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A TORELLI-TYPE THEOREM FOR GRAVITATIONAL INSTANTONS

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1. Introduction

In an earlier paper [10], a construction was described which produced families of 4-dimensional hyper-Kähler manifolds (one family for each finite subgroup of SU(2)), the members of which were asymptotically locally Euclidean (ALE). Our purpose here is to demonstrate the completeness of this construction: we shall show that every ALE hyper-Kähler 4-manifold is isometric to a member of one of the families obtained in [10].

For us, a Riemannian 4-manifold is ALE if it has just one end and if some neighborhood of infinity has a finite covering space $\tilde{V}$ diffeomorphic to the complement of the unit ball in $\mathbf{R}^4$; the Riemannian metric g^{ij} on $\tilde{V}$ is required asymptotically to approximate the Euclidean metric δ^{ij} on $\mathbf{R}^4$, so that in the natural coordinates x_i one has

$$g^{ij} = \delta^{ij} + a^{ij}$$

with $\partial^p a^{ij} = O(r^{-4-p})$, $p \geq 0$, where $r^2 = \sum x_i^2$ and ∂ denotes differentiation with respect to the coordinates x_i. We recall that a hyper-Kähler manifold carries three complex structures I, J, K and that these give three (closed) Kähler 2-forms $\omega_1, \omega_2, \omega_3$. With this notation, the main result of [10] is the following. Let Γ be a finite subgroup of SU(2) and let X be the smooth 4-manifold underlying the minimal resolution of the complex quotient singularity $\mathbf{C}^2/\Gamma$.

Theorem 1.1. *Let three cohomology classes $\alpha_1, \alpha_2, \alpha_3 \in H^2(X;\mathbf{R})$ be given which satisfy the nondegeneracy condition*

(*) *for each $\Sigma \in H_2(X;\mathbf{Z})$ with $\Sigma \cdot \Sigma = -2$ there exists $i \in \{1,2,3\}$ with $\alpha_i(\Sigma) \neq 0$.*

Then there exists on X an ALE hyper-Kähler structure for which the cohomology classes of the Kähler forms $[\omega_i]$ are the given α_i.

The results of this paper were announced in [10]. They comprise the following two theorems, which will be proved in §§2 and 3, respectively.

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Theorem 1.2. *Every ALE hyper-Kähler 4-manifold is diffeomorphic to the minimal resolution of $\mathbf{C}^2/\Gamma$ for some $\Gamma \subset \mathrm{SU}(2)$, and the cohomology classes of the Kähler forms on such a manifold must satisfy condition* $(*)$.

Theorem 1.3. *If X^1 and X^2 are two ALE hyper-Kähler 4-manifolds and if there is a diffeomorphism $X^1 \to X^2$ under which the cohomology classes of the Kähler forms agree, then X^1 and X^2 are isometric by an isometry which respects I, J, and K.*

The author owes several ideas in the proofs of these results to conversations with N. J. Hitchin. Theorem 1.2 was essentially known to him, while Theorem 1.3 generalilzes a conjecture made in [6], from which this work originated. Following the description given there, we can give a more concrete version of Theorem 1.3. If a basis $\Sigma_1, \cdots, \Sigma_r$ for the lattice $H_2(X; \mathbf{Z})$ is chosen, then for each hyper-Kähler structure, one can form the period matrix Ω whose entries are obtained by evaluating the three Kähler forms ω_i on the cycles Σ_j:

$$\Omega_{ij} = \int_{\Sigma_j} \omega_i.$$

What the theorem says is that the hyper-Kähler structure is determined once the period matrix is known; this is the sense in which it relates to the classical Torelli theorem for Riemann surfaces. A closer cousin is the corresponding result for hyper-Kähler metrics on the $K3$ surface; the most significant difference here is that, whereas the compactness of the $K3$ surface forces the three cohomology classes $[\omega_i]$ to be orthogonal, the period matrix for an ALE space is constrained only by the nondegeneracy condition $(*)$. Of course, Ω is well defined only to within an isometry of the homology lattice.

At many points in §2 our proof runs parallel to the proof given in [1] that every finite-action, self-dual solution to the Yang-Mills equations on $\mathbf{R}^4$ arises from the monad construction of Atiyah, Drinfeld, Hitchin and Manin. Since a hyper-Kähler 4-manifold is an (anti)-self-dual solution to Einstein's equations, the results of this paper constitute, perhaps, a gravitational analogue of the ADHM classification.

