliptic curves. Then*

$$\bar{c}_1^2(\bar{X},D) = 3\bar{c}_2(\bar{X},D).$$

We digress to briefly discuss the logarithmic Chern classes and then apply them to calculate $c_1^2(\bar{X})$, $c_2(\bar{X})$.

Let $\Omega^q_{\bar{X}}$ be the sheaf of holomorphic q-forms on the compactification $\bar{X}$, and suppose D is given in local coordinates z_1, z_2 by $z_1 = 0$. The sheaf of *logarithmic* 1*-forms along D is* $\Omega^1_{\bar{X}}\{dz_1/z_1\}$, and that of *logarithmic* 2*-forms along D* is $\Omega^2_{\bar{X}}\{dz_1/z_1 \wedge dz_2\}$. Sections of these sheaves are q-forms with poles of order ≤ 1 along D. For details see [43, Chapter 11]. The sheaf is denoted $\Omega^q_{\bar{X}}(\log D)$, and the *logarithmic Chern classes* are

$$\bar{c}_i(\bar{X},D) := (-1)^i c_i(\Omega^q_{\bar{X}}(\log D)).$$

These are thought of as "Chern classes of the noncomplete X". In fact, the sheaves $\Omega^q_{\bar{X}}(\log D)$ do turn out to be intrinsic to X, i.e., independent of the desingularization [43, Theorem 11.1].

Now assume D consists of disjoint components each of which is a complex torus. To calculate $\bar{c}_j(\bar{X},D)$ use the standard exact sequence

$$0 \to \Omega^1_{\bar{X}} \to \Omega^1_{\bar{X}}(\log D) \to j_*\mathcal{O}_D \to 0,$$

where $j\colon D \hookrightarrow \bar{X}$ is the inclusion. Then for the total Chern classes we have

$$c(\Omega^1_{\bar{X}}(\log D)) = c(\Omega^1_{\bar{X}})c(j_*\mathcal{O}_D) = (1 - c_1 + c_2)(1 + D + D^2),$$

so using adjunction

$$\overline{c_1}(\bar{X},D) = c_1(\bar{X}) - (D),$$
$$\overline{c_2}(\bar{X},D) = c_2(\bar{X}) - c_1(D) = c_2(\bar{X}).$$

This in turn yields for $\overline{c_1^2}(\bar{X},D)$

$$\begin{aligned}\overline{c_1^2}(\bar{X},D) &= (c_1(\bar{X}) - (D))^2 = c_1^2(\bar{X}) - 2c_1(\bar{X})\cdot(D) + (D)^2 \\ &= c_1^2(\bar{X}) - 2(c_1(D) - (D)^2) + (D)^2 = c_1^2(\bar{X}) + 3(D)^2.\end{aligned}$$

Putting everything together we get

Proposition 3.2.2. *Assume $X \subset \bar{X}$ as above, $\bar{X}$ an algebraic surface. Then*

$$c_1^2(\bar{X}) = 3c_2(\bar{X}) - 3(D)^2, \qquad c_2(\bar{X}) = c_2(X) = \mathrm{vol}(X).$$

3.3. Picard modular groups. It remains to find a suitable family of discrete subgroups. By the formula above, c_1^2 and c_2 will be computed as soon as we know:

$$(1)\ \mathrm{vol}(X), \quad (2)\ \sharp \text{ cusps}, \quad (3)\ T_\kappa^2 = -2|\Lambda_\kappa|/q_\kappa.$$

Let $K = \mathbf{Q}(\sqrt{-d})$ be an imaginary quadratic field and $\mathscr{O}_K = \mathbf{Z} \oplus \varepsilon\mathbf{Z}$ its ring of integers, where

$$\varepsilon = \sqrt{-d}, \qquad d \equiv 1, 2\ (4),$$
$$\varepsilon = \frac{1+\sqrt{-d}}{2}, \qquad d \equiv 3\ (4),$$

d being a square-free natural number. The *Picard modular group* is defined

$$\Gamma^{(d)} := SU((2,1), \mathscr{O}_K),$$

consisting of matrices in $SU(2,1)$ with coefficients in $\mathscr{O}_K$. Let $\mathbf{a} \subset \mathscr{O}_K$ be an ideal. The *main congruence subgroup* $\Gamma(\mathbf{a})$ of the ideal $\mathbf{a}$ is defined by the exact sequence

$$1 \to \Gamma(\mathbf{a}) \to \Gamma^{(d)} \to SU((2,1), \mathscr{O}_K/\mathbf{a}).$$

If the ideal $\mathbf{a} = m\mathscr{O}_K$, where m is a natural number, then $\Gamma(\mathbf{a}) =: \Gamma(m)$ is called a *natural congruence subgroup.*

Lemma 3.3.1 [35, *p.* 225]. *The natural congruence subgroups $\Gamma^{(d)}(m)$ are neat for all $m > 2$.*

The $\Gamma^d(m)$ give the desired series of groups. The Euler numbers of the quotients by the group $\Gamma^{(d)}$ are calculated in [36, V,3.3.7]:

$$c_2(\mathbf{B}^2/\Gamma^{(d)}) = |Z(\Gamma)| \frac{|D|^{5/2}}{32\pi^3} \cdot L(3, \chi_K), \tag{3.3.2}$$

where $L(s, \chi_K)$ is the L-series of the character $\chi_K(k) = (D/k)$ of the field K, D is the discriminant of K, and

$$|Z(\Gamma)| = \begin{cases} 3 & \text{if } K = \mathbf{Q}(\sqrt{-3}), \\ 1 & \text{otherwise,} \end{cases}$$

is the order of the intersection of Γ with the center of $SU(2,1)$. Now note that

$$\begin{aligned} c_2(\mathbf{B}^2/\Gamma^{(d)}(m)) &= \frac{[\Gamma : \Gamma(m)]}{|Z(\Gamma)|} \cdot c_2(\mathbf{B}^2/\Gamma^{(d)}) \\ &= [\Gamma : \Gamma(m)] \cdot \frac{|D|^{5/2}}{32\pi^3} L(3, \chi_K). \end{aligned} \tag{3.3.3}$$

(The factor $|Z(\Gamma)|$ is due to the fact that $SU((2,1)\mathcal{O}_{\mathbf{Q}\sqrt{-3}})$ does not act freely.) Let $T(m) \subset \mathbf{B}^2/\Gamma(m)$ denote the whole compactifying curve. Then the formula

$$(T(m))^2 = -[\Gamma : \Gamma(m)]\frac{t}{\gamma} \cdot \frac{h(K)}{m^2} \tag{3.3.4}$$

is derived, where

$$\frac{t}{\gamma} = \begin{cases} 1 & d = 1, \\ \frac{1}{6} & d = 3, \\ 2 & d \equiv 1, 2\ (4), d \neq 1, \\ \frac{1}{2} & d \equiv 3\ (4), d \neq 3, \end{cases}$$

and $h(K)$ is the class number of the field K. Inserting (3.3.2), (3.3.3) and (3.3.4) into 3.2.2 yields

Theorem 3.3.5 [35, 4.12]. *Let $\Gamma^{(d)}$ be the Picard modular group of the field $K = \mathbf{Q}(\sqrt{-d})$ and $\Gamma(m) \subset \Gamma^{(d)}$ the natural congruence subgroup, D the discriminant and χ_K the character corresponding to (D/k). Then*

$$c_1^2(\overline{\mathbf{B}^2/\Gamma(m)}) = [\Gamma : \Gamma(m)]\left(\frac{3 \cdot |D|^{5/2}}{32\pi^3}L(3, \chi_K) - \frac{t}{\gamma}\frac{h(K)}{m^2}\right),$$

$$c_2(\overline{\mathbf{B}^2/\Gamma(m)}) = [\Gamma : \Gamma(m)] \cdot \frac{|D|^{5/2}}{32\pi^3}L(3, \chi_K).$$

This implies

$$\frac{c_1^2(\overline{\mathbf{B}^2/\Gamma(m)})}{c_2(\overline{\mathbf{B}^2/\Gamma(m)})} = 3 - \frac{t}{\gamma}\frac{|Z(\Gamma)| \cdot h(K)}{c_2(\mathbf{B}^2/\Gamma^{(d)}) \cdot m^2}. \tag{3.3.6}$$

So for fixed K and $m \geq 3$ we have a monotonously increasing sequence

$$\frac{c_1^2(\overline{\mathbf{B}^2/\Gamma(m)})}{c_2(\overline{\mathbf{B}^2/\Gamma(m)})},$$

and more generally

$$\lim_{|D|\to\infty} \frac{c_1^2(\overline{\mathbf{B}^2/\Gamma(m)})}{(\overline{\mathbf{B}^2/\Gamma(m)})} = 3 \quad \text{for fixed } m > 2,$$

$$\lim_{m\to\infty} \frac{c_1^2(\overline{\mathbf{B}^2/\Gamma(m)})}{c_2(\overline{\mathbf{B}^2/\Gamma(m)})} = 3 \quad \text{for fixed } K.$$

Remark 3.3.7. Some of these surfaces have been constructed "from the bottom up", as branched coverings of abelian surfaces, by Hirzebruch [27], [38].

4. Constructions yielding ball quotients

In this paragraph we discuss a variety of constructions which have been used to construct ball quotients "from the bottom up", that is, as coverings of known surfaces. So in this section we return to the problem of finding certain special divisors to use as branch locus for a finite morphism.

4.0. The Miyaoka-Yau inequality. Let S be an algebraic surface of general type. Then its Chern numbers satisfy the inequality

$$c_1^2(S) \leq 3c_2(S).$$

An algebro-geometric proof of this is given by Miyaoka in [62]. Furthermore,

$$c_1^2(S) = 3c_2(S) \Leftrightarrow S \text{ is a ball quotient.}$$

A proof of this under the assumption K_S ample as a consequence of his solution of Calabi's conjecture is given by Yau in [91]. The proof of Calabi's conjecture is in [92]. Miyaoka [62] proved the general case where S is of general type, and K_S is not ample.

This is the criterion most often used to check whether a given S is indeed a ball quotient.

4.1. Coverings of elliptic modular surfaces.

4.1.1. Elliptic modular surfaces. An elliptic surface $\pi\colon X \to E$ is a surface X together with a morphism π onto a curve E, such that the generic fiber is an elliptic curve. Elliptic surfaces were studied in great detail by Kodaira in the sixties [55], [56]. A very special class of elliptic surfaces, with connections to number theory, were considered by Shioda [83]. They can be constructed as follows: Let $\Gamma(N) \subset SL_2(\mathbf{Z})$ be the main congruence subgroup of level $N \geq 3$ (see [80]), and $X(N) = \Gamma(N)\backslash\mathscr{H}$ the modular curve of level N. $X(N)$ is not compact, but can be compactified by adding a finite number of cusps. Now let

$$G = \Gamma(N) \rtimes \mathbf{Z} \oplus \mathbf{Z}$$

be the semidirect product where $\Gamma(N) \subset SL_2(\mathbf{Z})$ acts in an obvious manner on $\mathbf{Z} \oplus \mathbf{Z}$. G acts properly discontinuously on $\mathscr{H} \times \mathbf{C}$, as follows:

$$G \times (\mathscr{H} \times \mathbf{C}) \to \mathscr{H} \times \mathbf{C}$$
$$((\gamma, n, m), (z, w)) \mapsto (\gamma z, (cz+d)^{-1}(z + nw + m))$$

where $\gamma \in \Gamma(N)$ is written $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. The quotient

$$G\backslash\mathscr{H} \times \mathbf{C} = E(N)$$

is an elliptic fiber space $E(N) \xrightarrow{\pi} X(N)$ of elliptic curves with a level N

structure. $E(N)$ can be smoothly compactified, $\bar{\pi}: E(\bar{N}) \to X(\bar{N})$, such that the fiber $\bar{\pi}^{-1}(*)$ is a Kodaira fiber of type I_N [55, §6] for any cusp $* \in \overline{X(N)} - X(N) = \Sigma$. In [2] this is done as an example of the theory of compactification by means of toroidal embeddings. The compactification is also the desingularization of quotient singularities on the closure

$$\overline{G\backslash\mathscr{H} \times \mathbf{C}}.$$

The number of exceptional fibers I_N on $\overline{E(N)}$ is equal to the number of cusps of $\Gamma(N)$, which is

$$\mu(N) = \frac{1}{2}N^2 \prod_{p|N}\left(1 - \frac{1}{p^2}\right).$$

The irregularity of $\overline{E(N)}$ is equal to the genus of $\overline{X(N)}$ [55, §11], which is

$$g(\overline{X(N)}) = 1 + \frac{N\mu(N)}{12} - \frac{\mu(N)}{2}.$$

The formula for the canonical divisor is [55, 12.1]

$$K_{\overline{E(N)}} = \bar{\pi}^*(D),$$

where $D \subset \overline{X(N)}$ is a divisor of degree $2g(\overline{X(N)}) - 2 + \chi(\overline{E(N)})$, χ the arithmetic genus. In fact, as Shioda shows, D can be taken as the divisor associated to the line bundle of cusp forms of weight 3 with respect to $\Gamma(N)$ [82, I]. Since the self-intersection of a fiber vanishes we get $c_1^2(\overline{E(N)}) = 0$ and in addition

$$c_2(\overline{E(N)}) = N \cdot \mu(N).$$

4.1.2. The cyclic covers. The following basic result will be applied shortly to construct cyclic coverings of degree d (for cyclic coverings see [28]). This construction is due to R. Livné [58], which is the basic reference for the rest of §4.1.

Lemma 4.1.3 [58, §1.3]. *Let V be a smooth algebraic variety, $d \geq 2$ an integer, and D a reduced, effective divisor on V divisible by d in* Pic(V). *Then the following hold.*

(a) *There exist cyclic d-sheeted covers $\pi_\Delta: W_\Delta \to V$, totally branched along D and nowhere else.*

(b) *The covers $\pi_\Delta: W_\Delta \to V$ are in* 1-1 *correspondence with dth roots of D, i.e., classes $\Delta \in$* Pic(V) *with $d\Delta \equiv D$ (linear equivalence).*

(c) *Letting $f \in \mathscr{M}(V)$ be a rational function with $(f) = d\Delta - D$, we have $\mathscr{M}(W_\Delta) = \mathscr{M}(V)(\sqrt[d]{f})$. W_Δ is irreducible unless $k\Delta = 0 \in$* Pic(V) *for some $k|d$, $k \neq d$.*

(d) *The Euler number of the covering is given by*

$$e(W_\Delta) = de(V) - (d-1)e(D).$$

(e) W_Δ *is smooth* $\Leftrightarrow$ D *is smooth. In this case*

$$K_{W_\Delta} = \pi_\Delta^*(K_V) + (d-1)\tilde{D},$$

where $\tilde{D}$ *is the reduced inverse image of* D.

The divisors used to apply this will come from sections of the fibration $\bar{\pi}\colon \overline{E(N)} \to \overline{X(N)}$. The group of sections of $\bar{\pi}$ is isomorphic to $(\mathbf{Z}_N)^2$: there are N^2 section, each of order N (in the group of sections).

Let $D = \sum_1^{N^2} D_i$ be the sum of all N^2 sections, and

$$\theta_i = \begin{cases} \sum_{x\in\Sigma} \theta_{x,i} + \theta_{x,N-i}, & 2i \neq N, \\ \sum_{x\in\Sigma} \theta_{x,i}, & 2i = N, \end{cases}$$

where $\Sigma = \overline{X(N)} - X(N)$ denotes the set of cusps, and $\theta_{x,i}$ the N components of the singular fiber over $x \in \Sigma$.

Lemma 4.1.4 [58, 1.4]. *Let* $K = [N/2]$, *and*

$$D^1 = N^2 D_1 + \bar{\pi}^*(\rho) - \frac{N}{2}\sum_{i=1}^{K} i(N-i)\theta_i,$$

where $\rho = (N^2-1)\cdot\rho_1$, *and* ρ_1 *is the divisor on* $X(N)$ *of degree* $N_\mu(N)/12$ *which is associated to the line bundle of modular forms of weight one, and* D_1 *is the zero section (identity in the group of sections). Then*

(a) $D^1 \approx D$ *(linear equivalence).*

(b) D^1 *is divisible by*

$$\operatorname{num}\left(\frac{N}{2}\right) = \begin{cases} N, & N \text{ odd}, \\ N/2, & N \text{ even}. \end{cases}$$

This, together with the result stated above on cyclic covers, yields

Theorem 4.1.5 [58, 1.5]. *Let* $d \mid \operatorname{num}(N/2)$, $N \geq 4$. *Then there exist d-fold cyclic branded covers,* $S_d(N)_\Delta$, *of* $E(N)$ *with branch locus the union of sections. The Chern numbers are given by*

$$c_1^2(S_d(N)_\Delta) = \frac{(d-1)N^2\mu(N)}{12}\left(6(N-4) - (d-1)\frac{N}{d}\right),$$

$$c_2(S_d(N)_\Delta) = dN\mu(N) + (d-1)N^2\mu(N)\frac{N-6}{6}$$

for any Δ *fulfilling* 4.1.3(b), *i.e.,* $d\Delta \equiv D$.

Now among all the $S_d(N)_\Delta$ for fixed d there is a unique Δ for which every automorphism of $E(N)$ lifts to one on $S_d(N)_\Delta$ [58, 2.3]. This unique covering is denoted simply by $S_d(N)$.

Plugging in different N and d in 4.1.5 it is seen that $c_1^2(S_d(N)) = 3c_2(S_d(N))$ for $(N,d) = (7,7)$, $(8,4)$, $(9,3)$ and $(12,2)$, so by Yau's theorem stated in §4.0 these are compact ball quotients. Furthermore, $S_5(5)$ is nonminimal, and by blowing down the exceptional curves one gets a minimal surface of general type, which we also denote by $S_5(5)$, with $c_1^2(S_5(5)) = 3c_2(S_5(5))$, and so it is also a ball quotient.

We will be meeting these surfaces again in the sections that follow: $S_5(5)$ and $S_3(9)$ in §§4.2 and 4.3, and $S_7(7)$ in §4.4.

4.2. Coverings defined by differential equations. Let M be a connected complex manifold and Γ a properly discontinuous group acting on M (holomorphically). If $X = M/\Gamma$ is the quotient space, a function $b\colon X \to \mathbf{N}$ is defined by

$$\begin{aligned} b\colon X &\to \mathbf{N} \\ x &\mapsto |G_z|, \end{aligned}$$

where G_z is the isotropy group of $z \in \pi^{-1}(x)$, $\pi\colon M \to X$. b describes the branching behavior of π. The pair (X, b) is called an *orbifold* ([93], [90]), and $\pi\colon M \to X$ ramifies along $\operatorname{supp}(b-1)$ with branching degree given by b.

Conversely, suppose we are given a pair (X, b), X a normal analytic space and $b\colon X \to \mathbf{N}$ a function. If there exists a pair (M, Γ) such that $M/\Gamma = (X, b)$ as above, (X, b) is called a *uniformizable orbifold.* Let (X, b) be such a uniformizable orbifold, and (M, Γ) the universal uniformization. Denote by ϕ an inverse mapping (many-valued) of $\pi\colon M \to X$. Such a ϕ is called a uniformizing map for (X, b). If Γ is a subgroup of some Lie group (in particular if M is hermitian symmetric), then by the theory of Schwarzian derivatives [71], there exists a unique Fuchsian differential equation (E) (one with regular singular points) such that solutions of (E) give the uniformizing map. Thus the "Riemann surface" associated to the many-valued function ϕ, the solution of a differential equation, is a branched cover of X.

4.2.1. The hypergeometric differential equation. This was applied a long time ago by Picard [75] in the case that the differential equation (E) is the hypergeometric differential equation F_1 of Appell ([1], [11, §5]) to construct discrete subgroups in $PSU(2,1)$. However, the proofs he gave were not entirely correct, as noted by Mostow, and he [68], Deligne [10] and Terada [85] recently have given correct proofs of Picard's statements.

Appell's hypergeometric function $F_1(\alpha,\beta',\beta,\gamma,x,y)$ is defined for $(x,y)\in \mathbf{C}^2$ and $\alpha,\beta',\beta,\gamma\in\mathbf{C}$ by the series [11, 5.7.1]

$$F_1(\alpha,\beta',\beta,\gamma,x,y)=\sum\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\alpha)_{m+n}m!n!}x^m y^n,$$
$$(\varphi)_m := \Gamma(\varphi+m)/\Gamma(\varphi).$$

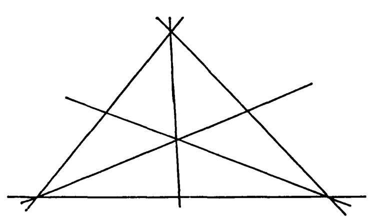
F_1 has the integral representation [11, 5.8.2]:

$$F_1(\alpha,\beta,\beta',\gamma,x,y)$$
$$=\frac{\Gamma(\gamma)}{\Gamma(\gamma)\Gamma(\gamma-\alpha)}\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-uy)^{-\beta'}\,du,$$
$$\operatorname{Re}\alpha>0,\qquad \operatorname{Re}(\gamma-\alpha)>0,$$

which gives explicitly the many-valuedness of this function. Also, F_1 fulfills the following system of second order linear partial differential equations in $\mathbf{P}^2$:

$$AP_1\begin{cases} x(1-x)\dfrac{\partial^2 z}{\partial x^2}+y(1-x)\dfrac{\partial^2 z}{dx\,dy} \\ \qquad +[\gamma-(\alpha+\beta+1)x]\dfrac{\partial z}{\partial x}-\beta y\dfrac{\partial z}{\partial y}-\alpha\beta z=0, \\ y(1-y)\dfrac{\partial^2 z}{\partial y^2}+x(1-y)\dfrac{\partial^2 z}{\partial x\partial y} \\ \qquad +[\gamma-(\alpha+\beta'+1)y]\dfrac{\partial z}{\partial y}-\beta' x\dfrac{\partial z}{\partial x}-\alpha\beta' z=0. \end{cases}$$

This system of equations has three linearly independent solutions ([1, Chapter III], [11, 5.9]) ω_1, ω_2, ω_3, each of which is many-valued with ramification along the singular locus of (AP_1), which is the following arrangement of lines in $\mathbf{P}^2$: $L=\{x_0x_1x_2(x_1-x_2)(x_0-x_1)(x_2-x_0)=0\}$.



Projectivizing the solutions ω_i defines a many-valued map

$$\phi:\mathbf{P}^2\to\mathbf{P}^2$$
$$x\mapsto[\omega_1(x):\omega_2(x):\omega_3(x)]$$

which is the uniformizing map of $\pi: M \to X$. The idea is to show that the lift of ϕ to the universal cover is bimeromorphic onto the ball $\mathbf{B}^2 \subset \mathbf{P}^2$. However, there are difficulties at the four triple points of L, so we must desingularize. Doing this in the usual manner (by blowing up the four points in question) defines ϕ on a Del Pezzo surface.

To state the results on ϕ it is convenient to introduce the following notation. The proper transform of L together with the four blown up points is a very symmetric configuration of ten (-1) curves. There is a unique way to label them: E_{ij}, $i, j \in \{0, 1, 2, 3, 4\}$, such that $E_{ij} \cap E_{kl} \neq \phi \Leftrightarrow \{i, j\} \cap \{k, l\} = \varnothing$.

Let $\hat{b}(D)$ denote the branching degree of ϕ along the divisor $D \subset \hat{\mathbf{P}}^2$, where $\hat{\mathbf{P}}^2 =$ Del Pezzo surface. The result, proved in [10] and [85], is

Theorem 4.2.2. *Set* $\mu_1 = \beta$, $\mu_2 = \beta'$, $\mu_3 = 1 - \alpha$, $\mu_4 = 1 - \gamma - \alpha$, $\mu_0 = 1 - \gamma - 2\alpha$. *Then* $\phi = (\omega_1 : \omega_2 : \omega_3)$ *gives a uniformization, and* $(\hat{\mathbf{P}}^2, \hat{b})$ *is a uniformizable orbifold* $\Leftrightarrow$

$$\sum \mu_i = 2, \quad (1 - \mu_i - \mu_j)^{-1} \in \mathbf{Z} \cup \{\infty\},$$
$$b(E_{ij}) = (1 - \mu_i - \mu_j)^{-1} \quad \forall i, j \in \{0, 1, 2, 3, 4\}.$$

In these cases the universal uniformization (M, Γ) *is the complex* 2*-ball, and* $\Gamma \subset PSU(2, 1)$ *is discrete and properly discontinuous. If* $\mu_i + \mu_j = 1$, *then* E_{ij} *is covered by* (*the compactification*) *of a parabolic fixed point of* Γ *on* $\partial \mathbf{B}^2$. *If* $\mu_i + \mu_j > 1$ *for some* i, j, *then* E_{ij} *is covered on* M *by an elliptic fixed point of* Γ *on* $\mathbf{B}^2$.

There are 27 sets of $(\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$ which satisfy $(1 - \mu_i - \mu_j)^{-1} \in \mathbf{Z} \cup \{\infty\}$, $\sum \mu_i = 2$. Of these, seven yield co-compact groups acting freely, i.e., $\Gamma = \mathrm{Gal}(M \to \hat{\mathbf{P}}^2)$ has a subgroup of finite index $\Gamma^1 \subset \Gamma$ which acts fixed-point freely on $\mathbf{B}^2$ with compact quotient M^1, and $M \to \hat{\mathbf{P}}^2$ factorizes

$$\begin{array}{ccc} P : M & \longrightarrow & \hat{\mathbf{P}}^2 \\ & {\scriptstyle \pi^1} \searrow \quad \nearrow {\scriptstyle \pi} & \\ & M^1 & \end{array}$$

where π^1 is an infinite, unramified covering, and π is a finite branched covering with branch locus the union of the ten exceptional curves. In 11 cases, Γ is co-compact but does not act freely (i.e., it has elliptic fixed points). This occurs when $\mu_i + \mu_j > 1$ for some i, j. In the other nine cases, M^1 is not a compact quotient, as Γ^1 has fixed points on $\partial \mathbf{B}^2$, yielding cusps in M^1 which are resolved by elliptic curves (see the discussion in §3). There are even four cases where Γ^1 has both elliptic and parabolic fixed points. Also, some of the groups Γ^1 are found to be nonarithmetic (proofs in [10]), which was actually Mostow's original motivation for constructing these surfaces [68]. The branching degrees in all cases are listed in [33].

Remark 4.2.3. As is explained in [33, 4.1.5.6], for the 5-tuple $(\frac{3}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ the covering $M^1 \to \hat{\mathbf{P}}^2$ turns out to be an abelian variety. In this case ϕ is biholomorphic onto $\mathbf{C}^2 \subset \mathbf{P}^2$ (see also [10, §14]).

4.3. The constructions of Hirzebruch and Höfer. One of the examples of Mostow-Deligne, ♯4 in their list, with parameter values $\mu_i = \frac{2}{5}$ for all $i \in \{0, 1, 2, 3, 4\}$ has branching degrees

$$b(S_i) = \left(1 - \tfrac{2}{5} - \tfrac{2}{5}\right)^{-1} = 5$$

over all ten (-1) curves on the Del Pezzo surface $\hat{\mathbf{P}}^2$.

Hirzebruch realized that this surface could be constructed without using the uniformizing map of the hypergeometric differential equation, and he generalized this construction to yield an algebraic surface $Y(\mathscr{L}, n)$ for any line arrangement $\mathscr{L} \subset \mathbf{P}^2$ and any integer $n \geq 2$ [26]. In addition to the example of Mostow, he discovered two other arrangements, $\mathscr{L}_2$, $\mathscr{L}_3$, such that $Y(\mathscr{L}_2, 3)$, $Y(\mathscr{L}_3, 5)$ fulfill $c_1^2 = 3c_2$, and are therefore ball quotients.

4.3.1. The Kummer extension. Let $\mathscr{L} = \bigcup_{i=1}^k L_i$ be an arrangement of lines in $\mathbf{P}^2$, $L_i = \{l_i(x) = 0\}$ for linear forms l_i. Thus $\mathscr{L}$ is the zero set of $l_1 \dots l_k$. Since the quotients l_i/l_j are meromorphic functions, one can adjoin their nth roots to the function field $\mathscr{M}(\mathbf{P}^2)$ of $\mathbf{P}^2$:

$$\mathscr{A} = \mathscr{M}(\mathbf{P}^2)[\sqrt[n]{l_2/l_1}, \cdots, \sqrt[n]{l_k/l_1}].$$

This is a Kummer extension of the field $\mathscr{M}(\mathbf{P}^2)$ and has Galois group $(\mathbf{Z}_n)^{k-1}$. The function field $\mathscr{A}$ defines the birational equivalence class of a branched cover $X(\mathscr{L}, n) \to \mathbf{P}^2$, branched of degree n along all k lines of $\mathscr{L}$ and singular over all points of $\mathscr{L}$, which are not normal crossing, i.e., where more than two of the L_i pass.

The singularities of X are resolved by resolving the branch locus. Let $\tau(p) = \sharp\{L_i | p \in L_i\}$. Then blow up $\mathbf{P}^2$ at all points p such that $\tau(p) > 2$. Denoting this blow-up by $\hat{\mathbf{P}}^2$ we get a commutative diagram

$$\begin{array}{ccc} Y(\mathscr{L}, n) & \longrightarrow & X(\mathscr{L}, n) \\ \downarrow & & \downarrow \\ \hat{\mathbf{P}}^2 & \longrightarrow & \mathbf{P}^2 \end{array}$$

where $Y \to X$ is the minimal resolution of singularities of X. $Y \to \hat{\mathbf{P}}^2$ is branched along the proper transforms of the L_i, as well as along the exceptional curves $E_j \subset \hat{\mathbf{P}}^2$. The branching degree is n everywhere.

4.3.2. Three ball quotients. Let $t_r = \sharp r$-fold points$= |\{p \in \mathbf{P}^2 | \tau(p) = r\}|$, $f_0 = \sum_{r\geq 2} t_r$, and $f_1 = \sum_{r \geq 2} r t_r$. The Chern numbers of $Y(\mathscr{L}, n)$ are easy to calculate in terms of these combinatorial data of $\mathscr{L}$ ([26], here one also finds a complete discussion of the notation and terminology for line arrangements):

Lemma 4.3.3.

$$c_1^2(Y(\mathcal{L},n)) = n^{k-3}[n^2(9-5k+3f_1-4f_0) + 4n(k-f_1+f_0)+f_1-f_0+k+t_2],$$

$$c_2(Y(\mathcal{L},n)) = n^{k-3}[n^2(3-2k+f_1-f_0)+2n(k-f_1+f_0)+f_1-t_2].$$

The reader may verify $c_1^2 = 3c_2$ for the following three arrangements:

I. The simplicial arrangement $A_1(6)$ [19] which is the arrangement associated with the hypergeometric differential equation of the last section, $k=6$, $f_0=7$, $t_2=3$ and for $n=5$:

$$c_1^2(Y(A_1(6),5)) = 3c_2(Y(A_1(6),5)).$$

II. The arrangement $\mathcal{L}_{25}$ of the 12 lines of the Hasse pencil, defined by the reflection group ♯25 in the Shepard-Todd List [81], $k=12$, $t_2=12$, $t_4=9$ and for $n=3$:

$$c_1^2(Y(\mathcal{L}_{25},3)) = 3c_2(Y(\mathcal{L}_{25},3)).$$

III. The arrangement $A_3^0(3)$ coming from the nine inflection points of a smooth cubic by dualizing, $k=9$, $t_3=12$, and for $n=5$:

$$c_1^2(Y(A_3^0(3),5)) = 3c_2(Y(A_3^0(3),5)).$$

The first two of these are closely related to the ball quotients constructed as cyclic coverings by Livné as in §4.1. This was proved by Hirzebruch and Ishida [45]. The surface $Y(A_1(6),5)$ admits an action of $(\mathbf{Z}_5)^5$. There are many subgroups $G \subset (\mathbf{Z}_5)^5$ of order 25, which operate freely. Dividing Y by such a subgroup gives a surface with $c_1^2 = 3c_2 = 225$, and G can be chosen in such a way that $Y(A_1(6),5)/G = S_5(5)$, the ball quotient constructed by Livné. Similarly $Y((\mathcal{L}_{25},3))$ admits an action of $(\mathbf{Z}_3)^{11}$, and there are subgroups $G \subset (\mathbf{Z}_3)^{11}$ of order 3^5, which operate freely, the quotient being $S_3(9)_\Delta$ for the unique $\Delta \in \mathrm{Pic}(\Gamma(9))$ such that all automorphisms of $\Gamma(9)$ lift to $S_3(9)_\Delta$ [58, §2.3, Theorem 4].

The third example turns out to be just a different realization of the surface ♯24 in the Deligne-Mostow list, as is explained in [33, 5.2.5].

4.3.4. Höfer's theorem. T. Höfer, in his doctoral thesis in Bonn [33], has generalized Hirzebruch's results along the following lines. Let $\mathcal{L} \subset \mathbf{P}^2$ and $\hat{\mathbf{P}}^2$ be as above. Consider a Galois branched cover $\pi\colon Y \to \hat{\mathbf{P}}^2$ such that the branching degrees of π are n_i along L_i and m_j along E_j. Two natural questions arise: do such coverings exist?, and if they do, are there necessary and sufficient conditions (on $\mathcal{L}$, n_i, m_j) for the corresponding Y to be a ball quotient?

First, assume $Y \to \hat{\mathbf{P}}^2$ exists with given branching degrees.

Set $x_i = (n_i - 1)/n_i$, $y_j = (-m_j - 1)/m_j$, $d = \deg \pi$. Höfer derives the following formula:

$$\frac{3c_2(Y) - c_1^2(Y))}{d} = \frac{1}{4}\{{}^t xAx + \sum_{j=1}^{s} (2y_j + \sum_{p_j \in L_i} x_i)^2\}, \tag{4.3.5}$$

where $A = (A_{ij})$ is the matrix with entries

$$A_{ij} = \begin{cases} 3\sigma_i - 4, & i = j, \\ 2, & i \neq j,\ p = L_i \cap L_j,\ \tau(p) = 2, \\ -1, & i \neq j,\ p = L_i \cap L_j,\ \tau(p) \geq 3, \end{cases}$$

and $\sigma_i = \sharp\{p \in L_i | \tau(p) \geq 3\}$.

Let $\tau_i = \sharp\{p \in L_i | \tau(p) \geq 2\}$. An arrangement $\mathscr{L}$ is called *homogeneous* if $3\tau_i = k+3$ for all $i = 1, \cdots, k$. Höfer lists all possible weights n_i, m_j for all known homogeneous arrangements $\mathscr{L}$ such that the covering Y (if it exists) is a ball quotient. To state Höfer's main result, we introduce some notation. Set

$$G_i(x,y) = 2(\sigma_i - 1)x_i + \sum_{i \neq k} x_k + \sum_{p_j \in L_i} y_j, \qquad P_j(x,y) = 2y_j + \sum_{p_j \in L_i} x_i.$$

Let $\bar{L}_i$ be the reduced divisor in Y covering L_i in $\hat{\mathbf{P}}^2$, and $\bar{E}_j$ the reduced divisor covering the exceptional E_j in $\hat{\mathbf{P}}^2$. Then [33, 2.5.1]

$$-e(\bar{L}_i) + 2(\bar{L}_i)^2 = \frac{d}{n_i} G_i(x,y), \qquad -e(\bar{E}_j) + 2(\bar{E}_j)^2 = \frac{d}{m_j} P_j(x,y).$$

For a curve $C \subset Y$, the expression $2C^2 - e(C)$ has the following importance. If Y is a ball quotient, and $C \subset Y$ is totally geodesic, covered by a disc in $\mathbf{B}^2$ under $\mathbf{B}^2 \to Y$, then necessarily $2C^2 = e(C)$ (see [33, 1.2.4]), which is a relative version of Hirzebruch proportionality. Thus, the expression $2C^2 - e(C)$ is the deviation from fulfilling the proportionality. Höfer's main result states that this condition, if satisfied for all $\bar{L}_i$, $\bar{E}_j$ is also sufficient for Y to be a ball quotient. Using the above formula, a calculation gives

Theorem 4.3.6 [33, 3.5.2]. *Assume $Y \to \hat{\mathbf{P}}^2$ exists, with $\bar{L}_i$, $\bar{E}_j \subset Y$ as above. Then*

$$\left\{ \begin{matrix} Y \text{ is a (compact) ball} \\ \text{quotient} \end{matrix} \right\} \Leftrightarrow G_i(x,y) = P_j(x,y) = 0 \quad \forall_{i,j}.$$

By a slight modification, the theorem stays true while accommodating either elliptic or parabolic fixed points.

Elliptic: Set $m_j^* = -m_j$.

Parabolic: Set $m_j^* = \infty$.

In the elliptic case, some of the $\bar{E}_j$ are exceptional curves of the first kind, and can be blown down (Y was not minimal to begin with). In the parabolic case, the $\bar{E}_j$ are elliptic curves, which are the desingularization of cusps as discussed in §3. Set D =union of $\bar{E}_j$ which are elliptic curves. Then plugging m_j^* formally into the formula for $G_i(x,y)$ and $P_j(x,y)$, the corresponding result is:

$$\left\{ \begin{array}{c} Y - D \text{ is a noncompact} \\ \text{ball quotient} \end{array} \right\} \Leftrightarrow G_i(x,y), P_j(x,y) = 0 \quad \forall_{i,j}.$$

The proof of this fact uses a result of R. Kobayashi on noncompact ball quotients (possibly singular) [52] (see also §4.5).

Applying this result to the known homogeneous arrangements yields the following list of ball quotients (copied from [33, 5.3.3]):

Arrangement	k	$n_i's$	$m^*_{(3)}$	$m^*_{(4)}$	$m^*_{(5)}$	Method of Const.
$\mathscr{L}_{23}$	15	2	-4		4	(2)
(icosahedral)		5	5		1	(2)
$\mathscr{L}_{24}$	21	2	-4	∞		(2)
		3	∞	3		(1)
		4	8	2		(2)
$\mathscr{L}_{25}$	12	2		∞		(1)
(Hesse)		3		3		(1)
		4		2		(3)
$\mathscr{L}_{26}$	21	2, 2		∞	4	(2)
		2, 3		∞	2	(2)
		3, 9		3	1	(2)
		4, 2		2	2	(2)
		4, 6		2	1	(2)
$\mathscr{L}_{27}$	45	2	-4	∞	4	(2)

Here $\mathscr{L}_i$ means the arrangement defined by the unitary reflection group with classification number i in [81], and $m^*_{(\gamma)} = m_j^*$ for all γ-fold points p_j.

Höfer proves the existence of all listed coverings $Y \to \hat{\mathbf{P}}^2$ with the given branching degrees, utilizing one of the following three methods:

(1) Direct construction as Kummer extension (Hirzebruch's construction).

(2) Finding a normal subgroup $N \triangleleft \pi_1(\mathbf{P}^2 - \mathcal{L})$.

(3) Find a differential equation with uniformization as discussed in §4.2.

Remark 4.3.7. As it turns out, the existence of all of these examples follows from Kobayashi's Theorem 4.5.2, below, which was however proven later.

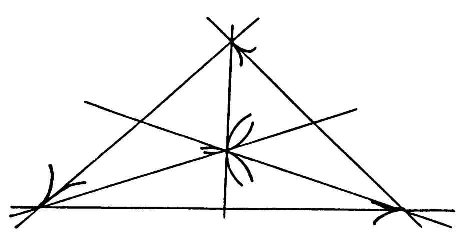
Remark 4.3.8. This theorem can also be used to give a different proof of the result 4.2.2 above, assuming the existence of a covering say by Kobayashi's theorem.

4.4. A $K3$ surface which is a ball quotient. In this section, we describe a ball quotient S_7 which is a covering of an elliptic $K3$ surface S which has three fibers of type I_7 and three of type I_1, due to I. Naruki [70]. Actually we will work just backward to the way he did. We start by constructing a double cover $S \to \mathbf{P}^2$ such that S is a $K3$ surface of the mentioned type. Then S_7 is constructed as a branch covering of S and finally S is identified with a covering of the ball quotient $S_7(7)$ constructed by Livné (see §4.1 and [58]).

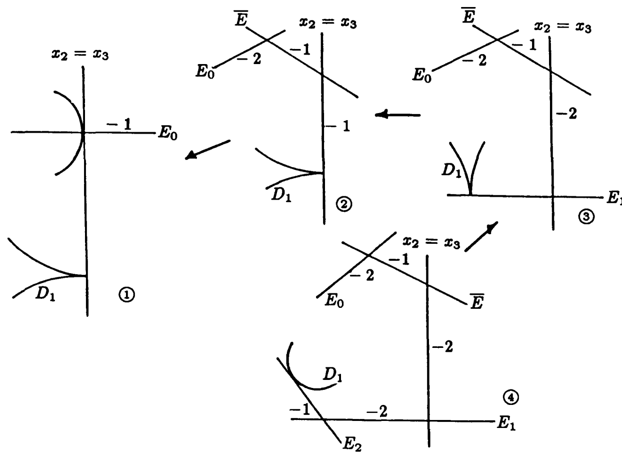
Let $D \subset \mathbf{P}^2$ be the singular sextic curve defined by

$$(x_1^2x_2 + x_2^2x_3 + x_3^2x_1 - 3x_1x_2x_3)^2 - 4x_1x_2x_3(x_1 - x_2)(x_2 - x_3)(x_3 - x_1) = 0.$$

It can be checked that this sextic has three singularities of type A_4 at the vertices $x_i = x_j = 0$ $(i, j \in (1, 2, 3))$ and a D_4 singularity at $(1 : 1 : 1)$ (of $x_1 = x_2 = x_3$) as drawn:



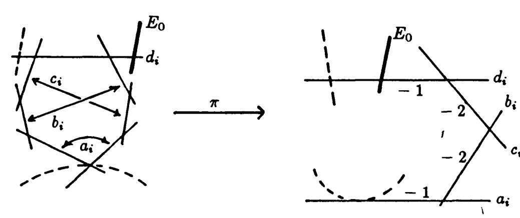
Let $X \to \mathbf{P}^2$ be the double cover. We resolve singularities in the standard way. First, blow up $(1 : 1 : 1)$, and let E_0 denote the exceptional curve. Then blow up the three vertices of the triangles four times. Consider for example the cusp D_1. Then the picture is



Let $S \to X$ be the resolution of singularities such that

$$\begin{array}{ccc} S & \longrightarrow & X \\ \downarrow & & \downarrow \\ \hat{\mathbf{P}}^2 & \xrightarrow{\ \rho\ } & \mathbf{P}^2 \end{array}$$

is a fiber square, where $\rho: \hat{\mathbf{P}}^2 \to \mathbf{P}^2$ is the modification mentioned above. The cover $\sigma \to E_0$ of the exceptional $\mathbf{P}^1$ is still a $\mathbf{P}^1$, and S fibers over it: $S \to \mathbf{P}^1$. We claim S is an elliptic $K3$-surface with three I_7's and three I_1's. S is obviously elliptic $K3$, since the sextic has only rational double points (see [6, §3]), and it remains to spot the singular fibers. The three I_1's are $\pi^{-1}(l_i)$ where l_i is a line tangent to D through $(1, 1, 1)$. There are class $D = 6 \cdot 5 - 4 \cdot 6 = 6$ such lines, the three lines $x_i = x_j$ and three others. The lines $x_i = x_j$ are components of the I_7 fibers, whereas the other three, tangent to the branch locus, have a simple node in the covering. At each A_4-cusp we have after resolution the following curve configuration:



(4.4.1)

The surface S has seven sections: one totally ramifying along E_0, and two each doubly covering the lines $x_i = 0$, $i = 1, 2, 3$. On S we have a configuration of 28 (-2) curves, which have a great deal of symmetry. In fact, Naruki shows that the combinatorial automorphism group is $PGL(2, \mathbf{F}_7)$, where $\mathbf{U}_7$ is the field of order 7 [70, Proposition 5.1]. We now sketch the idea of Naruki's proof of the following

Theorem 4.4.2. *There exists a Galois cover, of degree* 7^5, $\pi\colon \bar{S} \to S$, *branched of order* 7 *along all* 28 (-2) *curves, such that* $\bar{S}$ *is a smooth compact ball quotient,* π *factors as follows:*

$$\begin{array}{ccc} \pi\colon \bar{S} & \longrightarrow & S \\ \downarrow & & \uparrow \\ S_7(7) & \longrightarrow & E(7) \end{array}$$

where $S_7(7) \to E(7)$ *is the cyclic cover of the elliptic modular surface of degree* 7 *which is a smooth compact ball quotient. Both vertical covering maps have degree* 7^2.

Remark 4.4.3. A corresponding factorization has been proven for the surface $\pi\colon Y(A_1(6), 5) \to \mathbf{P}^2$ constructed by Hirzebruch, in [45].

To describe $\bar{S}$, we start with the Hermitian metric for $m \in \mathbf{N}$,

$$H_m(z) := z_1\bar{z}_1 + z_2\bar{z}_2 - (\zeta + \bar{\zeta})z_3\bar{z}_3, \quad \zeta = \exp(2\pi i/m),$$

and the corresponding special unitary group

$$SU_m = \{(3\times 3) \text{ matrices of determinant}_i = 1, \text{unitary with respect to } H_m\}.$$

This group is naturally isomorphic to the standard $SU(2, 1)$, and the corresponding ball

$$B_m = \{z \in \mathbf{C}^2 | H_m(z) < 0\}$$

is naturally isomorphic to the standard 2-ball $\mathbf{B}^2$. The group SU_m has a special discrete subgroup,

$$\Gamma_m = \{A \in SU_m | A_{ij} \text{ is an integer in the cyclotomic field } \mathbf{Q}(\zeta)\}.$$

In [5] it is proved that for $m = 5, 7, 8, 12$, Γ_m acts properly discontinuously on B_m with compact quotient B_m/Γ_m. We consider some subgroups of Γ_7. The principle ideal generated in $\mathbf{Q}(\zeta)$ by $(1 - \zeta)$ is prime; denote it by $\mathscr{P}$, and set

$$\Gamma_7' = \{A \in \Gamma_7 | A \equiv 1 \pmod{\mathscr{P}}\}, \qquad \Gamma_7'' = \{A \in \Gamma_7 | A \equiv 1 \pmod{\mathscr{P}^2}\}.$$

The combinatorial automorphism group $PGL(2, \mathbf{F}_7)$ of the 28 (-2)-curves on S is isomorphic to Γ_7/Γ_7'.