2. The twistor space

Throughout this section, X will denote an arbitrary ALE hyper-Kähler 4-manifold (we do not assume that X is one of the spaces constructed in [10]) and $\overline{X}$ will denote the topological one-point compactification $\overline{X} = X \cup \{\infty\}$. Although it is not a manifold, the ALE condition allows us to give $\overline{X}$ the structure of an *orbifold* (or *V-manifold* in Satake's terminology [12]) as follows. Let U' be a neighborhood of infinity in X having a finite covering $\tilde{U}'$ with coordinates x_i as in §1, and let U and $\tilde{U}$ be obtained from these by adjoining $\{\infty\}$:

$$U = U' \cup \{\infty\}, \qquad \tilde{U} = \tilde{U}' \cup \{\infty\}.$$

We will have $U = \tilde{U}/\Gamma$ where Γ is the finite group of covering transformations. Since $\tilde{U}'$ is Euclidean at infinity, the space $\tilde{U}$ is a topological manifold, and we can make it a smooth manifold by declaring the coordinates $y_i = x_i/r^2$ to be smooth. The Riemannian metric on $\tilde{U}'$ extends to $\tilde{U}$ after a conformal change: we put $\overline{g} = \phi^2 g$ where $\phi\colon X \to \mathbf{R}^+$ is smooth and equal to $1/r^2$ outside some compact set; then in the coordinates y_i this metric has components

$$\overline{g}_{ij} = \delta_{ij} + O(|y|^{-4})$$

and therefore extends to $\tilde{U}$ as a metric of class C^3 (there being similar decay in the derivatives). The action of Γ on $\tilde{U}$ preserves the metric and is therefore of class $C^{3,\alpha}$ for $\alpha < 1$ (since harmonic coordinates are of this class).

Thus $\overline{X}$ is an orbifold (of class $C^{3,\alpha}$, though this is hardly important) with a finite quotient singularity at ∞ modelled on $\tilde{U}/\Gamma$; we may regard $\overline{g}$ as an orbifold metric on $\overline{X}$.

The Riemann curvature tensor of a hyper-Kähler 4-manifold is anti-self-dual with respect to the orientation associated with the complex structures. This means that the metric is Ricci-flat and *conformally anti-self-dual* (that is, the Weyl tensor is anti-self-dual [2]). Since this last condition is a conformally invariant one, it is satisfied also by $\overline{g}$, and one sees that $\overline{X}$ is a conformally anti-self-dual orbifold.

In the coordinates y_i on $\tilde{U}$, the extra point ∞ is at the origin, and by means of its action on the tangent space at this point, we may identify Γ with a subgroup of SO(4). Let $\tilde{U}$ be given the orientation appropriate to the orientation of X and let the $\frac{1}{2}$-spin bundles be labelled V^+ and V^- accordingly. We should remark that the hyper-Kähler condition ensures that X is a spin manifold and that V^+ is flat and globally trivial.

Lemma 2.1. *The group Γ lies in the subgroup* $\mathrm{SU}(2) \subset \mathrm{SO}(4)$ *which acts trivially on V^- and nontrivially on V^+ at the fixed point.*

Proof. To leading order, Γ acts linearly on the coordinates x_i and y_i by one and the same representation $\rho\colon \Gamma \to \mathrm{SO}(4)$. Since X is hyper-Kähler, the $\frac{1}{2}$-spin bundle V^+ is trivial on $\tilde{U}'$ and the trivialization is invariant under Γ. So $\rho(\Gamma)$ acts trivially on V^+ at the origin of the x coordinates. The y coordinates differ from the x coordinates by an orientation-reversing diffeomorphism, so that in the orientation appropriate to X, the group $\rho(\Gamma)$ acts trivially on V^- at the origin of the y coordinates.

To study the spaces X and $\overline{X}$ we shall exploit Penrose's nonlinear gravitation construction. What we need can be found in [2] for the conformally anti-self-dual case, and in [8] for the case of hyper-Kähler manifolds.

Recall that when X is a conformally anti-self-dual 4-manifold with a spin structure (as it is in our case), its *twistor space* may be defined as the projectivized $\frac{1}{2}$-spin bundle $Z = \mathbf{P}(V^+)$. This Z carries an integrable complex structure and an anti-holomorphic involution $\tau\colon Z \to Z$ which depend only on the conformal class of the metric. The fibers of the projection $\mathbf{P}(V^+) \to X$ are the *twistor lines*: they are holomorphic rational curves in Z which are preserved by τ. The dual of the tautological bundle on $\mathbf{P}(V^+)$ is a holomorphic line bundle on Z whose sheaf of sections we denote by $\mathscr{O}(1)$.