The surface $\bar{S}$ which we are looking for is B_7/Γ_7'', and S itself is isomorphic to B_7/Γ_7', so the Galois group of $\pi\colon \bar{S} \to S$ is Γ_7'/Γ_7''. $\pi_1(S_7(7))$ is a discrete subgroup of $SU(2,1)$, and

$$S_7(7) = \mathbf{B}^2/\pi_1(S_7(7)).$$

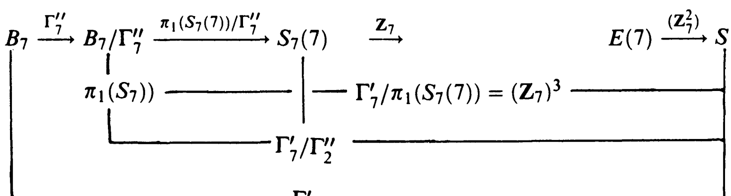
The proof of the theorem is based on the following exact sequences:

$$1 \to \pi_1(S_7(7)) \to N(7) \to \mathrm{Aut}(S_7(7)) \to 1,$$
$$1 \to \mathbf{Z}_7 \to \mathrm{Aut}(S_7(7)) \to \mathrm{Aut}(E(7)) \to 1,$$

and the fact that $\mathrm{Aut}(E(7)) \cong (\mathbf{Z}_7)^2 \times SL(2, \mathbf{F}_7)$. There is also a natural map $N(7) \to \mathrm{Aut}(E(7))$, and Naruki then proves

Lemma 4.4.4. *The group* Γ_7' *contains the kernel of* $N(7) \to \mathrm{Aut}(E(7))$, *and* Γ_7'' *is a subgroup of* $\pi_1(S_7(7))$.

This yields the following sequence of coverings:



Remark 4.4.5. The description of the coverings given shows that $B_7/\Gamma_7'' \to \hat{\mathbf{P}}^2$ is a Galois cover, branched over (the proper transform of) a curve of degree 12 in $\mathbf{P}^2$, the product of the singular sextic D above with the six lines of the arrangement $A_1(6)$. Here $\hat{\mathbf{P}}^2$ means $\mathbf{P}^2$ blown up to resolve the singularities of the sextic D, as above. Thus we have an example of a ball quotient which is a Galois cover of $\mathbf{P}^2$ over a curve which is not a line arrangement, a rare phenomenae. A similar example of a $K3$ surface which is the compactification of the ball is described in [38, VI].

4.5. The Miyaoka-Kobayashi inequality. We just mention briefly a generalization of the Miyaoka-Yau inequality $c_1^2(S) \leq 3c_2(S)$ for S a smooth surface of general type, the inequality being due to Miyaoka [63], and an implication (equality $\Rightarrow$ covered by the ball) being due to R. Kobayashi [52], [53]. The generalization consists in allowing S to be noncompact and to have quotient singularities.

A *Satake V-manifold* is a complex analytic space with at worst isolated quotient singularities. Let ds^2 be a Riemannian metric defined over the regular (smooth) part of a V-manifold X. ds^2 is a *V-metric* if it can be defined locally as the quotient of a metric which is smooth on the uniformization.

Kobayashi's first theorem gives an analogue to the existence of a Kähler-Einstein metric on a manifold with ample canonical bundle:

Theorem 4.5.1 [53, *Theorem* **1**]. *Let X be a minimal V-surface, and $C \subset X$ a divisor with at worst normal crossings singularity. Let $\bar{X} \to X$ be the minimal resolution of singularities, and $E = \sum_i E_i$ the exceptional divisor. Let $\mu_i \in \mathbf{Q}$ be defined by the condition that $K_{\bar{X}} + \sum \mu_i E_i$ is trivial near E. Assume*:

(1) $(K_{\bar{X}} + C) \cdot C_j \geq 0 \; \forall C_j \subset C$.

(2) $\kappa(K_{\bar{X}} + D + C) = 2$.

(3) *Every (-2)-curve $F \not\subset$ supp(E), which meets E, meets it in a component E_i which has $\mu_i > 0$.*

Then there exists a unique complete, Ricci-negative Einstein-Kähler V-metric on $X - C$ with finite volume.

In this more general setup one also gets a result corresponding to the famous Yau inequality, i.e., $c_1^2(S) \leq 3c_2(S)$ if K_S is ample, equality holding if and only if S is covered by the 2-ball. For smooth, noncompact surfaces X with compactification $\bar{X}$, $\bar{X} = X \cup D$, where $D = \sum D_i$ is a disjoint union of elliptic curves, the inequality

$$\bar{c}_1^2(\bar{X}, D) \leq 3\bar{c}_2(\bar{X}, D)$$

is proven in [52], with equality holding if and only if X is covered by the ball. Here one uses the logarithmic Chern classes, and the result is formulated precisely as in the noncompact case. The more complicated V-surface version is proved in [53] as a corollary of the above.

Theorem 4.5.2 [53, *Theorem* 2]. *Let X, C, $\bar{X}$, E be as above, fulfilling* (1)–(3) *in the above theorem. Then*

$$\left(K_{\bar{X}} + \sum M_i E_i + C\right)^2 \leq 3\left\{e(\bar{X}) - e(E) - e(C) + \sum_{p \in \operatorname{sing} X} \frac{1}{|G_p|}\right\},$$

where G_p is the subgroup of $U(2)$ corresponding to p. Moreover, equality holds if and only if $X - C$ is a quotient of the ball by $\Gamma \subset SU(2,1)$ which has isolated fixed points, $\bar{X} - C$ being the minimal resolution of $\mathbf{B}^2/\Gamma$.

The inequality part of the theorem was proved by Miyaoka, using algebro-geometric methods [63].

Examples 4.5.3. This example is due to Hirzebruch. Let X' be the complete intersection of the two Fermat hypersurfaces

$$F(x_0 : \cdots : x_4) = \sum x_i^5 = 0, \qquad G(x_0 : \cdots : x_4) = \sum x_i^{15} = 0.$$

Then X' has 50 singularities each of which is resolved in a smooth curve of genus 6 with self-intersection -5, and 1875 A_4-singularities. Let $\rho' \colon X \to X'$ be the resolution of the former 50 singularities and $\rho \colon \bar{X} \to X$ be the minimal resolution of the A_4's. Hirzebruch calculated

$$3c_2(\bar{X}) - c_1^2(\bar{X}) = 27{,}000,$$

while the contribution of the A_4's is $e(E_p) - 1/|G_p| = \frac{24}{5}$, so for X the above inequality becomes

$$3c_2(\bar{X}) - c_1^2(\bar{X}) \leq 3 \cdot 1875 \cdot \tfrac{24}{5} = 27{,}000,$$

so X is the quotient of $\mathbf{B}^2$ by a discrete group with isolated fixed points.

Our final example is due to Kes Ivinskis [46]. Let $\mathscr{L}_{23}$ be the icosahedral arrangement of 15 lines as in §4.3. Since $\mathscr{L}_{23} \approx 15 \cdot [\mathrm{H}]$ (linearly equivalent to 15 times the hyperplane class), we can take the Nth cyclic cover for any N dividing 15. Let $X(N)$ denote the cyclic cover of degree N branched over the arrangement $\mathscr{L}_{23}$. Then $X(5)$ has 30 singularities of the type $z_1^5 = z_2^3 z_3$ and 15 A_4 singularities. Ivinskis calculates $c_2(\bar{X}) = 195$, $c_1^2(\bar{X}) = 105$ and

$$3c_2(\bar{X}) - c_1^2(\bar{X}) = \sum_{p \in \operatorname{sing} X} \left(e(E_p) - \frac{1}{|G_p|} \right),$$

so X is covered by the ball. It can be shown that X is a quotient of one of the examples of Höfer listed in §4.3.

5. Fiber products

We now come to the easiest, and in one sense most powerful method, of constructing large families of surfaces described by A. Sommese [84]. Let $f \colon S \to C$ be a surface together with a holomorphic map onto a curve C. Denote by $\Sigma \subset C$ the singular locus of f, i.e., where $\operatorname{rk} f < 2$. Let $\pi \colon C' \to C$ be a branched Galois covering such that the ramification locus of π is disjoint from Σ. We then form the fiber product:

$$\begin{array}{ccc} S' & \longrightarrow & S \\ \downarrow & & \downarrow \\ C' & \longrightarrow & C \end{array}$$

5.1. Fiber products. The following is well known (or easy to prove).

Lemma 5.1.1 [84, 2.1]. *Assume $e(S) > 0$, and F is a typical fiber of $f: S \to C$. Then S' is minimal if S is, and*

$$\frac{c_1^2(S')}{c_2(S')} = \frac{dc_1^2(S) - 2\rho e(F)}{dc_2(S) - \rho e(F)},$$

where d and ρ are the sheet number and ramification number, respectively, of $C' \to C$.

The sheet number d is the degree of f. The ramification number is $\rho = \sum_{z \in C}(e_z - 1)$, where e_z = degree of ramification of f at z.

5.2. Density results. The idea is now to consider the *quotient* c_1^2/c_2, instead of the pair (c_1^2, c_2). By taking successive fiber products one gets a family $(S')_n$ with very good control on the growth of c_1^2/c_2.

Lemma 5.2.1 [84, 2.2]. *Assume S, C and F as above. Assume $g(F) > 1$ and $g(C) > 0$. The closure of the set of ratios $c_1^2(S')/c_2(S')$ as S' varies over all branched covers $C' \to C$ as above is the interval*

$$[c_1^2(S)/c_2(S), 2], \qquad c_1^2/c_2 < 2,$$

$$[2, c_1^2(S)/c_2(S)], \qquad c_1^2/c_2 > 2.$$

The proof of this is quite easy. Assume for simplicity $c_1^2/c_2 > 2$. Then

$$\frac{c_1^2(S')}{c_2(S')} = \frac{c_1^2(S)}{c_2(S)} + \left(\frac{2 - c_1^2(S)}{c_2(S)}\right)\left(\frac{-\rho e(F)}{dc_2(S) - \rho e(F)}\right) \geq 2, \tag{5.2.2}$$

$$\frac{c_1^2(S')}{c_2(S')} = 2 + \frac{d(c_1^2(S) - 2c_2(S))}{dc_2(S) - \rho e(F)} \leq \frac{c_1^2(S)}{c_2(S)}. \tag{5.2.3}$$

To prove the density it suffices by (5.2.2) to show that for every $p/q \in [0, 1]$, there exists a d-sheeted covering with branch number ρ and

$$\frac{-\rho e(F)}{dc_2(S) - \rho e(F)} = \frac{p}{q}.$$

In fact, take a $d' = -(q-p)e(F)$-sheeted unbranched cover $C' \to C$ of C (which is possible since $g(C) > 0$) and then a double cover $C'' \to C'$ branched at $\rho' = 2pc_2(S)$ generic points. Then for the composed cover $C'' \to C$ we have $d = 2(-(q-p)e(F))$, $\rho = 2pc_2(S)$ and

$$\frac{-pe(F)}{dc_2(S)-pe(F)} = \frac{-2pc_2(S)e(F)}{2(-(q-p)e(F)c_2(S)) - 2pc_2(S)e(F)}$$
$$= \frac{+p}{(q-p)+p} = \frac{p}{q}.$$

5.2.1 yields as a consequence:

Theorem 5.2.4 [84, 2.3]. *The closure, R, of the set of ratios $c_1^2(S)/c_2(S)$ of the Chern numbers of minimal models of general type surfaces is $[\frac{1}{5}, 3]$. In fact, for any $p/q \in [\frac{1}{5}, 3]$, there is a minimal surface of general type with $c_1^2/c_2 = p/q$.*

By 5.2.1 this is proved if we have one example of a surface S with $c_1^2 = 3c_2$ (i.e., a ball quotient) which fibers holomorphically onto a curve C of genus > 0, and one example of a minimal surface S' with $c_1^2 = \frac{1}{5}c_2$, which fibers holomorphically onto a curve C' with $g(C') > 0$. For S we can take, for example, the surface ♯4 in the [10]-list (i.e., Hirzebruch's $Y(A_1(6), 5)$ which is also a Galois cover of Livné's surface $S_5(5)$ [45]). A surface S' as above is easy to construct. (See e.g. [84].)

Remark 5.2.5. It is a generalization of this approach we will apply below in the 3-dimensional case to prove our density results (§8).

Part II. Some 3-Fold Geography

6. Fermat covers

In this section we describe a construction of algebraic three-folds, which yields as examples 3-dimensional analogues of the algebraic surfaces with positive index.

6.1. The construction. Corresponding to any arrangement $\mathscr{L}$ of planes in $\mathbf{P}^3(\mathbf{C})$, and any natural numbers $n \geq 2$, we construct a singular cover $X(\mathscr{L}, n)$ of $\mathbf{P}^3(\mathbf{C})$, branched along the given arrangement $\mathscr{L}$ with branching degree n everywhere. These coverings might best be described as 'iterated cyclic coverings'. We prefer to view them as singular complete intersections of Fermat hypersurfaces (whence the name Fermat covers). (See [41, §1].)

6.1.1. Combinatorial Data. Let $H_1, \cdots, H_k$ be k hyperplanes in $\mathbf{P}^3(\mathbf{C})$, given by linear forms $l_1, \cdots, l_k$. The *arrangement* $\mathscr{L} = \bigcup_{i=1}^k H_i$ is the zero set of the product $l_1 \cdots l_k$. Let t_q^1 denote the number of *q-fold lines* of the arrangement, i.e., one-dimensional linear subspaces of $\mathbf{P}^3(\mathbf{C})$ through which q of the H_i pass, and let t_p denote the number of *p-fold points* of the arrangement, i.e., points through which exactly p of the H_i pass. The set of all t_q^1's and t_p's, $q \geq 2$, $p \geq 3$, is called the *combinatorial data* of the given arrangement $\mathscr{L}$. The arrangement $\mathscr{L}$ is said to be in *general position* (in the combinatorial sense) if $t_q^1 = 0$ for all $q \geq 3$ and $t_p = 0$ for all $p \geq 4$. In this case the formulas

$$t_2^1 = \binom{k}{2}, \qquad t_3 = \binom{k}{3}$$

will hold. We speak of *singular lines and points of* $\mathscr{L}$ if $t_q^1 \neq 0$ for some $q \geq 3$ and $t_p \neq 0$ for some $p \geq 4$, respectively. The following formula for the t_q^1 will hold:

$$\sum_{q \geq 2} t_q^1 \binom{q}{2} = \binom{k}{2}.$$

If there are no singular lines, we get the following formula for the number of singular points:

$$\sum_{p \geq 3} t_p \binom{p}{3} = \binom{k}{3}.$$

If we admit both singular lines and singular points, we must consider the data t_{pq}: $\sharp\{$intersections of a q-fold line with a p-fold point$\}$. In this case we have the following formula for the combinatorial data:

$$\sum_{p \geq 3} t_p \binom{p}{3} - \sum_{q \geq 3} \left\{ \sum_{p} t_{pq} - t_q^1 \right\} \binom{q}{3} = \binom{k}{3}.$$

Two arrangements will be considered equivalent if they have the same combinatorial data, t_q^1, t_p and t_{pq}, and we call this equivalence class the *combinatorial type* of the arrangement.

6.1.2. The Kummer extension. Let $\mathscr{L}$ be an arrangement in $\mathbf{P}^3$ given by the linear forms $l_1, \cdots, l_k$. We assume from now on that $t_k^1 = t_{k-1}^1 = t_k = 0$ holds for the combinatorial data of $\mathscr{L}$. The *quotients* $l_2/l_1, \cdots, l_k/l_1$ are *meromorphic functions* on $\mathbf{P}^3$, and we can form the algebraic field extension $\mathscr{A}$ of the function field on $\mathbf{P}^3$ obtained by taking the nth roots:

$$\mathscr{A} := \mathbf{C}(x_1/x_0, x_2/x_0, x_3/x_0) \left[(l_2/l_1)^{1/n}, \cdots, (l_k/l_1)^{1/n} \right].$$

Here $\mathbf{C}(x_0/x_0, x_2/x_0, x_3/x_0)$ is the function field of $\mathbf{P}^3$, and $\mathscr{A}$ is a Kummer extension of this field and defines the function field (birational equivalence class) of a ramified covering $X = X(\mathscr{L}, n)$ of $\mathbf{P}^3$:

$$\pi : X \to \mathbf{P}^3.$$

π has covering degree n^{k-1}, and X is ramified along the arrangement $\mathscr{L}$ with branch locus the arrangement $\mathscr{L}$ and branching degree n along each plane H_i. The Galois group is $(\mathbf{Z}_n)^{k-1}$. X is smooth except for the loci lying over the singular lines and singular points of the arrangement $\mathscr{L}$. The singularities of X are as follows:

singularity of $\mathscr{L}$	*singularities of* X *lying over those of* $\mathscr{L}$
p-fold point	n^{k-p-1} singular points
q-fold point line	n^{k-q-1} singular curves

The singularities of X are orbits of the Galois group acting on X. If the arrangement $\mathscr{L}$ is in general position, then $X(\mathscr{L}, n)$ is smooth for all $n \geq 2$, and is in fact a *smooth complete intersection* of $k-4$ Fermat hypersurfaces in $\mathbf{P}^{k-1}$. $X(\mathscr{L}, n)$ is therefore a *degeneration* of a nonsingular complete intersection. For details and proof of these statements see [41].

6.2. Resolution of singularities. In this section we describe a resolution of singularities of $X(\mathscr{L}, n)$ and get a smooth, projective 3-fold $Y(\mathscr{L}, n)$ which is a branched cover of some blow-up of $\mathbf{P}^3$.

6.2.1. Near-pencil singular points. As it turns out, not all singularities of the arrangement $\mathscr{L}$ (and hence of X) are on equal footing.

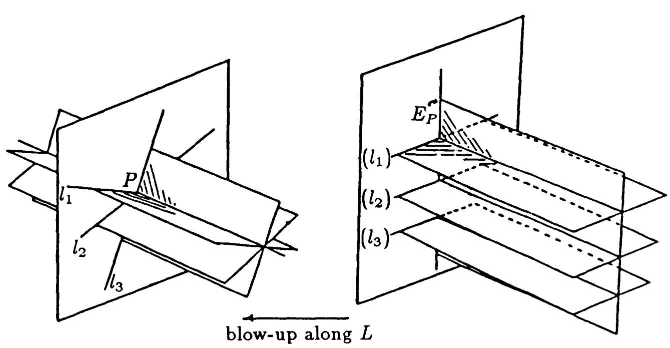


FIGURE 6.1 FIGURE 6.2

Definition. Let P be a p-fold point of the arrangement $\mathscr{L}$. P is called a *near-pencil* singular points $\Leftrightarrow$ there is a $(p-1)$-fold line L passing through P. The picture is as in Figures 6.1 and 6.2.

The reason these singularities must be treated separately is that in the process of resolution of singularities (6.2.2), they are resolved "en passant" in the process of resolving the singular line L passing through P (Figure 6.2).

If we blow up $\mathbf{P}^3$ at such a near pencil point, the exceptional divisor is a $\mathbf{P}^2$, and the *induced arrangement* (i.e., proper transform of all H_i through P) is of the following type. Such an arrangement of lines in $\mathbf{P}^2$ is called a near-pencil arrangement, which explains the name for these singularities.

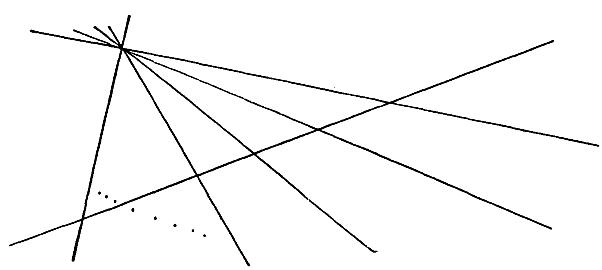


FIGURE 6.3

6.2.2. Embedded resolution. To resolve the singularities of X we use the standard method of resolving the singularities of the branch locus $\mathscr{L} \subset \mathbf{P}^3$. This is done in two steps:

$$\hat{\mathbf{P}}^3 \xrightarrow{\rho_2} \hat{\mathbf{P}}^3_1 \xrightarrow{\rho_1} \mathbf{P}^3, \qquad \rho = \rho_1 \circ \rho_2$$

Here ρ_1 = blow-up of $\mathbf{P}^3$ at all singular points *which are not near pencil singularities*, ρ_2 = blow-up of $\hat{\mathbf{P}}^3_1$ along the (proper transforms of) singular lines of the arrangement. Once this is done we have a new branch locus

$$\hat{\mathbf{P}}^3 \supset \tilde{\mathscr{L}} = \{\text{proper transforms of } \mathscr{L}\} \cup \text{ exceptional divisors.}$$

The picture is as in Figure 6.4.

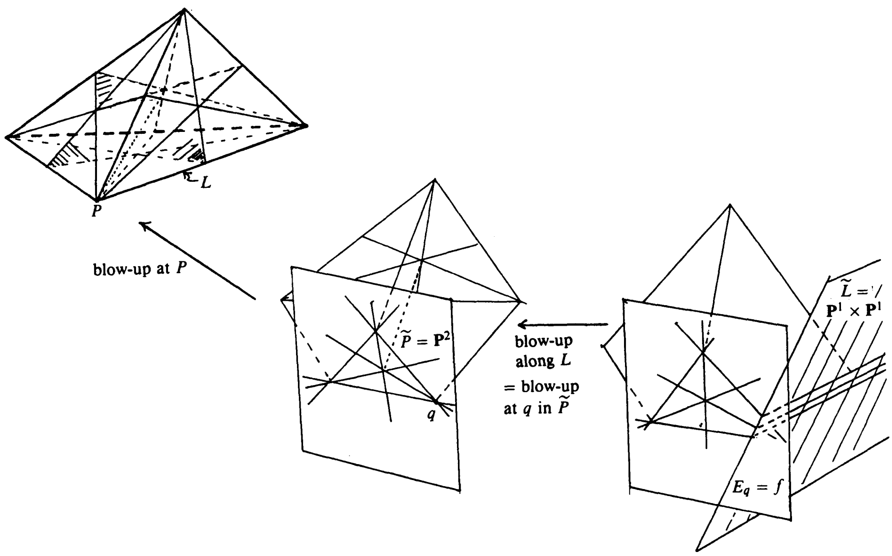


FIGURE 6.4

Now let Y be the Fox completion (= unique completion with a given branching behavior, see [15]) of the lift of $X - \operatorname{sing} X$ making the following diagram commute:

$$\begin{array}{ccc} Y & \longrightarrow & X \\ {\scriptstyle \hat{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ \hat{\mathbf{P}}^3 & \xrightarrow{\rho} & \mathbf{P}^3 \end{array}$$

We get a smooth projective 3-fold $Y(\mathscr{L}, n)$ covering $\hat{\mathbf{P}}^3$ and ramifying along $\tilde{\mathscr{L}}$. The branching degree of $\hat{\pi}\colon Y \to \hat{\mathbf{P}}^3$ is n along (the proper transforms of) all k-planes and along all exceptional divisors.

If we view X as a singular complete intersection of Fermat hypersurfaces, it is easy to see that this resolution is just the embedded resolution of $X(\mathscr{L}, n)$ in $\mathbf{P}^{k-1}$, which is the "canonical resolution" in the sense of toroidal embeddings.

6.2.3. Exceptional divisors. Let P be a p-fold point of the arrangement. P is resolved by a $\mathbf{P}^2$. In this $\mathbf{P}^2$ we have an induced line arrangement. Let $\tilde{P}$ be one of the n^{k-p-1} singular points of X covering P. Then, as is easy to see, $\tilde{P}$ is resolved by an exceptional surface S which is a Fermat cover of $\mathbf{P}^2$, branched along the induced line arrangement (see [26] and §4.3 above for details on this the surface case).

Let L be a q-fold line of the arrangement. L is resolved by a $\mathbf{P}^1 \times \mathbf{P}^1$ since the normal bundle of the proper transform of $\tilde{L}$ is $\mathcal{O}(1-\sigma) \oplus \mathcal{O}(1-\sigma)$, where $\sigma = \sharp\{\text{points blown up on } L\}$. Let $\tilde{L}$ be one of the n^{n-q-1} singular curves on X covering L. Then $\tilde{L}$ is resolved by a *product*

$$C_1 \times C_2 \to \mathbf{P}^1 \times \mathbf{P}^1,$$

where $C_1 \to \mathbf{P}^1$ is a Fermat cover of degree n^{q-1}, branched at q points, and $C_2 \to \mathbf{P}^1$ is a Fermat cover of degree $n^{\Sigma-1}$ branched at $\Sigma = \{\sigma + \sharp\text{planes which } L \text{ meets transversally}\}$ points.

6.3. Induced fiberings. First consider a p-fold point p_j of the arrangement L. Blowing up at p_j yields an exceptional $\mathbf{P}^2$, and $\hat{\mathbf{P}}^3_j = \{\mathbf{P}^3 \text{ blown up at } p\}$ fibers over the exceptional $\mathbf{P}^2$:

$$\begin{array}{ccc} \hat{\mathbf{P}}^3_j & \longrightarrow & \mathbf{P}^2 \\ \cup & & \Cup \\ l_x & \longmapsto & x, \end{array} \qquad l_x = \text{unique line with segment } x \text{ at } p_j.$$

We have a commutative diagram ($\hat{\mathbf{P}}^3$ now as in §6.2; blown up at all actual singularities):

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & \hat{\mathbf{P}}^3 \\ {\scriptstyle\hat{p}}\downarrow & & \downarrow{\scriptstyle p} \\ \pi_j^{-1}(\mathbf{P}^2) & \xrightarrow[\pi_j]{} & \mathbf{P}^2 \end{array}$$

Corresponding to the composition $\pi_j \circ \hat{p}\colon Y \to \mathbf{P}^2$ we have the Remmert-Stein factorization $Y \xrightarrow{f} S \xrightarrow{g} \mathbf{P}^2$, where S is a surface, f has connected fibers and g is finite-to-one. It is easy to see that g is just a Fermat cover (the resolving divisor, see §6.2.3), branched along the induced arrangement in $\mathbf{P}^2$. Recall that $\pi^{-1}(\mathbf{P}^2)$ consists of n^{k-p-1} disjoint components. These are *sections* of the map f. The generic fiber of f is a covering of $\mathbf{P}^1 = l_x$, branched at $k - p + 1$ points. However, it is important to note that f is in general *not flat*. Indeed, if l_x is a line in $\mathbf{P}^3$ passing through p_j and another (actual) singular point, then the exceptional divisor (of the other point) lies in the fibers of Y covering l_x. If we wish to have a flat map f, then we only resolve part of the branch locus (so the covering has singularities in the fibers, instead of exceptional divisors), and perform some other resolution of singularities (this is somewhat vague, but we will not need f flat in this paper).

Now consider a q-fold line l_μ of the arrangement. Blowing up along l_μ with exceptional divisor $\mathbf{P}^1 \times \mathbf{P}^1$, $\hat{\mathbf{P}}^3_\mu = \{\mathbf{P}^3$ blown up along $l_\mu\}$ fibers over $\mathbf{P}^1$:

$$\begin{array}{ccc} \hat{\mathbf{P}}^3_\mu & \longrightarrow & \mathbf{P}^1 \\ \cup & & \Psi \\ H_t & \mapsto & t, \end{array} \qquad H_t = \text{(element of) unique plane through } l_\mu \text{ with tangent } t \text{ there.}$$

Once again, we consider the diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & \hat{\mathbf{P}}^3 \\ {\scriptstyle \hat{\rho}}\downarrow & & \downarrow{\scriptstyle \rho} \\ \pi_\mu^{-1}(\mathbf{P}^1) & \xrightarrow[\pi_\mu]{} & \mathbf{P}^1 \end{array}$$

and the Remmert-Stein factorization of $\pi_\mu \circ \hat{\rho}$:

$$Y \xrightarrow{f} C \xrightarrow{g} \mathbf{P}^1.$$

Here f is flat and has generic fibers which are Fermat coverings of $\mathbf{P}^2$ $(= H_t)$, branched along $k - q + 1$ planes, and g is a Fermat cover of $\mathbf{P}^1$ branched at q points.

Remark. Everything we have done in §§6.1–6.3 can be done in arbitrary dimension [41].

6.4. Calculation of Chern numbers. The smooth algebraic 3-fold $Y(\mathscr{L}, n)$ has three Chern numbers, $c_1^3 = -K_Y^3$, $c_1c_2 = 24\chi(Y, \mathscr{O}_Y)$, where $\chi(Y, \mathscr{O}_Y)$ is the arithmetic genus, and $c_3 = \text{Euler}(Y)$, the Euler-Poincaré characteristic of Y. Using standard methods (basically using adjunction to determine the *classes* $c_1(Y)$, $c_2(Y)$ in the form $c_1(Y) = \pi^*(c_Y^1)$, $c_2(Y) = \pi^*(c_Y^2)$ for homology classes c_Y^1, c_Y^2 on $\hat{\mathbf{P}}^3$, and then using (known) products on $\hat{\mathbf{P}}^3$ to calculate $c_1^3(Y) = \deg \pi \cdot ((c_Y^1)^3)$, $c_1c_2(Y) = \deg \pi(c_Y^1 \cdot c_Y^2))$, these numbers can be calculated for $Y(\mathscr{L}, n)$ as an expression in the combinatorial data and the branching degree n. This is done in [41]. Dividing by n^{k-4}, the result is a cubic polynomial

$$\begin{aligned} c_1^3/n^{k-4} &= A_{13}n^3 + B_{13}n^2 + C_{13}n + D_{13}, \\ c_1c_2/n^{k-4} &= A_{12}n^3 + B_{12}n^2 + C_{12}n + D_{12}, \\ c_3/n^{k-4} &= A_3n^3 + B_3n^2 + C_3n + D_3, \end{aligned}$$

where the coefficients are given by the following formulas:

$$A_{13} = \Bigg\{(4-k)^3 - \overline{\sum}(3-p)^3 t_p - 2\sum_{q\geq 3}(2-q)^3\left[\overline{\sum} t_{pq} - t_q^1\right]$$

$$- 3(r-k)\sum_{q\geq 3}(2-q)^2 t_q^1 + 3\overline{\sum}(3-p)(2-q)^2\cdot t_{pq}\Bigg\},$$

$$B_{13} = \Big\{3k(r-k)^2 - 3\overline{\sum}(3-p)^2(p-1)t_p$$

$$- 3\sum_{q\geq 3}(2-q)^2(q-1)\left[\overline{\sum} t_{pq} - t_q^1\right]$$

$$- 3\sum_{q\geq 3}[2(4-k)(2-q)(q-1) + k(2-q)^2]t_q^1$$

$$+3\overline{\sum}[2(3-p)(2-q)(q-1) + (p-1)(2-q)^2]t_{pq}\Big\},$$

$$C_{13} = \Big\{3k^2(4-k) - 3\overline{\sum}(3-p)(p-1)^2 t_p$$

$$- 3\sum_{q\geq 3}(2-q)(q-1)^2\left[\overline{\sum} t_{pq} - t_q^1\right]$$

$$- 3\sum_{q\geq 3}[(4-k)(q-1)^2 + 2k(2-q)(q-1)]t_q^1$$

$$+ 3\overline{\sum}[(3-p)(q-1)^2 + 2(p-1)(2-q)(q-1)]t_{qp}\Big\},$$

$$D_{13} = \Big\{k^3 - \overline{\sum}(p-1)^3 t_p - \sum_{q\geq 3}(q-1)^3\left[\overline{\sum} t_{pq} - t_q^1\right]$$

$$- 3k\sum_{q\geq 3}(q-1)^2 t_q^1 + 3\overline{\sum}(p-1)(q-1)^2 t_{pq}\Big\},$$

$$A_{12} = \Bigg\{(4-k)\left[6 - 3k + t_2^1 + \sum_{q\geq 3}(q-1)t_q^1\right]$$

$$+ \overline{\sum}(3-p)\left[-t_{p_2} + (2p-3)t_p - \sum_{q\geq 3}(q-1)t_{pq}\right]$$

$$+ \sum_{q\geq 3}(2-q)\left[\overline{\sum} t_{pq} + t_{q+1,q} + 2(q-2)t_q^1 + (1-q)\left[\sum_{p\geq 4} t_{pq}\right]\right]\Bigg\},$$

$$
\begin{aligned}
B_{12} = \Bigg\{ & k\left[6 - 3k + t_2^1 + \sum_{q\geq 3}(q-1)t_q^1\right] \\
& + (4-k)\left[3k - 2t_2^1 - 2\sum_{q\geq 3}(q-1)t_q^1\right] \\
& + \overline{\sum}\left[(3-p)\left[3(1-p)t_p + 2\sum_{q\geq 3}(q-1)t_{pq} + 2t_{p_2}\right]\right. \\
& \qquad \left. - (p-1)\left[-t_{p_2} + (2p-3)t_p - \sum_{q\geq 3}(q-1)t_{pq}\right]\right] \\
& + \sum_{q\geq 3}\left[(q-1)\left[\overline{\sum} t_{pq} + t_{q+1,q} + 2(q-2)t_q^1 + (1-q)\sum_{p\geq 4} t_{pq}\right]\right. \\
& \qquad + (2-q)\left[-qt_q^1 + (1-q)\overline{\sum} t_{pq} - t_{q+1,q}\right. \\
& \qquad\qquad \left.\left. - 2(1-q)\left(\sum_{p\geq 4} t_{pq}\right)\right]\right]\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
C_{12} = \Bigg\{ & k\left[3k - 2t_2^1 - 2\sum_{q\geq 3}(q-1)t_q^1\right] + (4-k)\left(t_2^1 + \sum_{q\geq 3} qt_q^1\right) \\
& + \overline{\sum}\left[(p-1)\left[3(1-p)t_p - 2\sum_{q\geq 3}(1-q)t_{pq} + 2t_{p_2}\right]\right. \\
& \qquad \left. + (3-p)\left[pt_p - t_{p_2} - \sum_{q\geq 3} qt_{pq}\right]\right] \\
& + \sum_{q\geq 3}\left[(2-q)\left[(1-q)\left(\sum_{p\geq 4} t_{pq}\right) + q\left(\overline{\sum} t_{pq} - t_q^1\right)\right]\right. \\
& \qquad + (q-1)\left[-qt_q^1 + (1-q)\overline{\sum} t_{pq} - t_{q+1,q}\right. \\
& \qquad\qquad \left.\left. - 2(1-q)\left(\sum_{p\geq 4} t_{pq}\right)\right]\right]\Bigg\},
\end{aligned}
$$

$$D_{12} = \left\{ k \left[t_2^1 + \sum_{q\geq 3} q t_q^1 \right] + \overline{\sum}(p-1) \left[p t_p - t_{p_2} - \sum_{q \geq 3} q t_{pq} \right] \right.$$
$$\left. + \sum_{q\geq 3}(q-1) \left[(1-q) \left(\sum_{p\geq 4} t_{pq} \right) + q \left(\overline{\sum} t_{pq} - t_q^1 \right) \right] \right\},$$

$$A_3 = 4 - 3k + 2t_2^1 - t_3 + 2\sum_{q\geq 3}(q-1)t_q^1 + \overline{\sum}(p-1)t_p$$
$$+ \sum_{\substack{q\geq 3\\ p\geq 4}}(1-q)t_{pq} - \overline{\sum} t_{p_2},$$
$$B_3 = 3k - 4t_2^1 + 3t_3 - 4\sum_{q\geq 3}(q-1)t_q^1 - 3\overline{\sum}(p-1)t_p$$
$$+ 3\sum_{\substack{q\geq 3\\ p\geq 4}}(q-1)t_{pq} + 3\overline{\sum} t_{p_2},$$
$$C_3 = 2t_2^1 - 3t_3 + 2\overline{\sum} p t_p + 2\sum_{q\geq 3} q t_q^1 - \sum_{\substack{q\geq 3\\ p\geq 4}}(3q-2)t_{pq} - 3\overline{\sum} t_{p_2},$$
$$D_3 = t_3 + \sum_{\substack{q\geq 3\\ p\geq 4}} q t_{pq} + \overline{\sum} t_{p_2},$$

where $\overline{\sum}$ means the summation extended only over those p-fold singular points which are not near-pencil singular points (see 6.2.1).

6.5. Minimality. In this section we state some results proved in [41] and needed in the sequel. To state them, we need some preliminaries.

6.5.1. For varieties X with certain mild singularities (rational **Q**-Gorenstein) it makes sense to define a canonical divisor K_X as a **Q**-divisor (i.e., an element in $\mathrm{Div}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$). This is done as follows. Rational **Q**-Gorenstein implies that at any singular point $x \in X$ there is a number $r > 0$ such that $\omega_X^{[r]}$ is invertible (ω_X the dualizing sheaf [21, II.6], and $\omega_X^{[r]}$ means r-fold tensor product of the double dual; see [76]). The minimum such r is called the *index* of the singular point ([76], [77], [47], [48], [51] are general references for this material). Then K_X is defined to be the unique Weil divisor (**Q**-divisor) with $\omega_X^{[r]} = \mathscr{O}_X(rK_X)$. So K_X has denominator r.

6.5.2. A Weil divisor $D \subset X$ is called *nef* if $D \cdot C \geq 0$ for all effective curves $C \subset X$. A 3-fold X is called a *minimal model* if X has only rational **Q**-Gorenstein singularities and K_X is nef. For more on these concepts see, in addition to the above mentioned works, [49], [50]. Since Mori solved the flip conjecture last year the existence of minimal models of 3-folds is known. But minimal models need not be unique when they exist.

6.5.3. A (Cartier) divisor $D \subset X$ is called *ample* if some multiple of it is a hyperplane section (i.e., some power $\mathscr{L}^k$ of the line bundle $\mathscr{L} = [D]$ corresponding to D embeds X as described in [18, I.1] and [21, II,7]). P. M. H. Wilson has proved [89] for 3-folds that K_X ample $\Leftrightarrow K_X \cdot C > 0$ for all irreducible curves C. So K_X nef is somewhere between ample and not numerically effective (as studied by Mori [65]). By definition, K_X ample implies the pluricanonical map φ_m is an embedding for some m. The image of a pluricanonical map such that φ_m is an embedding is called the *canonical model.* It is a smooth, unique minimal model.

6.5.4. If K_X is not ample, the canonical model will have singularities, perhaps not surprisingly called canonical singularities. These singularities are studied in [76], [77], [78], [66]. They are characterized roughly as follows:

(i) terminal, (ii) not terminal.

The terminal singularities have the property that for any resolution $\tilde{X} \to X$ of the terminal singularity there is a curve C on the exceptional divisor with $K_{\tilde{X}} \cdot C < 0$. Since one can not resolve them without losing the nef property, one just leaves them alone. The nonterminal singularities are those which can be resolved with $K_{\tilde{X}}$ nef. Resolve these (there is of course no unique way of doing this) and we get a minimal model as above.

6.5.5. These things can be (and are in [41]) checked for Fermat covers. Since K_Y can be written in such a convenient form, $K_Y = \pi^*(K^Y)$, K^Y a rational linear combination of branch divisors, complete results can be obtained. In the following assume the arrangement $\mathscr{L}$ which Y is associated with fulfills:

The arrangement contains eight planes,

(1) no more than three of which pass through a line;

(2) no more than five of which pass through a point.

Also assume $t_k^1 = t_{k-1}^1 = t_{k-2}^1 = t_{k-3}^1 = 0$, where k = number of planes in $\mathscr{L}$ and for $n = 2$, $t_{k-4}^1 = t_{k-5}^1 = 0$.

Theorem 6.5.6. *Let $Y = Y(\mathscr{L}, n)$ be a Fermat cover of degree n, and assume*:

$$\textit{for } n = 2, \quad t_4 = t_5 = 0, k > 8, t_3^1 = 0,$$
$$\textit{for } n = 3, \quad t_4 = 0.$$

Then K_Y *is ample.*

The proof of the theorem uses the criterion $K_Y \cdot C > 0$ for all curves C of 6.5.3 and the observation that

$$\pi(C) \cdot K^Y > 0 \Rightarrow C \cdot K_Y > 0,$$

where $K_Y = \pi^*(K^Y)$ for $\pi\colon Y \to \hat{\mathbf{P}}^2$. The former condition is easily checked.

Theorem 6.5.7. *The canonical singularities which can occur on a Fermat cover* Y *are*

(i) *terminal* $n = 2, 4$-*fold point,*

(ii) *not terminal* $n = 2, 5$-*fold point,* 3-*fold line,*
$n = 3, 4$-*fold point.*

Corollary 6.5.8. *For Fermat covers* Y,

$$\left\{\begin{array}{c} Y \textit{ has no canonical} \\ \textit{singularities} \end{array}\right\} \Leftrightarrow K_Y \textit{ is ample.}$$

We introduce partial resolutions of the singular Fermat cover X:

$$Y \to X'' \to X' \to X,$$

where $X' \to X$ resolves all singularities which are not of the types listed in 6.5.7, $X'' \to X'$ resolves the nonterminal canonical singularities, and $Y \to X''$ resolves everything (the singularity for $n = 2, 4$-fold point is an ordinary double point, $x^2 + y^2 + w^2 + z^2 = 0$, and the resolving divisor is a quadric surface $\mathbf{P}^1 \times \mathbf{P}^1$. $Y \to X''$ blows all these quadric surfaces down). Notice that the construction leading to X'' is unique, starting with the function field of X, a birational invariant. Summing up we have [41, Theorem 2.4.10].

Fact 6.5.9. X'' is a unique minimal model of X. Furthermore,

(i) $K_Y \cdot C = 0 \Leftrightarrow C = \mathbf{P}^1$ is on the exceptional locus of $X'' \to X'$.

(ii) $K_Y \cdot C < 0 \Leftrightarrow C = \mathbf{P}^1$ is on the exceptional locus of $Y \to X''$.

(iii) $K_Y \cdot C > 0$ otherwise.

7. Interesting Fermat covers of general type

In this section we introduce our "zoo" of examples we have found by means of the construction of §6. But before we begin, it is necessary to say a few words about the birational behavior of Chern numbers of 3-folds. As opposed to the surface case where there exist unique minimal models, in the 3-fold case there is the difficulty of choosing a model (often times singular) which has certain desired properties. Only if we fix a unique model in each birational equivalence class does it make sense to speak of *the* Chern numbers of the manifold.

7.1. Birational behavior of Chern numbers. Let X be a smooth algebraic 3-fold, $p \in X$ a point and $C \subset X$ a curve. Let

$$\rho_p\colon X_p \to X \qquad (\rho_c\colon X_C \to X)$$

be the blow-up of X at p (along C, respectively). Then

$$\begin{aligned} c_1^3(X_p) &= c_1^3(X) - 8, & c_1^3(X_C) &= c_1^3(X) + c_1(N_X C), \\ c_1c_2(X_p) &= c_1c_2(X), & c_1c_2(X_C) &= c_1c_2(X), \\ c_3(X_p) &= c_3(X) + 2, & c_3(X_C &= c_3(X) + c_1(C). \end{aligned} \tag{7.1.1}$$

Thus the quotient $c_1^3/(c_1c_2)$ behaves erratically under blow-ups and we cannot deduce for example, as in the surface case, that a 3-fold (smooth) with $c_1^3 = \frac{8}{3}c_1c_2$ is automatically (relatively)[3] minimal.

7.1.2. As discussed in §6.5.1, if X is **Q**-Gorenstein, then K_X^3 can be defined as a rational number. Since χ, the arithmetic genus ($= c_1c_2/24$ for smooth X), is a birational invariant, we can define $\chi(X) = \chi(\bar{X})$ for any nonsingular model $\bar{X}$ of X. $c_3(X) = e(X)$ is just the topological Euler number of X. Thus, on the set of **Q**-Gorenstein spaces X there is a well-defined map

$$X \mapsto [-K_X^3, 24\chi(X), e(X)]. \tag{7.1.3}$$

As remarked by Kawamata, this map is well defined on birational equivalence classes of minimal models: $e(X)$ is invariant under flips, K_X *is* unique in codimension one (so K_X^3 is the same for any two minimal models), and χ is a birational invariant.

7.1.4. For the rest of this paper we shall be discussing 3-fold geography. By §§6.5.2, 6.5.3 and the remark in §7.1.2, this is a well-defined concept (with $[-K_X^3, 24\chi(X), e(X)] \in \mathbf{Q}^3$) for any minimal model. Since minimal models exist, we get a well-defined map

$$\left\{\begin{array}{c}\text{birational equivalence classes of}\\ \text{minimal models of general type}\\ \text{algebraic 3-folds}\end{array}\right\} \to \mathbf{Q}^3$$

$$X \mapsto [-K_X^3, 24\chi(X), e(X)]$$

which reduces to $X \mapsto [c_1^3(X), c_1c_2(X), c_3(X)]$ for any smooth minimal model (and therefore takes values in $\mathbf{Z}^3$ in that case).