The same constructions can be made when the conformally anti-self-dual space is an orbifold rather than a manifold. For example, the twistor space of $U = \tilde{U}/\Gamma$ can be defined to be the complex orbifold $W = \tilde{W}/\Gamma$, where $\tilde{W}$ is the twistor space of $\tilde{U}$. (Note that Γ will act on $\tilde{W}$ biholomorphically.) In this way one may construct the twistor space $\overline{Z}$ of the compactification $\overline{X} = X \cup \{\infty\}$: it is a complex orbifold containing a singular line l_∞ lying over $\infty \in \overline{X}$. If $\tilde{l}_\infty \subset \tilde{W}$ denotes the nonsingular twistor line over $\infty \in \tilde{U}$ (a copy of $\mathbf{C}P^1$), then we will have $l_\infty = \tilde{l}_\infty/\Gamma$, and it follows from Lemma 2.1 that Γ acts on $\tilde{l}_\infty$ by the standard action of SU(2) on $\mathbf{C}P^1$. The complex manifold $Z = \overline{Z}\backslash l_\infty$ is the twistor space of X, and the sheaf $\mathscr{O}(1)$ can be extended from Z to $\overline{Z}$ by defining its local sections on $\tilde{W}/\Gamma$ to be the Γ-invariant local sections on $\tilde{W}$.

To summarize, Z is the twistor space of the conformally anti-self-dual manifold (X, g), and $\overline{Z}$ is the twistor space of the orbifold $(\overline{X}, \overline{g})$. The following vanishing theorem is the key to the structure of these two complex spaces.

Lemma 2.2. $H^1(\overline{Z}, \mathscr{O}(-1)) = 0$.

Proof. Suppose not. Then by the Penrose transform (see [7]), we obtain on $\overline{X}$ a nonzero solution $\overline{\psi}$ to the orbifold Dirac equation $D\overline{\psi} = 0$. (Here D is the Dirac operator acting on sections of V^-. We remark that the Penrose transform needs no modification for orbifolds.) By conformal invariance [7], the spinor $\overline{\psi}$ gives rise to a nonzero solution of the Dirac equation on X satisfying the decay conditions

$$|\psi| = O(r^{-3}), \qquad |\nabla\psi| = O(r^{-4}).$$

The Weitzenböck formula for the Dirac operator says that

$$D^*D = \nabla^*\nabla + \tfrac{1}{4}S,$$

where S is the scalar curvature. Since $S = 0$ on X, we have $\nabla^*\nabla\psi = 0$. So, following the usual Bochner vanishing argument, we find

$$0 = \int_{r\leq R} (\nabla^*\nabla\psi, \psi) = \int_{r\leq R} |\nabla\psi|^2 + \int_{r=R} (\nabla\psi, \psi) = \int_{r\leq R} |\nabla\psi|^2 + O(R^{-4}).$$

Letting $R \to \infty$ one sees that $\nabla\psi = 0$ and hence $\psi = 0$, a contradiction.

The twistor space of a hyper-Kähler 4-manifold possesses two additional structures which are not present when the manifold is merely conformally anti-self-dual. The first is a holomorphic fibration $\pi: Z \to \mathbf{C}P^1$ of which the twistor lines are sections. If one identifies $\mathbf{C}P^1$ with $S^2 \subset \mathbf{R}^3$, then for each $a = (a_1, a_2, a_3) \in S^2$, the fiber $Z_a = \pi^{-1}(a)$ is the complex surface obtained by equipping Z with the complex structure

$$I_a = a_1 I + a_2 J + a_3 K.$$

For each of these complex structures, there is a holomorphic symplectic form on X (unique to within a complex scalar multiple) which is a linear combination of the three Kähler forms. (For the complex structure I, the holomorphic 2-form is $\omega_2 + i\omega_3$.) Globally these fit together to give a holomorphic section [8]

$$\omega \in H^0(Z, \Lambda^2 T_F^* \otimes \mathscr{O}(2)),$$

where T_F^* is the tangent space to the fibers, the kernel of $d\pi$. This *twisted vertical 2-form* is the second piece of additional data.