[3]The adjective "relatively" denotes minimal in the sense of containing nothing which can be blown down smoothly as opposed to the previously used meaning.

7.2. Zones. For the remainder of this paper we will be considering the triple $(c_1^3(X), c_1c_2(X), c_3(X))$ determined by an algebraic 3-fold X of general type, either assuming K_X is ample or X is a smooth minimal model. Actually we consider only the ratios $[c_1^3(X) : c_1c_2(X) : c_3(X)]$ as determining a point in homogeneous coordinates in the rational projective plane $\mathbf{P}^2(\mathbf{Q})$. The idea is to determine for which $[x_0 : x_1 : x_2]$ in $\mathbf{P}^2(\mathbf{Q})$ there corresponds an actual algebraic 3-fold of general type. To organize the results somewhat, we introduce in this section different zones. To draw pictures we will be working in the affine chart $c_1c_2 = x_1 \neq 0$.

7.2.1. Forbidden zones. Let X be a smooth, minimal model of an algebraic 3-fold (smooth, minimal models do not in general exist). Then

$$\chi(X, \mathcal{O}_X) = \frac{c_1c_2(X)}{24} \leq 0, \qquad K_X^3 > 0. \tag{7.2.2}$$

The second statement is immediate from X general type and K_X nef, and the first is due to $c_1(X) = -K_X$ which implies

$$c_1c_2(X) = -K_X \cdot c_2(X) \leq 0,$$

since K_X is nef and $c_2(X)$ is numerically nonnegative [64, 6.6–6.7] under this assumption. Assume $K_X \cdot c_2(X) = 0$. Then either $c_2(X)$ is represented via Poincaré duality by a curve which maps to a point in the (pluri-)canonical map, or $c_2(X)$ is zero in $H^4(X, \mathbf{Z})$. In the former case, since X is smooth and minimal, the canonical model of X has only canonical singularities which are not terminal.

Now assume K_X is ample. Then the inequality above becomes $c_1c_2 < 0$ and (7.2.2) yields $c_1^3/(c_1c_2) < 0$. The zone $c_1^3/(c_1c_2) \leq 0$ is forbidden for X with K_X ample. (The zone $c_1^3/(c_1c_2) < 0$ is forbidden for smooth minimal models.)

Now, again assuming K_X ample, deep results of Yau [92] quoted in the introduction imply

$$-c_1^3 \leq -\tfrac{8}{3}c_1c_2.$$

So for X with K_X ample, $c_1^3/(c_1c_2) > \frac{8}{3}$ is a forbidden zone. Furthermore, if $c_1^3 = \frac{8}{3}c_1c_2$, then by Yau's theorem X is a ball quotient and by Hirzebruch proportionality (0.2) $c_1c_2 = 6c_3$. This describes the forbidden zones in the $c_1c_2 \neq 0$ chart. Globally, the strip of allowable $[x_0 : x_1 : x_2]$, $0 < x_0/x_1 \leq \frac{3}{8}$, is bounded by two projective lines which meet at infinity. Our job is to find where in this strip there correspond actual 3-folds.

7.2.3. Line CP (Cartesian product). Consider any 3-fold of the type $S \times C$, where both the surface S and the curve C are of general type. The Chern numbers of such a Cartesian product will lie on the line segment in $\mathbf{P}^2(\mathbf{Q})$ joining $[9 : 4 : 1]$ and $[3 : 6 : 5]$. We call this line CP. It follows from the results of Sommese [84] (see §5) that for any rational point on this line, there is actually an algebraic 3-fold of the type $S \times C$ with the given point as its Chern number ratio. (For more details on this, see §8.)

7.2.4. Zone SCI (smooth complete intersection). Let $X \subset \mathbf{P}^4$ be a smooth hypersurface of degree n. Using adjunction, it is easy to calculate the Chern classes and Chern numbers of X:

$$\begin{aligned} c_1(X) &= 5(5-n)H|_X, \\ c_2(X) &= (10-5n+5n^2)H^2|_X, \\ c_3(X) &= (10-10n+(5-n)n^2)H^3|_X, \end{aligned}$$

where H is a hyperplane in $\mathbf{P}^4$, and $H|_X$ means H restricted to X. This yields for the Chern numbers of X:

$$\begin{aligned} c_1^3(X) &= -n^4+15n^3-75n+125n, \\ c_1c_2(X) &= -n^4+10n^3-35n+50n, \\ c_3(X) &= -n^4+5n^3-10n^2+10n. \end{aligned}$$

Therefore, for $n \to \infty$ we have

$$[c_1^3 : c_1c_2 : c_3] \to [1:1:1].$$

Similar, but more complicated calculations could be made to calculate the limits of Chern numbers of complete intersections of several hypersurfaces; alternatively, use the formula for Fermat covers in §6.4 by setting all singular combinatorial data $t_p = t_q(1) = t_{pq} = 0$ for all $p \geq 4$, $q \geq 3$. This yields as limits of the Chern numbers of complete intersections of $k-4$ hyperplanes in $\mathbf{P}^{k-1}$ of degree n, as $n \to \infty$:

$$\begin{aligned} c_1^3 &\xrightarrow[n\to\infty]{} A_{13} = -k^3+12k^2-48k+64, \\ c_1c_2 &\xrightarrow[n\to\infty]{} A_{12} = -\tfrac{1}{2}k^3+\tfrac{11}{2}k^2-20k+24, \\ c_3 &\xrightarrow[n\to\infty]{} A_3 = -\tfrac{1}{6}k^3+\tfrac{3}{2}k^2-\tfrac{13}{3}k+4. \end{aligned}$$

In particular,

$$\begin{aligned} &\text{for } k=2 \quad [c_1^3 : c_1c_2 : c_3] \to [4:3:2], \\ &\text{for } k=3 \quad [c_1^3 : c_1c_2 : c_3] \to [27:18:10], \\ &\text{for } k\to\infty \quad [c_1^3 : c_1c_2 : c_3] \to [6:3:1]. \end{aligned}$$

Notice that the last point is the one in $\mathbf{P}^2(\mathbf{Q})$ represented by Cartesian products of the form $C_1 \times C_2 \times C_3$ for three curves of general type C_1, C_2 and C_3. It would appear that all of these limit points lie on a smooth curve, but I do not know the equation of the curve. Note furthermore that *all* nonsingular complete intersections lie in the geometric 3-sided figure formed by 1. Line CP, 2. The line $c_1^3 = 0$ and 3. The above mentioned curve. We call this zone the SCI zone (see the map in the appendix).

7.2.5. Zone E (empty or exotic). Consider the zone between $c_1^3/(c_1c_2) = \frac{8}{3}$ and the curve mentioned in 2.1.3 above. I know of no examples of algebraic 3-folds with ratios of Chern number lying here. Furthermore, calculations show that by introducing singularities to known examples (for instance to nonsingular complete intersections) we do not get ratios of Chern numbers lying here (see §10.2.2 below). I conjecture there are none, which is why I term this the empty zone, or alternatively, if there are any, they would seem to be exotic.

7.2.6. Zone F (Fermat). I define Zone F to be delineated by $c_1^3/(c_1c_2) = \frac{9}{4}$, $c_3 = 0$, and the extension of the curve used in §§7.2.4 and 7.2.5. This zone is the 3-dimensional analogue of the surface zone $c_1^2 > 2c_2$ (surfaces of positive index). There are lots of Fermat covers in this zone; compare the examples listed in my thesis [41], and see also §7.5 below.

7.2.7. Zone AC (algebraic cycle). This is the zone $c_3 > 0$, in other words, $c_3/c_1c_2 < 0$. Writing down the Euler number as sum of Hodge numbers,

$$\begin{aligned} c_3 &= 2 - 2b_1 + 2b_2 - b_3 = 2 - 4h^{1,0} + 2(2h^{2,0} + h^{1,1}) - (2h^{3,0} + 2h^{2,1}) \\ &= 2 - 4h^{1,0} + 4h^{2,0} - 2h^{3,0} + 2h^{1,1} - 2h^{2,1}, \end{aligned}$$

we get

$$\begin{aligned} c_3 > 0 \Leftrightarrow 4\chi(\mathcal{O}_X) + 2h^{1,1} &> 2h^{2,1} + 2h^{3,0} + 2 \\ 2\chi + h^{1,1} &> h^{2,1} + h^{3,0} + 1. \end{aligned}$$

Since probably χ and the geometric genus $h^{3,0}$ will be comparatively small, we imply as the meaning of this inequality

$$(*) \qquad h^{1,1} \text{ is large compared with } h^{2,1}$$

or in other words, there are lots of algebraic (or transcendental) cycles. We warn the reader that this conclusion might be quite wrong, but because of it we call the zone $c_3 > 0$ the 'algebraic cycle' zone.

7.3. Positive Euler number. Actually it was a surprise to the author that there exist any algebraic 3-folds of general type with positive Euler number. This was because in analogy to the surface case (where $c_1^2 > 0$, $c_2 > 0$), for smooth complete intersection 3-folds $c_1^3 < 0$, $c_1c_2 < 0$ and $c_3 < 0$. It seemed natural that all Chern numbers in even (odd) dimensions might be positive (negative). Conclusions of this type are always dangerous. Examples of positive Euler number will be used in §8.2, so we now discuss particular examples of Fermat covers with $c_3 > 0$ in more detail.

Examples 7.3.1. Let $\mathscr{L}$ be the arrangement in $\mathbf{P}^3$ consisting of the six facet planes of the cube. This arrangement has four-fold points. The singularities on $X(\mathscr{L}, 4)$ are resolved on $Y(\mathscr{L}, 4)$ by nonsingular quartic surfaces S (in $\mathbf{P}^3$) (see §6.2.3). As discussed in §6.3, $Y(\mathscr{L}, 4)$ fibers over S:

$$f\colon Y \to S.$$

The generic fiber of f is a Fermat cover of the generic line in $\mathbf{P}^3$ passing through the 4-fold point S resolved, a Fermat quartic curve (of genus 3). K_Y is ample (criteria for this are given in §6.5). The Chern numbers of Y are

$$c_1^3 = -5 \cdot 4^2, \quad c_1c_2 = -12 \cdot 4^2, \quad c_3 = 10 \cdot 4^2.$$

To determine the degeneracy locus of the map f, we need only determine the locus (in the exceptional $\mathbf{P}^2$ in the branch locus $\tilde{\mathscr{L}}$ in the notation of §6) of all lines in $\mathbf{P}^3$ which pass through the 4-fold point, and in addition, through the line in $\mathbf{P}^3$ which is the intersection of the two remaining planes (i.e., those *not* passing through the 4-fold point). This is a line on the exceptional $\mathbf{P}^2$, and it is covered on S by the intersection of S with another quartic in the $\mathbf{P}^3$ in which S is a hypersurface. However, this intersection will not be smooth, as is seen as follows. S covers the exceptional $\mathbf{P}^2$ with branch locus four lines:

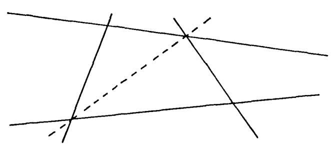


FIGURE 7.1

The dotted line indicates the image of the singular locus on the $\mathbf{P}^2$, so its cover on S has eight singularities which are locally of the form $y^4 + x^4 = 0$. Also, the Euler number is calculated to be 8.

The generic singular fiber of f is the singular quartic in Figure 7.2, where each component is a rational curve with normal bundle $\mathscr{O} \oplus \mathscr{O}(-4)$. The singularities in the fibers over the singular points of the degeneracy locus are resolved by Fermat quartic surfaces.

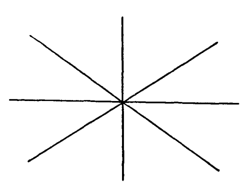


FIGURE 7.2

Of course, similar statements will hold for any degree n. For the Fermat cover $Y(\mathscr{L}, S)$ of degree 5 we also have positive Euler number. The Chern numbers in this case are

$$c_1^3 = -40 \cdot 5^2, \quad c_1c_2 = -48 \cdot 5^2, \quad c_3 = 6 \cdot 5^2.$$

In this case Y fibers over a quintic surface S, which is a surface of 'special general type' (low quotient c_1^2/c_2), and K_Y is again ample.

Example 7.3.2. Let $\mathscr{L}$ be the arrangement consisting of the 12 facet planes of the dodecahedron. $\mathscr{L}$ has 15 4-fold points and 12 5-fold points, and no singular lines. Let $Y = Y(\mathscr{L}, 2)$ be the smooth Fermat cover. Y has two kinds of fiberings corresponding to the 4- and 5-fold points:

$$\begin{array}{ccc} & Y & \\ f_2 \swarrow & & \searrow f_1 \\ S_2 & & S_1 \end{array}$$

where S_1 is a quadric surface $\mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$ resolving divisor of an ordinary node, and S_2 is a $(2,2)$-complete intersection in $\mathbf{P}^4$, which is $\mathbf{P}^2$ blown up five times. The fibers of f_1 are curves of genus 49 and those of f_2 are curves of genus 17.

The Chern numbers of Y are

$$c_1^3 = -79 \cdot 2^8, \quad c_1c_2 = -72 \cdot 2^8, \quad c_3 = 10 \cdot 2^8.$$

However note that Y is not the minimal model of §6.5; we must blow down all the quadric surfaces to nodes. This reduces the Euler number by three per node. There are $15 \cdot 2^{12-5} = 15 \cdot 2^7$ such nodes. Let $\tilde{Y}$ denote the blown down variety. Then $c_3(\tilde{Y}) = c_3(Y) - 3 \cdot 15 \cdot 2^7 = -25 \cdot 2^7$ is no longer positive. This example illustrates how important it is to specify the model being used.

Example 7.3.3. Let $\mathscr{L}$ be the arrangement of eight planes consisting of the six facet planes of a cube, the plane at infinity and one further plane passing through three of the corners of the cube. The combinatorial data of this arrangement are $t_2^1 = 28$, $t_5 = 3$, $t_4 = 3$ and $t_3 = 14$. Let Y be the Fermat cover for $n = 3$. Then Y has again two kinds of fiberings corresponding to the two kinds of singular points (cf. §6.3):

$$\begin{array}{ccc} & Y & \\ f_2 \swarrow & & \searrow f_1 \\ S_2 & & S_1 \end{array}$$

where S_1 is a smooth cubic surface in $\mathbf{P}^3$ and S_2 is a $(3,3)$ complete intersection in $\mathbf{P}^4$. S_2 is again a surface of 'special' general type, meaning its quotient c_1^2/c_2 is near the lower bound of $1/5$. The generic fibers of f_1

are curves of genus 55, and those of f_2 are curves of genus 10. The Chern numbers of Y are

$$c_1^3 = -40 \cdot 3^2, \quad c_1c_2 = -40 \cdot 3^2, \quad c_3 = 14 \cdot 3^2.$$

Here again K_Y is not ample. However, K_Y is nef, so Y is a minimal model.

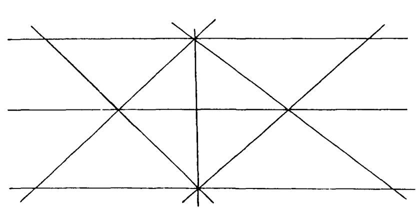
Example 7.3.4. Now let $\mathscr{L}$ be the simplicial arrangement $A_1^3(10)$ consisting of the four facet and six symmetry planes of the tetrahedron. For more on simplicial arrangements in 3-space see [20] and [41, 2.1.2]. This arrangement is pictured in Figure 6.4. It is the 3-dimensional arrangement corresponding to the singular locus of the hypergeometric differential equation in three variables. For homogeneous coordinates $[x_0 : x_1 : x_2 : x_3]$ on $\mathbf{P}^3$,

$$\mathscr{L} = \{x_0x_1x_2x_3(x_0 - x_1)(x_2 - x_3)(x_3 - x_1)(x_2 - x_0)(x_3 - x_0)\}.$$

The Fermat cover $Y(\mathscr{L}, 3)$ is a compactification of a ball quotient, one of the seven examples of [10] in dimension 3 ([41, 4.6.3], see also §7.6 below). We consider here $Y(\mathscr{L}, 2)$. Y has fiberings

$$\mathbf{P}^1 \xleftarrow{\;f\;} Y \xrightarrow{\;g\;} S$$

where S is the elliptic modular surface $\Gamma(4)$, a $K3$-surface [83]. S is a Fermat cover of $\mathbf{P}^2$ branched along $A_1(6)$ in the notation of §4.3, which is $K3$. The 16 (-2)-curves covering the exceptional $\mathbf{P}^1$'s resolving the 3-fold points of the arrangement are the 4^2 sections of $\Gamma(4)$, and the $6 \cdot 4 = 24$ components of the six singular fibers of $\Gamma(4)$ cover the six lines of $A_1(6)$. The map g has as generic fiber a curve of genus 5, while f, mapping onto $\mathbf{P}^1$, has fiber the Fermat cover of $\mathbf{P}^2$ along the following line arrangement (for $n = 2$):



This is a surface of general type with $c_1^2 = 2 \cdot 2^5$ and $c_2 = 4 \cdot 2^5$ and ratio $c_1^2/c_2 = \frac{1}{2}$. Thus the surface is of 'somewhat special' general type. Y has the Chern numbers

$$c_1^3 = -3 \cdot 2^6, \quad c_1c_2 = -3 \cdot 2^6, \quad c_3 = 7 \cdot 2^6.$$

The fiber space $g: Y \to S$ has a beautiful structure which can be described quite explicitly. The generic fiber, a (Fermat) cover of $\mathbf{P}^1$ branched along five points, is a curve of genus 5. The singular locus on S is the union of the six singular fibers and the 16 sections. This is the total inverse image of the ten (-1) curves on $\hat{\mathbf{P}}^2$ (after resolution, see §4.2). Studying the description of the induced fiberings (§6.3) we find in this case: the fiber of $Y \to S$ over the smooth part of the I_4 fibers is an elliptic curve; the fiber over the 16 sections and the double points of the fibers I_4 are singular fibers of type I_4. Thus, denoting by θ_i one of the components of one of the I_4 fibers of S, we see

$$g|_{g^{-1}(\theta_i)}: Y|_{g^{-1}(\theta_i)} \to \theta_i$$

is itself an elliptic modular surface $\Gamma(4)$! We leave it as an exercise for the reader to verify the Euler number $c_3(Y)$ using this explicit description of the fiber space $g: Y \to S$.

Recently this 3-fold has been identified with the compactification of the Siegel modular 3-fold of level 4, defined as follows (see [42]): the Siegel upper half-space of degree (genus) 2 is

$$\mathcal{S}_2 = Sp(2, \mathbf{R})/U(2).$$

Notice that this bounded symmetric domain is isomorphic to the domain $\mathbf{E}^3$ (noncompact dual of the quadric hypersurface in $\mathbf{P}^4$) mentioned in §10.1 below by the exceptional isomorphisms of Lie algebras:

$$sp(2, \mathbf{R}) \cong so(5), \qquad u(2) \cong su(2) + u(1) \cong so(3) + so(2).$$

The Siegel modular group is $Sp(2, \mathbf{Z})$, and the main congruence subgroup of level N, $\Gamma(N) \subset Sp(2, \mathbf{Z})$, is defined by the sequence

$$1 \to \Gamma(N) \to Sp(2, \mathbf{Z}) \to Sp(2, \mathbf{Z}_N) \to 1.$$

$\Gamma(N)$ acts properly discontinuously and freely for all $N \geq 3$. The quotient $\Gamma(N)\backslash\mathcal{S}_2$ can be compactified to a normal algebraic variety with 1-dimensional singular locus [3]. In 1967, Igusa (Math. Ann. **168**, 228–260) showed that the monoidal transformation along the singular locus desingularizes the singular Baily-Borel compactification. We denote this smooth 3-fold by $X(N)$. Then $Y \cong X(4)$, where Y is the example discussed above. We just sketch the argument.

Consider the group $\Gamma(2) \subset Sp(2, \mathbf{Z})$. Since $-I \in \Gamma(2)$, $P\Gamma(2)$ acts on $\mathcal{S}_2$ (with fixed points), the quotient, however, is smooth (also due to Igusa). It has been known for nearly a century (see for example Lee-Weintraub, Topology **24** (1985), 391–410 for some references) that the compactification of this modular 3-fold is isomorphic to the desingularization of the Segre Cubic, the cubic 3-fold given in $\mathbf{P}^5$ by the equations

$$\sum_{i=0}^{5} x_i = 0; \qquad \sum_{i=0}^{5} x_i^3 = 0.$$

(This is the unique cubic 3-fold with ten ordinary double points, the maximal possible number of such.) On the other hand, it is easy to see that this desingularization is the same as $\hat{\mathbf{P}}^3$ above, i.e., $\mathbf{P}^3$ blown up at five points and along ten lines of the arrangement $A_1^3(10)$. Since $P\Gamma(4) \subset P\Gamma(2)$, we have a covering $X(4) \to \hat{\mathbf{P}}^3$ $(= X(2))$, branched along the compactification divisors of $X(2)$ (= proper transform of the ten planes of $A_1^3(10)$ together with the five exceptional $\mathbf{P}^2$'s of blown-up points) as well as along Humbert surfaces (= ten exceptional $\mathbf{P}^1 \times \mathbf{P}^1$'s). The degree of the cover is $d = [P\Gamma(2) : P\Gamma(4)] = 2^9$; in fact the Galois group is $P\Gamma(2)/P\Gamma(4) \cong (\mathbf{Z}_2)^9$, which is the Galois group of $Y \to \hat{\mathbf{P}}^3$! This suffices to see $Y \cong X(4)$.

This description also explains the appearances of the elliptic modular surfaces $\Gamma(4)$ on Y; they are precisely the compactification divisors which are not disjoint since $Sp(2, \mathbf{R})$ has rank 2.

Steve Weintraub and Ronnie Lee have calculated the Hodge numbers of this Seigel variety. The Hodge diamond is

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 6 & & 226 & & 6 & \\
15 & & 0 & & 0 & & 15 \\
 & 6 & & 226 & & 6 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

Furthermore, $h^{1,1}$ *is* generated by algebraic cycles (Humbert surfaces and compactification divisors). Hence this example certainly does support our supposition that there are lots of algebraic cycles, and is of great importance in §8 below. Finally we remark that although K_Y is not ample, it is nef, so Y is a minimal model.

7.3.5. Other examples. There are two other examples of general type 3-folds with positive Euler number, Fermat covers of the arrangements $\mathrm{Ceva}^3(2)$ and $\mathrm{Ceva}^3(2, 1)$, respectively (see [41, 2.1.3]), for $n = 2$. These are arrangements of 12 and 13 planes respectively. Since we will not use these examples, we just list the Chern numbers:

$\mathrm{Ceva}^3(2)$	$\mathrm{Ceva}^3(2,1)$
$c_1^3 = -52 \cdot 2^8$	$c_1^3 = -92 \cdot 2^9$
$c_1c_2 = -36 \cdot 2^8$	$c_1c_2 = -57 \cdot 2^9$
$c_3 = 8 \cdot 2^8$	$c_3 = 2 \cdot 2^9$

To end this section we gather the information of the above examples in tabular form.