Let $A(Z)$ denote the graded ring

$$A(Z) = \bigoplus_{k\geq 0} H^0(Z, \mathscr{O}(k)),$$

and let $A(\overline{Z})$ and $A(l_\infty)$ be similarly defined. By Hartog's theorem, sections of $\mathscr{O}(k)$ on Z extend to $\overline{Z}$, so that $A(\overline{Z}) = A(Z)$, and there is therefore a restriction map $A(Z) \to A(l_\infty)$. Via the holomorphic fibration $\pi: Z \to \mathbf{C}P^1$, we can pull back a basis u, v for $H^0(\mathbf{C}P^1, \mathscr{O}(1))$ to obtain two sections of $\mathscr{O}(1)$ on Z, also denoted by u and v, which generate an ideal $I \subset A(Z)$.

Proposition 2.3. *The following sequence is exact:*

$$0 \to I \to A(Z) \to A(l_\infty) \to 0.$$

Proof. Let us first prove that the sequence is exact at the middle term. Let $\mathscr{I}_k \subset \mathscr{O}(k)$ be the subsheaf consisting of sections which vanish on l_∞. What we want to prove is that $\mathscr{I}_k$ is generated by u and v, or that $\mathscr{I}_k$ is the image of

$$\mu: \mathscr{O}(k-1) \oplus \mathscr{O}(k-1) \to \mathscr{O}(k)$$
$$(s, t) \mapsto us + vt.$$

Since u and v are nonvanishing on Z, we need only look at a neighborhood of l_∞. For this purpose we may take the neighborhood W to be the twistor space of $U \subset \overline{X}$, so that $W = \tilde{W}/\Gamma$, where $\tilde{W}$ is the twistor space of $\tilde{U}$. Then u and v lift to sections $\tilde{u}$ and $\tilde{v}$ of $\mathscr{O}(k)$ on $\tilde{W}$, and we must show that they generate the ideal sheaf of $\tilde{l}_\infty$.

This is a matter which depends only on the values of $\tilde{u}$ and $\tilde{v}$ and their first derivatives at points of $\tilde{l}_\infty$. It is therefore enough to look at the case $X = \mathbf{R}^4$, for since $\tilde{u}$ and $\tilde{v}$ are determined by the metric alone, they will resemble the flat case to fourth order near infinity. (To see how $\tilde{u}$ and $\tilde{v}$ are related to the metric, observe that in the hyper-Kähler manifold X they can be interpreted as the two covariant-constant sections of $(V^+)^*$. In this conformal model the Christoffel symbols have order r^{-5}, and so $\tilde{u}$ and $\tilde{v}$ are Euclidean to order r^{-4}.) Now, in the $\mathbf{R}^4$ case, $\overline{X}$ is conformally the 4-sphere, its twistor space $\overline{Z}$ is $\mathbf{C}P^3$, and Z is $\mathbf{C}P^3 \backslash l_\infty$. In suitable homogeneous coordinates $[u, v, s, t]$, the projection $\pi\colon Z \to \mathbf{P}_1$ is given by

$$[u, v, s, t] \mapsto [u, v],$$

and l_∞ is the line defined by $u = v = 0$, which is what we wanted to prove. So the sequence is exact at the middle term.

Now we must prove the surjectivity of the restriction map $A(Z) \to A(l_\infty)$. By the above arguments we have two short exact sequences of sheaves on $\overline{Z}$:

(A) $0 \to \mathscr{J}_k \to \mathscr{O}(k) \to \mathscr{O}_{l_\infty}(k) \to 0$,

(B) $0 \to \mathscr{O}(k-2) \xrightarrow{\lambda} \mathscr{O}(k-1) \oplus \mathscr{O}(k-1) \xrightarrow{\mu} \mathscr{J}_k \to 0$,

where $\lambda\colon s \mapsto (vs, -us)$. To prove surjectivity we must show that $H^1(\overline{Z}, \mathscr{J}_k) = 0$. In fact we shall prove two assertions for all $k \geq 0$:

(Ψ_k) $H^1(\overline{Z}, \mathscr{J}_k) = 0$,

(Φ_k) $H^1(\overline{Z}, \mathscr{O}(k)) = 0$.