Example	Structure of Fiber	Yau quotient $c_1^3/(c_1c_2)$
1. $n = 4 \begin{cases} Y \to S \\ S\ K3 \text{ surface} \end{cases}$	$\begin{cases} \text{Fermat quartic curve} \\ \subset \mathbf{P}^2, g = 3 \end{cases}$	$5/12 = .42$
$n = 5 \begin{cases} Y \to S \\ S \text{ quintic surface} \end{cases}$	$\begin{cases} \text{Fermat quintic curve} \\ \subset \mathbf{P}^2, g = 6 \end{cases}$	$.83$
2. $Y \to S_1$, $Y \to S_2$	$g = 17$	1.09
$S_1 = \mathbf{P}^1 \times \mathbf{P}^1, S_2$ del Pezzo surface	$g = 49$	
3. $Y \to S_1$, $Y \to S_2$	$g = 10$	1
S_1 cubic surface, $S_2(3,3)$ complete inter $\subset \mathbf{P}^4$	$g = 55$	
4. $Y \xrightarrow{f_1} S$, $Y \xrightarrow{f_2} \mathbf{P}^1$	$g = 5(\text{fiber of } f_1)$ $c_1^2/c_2 = \frac{1}{2}(\text{fiber of } f_2)$	1
$S = \Gamma(4)$, $K3$-surface	$c_1^2/c_2 = \frac{1}{2}(\text{fiber of } f_2)$	
5. $\mathrm{Ceva}^3(2)$		1.44
$\mathrm{Ceva}^3(2,1)$		1.61

7.4. Dual fibering structures. In this section we introduce an interesting phenomenon, which will not be used, however, in the rest of this paper. Let Y be an algebraic manifold of dimension n, and $D_1 \subset Y$ and $D_2 \subset Y$ two subvarieties of dimensions n_1 and n_2, respectively, with $n_1 + n_2 = n$. Suppose Y has two structures of fiber space:

$$D_2 \xleftarrow{f_2} Y \xrightarrow{f_1} D_1$$

Definition 7.4.1. We say Y has a *dual fibering structure* if the fiber of f_1 is isomorphic to D_2 and the fiber of f_2 is isomorphic to D_1, but Y is *not birational* to the Cartesian product $D_1 \times D_2$.

We can prove the existence of many such Y.

Theorem 7.4.2. *Given any curve C and surface S which are Fermat coverings:*

$$C \to \mathbf{P}^1, \qquad S \to \hat{\mathbf{P}}^2,$$

each of degree n, there exists a Fermat cover $Y \to \hat{\mathbf{P}}^3$ of degree n such that Y has a dual fibering structure.

Proof. Let $\Lambda \subset \mathbf{P}^2$ be the line arrangement with which the cover S is associated (as in §4.3). The *near-pencil* arrangement $\mathscr{L} \subset \mathbf{P}^3$ associated with Λ is defined as follows. Let $\mathbf{P}^2 \subset \mathbf{P}^3$ be any hyperplane in $\mathbf{P}^3$, and $\Lambda \subset \mathbf{P}^2$ the given line arrangement. Take any point $p \in \mathbf{P}^3$ not on the given $\mathbf{P}^2$. $\mathscr{L}$ is defined as the arrangement consisting of (1) the given $\mathbf{P}^2$ and (2) all planes passing through p and a line of $\Lambda \subset \mathbf{P}^2$. The picture is as in Figure 7.5.

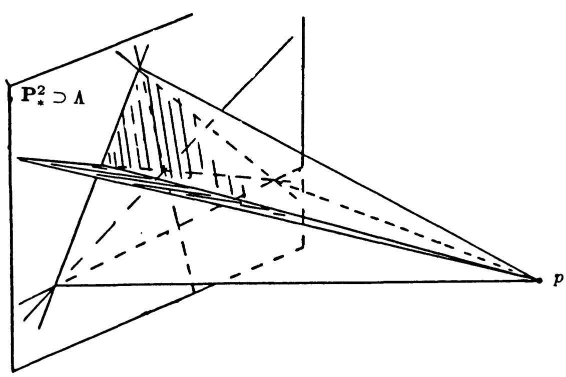


FIGURE 7.5

Let ν be any natural number $\nu \geq 1$. Add any ν planes $H_1, \cdots, H_\nu$ to $\mathscr{L}$, all of them through *one* of the lines of Λ. For any n, let Y be the Fermat cover associated with the arrangement $\mathscr{L} \cup H_1 \cup \cdots \cup H_\nu$. Y has two fiberings (see §6.3)

$$\begin{array}{ccc} & Y & \\ {}^{f_1}\swarrow & & \searrow^{f_2} \\ C & & S \end{array}$$

where $S \to \hat{\mathbf{P}}^2$ covers the 'pencil-point' p of the arrangement, while C is a cover of $\mathbf{P}^1$ branched at $(\nu + 2)$ points. It is clear that the fiber of f_1

is isomorphic to S, while the fiber of f_2 is isomorphic to the curve C. It remains to show that Y is not birational to the Cartesian product $S \times C$. To see this, it is sufficient to notice that the fiber of f_1 covers S with as many sheets as it meets the fiber of f_2, which is $n^{\nu+1}$ times. q.e.d.

It is interesting to note that for growing n the Chern numbers do converge to those of a Cartesian product (see question 10.2.5 in §10 below).

Example 7.4.3. Consider the arrangement drawn in Figure 7.5, the near pencil arrangement associated with $A_1(6)$. Set $\nu = 2$ and let Y be the Fermat cover for $n = 2$. Then Y has the structure of elliptic 3-fold over the elliptic modular surface $\Gamma(4)$, with singular locus a disjoint union of four components of singular fibers. Here we have fibers of type I_4 once again. Since the branching locus consists of nine planes, this example is "honestly elliptic", i.e., $\kappa(Y) = 2$.

Example 7.4.4. Take the same arrangement, $\nu = 1$ and $n = 3$. Let Y be the Fermat cover, and $Y \to S$ be a fiber space over the compactification of a ball quotient (♯2 in the [10]-list), with fiber an elliptic curve $(x^3+y^3+w^3 = 0)$. Y also fibers over an elliptic curve, with fiber this compactification of a ball quotient. So here once again (by additivity of Kodaira dimension) $\kappa(Y) = 2$.

7.5. Zone F; characteristic ratios. As mentioned above, Fermat covers yield examples of 3-folds which are in my opinion the 3-dimensional analogue of surfaces with positive signature. These are 3-folds X whose Chern numbers determine points in the Zone F defined above. The analogy to surfaces with positive signature is as follows: if $\tau(S) > 0$ for a surface S, then the quotient c_1^2/c_2 is larger than the corresponding quotient for $\mathbf{B}^1 \times \mathbf{B}^1$-quotients. Zone F was defined by $c_1^3/(c_1c_2) > \frac{9}{4}$, which is the corresponding value for (compact, smooth) $\mathbf{B}^2 \times \mathbf{B}^1$-quotients. In this section we give some examples.

7.5.1. The characteristic ratios of an arrangement. In §6.4 we have calculated the Chern numbers $c_1^3(Y(\mathscr{L}, n))$, $c_1c_2(Y(\mathscr{L}, n))$ and $c_3(Y(\mathscr{L}, n))$ for $Y = Y(\mathscr{L}, n) \to \hat{\mathbf{P}}^3$ a smooth Fermat cover of degree n associated with the arrangement $\mathscr{L} \subset \mathbf{P}^3$, and given these formula as cubic polynomials in n. The characteristic ratios of the arrangement $\mathscr{L}$ are the ratios of the leading coefficients A_{13}, A_{12} and A_3 of these cubic polynomials. Only two of them are independent:

$$\begin{aligned}
\gamma_1 &:= \frac{A_{13}}{A_{12}} = \lim_{n\to\infty} \frac{c_1^3(Y(\mathscr{L}, n))}{c_1c_2(Y(\mathscr{L}, n))}, \\
\gamma_2 &:= \frac{A_3}{A_{12}} = \lim_{n\to\infty} \frac{c_3(Y(\mathscr{L}, n))}{c_1c_2(Y(\mathscr{L}, n))}.
\end{aligned} \tag{7.5.2}$$

Thus, plotting (γ_1, γ_2) in the $c_1c_2 \neq 0$ chart of $\mathbf{P}^2(\mathbf{Q})$ will yield accumulation points of Chern numbers of 3-folds (here K_Y will be ample for $n > 3$ at least). In [41] these ratios are listed for about all interesting arrangements known.

Examples 7.5.3. The arrangement for which the ratio γ_1 is highest known is the arrangement named $\text{Ceva}^3(3)$ in [41]. This is an arrangement of 18 planes defined by an imprimitive reflection group. Whereas in Zone F γ_1 is large, γ_2 will be small. $\text{Ceva}^3(3)$ has largest known γ_1 and smallest known γ_2. They are

$$\gamma_1 = \frac{995}{414} = 2.4034\cdots, \qquad \gamma_2 = \frac{90}{414} = .2174.$$

The covering $Y(\mathscr{L}, 5)$ is very interesting. The arrangement $\mathscr{L}$ has 6-fold points and 9-fold points. The induced arrangements in the exceptional $\mathbf{P}^2$'s are $A_1(6)$ and $A_3^0(3)$ as described in §4.3.2. Thus for $n = 5$ all those exceptional $\mathbf{P}^2$'s are covered by ball quotients which lie on $Y(\mathscr{L}, 5)$ as totally geodesic manifolds. The Chern numbers of $Y = Y(\mathscr{L}, 5)$ are

$$\text{(7.5.4)} \qquad \begin{aligned} &c_1^3(Y) = -63564.5^{14}, \quad c_1c_2(Y) = -24912.5^{14}, \\ &c_1^3/(c_1c_2) = 2.55, \quad c_3(Y) = -4542.5^{14}. \end{aligned}$$

There is also an interesting coincidence of ratios: the three arrangements $\mathscr{L} = A_1^3(10)$, $\mathscr{L} = A_1^3(16) = \text{Ceva}^3(2,4)$ and $\mathscr{L} = G_{25,920}$ (see [41, 2.6]) all have

$$\text{(7.5.5)} \qquad \gamma_1 = \frac{61}{26} = \frac{488}{208} = \frac{13176}{5616}, \qquad \gamma_2 = \frac{6}{26} = \frac{48}{208} = \frac{1296}{5616},$$

where the three numbers are the leading coefficients for the different arrangements. Other examples can be found in [41, 2.6]. The points in $\mathbf{P}^2(\mathbf{Q})$ determined by (7.5.4) and (7.5.5) are drawn below in the atlas.

The single example of Fermat cover with $c_1^3/(c_1c_2)$ highest is for the arrangement $\text{Ceva}(2) = A_1^3(12)$ for $n = 4$. Here $c_1^3/(c_1c_2) = 3564/1380 = 2.58$ and $c_3/(c_1c_2) = 240/1380 = 4/21 = .19$.

7.6. Ball quotients. As in the surface case it is interesting to try to use our constructions to yield ball quotients. Let $Y \to \hat{\mathbf{P}}^N$ be as above, $B = \sum B_i$ the *reduced* ramification locus in Y, and B_i the irreducible components. Let $D \subset B$ be the subset of components which are complex tori. Assume also the different components of D are pairwise disjoint. The following higher dimensional analogue of Höfer's Theorem 4.3.6 above was proved in [41]:

Theorem 7.6.1. *Set Y, B, D as above and $N = \dim Y$. Then, if $N \geq 3$,*

$$\left\{ \begin{matrix} Y - D \text{ is a ball} \\ \text{quotient} \end{matrix} \right\} \Leftrightarrow \left\{ \begin{matrix} \text{Each } B_i - D \text{ is a} \\ \text{subball quotient} \end{matrix} \right\}.$$

B_i is a subball quotient if:

(i) *$B_i - D$ is a ball quotient ($(N-1)$-dimensional),*

(ii) $\bar{c}_j(Y,D)|_{B_i - D} = ((N+1)/(N+1-j))\bar{c}_j(B_i, B_i \cap D)$.

As a corollary we get:

Corollary 7.6.2. *There are no Fermat covers $Y \to \hat{\mathbf{P}}^N$, $N \geq 3$, such that Y is a compact ball quotient.*

This follows from that for $N = 3$, if the arrangement has singular lines, in resolution of singularities we get components in the branch locus of the type $C_1 \times C_2$, C_i a curve. So by 7.6.1, Y cannot be a ball quotient, since $C_1 \times C_2$ is a $\mathbf{B}^1 \times \mathbf{B}^1$-quotient. If the arrangement has no singular lines, it is easy to prove that no covering can be a ball quotient (see [41, 4.6.2]).

The following are, however, two examples of Fermat covers which are noncompact ball quotients, and have appeared in the [10] list:

$$\begin{gathered} \mathscr{L} = A_1^3(10), \quad n = 3, \quad c_1^3 = -172 \cdot 3^6, \\ c_1c_2 = -72 \cdot 3^6, \quad D^3 = -60 \cdot 3^6, \quad \text{so } \bar{c}_1^3(Y,D) = \tfrac{8}{3}\overline{c_1c_2}(Y,D), \\ \mathscr{L} = A_1^3(12), \quad n = 3, \quad c_1^3 = -896 \cdot 3^8, \\ c_1c_2 = -360 \cdot 3^8, \quad D^3 = -192 \cdot 3^8, \quad \text{so } \bar{c}_1^3(Y,D) = \tfrac{8}{3}\overline{c_1c_2}(Y,D). \end{gathered} \tag{7.6.3}$$

The first example above has recently been identified with the compactification of the Picard modular 3-fold corresponding to the group $SU((3,1), \mathscr{O}_{\sqrt{-3}}(1-\rho)^2), \rho = e^{2\pi i/3}$ (see [42]).

Just as in the surface case we can admit more general coverings $Y \to \hat{\mathbf{P}}^N$ with arbitrary branching degrees along the branch divisors. This was done in [41], and using 7.6.1 the following result was proven (here $N = 3$):

Theorem 7.6.4 [41, 4.6.2]. *Besides the seven examples of Deligne-Mostow* [10] *there is precisely one Galois covering $Y \to \hat{\mathbf{P}}^3$ ramifying along a known arrangement and the exceptional divisors introduced in the resolution of singularities, such that Y is the compactification of a ball quotient.*

For the proof we refer the reader to [41]; here we just describe the new example mentioned. The arrangement is $A_1^3(24)$; the arrangement defined in $\mathbf{R}^3$ by symmetry planes of the regular 24-cell in $\mathbf{R}^4$ [20], or the arrangement defined in $\mathbf{P}^3(\mathbf{C})$ by the unitary reflection group of order 576. $A_1^3(24)$ may also be described as follows: Start from the cube and the octahedron inscribed into it, use the six facet planes of the cube, the eight facet planes of the octahedron, the nine symmetry planes and the plane at infinity. This is sometimes called the desmic figure. All line arrangements induced in the 24 planes are $A_2^2(13)$ in [19]. The arrangement has 3-fold and 4-fold lines, and 9-fold points.

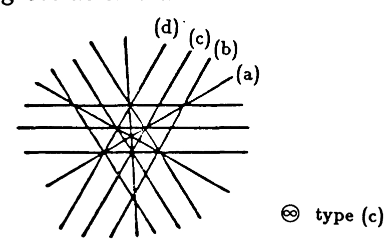
Theorem 7.6.5. *Consider the arrangement* $A_1^3(24)$ *with the following branching degrees*:

2	*24 planes*
2	*exceptional* $\mathbf{P}^2$*'s*
-4	*exceptional* $\mathbf{P}^1 \times \mathbf{P}^1$*'s over 3-fold lines*
∞	*exceptional* $\mathbf{P}^1 \times \mathbf{P}^1$*'s over 4-fold lines.*

The covering $Y \to \hat{\mathbf{P}}^3$ *with these branching degrees has a disjoint union* $D = \sum D_i$ *of complex tori over the 4-fold lines. Over the 3-fold lines are* $\mathbf{P}^1$*-bundles which are exceptional and may be blown down. Let* $p\colon Y \to \bar{Y}$ *be this blown-down. Then* $\bar{Y} - p(D)$ *is a ball quotient.*

Proof. By 7.6.1 we must show that the divisors B_i covering the 24 branch planes and the exceptional $\mathbf{P}^2$'s are subball quotients. This follows from the following two lemmas.

Lemma 7.6.6. *Let* $\mathscr{L}$ *be the line arrangement* $A_2^2(13)$ *and branching degrees as stated.*

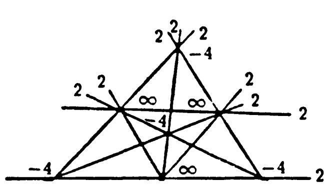


branching degree	
type (a):	2
type (b):	2
type (c):	4
type (d):	∞
3-fold pts.:	-4
4-fold pts.:	2

Then there is a covering $D \to \hat{\mathbf{P}}^2$ *which is the compactification of a ball quotient.*

Proof. Use the formula (4.3.5). The existence of the covering follows from 4.5.2.

Lemma 7.6.7. *Consider the arrangement* $A_2^1(9)$ *with branch degrees*



lines:	2
3-fold pts.:	-4
4-fold pts.:	∞

Then there is a covering $D \to \hat{\mathbf{P}}^2$ *which is the compactification of a ball quotient.*

Proof. Same as above. (This is one of the 27 examples in the 2-dimensional list in [10].) The fact that these divisors are subball quotients is just as easily checked [41, 4.6.3, Corollary 4.2.4]).

The existence of this covering follows from the 3-dimensional analogue of 4.5.2, also due to R. Kobayashi [41, Theorem 3.3.2].

8. Fiber products and density results

In this section we come to the main results of this paper. Our goal, as mentioned in the introduction, was to give a 3-dimensional generalization of Sommese's results in [84] (see §5).

8.1. Fiber products. We use the same set-up as in [84]. Let Y be an algebraic 3-fold possessing a fibering to a curve:

$$f\colon Y \to C.$$

Let $\pi\colon C' \to C$ be a finite branched cover of degree d and ramification degree $\rho = \sum(e_p - 1)$, where e_p is the degree of π at the point p. Let $Y' = Y \times_\pi C'$ be the fiber product:

$$\begin{array}{ccc} Y' & \xrightarrow{\bar{\pi}} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ C' & \xrightarrow{\pi} & C \end{array}$$

Assume that $C' \to C$ is branched away from the degeneracy locus of f. Let S be a generic fiber.

Lemma 8.1.1. *Set $\bar{\pi}\colon Y' \to Y$ as above. Then*

$$\begin{aligned} c_1^3(Y') &= dc_1^3(Y) - 3\rho c_1^2(S), \\ c_1c_2(Y') &= dc_1c_2(Y) - \rho[c_1^2(S) + c_2(S)], \\ c_3(Y') &= dc_3(Y) - \rho c_2(S). \end{aligned}$$

Furthermore, $K_{Y'}$ is ample if K_Y is.

The proof is standard.

In what follows we shall be interested in the ratios $c_1^3/(c_1c_2)$ and $c_3/(c_1c_2)$ of Y, resp. Y'. Set for convenience $c_1^2 = c_1^2(S)$ and $c_2 = c_2(S)$. An easy calculation gives

Lemma 8.1.2.

$$\frac{c_1^3(Y')}{c_1c_2(Y')} = \frac{c_1^3(Y)}{c_1c_2(Y)} + \left(\frac{3c_1^2}{c_1^2 + c_2} - \frac{c_1^3(Y)}{c_1c_2(Y)}\right)\left(\frac{-\rho[c_1^2 + c_2]}{dc_1c_2(Y) - \rho[c_1^2 + c_2]}\right),$$

$$\frac{c_3(Y')}{c_1c_2(Y')} = \frac{c_3(Y)}{c_1c_2(Y)} + \left(\frac{c_2}{c_1^2 + c_2} - \frac{c_3(Y)}{c_1c_2(Y)}\right)\left(\frac{-\rho[c_1^2 + c_2]}{dc_1c_2(Y) - \rho[c_1^2 + c_2]}\right).$$

The practical application of the above formula is

Lemma 8.1.3. *Suppose $g(C) \geq 1$. Then by taking fiber products corresponding to coverings $C' \to C$ as above, we can construct an algebraic 3-fold with*

$$c_1^3(Y')/c_1c_2(Y') = \alpha, \qquad c_3(Y')/c_1c_2(Y') = \beta$$

for any rational pair (α, β) in the interval with endpoints

$$A = \left(\frac{c_1^3(Y)}{c_1c_2(Y)}, \frac{c_3(Y)}{c_1c_2(Y)}\right), \qquad B = \left(\frac{3c_1^2}{c_1^2 + c_2}, \frac{c_2}{c_1^2 + c_2}\right).$$

Proof. Exactly as in [84, 2.2] (§5.2 above), it is sufficient to show that for any rational number $p/q \in [0, 1]$, we can find a covering $\pi\colon C' \to C$ with d and ρ such that

$$\frac{-\rho[c_1^2 + c_2]}{dc_1c_2(Y) - \rho[c_1^2 + c_2]} = \frac{p}{q}.$$

To do this, set $\rho = -pc_1c_2(Y)$, $d = (q - p)(c_1^2 + c_2)$. The existence of such a cover is assured by our assumption that $g(C) \geq 1$ (take a $d/2$-degree unbranched cover, then a double cover with ρ branch points). q.e.d.

Notice that the endpoint $B = (3c_1^2/(c_1^2 + c_2), c_2/(c_1^2 + c_2))$ of the line $\overline{AB}$ always lies on the line segment in the $(c_1^3/(c_1c_2), c_3/(c_1c_2))$ plane, which passes through the points $(\frac{9}{4}, \frac{1}{4})$ and $(2, \frac{1}{3})$. This is just the line CP discussed in §7.2.3 above! Thus we have Figure 8.1 (the dashed line indicates what we get by taking fiber products).

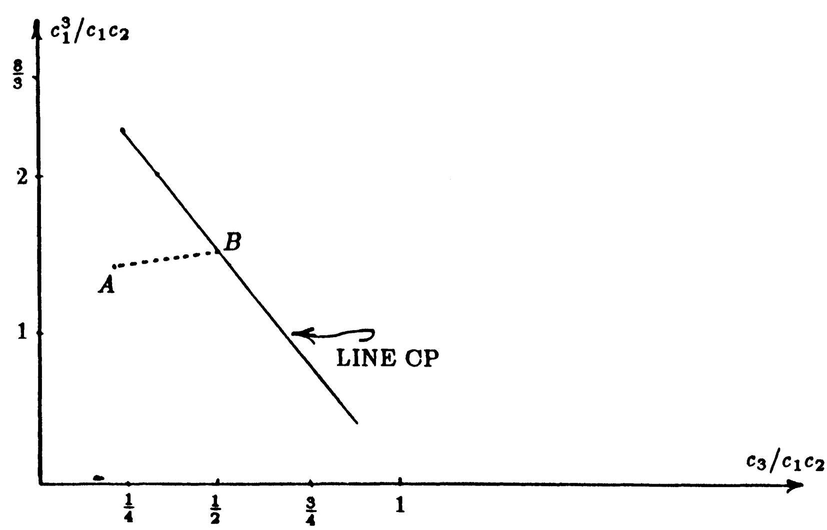


FIGURE 8.1

8.2. Density results. It is our goal to have some area in the $(c_1^3/(c_1c_2), c_3/(c_1c_2))$ plane dense in the sense of 3-fold geography, that is, for any rational point $(\alpha, \beta) \in D$ in some designated area D in the plane, we wish to know that there exists an algebraic 3-fold Y (with ample K_Y ample) such that Y has as its ratios of Chern numbers the given point (α, β). As it turns out, Fermat covers of $\mathbf{P}^3$ provide just the right structure to do this.