Since $H^0(\overline{Z}, \mathscr{J}_k) = 0$ for $k \leq 0$, the long exact sequence in cohomology coming from (B) gives

$$\Phi_{k-1} \Rightarrow \Phi_{k-2}, \qquad k \leq 0.$$

By Lemma 2.2 we already have Φ_{-1}, so Φ_k holds for all $k < 0$. Since the canonical sheaf of the orbifold $\overline{Z}$ is $\mathscr{O}(-4)$ (see [1]), Serre duality yields

$$H^2(\overline{Z}, \mathscr{O}(k)) = 0, \qquad k \geq -3.$$

(Serre duality for orbifolds is proved just as it is proved for complex manifolds: one chooses a Hermitian metric and then exploits the Hodge theory. The 'canonical sheaf' is in the sense of orbifolds; see [3].) Using the long exact sequence of (B) again, we deduce

$$\Phi_{k-1} \Rightarrow \Psi_k, \qquad k \geq -1.$$

Now l_∞ is the quotient of a projective line by a finite group, so $H^1(l_\infty, \mathscr{O}(k)) = 0$ for $k \geq -1$. The long exact sequence of (A) therefore yields

$$\Psi_k \Rightarrow \Phi_k, \qquad k \geq -1.$$

Using the last two implications and induction, one sees that Φ_k and Ψ_k hold for all $k \geq 0$. This proves the proposition.

The ring $A(l_\infty)$ is the Γ-invariant part of $A(\tilde{l}_\infty)$, and since the latter is just a polynomial ring in two variables (the affine coordinate ring of $\mathbf{C}^2$), it follows that $A(l_\infty)$ is the affine coordinate ring of $\mathbf{C}^2/\Gamma$, the Γ-invariant polynomials. If Y is the affine variety whose coordinate ring is $A(Z)$, then Proposition 2.3 can be interpreted as saying that there is a map $\phi\colon Y \to \mathbf{C}^2$ whose fiber $\phi^{-1}(0)$ is $\mathbf{C}^2/\Gamma$. Since $A(Z)$ is flat over $\mathbf{C}[u,v]$, this Φ is a *deformation* of $\mathbf{C}^2/\Gamma$. The grading of $A(Z)$ gives an action of $\mathbf{C}^*$ on Y, making ϕ a $\mathbf{C}^*$-*deformation* in the sense of [13]: that is to say, ϕ is $\mathbf{C}^*$-equivariant, and the fiber $\phi^{-1}(0)$ is equivariantly isomorphic to $\mathbf{C}^2/\Gamma$ with its obvious $\mathbf{C}^*$-action.

We can give a more concrete definition of Y. According to Klein [9], the ring $A(l_\infty)$ is generated by three homogeneous elements x, y, z subject to one relation $f(x,y,z) = 0$:

Group	Relation
C_k	$xy - z^k = 0$
D_k	$x^2 + y^2 z + z^{k+1} = 0$
T	$x^2 + y^3 + z^4 = 0$
O	$x^2 + y^3 + yz^3 = 0$
I	$x^2 + y^3 + z^5 = 0$

Thus the exact sequence of Proposition 2.3 shows that $A(z)$ is generated by elements $\{x, y, z, u, v\}$ subject to a relation

$$f(x,y,z) + u \cdot g(x,y,z,u,v) + v \cdot h(x,y,z,u,v) = 0 \tag{2.4}$$

for some polynomials g and h. This equation defines a hypersurface $Y \subset \mathbf{C}^5$, and the map $\phi = (u,v)\colon Y \to \mathbf{C}^2$ is a deformation of $\mathbf{C}^2/\Gamma$. If d_1, d_2, d_3 are the degrees of x, y, z, then the action of $\mathbf{C}^*$ on $\mathbf{C}^5$ given by

$$(x,y,z,u,v) \mapsto (\lambda^{d_1} x, \lambda^{d_2} y, \lambda^{d_3} z, \lambda u, \lambda v)$$

leaves Y invariant and makes ϕ a $\mathbf{C}^*$-deformation.

The quotient of $\mathbf{C}^5 \backslash 0$ by this action of $\mathbf{C}^*$ is a certain compact variety, a weighted projective space. Let $\overline{Z}^s = (Y\backslash 0)/\mathbf{C}^*$ be the image of Y in this weighted projective space, and define $Z^s = \overline{Z}^s \backslash l^s_\infty$ where $l^s_\infty = \{u = v = 0\} \subset \overline{Z}^s$. The functions x, y, z, u, v induce maps

$$\overline{\chi}\colon \overline{Z} \to \overline{Z}^s, \qquad \chi\colon Z \to Z^s.$$

The map $\overline{\chi}$ takes a neighborhood of l_∞ in $\overline{Z}$ isomorphically onto a neighborhood of l^s_∞ in $\overline{Z}^s$, while the map χ commutes with the projection to $\mathbf{C}P^1$:

$$\begin{array}{ccc} Z & \xrightarrow{\chi} & Z^s \\ \pi \big\downarrow & & \big\downarrow \pi^s \\ \mathbf{C}P^1 & = & \mathbf{C}P^1 \end{array}$$

On each fiber $Z_a = \pi^{-1}(a)$, the restriction $\chi_a \colon Z_a \to Z^s_a$ is proper and birational because $\overline{\chi}$ is an isomorphism near l_∞. Furthermore, since each surface Z_a has zero first Chern class, there can be no exceptional curves of the first kind in Z_a, and it follows that $\chi_a \colon Z_a \to Z^s_a$ is the minimal resolution. We can summarize the situation by saying that the diagram above is a *simultaneous resolution* of $\pi^s \colon Z^s \to \mathbf{C}P^1$ (see [5] for a definition) inducing the minimal resolution of each fiber.

What we have seen is that the twistor space Z of an ALE hyper-Kähler 4-manifold has a singular model Z^s which is obtained from the total space of a deformation $\phi \colon Y \to \mathbf{C}^2$ by removing the fiber $\phi^{-1}(0) = \mathbf{C}^2/\Gamma$ and then dividing by a $\mathbf{C}^*$-action. (In the case in which Γ is cyclic, this is essentially the description of the twistor space given by Hitchin [6], and was the starting point for the twistor construction of the multi-Eguchi-Hanson gravitational instantons.) This concrete description of Z, together with some results from the deformation theory of $\mathbf{C}^2/\Gamma$, will lead easily to the proof of Theorem 1.2 and 1.3.

Being a deformation of $\mathbf{C}^2/\Gamma$, the map ϕ will be the pull-back of the semi-universal deformation $\Psi \colon \mathscr{Y} \to \mathscr{T}$ by some map $t \colon \mathbf{C}^2 \to \mathscr{T}$:

$$\begin{array}{ccc} Y & \longrightarrow & \mathscr{Y} \\ \phi \big\downarrow & & \big\downarrow \Psi \\ \mathbf{C}^2 & \xrightarrow[t]{} & \mathscr{T} \end{array} \tag{2.5}$$

As in [10], we take for Ψ the $\mathbf{C}^*$-semi-universal deformation, so that all the maps in this diagram are homogeneous and globally defined [13].

Corollary 2.6. *X is diffeomorphic to $\widetilde{\mathbf{C}^2/\Gamma}$, the minimal resolution of* $\mathbf{C}^2/\Gamma$.

Proof. It is a special property of the singularities $\mathbf{C}^2/\Gamma$ that their semi-universal deformations admit simultaneous resolutions [5], [13]. From this property it follows that the minimal resolution of every fiber of Ψ is diffeomorphic to $\widetilde{\mathbf{C}^2/\Gamma}$; and because of the diagram (2.5), the same is true for the fibers of Y. But the latter are the spaces Z^s_a whose minimal resolutions are the surfaces Z_a, and by the nature of the twistor space, each Z_a is diffeomorphic to X.

The exceptional set in the minimal resolution $\widetilde{\mathbf{C}^2/\Gamma}$ is a union of rational curves, each with self-intersection -2, whose configuration is the dual of a certain simply-laced Dynkin diagram $\Delta(\Gamma)$, one of A_r, D_r, E_6, E_7 or E_8 according as Γ is cyclic, binary bihedral, tetrahedral, octahedral or icosohedral (see [13], for example). The second homology $H_2(X; \mathbf{Z})$ is therefore isomorphic to the corresponding root lattice in such a way that the classes Σ with $\Sigma \cdot \Sigma = -2$ correspond to the roots; and the cohomology $H^2(X; \mathbf{C})$ can be identified with the complex Cartan algebra h^c.

Lemma 2.7. *There are only finitely many points $a \in \mathbf{C}P^1$ for which Z_a^s is singular.*

Proof. Notice first that for any given (-2)-class $\Sigma \in H_2(X; \mathbf{Z})$, there is at most one complex structure I_a $(a \in S^2)$ for which Σ may be represented by a holomorphic curve in X. To see this, suppose for example that Σ is represented by a holomorphic curve P with respect to I. Then the Kähler form ω_1 must be positive on P, while the form $\omega_2 + i\omega_3$ must be zero because it is a holomorphic 2-form. Thus $[\omega_1](\Sigma) > 0$, while $[\omega_2](\Sigma) = [\omega_3](\Sigma) = 0$, and it is clear that the corresponding conditions cannot hold for any other complex structure.