Consider a Fermat cover $Y(\mathscr{L}, n) \to \hat{\mathbf{P}}^3$ which has fiberings onto *two* different curves (by §6.3 this means the arrangement $\mathscr{L}$ has two different types of singular lines l_1 and l_2, say):

$$C_2 \xleftarrow{\;f_2\;} Y \xrightarrow{\;f_1\;} C_1$$

Let S_i be the corresponding fiber of f_i.

Claim. *If l_1 and l_2 do not meet (in $\mathbf{P}^3$), then S_2 and S_1 fiber onto C_1 and C_2 respectively.*

Proof. The fiber S_2 is the Fermat cover of the generic plane passing through l_2. Since l_1 and l_2 do not meet, l_1 meets the generic plane through l_2 transversally. Figure 8.2 gives the picture for the line arrangement in the generic plane through l_2.

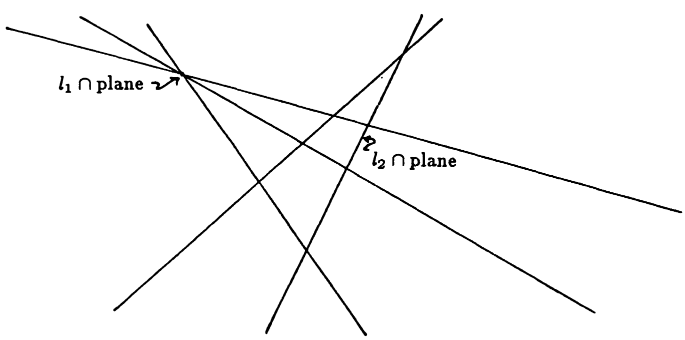


FIGURE 8.2

Thus by [26] or [84], S_2 fibers onto the covering of the exceptional fiber of the $\mathbf{P}^1 \times \mathbf{P}^1 = l_1$ blown-up (as in the resolution of singularities, §6.2), which is C_1. The other case is handled similarly.

First, we take coverings of $C_1 : \pi : C_1' \to C_1$. This induces (fiber product) coverings of S_2, as in the following diagrams:

$$\begin{array}{ccc} S_2' & \longrightarrow & S_2 \\ \downarrow & & \downarrow \\ C_1' & \xrightarrow{\pi} & C_1 \end{array}$$

The behavior of the Chern numbers of S_2' in this situation by taking all coverings of C_1 is described in §5: $c_1^2(S_2')/c_2(S_2') \to 2$.

The next step is to consider the diagram

$$\begin{array}{cccc} Y_1' & \xrightarrow{\tilde{\pi}} & Y & \\ & & & \searrow^{f_2} \\ \downarrow & & \downarrow & \quad C_2 \\ C_1' & \xrightarrow{\pi} & C_1 & \end{array}$$

where Y_1' is again the fiber product. Y_1' fibers over C_2 (by composition $f_2 \circ \tilde{\pi}$) with fiber S_2'. So we can take coverings of C_2 and the corresponding fiber products according to the following diagram:

$$\begin{array}{cccc} Y_2' & \longrightarrow & Y_1' & \\ & & & \searrow^{\tilde{\pi}} \\ \downarrow & & \downarrow & \quad Y \\ & & & \swarrow_{f_2} \\ C_2' & \longrightarrow & C_2 & \end{array}$$

Theorem 8.2.1. *Let Y be as above:*

$$\begin{array}{ccc} & Y & \\ \swarrow & & \searrow \\ C_2 & & C_1 \end{array} \qquad g(C_2) \geq 1, \quad g(C_1) \geq 1.$$

Then by taking fiber products of coverings $\pi_1 : C_1' \to C_1$ and $\pi_2 : C_2' \to C_2$ such that the branching locus of π_1, π_2 is disjoint from the degeneracy locus of f_1 and f_2, respectively, we can construct an algebraic 3-fold Y with

$$\left(\frac{c_1^3(Y)}{c_1c_2(Y)}, \frac{c_3(Y)}{c_1c_2(Y)} \right) = (\alpha, \beta)$$

for any rational pair (α, β) in the triangle $\triangle ABC$ with vertices

$$A = \left(\frac{c_1^3(Y)}{c_1c_2(Y)}, \frac{c_3(Y)}{c_1c_2(Y)} \right), \qquad B = \left(\frac{3c_1^2(S_1)}{c_1^2(S_1) + c_2(S_1)}, \frac{c_2(S_1)}{c_1^2(S_1) + c_2(S_1)} \right),$$

$$C = \left(\frac{3c_1^2(S_2)}{c_1^2(S_2) + c_2(S_2)}, \frac{c_2(S_2)}{c_1^2(S_2) + c_2(S_2)} \right),$$

where S_1, S_2 are the fibers of $Y \to C_1$ and $Y \to C_2$ respectively.

The proof of this is immediate applying Lemma 8.1.3 to the reasoning above.

We now apply this result to one example of Fermat cover to get the density result mentioned in the introduction. Let Y be Example 7.3.4, i.e., $Y \to \hat{\mathbf{P}}^3$ ramifying over the arrangement $A_1^3(10)$. Y has the structure of fiber space

$$\begin{array}{ccc} & Y & \\ {}^{f_1}\swarrow & & \searrow^{f_2} \\ \mathbf{P}^1 & & \mathbf{P}^1, \end{array}$$

the two fiberings arising (as in §6.3) from any two of the 3-fold lines, which we assume do not meet. Here we cannot apply Theorem 8.2.1 directly since both curves have genus zero (and therefore no unramified covers, whence the reasoning in the proof of Lemma 8.1.2 does not apply). To remedy this, let $E \to \mathbf{P}^1$ be a double cover with four branch points so that E is an elliptic curve. Let Y' be the corresponding fiber product as indicated in the following diagram:

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ E & \longrightarrow & \mathbf{P}^1 \end{array}$$

Using Lemma 8.1.1 we have

$$c_1^3(Y') = -9 \cdot 2^7, \quad c_1c_2(Y') = -9 \cdot 2^7, \quad c_3(Y') = 3 \cdot 2^7.$$

Y' has two fiberings:

$$\begin{array}{ccc} & Y' & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ E & & \mathbf{P}^1 \end{array}$$

and the fiber of π_2 is (as discussed above) the fiber product in the following diagram:

$$\begin{array}{ccc} F_2 & \longrightarrow & F_1 \\ \downarrow & & \downarrow \\ E & \longrightarrow & \mathbf{P}^1 \end{array}$$

where F_1 is the fiber discussed in 8.2. Using Lemma 5.1.1 this yields the Chern numbers of F_2, which is the fiber of π_2,

$$c_1^2(F_2) = 3 \cdot 2^6, \qquad c_2(F_2) = 9 \cdot 2^5.$$

Now we take a double cover of the remaining $\mathbf{P}^1$, branched at four points, which we denote by $E'' \to \mathbf{P}^1$. We now have the following diagram:

$$\begin{array}{ccccc} & & Y' & \leftarrow & Y'' \\ & \swarrow & \searrow & & \searrow \\ E & & \mathbf{P}^1 & \leftarrow & E'' \end{array}$$

The fiber F_1'' of $Y'' \to E$ has $c_1^2 = 7 \cdot 2^6$, $c_2 = 19 \cdot 2^5$, the fiber F'' of $Y'' \to E''$ has $c_1^2 = 3 \cdot 2^6$, $c_2 = 9 \cdot 2^5$, and for the Chern numbers of Y'' we get

$$c_1^3(Y'') = -36 \cdot 2^7, \quad c_1c_2(Y'') = -33 \cdot 2^7, \quad c_3(Y'') = -3 \cdot 2^7.$$

We can now apply Theorem 8.2.1 to

$$\begin{array}{ccc} & Y'' & \\ \swarrow & & \searrow \\ E & & E'' \end{array}$$

which yields (we formulate the result for simplicity in homogeneous coordinates on $\mathbf{P}^2(\mathbf{Q})$)

Theorem 8.2.2. *Let $[\alpha : \beta : \gamma]$ be a point inside the triangle $\triangle ABC$ with vertices*

$$A = [12 : 11 : 1], \quad B = [6 : 5 : 3], \quad C = [42 : 33 : 19].$$

Then there exists an algebraic 3-fold of general type X with $[c_1^3(X) : c_1c_2(X) : c_3(X)] = [\alpha : \beta : \gamma]$.

The reader might find it amusing and rewarding to try other examples to extend the results of Theorem 8.2.2. As it stands, 8.2.2 is only a qualitative result, not a quantitative one. For example, by taking a different covering of Y' we can get a result like the above for the triangle $\triangle ABC$ with vertices $A = [5 : 5 : -2]$, $B = [3 : 3 : 2]$, $C = [96 : 87 : 55]$. However, for most examples of Fermat covers, the procedure above will yield triangles so small that one would almost need a microscope to see them on the Atlas below. These triangles would almost always lie in the above two, which is why we have refrained from giving more results in this direction.

9. Degenerate arrangements

In this section we will be concerned with the following quite general

Problem. Given an arrangement $\mathcal{L} \subset \mathbf{P}^3$, what can be said about the relationship between

$$\left\{\begin{array}{c}\text{Restrictions on the}\\ \text{combinatorial data of } \mathcal{L}\end{array}\right\} \text{ and } \left\{\begin{array}{c}\text{the geographic location of}\\ [c_1^3(Y_n) : c_1c_2(Y_n) : c_3(Y_n)] \text{ for}\\ \text{Fermat covers } Y_n \to \hat{\mathbf{P}}^3 \text{ of}\\ \text{degree } n \text{ associated to } \mathcal{L}\end{array}\right\}?$$

In my thesis [41] I introduced the notion of degenerate arrangements, where such a relationship does indeed seem apparent.

9.1. Degenerate arrangements. Arrangements of the following kinds will be called *degenerate*.

point arrangement: $\Leftrightarrow \mathcal{L}$ has only singular points ($t_q^1 = 0,\ q \geq 3$).
line arrangement: $\Leftrightarrow \mathcal{L}$ has only singular lines (and of course near-pencil singular points (see 6.2.1)) but no other singular points.

An arrangement $\mathcal{L}$ is *both* a point arrangement and a line arrangement, iff it is the general position (in the combinatorial sense).

9.1.1. Point arrangements. In this case the combinatorial data are given by the following formula:

$$t_2^1 = \binom{k}{2}, \quad t_q^1 = 0, \quad q \geq 3, \quad \sum t_p \binom{p}{3} = \binom{k}{3}.$$

Interesting examples of point arrangements are given by the facet planes of any regular polyhedra.

9.1.2. Line arrangements. A line arrangement is characterized by the following data:

$$q_1, \cdots, q_s, \sigma, k = \sum q_i + \sigma, \quad t_2^1 = \binom{k}{2} - \sum \binom{q_i}{2},$$
$$t_3 = \binom{k}{3} - \frac{1}{3} \sum q_i(q_i - 1)(3k - 2q_i - 2).$$

There are s singular lines, through the ith one pass q_i planes plus σ additional planes in general position with respect to the others. Such an arrangement can always be realized over the reals. Choose one of the lines, say l_i. Then the generic plane through l_i has the induced arrangement of $k - q_i + 1$ lines, one q_j-fold point for each $j \neq i$, and each line of the induced arrangement passes through only one line. Such a line arrangement Λ in $\mathbf{P}^2$, i.e., one fulfilling the condition

(D) {each line of $\Lambda \subset \mathbf{P}^2$ passes through at most one singular point},
is the 2-dimensional analogue of our 3-dimensional "degenerate arrangements".

9.2. Fermat covers of degenerate arrangements.

Lemma 9.2.1. *Let* $\Lambda \subset \mathbf{P}^2$ *be a line arrangement fulfilling* (D) *above, which we assume has* $t_{k-1} = t_{k-2} = 0$,

$$S_n \to \hat{\mathbf{P}}^2 \text{ the associated Fermat cover of degree } n.$$

Then

(a) $c_1^2(S_n) < 2c_2(S_n)$,

(b) *the characteristic quotient* γ *of* Λ [26, *p. 135*] *fulfills* $\gamma \leq 2$.

Proof. To show (a) we use Hirzebruch's formula [26, pp. 124–125]

$$F(S_n) := 2c_2(S_n) - c_1^2(S_n) = n^2(k - 3 - f_1 + 2f_0) + f_1 + f_0 - k - 3t_2,$$

where $f_1 = \sum_{r\geq 2} rt_r$, $f_0 = \sum_{r\geq 2} t_r$. The condition (D) on the arrangement implies $f_1 \leq 2f_0 + k - 3$ since we are assuming $t_{k-1} = t_{k-2} = 0$, so the leading coefficient is positive. Thus $F(S_n) > 0$ if $F(S_2) > 0$ (now assume $k > 4$) and

$$\begin{aligned} F(S_2) &= 3k - 12 + 9f_0 - 3f_1 - 3t_2 \\ &= 3k - 12 + \sum(9 - 3r)t_r \\ &= 3\left(k - 4 + \sum(3 - r)t_r\right) \\ &> 0 \quad \text{by the assumption that } f_1 \leq 2f_0 + k - 3. \end{aligned}$$

(a) now implies (b). q.e.d.

Using this we can prove a 3-dimensional analogue.

Theorem 9.2.2. *Let* $\mathscr{L} \subset \mathbf{P}^3$ *be a line arrangement as defined above, and* $Y_n \to \hat{\mathbf{P}}^3$ *the associated Fermat cover of degree* $n \geq 3$. *Then*

(a) $c_1^3(Y_n)/c_1c_2(Y_n) < 2$,

(b) $A_{13}/A_{12} \leq 2$.

Proof. It is sufficient to prove (a). Here we use a formula developed in [41, 4.6.1]:

$$\begin{aligned} F(Y_n) &:= c_1^3(Y_n) - 2c_1c_2(Y_n) \\ &= \sum_{i=1}^{k} \lambda_i \left[(2c_2(\bar{H}_i) - c_1^2(\bar{H}_i)) - \left({}^{\bar{H}_i}|_{\bar{H}_i}\right)^2 \right] - \sum \mu_m \left({}^{\bar{L}_m}|_{\bar{L}_m}\right)^2, \end{aligned}$$

where $\bar{H}_i$, $\bar{L}_m$ denote the reduced ramification loci covering the k planes H_i of the arrangement and the exceptional $\mathbf{P}^1 \times \mathbf{P}^1$'s introduced in the resolution of the singular curves (6.2). We must show $F(Y_n) \geq 0$. Since the arrangement has no singular points, $({}^{\bar{L}_m}|_{\bar{L}_m})^2 < 0$. Therefore it suffices to show that

$$2c_2(\bar{H}_i) - c_1^2(\bar{H}_i) > \left({}^{\bar{H}_i}|_{\bar{H}_i}\right)^2.$$

This follows from Lemma 9.2.1 since $2c_2(\bar{H}_i) - c_1^2(\bar{H}_i) \geq 0$ whereas

$$\left({}^{\bar{H}_i}|_{\bar{H}_i}\right)^2 = \deg \pi \cdot \left(1 - \sum \sigma_{mi}\right),$$

so if $c_1^2 \neq 2c_2$, the expression in $F(Y_n)$ will be positive. q.e.d.

The conclusion is that from the viewpoint of geography, line arrangements are *as degenerate as general position* (for which exactly the same inequality will hold). They correspond to the natural boundary

$$c_1^3/(c_1c_2) \leq c_1^3(\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1)/(c_1c_2)(\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1) = 2.$$

Now suppose $\mathscr{L}$ is a point arrangement. The maximal quotients $c_1^3/(c_1c_2)$ that we know of are the following:

(i) The 20 facet planes of the iosahedron. For all $n \geq 5$, $c_1^3/(c_1c_2) > 2$, and $A_{13}/A_{12} = 2.07$.

(ii) Special 16_6-arrangement (see [41, pp. 21–22]). For all $n \geq 8$, $c_1^3/(c_1c_2) > 2$, and $A_{13}/A_{12} = 2.025$.

We notice that for point arrangements $c_1^3(Y(\mathscr{L}, n))/(c_1c_2)(Y(\mathscr{L}, n))$ is monotone increasing with n, i.e., the characteristic quotient A_{13}/A_{12} is really the maximal value. Following the general philosophy above, we wage the

Conjecture 9.2.3. *Let $\mathscr{L}$ be a point arrangement. Then*

$$c_1^3(Y_n)/(c_1c_2)(Y_n) \leq \tfrac{9}{4} = c_1^3(\mathbf{P}^2 \times \mathbf{P}^1)/(c_1c_2)(\mathbf{P}^2 \times \mathbf{P}^1)$$

for any $n > 2$.

This can be translated into a condition on the combinatorial data of the arrangement $\mathscr{L}$, and is equivalent to

Conjecture 9.2.4. *For any point arrangement $\mathscr{L} \subset \mathbf{P}^3$,*

$$40 + 3t_3 + 7t_4 + 8t_5 + 6t_6 + t_7 > 13k + 7t_8 + 18t_9 + \ldots.$$

Finally we would like to remark that in any dimension N there are as many kinds of degenerate arrangements as there are partitions of N, which makes the conjecture above, in an appropriate formulation, plausible in any dimension. The most degenerate arrangements are probably those with the following property:

$$(\mathrm{P})_1 \quad \{\text{each } (N-1)\text{-plane passes through only one singular } (N-1)\text{-dim locus}\}.$$

They probably will have Fermat covers with $c_1^N/c_1^{N-2}c_2 \leq 2$ as is the case for arrangements in general position. The least degenerate are probably those arrangements fulfilling

$$(\mathrm{P})_{N-1} \quad \{\text{the arrangement has no codim } 2 \text{ singular loci}\}.$$

Here we suspect an arrangement $\mathscr{L} \subset \mathbf{P}^N$ satisfying $(\mathrm{P})_{N-1}$ has associated Fermat covers for which

$$c_1^N/c_1^{N-2}c_2 \le c_1^N(\mathbf{P}^{N-1} \times \mathbf{P}^1)/c_1^{N-2}c_2(\mathbf{P}^{N-1} \times \mathbf{P}^1).$$

10. An atlas of 3-folds of general type

10.1. Legend. Below is a map of the area in the $[c_1^3/(c_1c_2), c_3/(c_1c_2)]$-plane which corresponds to known 3-folds. We remind the reader of our conventions of 7.1, that either

(i) Y have ample canonical bundle, or

(ii) Y is a smooth minimal model.

Fermat covers, for example, fulfill (ii) if they have no nodes (see 6.5.7 and the remark following 6.5.8). We now list different points, and their coordinates, which represent interesting examples.

♯	coord.	description	Example ♯
1–9	in Zone SCI	smooth complete intersections	
10	$(-.125, .83)$	desingularization of sing. $(5,5)$ complete intersection in $\mathbf{P}^5$	(7.3.1)
11	$(.08, 1.1)$	desingularization of sing. $(6,6)$ complete intersection in $\mathbf{P}^5$	(7.3.1)
12	$(.16, 1.2)$	desingularization of sing. $(7,7)$ complete intersection in $\mathbf{P}^5$	(7.3.1)
13	$(-2.3, 1)$	Fermat cover of $A_1^3(10)$	(7.3.4)
14	$(-.3, 1)$	fiber product of 13	Y' in (8.2.1)
15	$(-.2, 1)$	fiber product of 14	
16	$(.09, 1.1)$	fiber product of 14	Y'' in (8.2.1)
17	$(-.35, 1)$	Fermat cover of arrangement of 8 planes	(7.3.3)
18	$(-.22, 1.4)$	Fermat cover of $\mathrm{Ceva}^3(2)$	(7.3.5)
19	$(-.04, 1.6)$	Fermat cover of $\mathrm{Ceva}^3(2,1)$	(7.3.5)
20	$(.18, 2.55)$	Fermat cover of $\mathrm{Ceva}^3(3)$	(7.5.4)
21	$(.23, 2.35)$	Characteristic ratios $A_1^3(10)$, $A_1^3(16)$ and $\mathscr{L}_{G_{25,920}}$	(7.5.5)

The following zones were described in §7.2:

Zone AC, $c_3 > 0$.

Zone SCI bounded by: (1) the curve of §7.2.4 where lie the limit points for $d \to \infty$ of the Chern numbers $[c_1^3(Y^{d\cdot}) : c_1c_2(Y^d) : c_3(Y^d)]$, Y^d a complete intersection of k hyperplanes ($k \geq 1$), (2) the line CP, and (3) the line $c_1^3 = 0$ (Smooth Complete Intersections).

Zone E bounded by: (1) the curve of §7.2.4, and (2) the line $c_1^3 = \frac{8}{3}c_1c_2$, here no examples are known (Empty Zone?).

Zone F bounded by: (1) the line $c_1^3 = \frac{8}{3}c_1c_2$, (2) the line $c_1^3 = \frac{9}{4}c_1c_2$, (3) the line $c_3 = 0$, and (4) the "continuation" of the curve of §7.2.4 to $[c_1^3 : c_1c_2 : c_3] = [16 : 6 : 1]$. Here many examples are given by Fermat covers of $\mathbf{P}^3$ branched along simplicial arrangements or arrangements defined by unitary reflection groups.

Line CP passing through $[9 : 4 : 1]$ and $[6 : 3 : 1]$.

Other important landmarks are coordinates corresponding to compact, smooth quotients of hermitian symmetric domains. In dimension 3 there are four landmarks:

bounded domain	compact dual	coordinates
$\mathbf{B}^3$	$\mathbf{P}^3$	$[16 : 6 : 1]$
$\mathbf{B}^2 \times \mathbf{B}^1$	$\mathbf{P}^2 \times \mathbf{P}^1$	$[9 : 4 : 1]$
$\mathbf{B}^1 \times \mathbf{B}^1 \times \mathbf{B}^1$	$\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$	$[6 : 3 : 1]$
$\mathbf{E}^3$	$\mathbf{Q}^3$ hyperquadric	$[27 : 12 : 2]$

Finally, the dotted line $\cdots$ is the line $c_3/(c_1c_2) = -2 - 7c_1^3/(c_1c_2)$ which was mentioned in the introduction as a bound for Y such that $K_Y = i^*\mathcal{O}(1)$ for a canonical embedding $i\colon Y \subset \mathbf{P}^N$.