It follows that the number of points $a \in \mathbf{C}P^1$ for which Z_a contains a holomorphic (-2)-curve does not exceed the number of roots. The singularities in the fibers Z_a^s all have the form $\mathbf{C}^2/\hat{\Gamma}$ for some $\hat{\Gamma} \subset \mathrm{SU}(2)$ (this follows from the corresponding property for the fibers of Ψ; see [13]), and their minimal resolutions therefore contain (-2)-curves. So we deduce that the number of points $a \in \mathbf{C}P^1$ for which Z_a^s is singular is also bounded by the number of roots. This proves the lemma.

Being the twistor space of a hyper-Kähler manifold, Z carries a holomorphic section

$$\omega \in H^0(Z, \Lambda^2 T_F^* \otimes \mathscr{O}(2)).$$

Taking the homology class of ω on each fiber gives an element of $H^0(\mathbf{C}P^1, h^c \otimes \mathscr{O}(2))$, or alternatively a map $\tilde{p}\colon \mathbf{C}^2 \to h^c$ which is homogeneous of degree 2. Composing p with the projection $h^c \to h^c/W$ gives a map

$$p\colon \mathbf{C}^2 \to h^c/W. \tag{2.8}$$

Now ω also gives rise to a twisted vertical 2-form on the nonsingular part of Z^s via the map χ. From Z^s we can lift it to Y where it gives a vertical 2-form on the deformation $\phi\colon Y \to \mathbf{C}^2$. So p is nothing other than the period map for ϕ in the sense of [11].

Proof of Theorem 1.2, *completed.* It remains to show that the cohomology classes of the Kähler forms on X satisfy the nondegeneracy condition $(*)$. Suppose on the contrary that there is a Σ on which all three cohomology classes vanish. Then the image of $\tilde{p}$ lies in the kernel of a root, and the image of the period map p lies in the branch locus of the quotient map $h^c \to h^c/W$. As is explained in [10], there is an isomorphism $p_\Psi : \mathscr{V} \to h^c/W$ (in fact, the period map of the semi-universal deformation) with the property that $p = p_\Psi \circ t$, where t is the map in (2.5). Furthermore, this p_Ψ carries the discriminant locus $\mathscr{D} \subset \mathscr{V}$ onto the branch locus in h^c/W. It follows that the image of t lies in $\mathscr{D}$. But by the definition of the discriminant locus, this means that all the fibers of $\phi \colon Y \to \mathbf{C}^2$ are singular, and this contradicts Lemma 2.7.

3. Proof of Theorem 1.3

Let X^1 and X^2 be hyper-Kähler manifolds satisfying the hypotheses of Theorem 1.3. We aim to prove that they are isometric, and our strategy is to show that they have the same twistor space carrying the same real structure τ, the same family of twistor lines, and the same twisted 2-form. This will suffice, for it is a feature of the Penrose construction that the twistor space, together with these auxiliary structures, gives complete information about the metric (see [8] for the hyper-Kähler case).

So let Z^1 and Z^2 be the twistor spaces and let the holomorphic fibrations be

$$\pi^i \colon Z^i \to \mathbf{C}P^1 \qquad (i = 1, 2).$$

From the results of §2 we know that these are simultaneous resolutions of certain singular models

$$(\pi^s)^i \colon (Z^s)^i \to \mathbf{C}P^1 \qquad (i = 1, 2),$$

which in turn are quotients of two $\mathbf{C}^*$-deformations of $\mathbf{C}^2/\Gamma$,

$$\phi^i \colon Y^i \to \mathbf{C}^2 \qquad (i = 1, 2).$$

Since the cohomology classes of the three Kähler forms on X^1 and X^2 are equal, the deformations ϕ^1 and ϕ^2 have the same period map (2.8) and are therefore isomorphic by [10, Proposition (4.5)]. It follows that $(\pi^s)^1$ and $(\pi^s)^2$ are isomorphic too, and that π^1 and π^2 are simultaneous resolutions of one and the same map which we shall now denote by $\pi^s \colon Z^s \to \mathbf{C}P^1$. So the picture is as follows:

$$\begin{array}{ccccccc} Z^1 & \xrightarrow{\chi^1} & Z^s & \qquad & Z^2 & \xrightarrow{\chi^2} & Z^s \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbf{C}P^1 & = & \mathbf{C}P^1 & & \mathbf{C}P^1 & = & \mathbf{C}P^1 \end{array}$$

Both maps χ^i extend to the compactification obtained by adding the line l_∞.