10.2. Some open questions. In this section we state some questions and problems which in our opinion are relevant to further research in 3-fold geography.

10.2.1. Do the limit points (γ_k^1, γ_k^2),

$$\gamma_k^1 = \lim_{d\to\infty}(c_1^3(Y^d)/(c_1c_2)(Y^d)), \qquad \gamma_k^2 = \lim(c_3(Y^d)/(c_1c_2)(Y^d)),$$

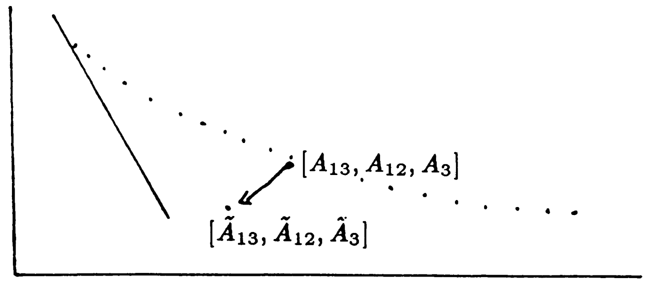
for Y^d a complete intersection of k hyperplanes of degree d lie on a smooth polynomial curve for $k \geq 1$?

10.2.2. For all known examples, introducing isolated singularities into smooth complete intersections moves the coordinates $[c_1^3 : c_1c_2 : c_3]$ (after resolving the singularities) "to the left", i.e., in the direction $c_3/(c_1c_2) \to -\infty$. If the singularities have one-dimensional components, then, in addition $c_1^3/(c_1c_2)$ seem to grow in magnitude. As an example of this, we can give the formulas corresponding to introducing a single singular point or curve into a smooth complete intersection. We use the formula of §6.4. Consider a smooth complete intersection of $k-4$ Fermat hypersurfaces in $\mathbf{P}^{k-1}$. The limits of $[c_1^1 : c_1c_2 : c_3]$ as the degrees approach ∞ are denoted $[A_{13} : A_{12} : A_3]$ (same notation as §6.4). Introducing a single p-fold point

($p \le k-5$) changes the combinatorial data as follows: $t_p = 1$, $t_3 = \binom{k}{3} - \binom{p}{3}$, $t_2(1) = \binom{k}{2}$ and $t_{p_2} = \binom{p}{2}$. Plugging this into the formula in §6.4 and letting $[\tilde{A}_{13}, \tilde{A}_{12}, \tilde{A}_3]$ denote the corresponding limits in this case yield

$$\tilde{A}_{13} = A_{13} + (p-3)^3, \quad \tilde{A}_{12} = A_{12} - \tfrac{1}{2}p^2 + \tfrac{3}{2}p, \quad \tilde{A}_3 = A_3 + \tfrac{1}{6}p^3 - p^2 + \tfrac{11}{6}p - 1.$$

Since A_{13}, A_{12} and A_3 are all negative, we see that $|A_{13}|$ and $|A_3|$ *decrease*, whereas $|A_{12}|$ increases. Thus we get a "specialization vector" something like



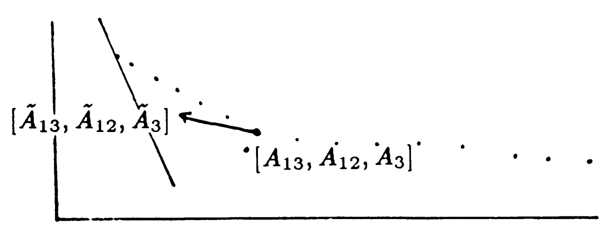
If we add a singular curve instead of point (i.e., a q-fold line ($q \le k-4$) of the arrangement), the combinatorial data are:

$$t_3 = \binom{k}{3} - \tfrac{1}{3}q(q-1)(3k-2q-2), \quad t_{pq} = 0, \quad p \ge 4,$$
$$t_2(1) = \binom{k}{2} - \binom{q}{2}.$$

Plugging this into the formula yields

$$\begin{aligned}
\tilde{A}_{13} &= A_{13} + 2(2-q)^3 - 3(4-k)(2-q)^2,\\
\tilde{A}_{12} &= A_{12} + (4-k)\left[(q-1) - \left(\tfrac{q}{2}\right)\right] + (2-q)[k+q-4],\\
\tilde{A}_3 &= A_3 + \tfrac{2}{3}q^3 - \left(\tfrac{7}{3}+k\right)q^2 + \left(\tfrac{11}{3}+k\right)q - 1.
\end{aligned}$$

In this case, $|A_{13}|$, $|A_{12}|$ and $|A_3|$ all decrease, $|A_{13}|$ as $-2q^3 + 3kq^2$, $|A_{12}|$ as $\frac{k}{2}q^2$ and $|A_3|$ as $\frac{2}{3}q^3$, $A_3/A_{12} \to \frac{4}{3}\frac{q}{k}$, $A_{13}/A_{12} \to (6k-4q)/k$. We get a "specialization vector" as follows:



Question. Is this true in general, i.e., can it be proven?

10.2.3. Is Zone E really empty? Is the bound really the curve 7.2.4?

10.2.4. If a three-fold Y with K_Y ample has $c_1^3/(c_1c_2) = \frac{8}{3}$, by Yau's theorem it is a smooth ball quotient. Thus $[16:6:1]$ is the *only* possible coordinate on the line $c_1^3 = \frac{8}{3}c_1c_2$. If the curve describing the bounds on the Chern numbers is smooth at $[16:6:1]$, then the line $c_1^3 = \frac{8}{3}c_1c_2$ is *tangent* to the area D of possible coordinates. Notice that the compactification of ball quotients always have $c_3/(c_1c_2) = \frac{1}{6}$, so that all lie on a vertical line. On the other hand, taking fiber products of ball quotients as in §8 leads to examples with $c_3/(c_1c_2) > \frac{1}{6}$.

Problem. Find examples of Y with

$$c_1^3/(c_1c_2) > \tfrac{9}{4}, \qquad c_3/(c_1c_2) < \tfrac{1}{6}.$$

10.2.5. Referring to the characteristic quotients $[A_{13} : A_{12} : A_3]$ of an arrangement described in §7.4 we make the following observation. In all known examples the characteristic exponents are all near the line CP, whereas for small n, the ratios $[c_1^3 : c_1c_2 : c_3]$ of all Y^n may lie far away. They all approach, however, the line CP for increasing n. This raises the following fascinating question:

If Y^n is any family of 3-folds of general type depending on a discrete parameter $n \in \mathbf{N}$ with $\chi(Y^n, \mathscr{O}_Y) \to \infty$ for $n \to \infty$, is

$$P_\infty := \lim_{n\to\infty} [c_1^3(Y^n) : c_1c_2(Y^n) : c_3(Y^n)]$$

a point minimizing the distance to the line CP?

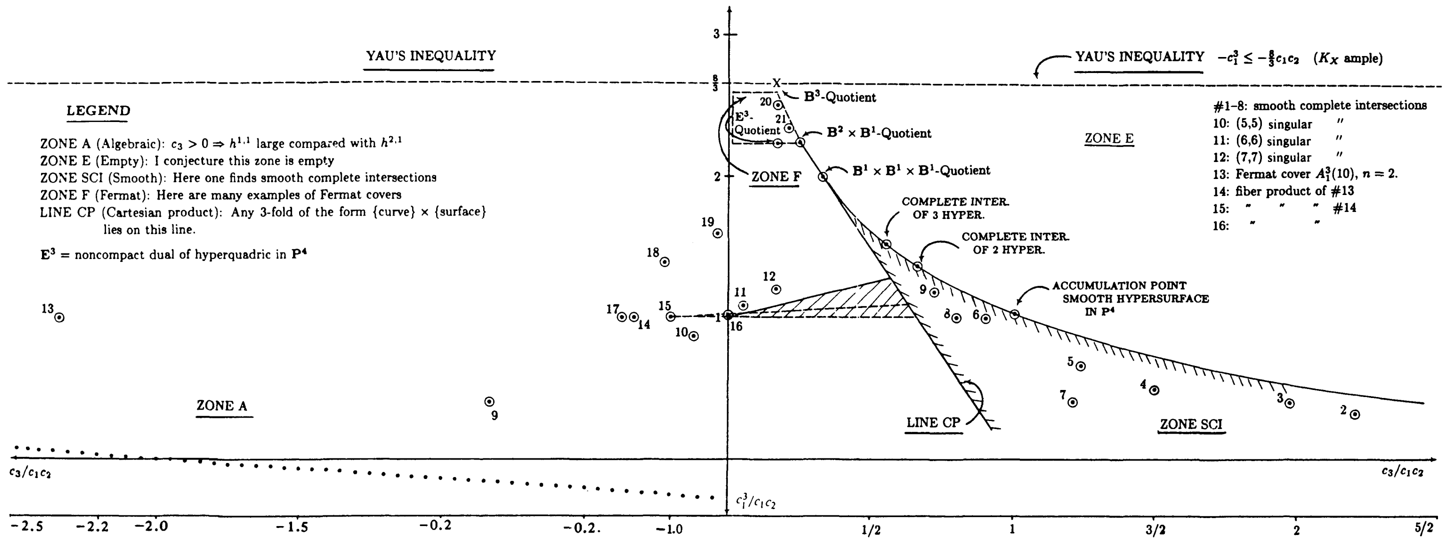
Here by minimizing we mean with finitely many exceptions (low n), P_∞ is nearest to the line of all $[c_1^3(Y^n) : c_1c_2(Y^n) : c_3(Y^n)]$ in the Euclidean metric on the affine chart $c_1c_2 \neq 0$. This question has an affirmative answer in the following cases:

(a) Y^n is a series obtained by taking fiber products as in §8.

(b) Y^n is a series of smooth complete intersections of k hypersurfaces, $k \geq 1$.

(c) Y^n is a Fermat cover of degree n.

10.2.6. In addition to what has already been mentioned there is the general problem of finding the bounds which must be satisfied by the coordinates of Y, say under the assumption K_Y ample.



ATLAS OF SMOOTH MINIMAL 3-FOLDS OF GENERAL TYPE
(in the $c_1c_2 \neq 0$ chart)

References

[1] Paul Appell & J. Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques*, Polynome d'Hermite, Gauthier-Villars, Paris, 1926.

[2] A. Ash, D. Mumford & Y. Tai, *Smooth compactification of locally symmetric varieties*, Math. Sci. Press, Brookline, 1975.

[3] W. Baily & A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. (2) **84** (1966) 442–528.

[4] W. Barth, C. Peters & A. van de Ven, *Compact complex surfaces*, Springer, Berlin, 1984.

[5] A. Borel & Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Ann. of Math. (2) **75** (1962) 485–535.

[6] E. Brieskorn, *Über die Auflösung gewisser Singularitöten von holomorphen Abbildungen*, Math. Ann. **166** (1966) 76–102.

[7] E. Calabi & E. Vesentini, *On compact, locally symmetric Kähler manifolds*, Ann. of Math. (2) **71** (1960) 472–507.

[8] Z. Chen, *On the geography of surfaces*, Math. Ann. **277** (1987) 141–164.

[9] S. Chern, F. Hirzebruch & J.-P. Serre, *On the index of a fibered manifold*, Proc. Amer. Math. Soc. **8** (1957) 587–596.

[10] P. Deligne & G. Mostow, *Monodromy of hypergeometric functions and non-lattice integral monodromy*, Inst. Hautes Études Sci. Publ. Math. **63** (1986) 5–89.

[11] A. Erdélyi, W. Magnus, F. Oberhettinger & F. Triconi, *Higher transcendental functions*, Vol. I, McGraw-Hill, New York, 1953.

[12] J.-M. Feustel, *Die negative K^2-Kontribution der Picardschen Modulflächen*, Math. Nachr. **106** (1982) 17–34.

[13] ——, *Zur groben Klassifikation der Picardschen Modulflächen*, Math. Nachr. **118** (1984) 215–251.

[14] J.-M. Feustel & R.-P. Holzapfel, *Symmetry points and Chern invariants of Picard modular surfaces*, Math. Nachr. **111** (1983) 1–34.

[15] R. H. Fox, *Covering spaces with singularities*, Lefschetz Symposium, Princeton University Press, Princeton, NJ, 1957.

[16] W. Fulton, *Intersection theory*, Springer, Berlin, 1984.

[17] D. Gieseker, *Global moduli for surfaces of general type*, Invent. Math. **43** (1977) 233–282.

[18] P. Griffiths & J. Harris, *Principles of algebraic geometry*, Wiley, New York, 1978.

[19] B. Grünbaum, *Arrangement of hyperplanes*, Proc. 2nd Louisiana Conf. on Combinatorics, Louisiana State University, Baton Rouge, LA, 1971, 41–108.

[20] B. Grünbaum & G. C. Shepard, *Simplicial arrangements in projective 3-space*, Mitt. Math. Sem. Giessen **166** (1984) 49–101.

[21] R. Hartshorne, *Algebraic geometry*, Springer, New York, 1977.

[22] F. Hirzebruch, *Automorphe formen und der Satz von Riemann-Roch*, Unesco. Sympos. Inter. Top. Alg., 1957.

[23] ——, *Komplexe Mannigfaltigkeiten*, Proc. Internat. Congr. Math., Cambridge University Press, Cambridge, 1958.

[24] ——, *New topological methods in algebraic geometry*, Springer, Berlin, 1966.

[25] ——, *Hilbert modular surfaces*, Enseignement Math. **19** (1973) 183–281.

[26] ——, *Arrangements of lines and algebraic surfaces*, Arithmetic and Geometry, Vol. II, Progress in Math., Vol. 36, Plenum Press, Boston, 1983, 113–140.

[27] ——, *On the Chern numbers of algebraic surfaces: an example*, Math. Ann. **266** (1984) 351–356.

[28] ——, *On the signature of ramified coverings*, Global Analysis, Papers in honor of K. Kodaira, Princeton University Press, Princeton, 1969.

[29] F. Hirzebruch & G. van der Geer, *Hilbert modular surfaces*, Presses de l'Université Montréal, Montréal, 1981.

[30] F. Hirzebruch & A. van de Ven, *Hilbert modular surfaces and the classification of algebraic surfaces*, Invent. Math. **23** (1974) 1–29.

[31] F. Hirzebruch & D. Zagier, *Number theory and the Atiyah-Singer index theorems*, Publish or Perish, Boston, 1974.

[32] ——, *Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus*, Invent. Math. **36** (1976) 57–113.

[33] T. Höfer, *Ballquotienten als verzweigte überlageregen der Projektiven Ebene*, Ph.D. thesis, Max-Planck Institut, Bonn, 1986.

[34] R.-P. Holzapfel, *A class of minimal surfaces in the unknown region of surface geography*, Math. Nachr. **98** (1980) 211–232.

[35] ——, *Invariants of arithmetic ball quotient surfaces*, Math. Nachr. **103** (1981) 117–153.

[36] ——, *Arithmetische Kugelquotienten Flächen* V/VI, Sek. Math., Seminar Berichte der Humboldt Univ. zu Berlin, 21, 1980.

[37] ——, *Arithmetic curves on ball quotient surfaces*, Ann. Global Analysis and Geometry, **1-2** (1983) 21–90.

[38] ——, *Chern numbers of algebraic surfaces—Hirzebruch's examples are Picard modular surfaces*, Math. Nachr. **126** (1986) 255–273.

[39] E. Horikawa, *Algebraic surfaces of general type with small* c_1^2, I, II, Ann. of Math. (2) **104** (1976).

[40] ——, *On algebraic surfaces with pencils of curves of genus two*, Complex Analysis and Algebraic Geometry, Cambridge University Press, New York, 1977.

[41] B. Hunt, *Coverings and ball quotients*, Bonner Math., Schriften 174, 1986.

[42] ——, *A Siegel modular 3-fold which is a Picard modular 3-fold*, Compositio Math.

[43] S. Iitaka, *Algebraic geometry*, Springer, Berlin, 1982.

[44] M. Inoue, *Some surfaces of general type with positive index*, Preprint.

[45] M. Ishida, *Hirzebruch's examples of surfaces of general type with* $c_1^2 = 3c_2$, Algebraic Geometry, Lecture Notes in Math., Vol. 1016, Springer, Berlin, 412–431, 1983.

[46] K. Ivinskis, *Normale Flächen und die Miyaoka-Kobayashi Ungleichung*, Diplomarbeit, Bonn, 1985.

[47] Y. Kawamata, *Elementary contractions of algebraic 3-folds*, Ann. of Math. (2) **119** (1984) 95–110.

[48] ——, *The cone of curves of algebraic varieties*, Ann. of Math. (2) **119** (1984) 603–633.

[49] ——, *Crepant blowing-ups of 3-dimensional canonical singularities and its application to degenerations of surfaces*, Ann. of Math. **127** (1988), 93–165.

[50] ——, *On the finiteness of generators of a pluricanonical ring for a 3-fold of general type*, Amer. J. Math. **106** (1984) 1503–1512.

[51] Y. Kawamata, K. Matsuda & K. Matsuki, *Introduction to the minimal model problem*, Advanced Studies in Math., Vol. 10, Kinokuniya, Tokyo, 283–361.

[52] R. Kobayashi, *Einstein-Kähler metrics on open algebraic surfaces of general type*, Tôhoku Math. J. **37** (1985) 43–77.

[53] ——, *Einstein-Kähler V-metrics on open Satake V-surfaces with isolated quotient singularities*, Math. Ann. **272** (1985) 385–398.

[54] S. Kobayashi & H. Wu, *Complex differential geometry*, Birkhäuser, Basel, 1983.

[55] K. Kodaira, *Compact, complex analytic surfaces*, I, II, III, Ann. of Math. (2) **71** (1963) 111–152; **77** (1963) 563–626; **78** (1963) 1–40.

[56] ——, *On the structure of compact, complex, analytic surfaces*, I-IV, Amer. J. Math. **86** (1964) 751–798; **88** (1966) 682–721; **90** (1968) 55–83; **90** (1968) 1048–1066.

[57] ——, *A certain type of irregular algebraic surfaces*, J. Analyse Math. **19** (1967) 207–215.

[58] R. Livné, *On certain covers of the universal elliptic curve*, thesis, Harvard University, 1981.

[59] J. Milnor, *On Ω^* and a complex analogue*, Amer. J. Math. **82** (1960) 505–521.

[60] R. Miranda, *Triple covers in algebraic geometry*, Amer. J. Math. **107** (1985) 1123–1159.

[61] Y. Miyaoka, *Algebraic surfaces with positive index*, Progress in Math., Vol. 39, Birkhäuser, Basel, 1983.

[62] ——, *On the Chern numbers of surfaces of general type*, Invent. Math. **42** (1977) 225–237.

[63] ——, *The maximal number of quotient singularities on surfaces with given numerical invariants*, Math. Ann. **68** (1984) 159–171.

[64] ——, *The Chern classes and Kodaira dimension of a minimal variety*, Advanced Studies in Math., Vol. 10, Kinokuniya, Tokyo, 1987, 449–477.

[65] S. Mori, *Three-folds whose canonical bundles are not numerically effective*, Ann. of Math. (2) **116** (1982) 133–176.

[66] ——, *On 3-dimensional terminal singularities*, Nagoya Math. J. **98** (1985) 43–66.

[67] G. Mostow, *Existence of a non-arithmetic lattice in $SU(2,1)$*, Proc. Nat. Acad. Sci. U.S.A. **75** (1978) 3029–3033.

[68] ——, *Existence of nonarithmetic monodromy groups*, Proc. Nat. Acad. Sci. U.S.A. **78** (1981) 5948–5950.

[69] D. Mumford, *Hirzebruch proportionality in the non-compact case*, Invent. Math. **42** (1977) 239–272.

[70] I. Naruki, *On a $K3$-surface which is a ball quotient*, Preprint, Max-Planck Inst., Bonn, 1986, 45–52.

[71] T. Oda, *On Schwarzian derivatives in several variables*, Kokyuroku, Res. Inst. Math. Kyoto University, (Japanese), **226** (1974).

[72] U. Persson, *Double coverings and surfaces of general type* (Proc. Algebraic Geometry, Norway, 1977), Lecture Notes in Math., Vol. 687, Springer, Berlin, 1978.

[73] ——, *Chern invariants of surfaces of general type*, Compositio Math. **43** (1981) 3–58.

[74] ——, *Horikawa surfaces with maximal Picard numbers*, Math. Ann. **259** (1982) 287–312.

[75] E. Picard, *Sur les fonctions hyperfucsiennes provanant des séries hypergéometriques de deux variables*, Ann. École Norm. Sup. **62** (1885), 357–384.

[76] M. Reid, *Canonical 3-folds*, Journées de Géometrie Algébrique d'Angers, Sijthoff and Noordhoff, Alphen, 1980.

[77] ——, *Minimal models of canonical 3-folds*, Advanced Studies in Pure Math., Vol. I, North-Holland, Amsterdam, 1983.

[78] ——, *Young person's guide to canonical singularities*, Algebraic Geometry, Bowdoin, 1985, Proc. Sympos. Pure Math., Vol. 46, Part 1, Amer. Math. Soc., Providence, RI, 1987, 345–416.

[79] ——, *On Bogomolov's theorem $c_1^2 \leq 4c_2$*, International Symposium on Algebraic Geometry, Kyoto University, 1978.

[80] B. Schoeneberg, *Elliptic modular functions*, Springer, Berlin, 1974.

[81] G. C. Shepard & J. Todd, *Finite unitary reflection groups*, Canad. J. Math. **6** (1954) 274–301.

[82] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Math. Soc. Japan, Iwanami Shoten, and Princeton University Press, Princeton, NJ, 1971.

[83] T. Shioda, *On elliptic modular surfaces*, J. Math. Soc. Japan **24** (1972) 20–59.

[84] A. Sommese, *On the density of ratios of Chern numbers of algebraic surfaces*, Math. Ann. **268** (1984) 207–221.

[85] T. Terada, *Fonctions hypergéométriques F_1 et fonctions automorphes*. I, J. Math. Soc. Japan **35** (1983) 451–475.

[86] R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954) 17–86.

[87] A. van de Ven, *On the Chern numbers of certain complex and almost complex manifolds*, Proc. Nat. Acad. Sci. U.S.A. **55** (1966) 1624–1627.

[88] ——, *On the Chern numbers of surfaces of general type*, Invent. Math. **36** (1976) 285–293.

[89] P. M. H. Wilson, *On complex algebraic varieties of general type*, Symposia Math., Vol. 26, Academic Press, New York, 1981.

[90] T. Yamazaki & M. Yoshida, *On Hirzebruch's examples of surfaces with $c_1^2 = 3c_2$*, Math. Ann. **266** (1984) 421–431.

[91] S.-T. Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), 1798–1799.

[92] ——, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation*. I, Comm. Pure Appl. Math. **31** (1978) 339–411.

[93] M. Yoshida, *Orbifold-uniformizing differential equations*, Math. Ann. **267** (1984) 125–142.

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