Simultaneous resolutions of singular maps are not always unique and we cannot deduce from these diagrams alone that Z^1 and Z^2 are isomorphic. We could settle the question by appealing to the results of [4] to show that there can only be one simultaneous resolution compatible with the known Kähler classes; but we shall not pursue this line, as the singular model contains all the information we need.

Via the maps χ^i, the space Z^s obtains two real structures τ^1 and τ^2, two twisted vertical 2-forms ω^1 and ω^2 (at least on the complement of the singular set), and two families of twistor lines $\mathscr{F}^1$ and $\mathscr{F}^2$, each of which fibers Z^s in some neighborhood of infinity.

The composite $\sigma = \tau^1 \circ \tau^2$ is a *holomorphic* transformation of Z^s and therefore produces an automorphism $A(\sigma)$ of the graded ring $A(Z^s)$. Since σ respects the holomorphic fibration over $\mathbf{C}P^1$, the automorphism $A(\sigma)$ fixes u and v. Further, since τ^1 and τ^2 both give the antipodal map on l_∞, the automorphism of $A(l_\infty)$ which σ induces is the identity. Applying the 5-lemma to the short exact sequence of Proposition 2.3, we deduce that $A(\sigma) = 1$ and hence $\tau^1 = \tau^2$.

The ratio of ω^1 and ω^2 is a holomorphic function on the nonsingular part of Z^s which extends, by Hartog's theorem, first to Z^s itself and then to $\overline{Z}^s$. Since $\overline{Z}^s$ is compact, this ratio is constant, and since the cohomology classes of ω^1 and ω^2 agree, the ratio must be 1. So $\omega^1 = \omega^2$.

For $i = 1, 2$, let U^i be a neighborhood of $\{\infty\}$ in $\overline{X}^i$, let $\tilde{U}^i$ be its nonsingular branched covering, and let W^i and $\tilde{W}^i$ be the twistor spaces of U^i and $\tilde{U}^i$. We may view W^1 and W^2 as neighborhoods of l_∞ in the singular model $\overline{Z}^s$, and by shrinking them somewhat, we may take it that they coincide. Since each $\tilde{W}^i$ restricts to give the universal covering of $W^i \backslash l_\infty$, there will be a diagram

$$\begin{tikzcd} \tilde{W}^1 \arrow[rr, "\nu"] \arrow[dr] & & \tilde{W}^2 \arrow[dl] \\ & W & \end{tikzcd}$$

in which ν is an isomorphism.

Being the twistor space of $\tilde{U}^i$, each $\tilde{W}^i$ has a real structure $\tilde{\tau}^i$, and we must show that ν preserves these: that is, $\tilde{\tau}^1 = \nu^{-1}\tilde{\tau}^2\nu$. Now if these two differ at all, then they differ by a covering transformation $\gamma \in \Gamma$, for we already know that τ^1 and τ^2 are the same on W. Furthermore, since both real structures give the antipodal map of $\tilde{l}_\infty$, the covering transformation γ must leave $\tilde{l}_\infty$ pointwise fixed. So the only possibility is $\Gamma = -1$, this being the only nontrivial element of SU(2) which acts trivially on $\mathbf{C}P^1$. To rule out this last possibility we recall from [8] that the twisted vertical 2-form on a twistor space must be compatible with the real structure in a strong sense, for it must give rise to a metric on X which is not only real but also *positive definite*. The point is that if the two real structures did differ by the action of $-1 \in \mathrm{SU}(2)$, then they would give rise to 'metrics' of opposite sign. (One can check this explicitly in the flat case when $\overline{Z} = \mathbf{C}P^3$.) These considerations show that ν must preserve the real structures τ^i.

The twistor lines of $\tilde{U}^i$ form a smooth family $\tilde{\mathscr{F}}^i$ in $\tilde{W}^i$ depending on four real parameters $(i = 1, 2)$. The line $\tilde{l}_\infty$ belongs to both families, and being a twistor line, it has normal bundle $\mathscr{O}(1) \oplus \mathscr{O}(1)$. A theorem of Kodaira implies that the universal deformation of $\tilde{l}_\infty$ in $\tilde{W}^i$ is a smooth family of four complex parameters (see [2]), and it follows that $\tilde{\mathscr{F}}^i$ consists of all those members of the universal family which are preserved by τ^i. Since ν carries $\tilde{l}_\infty$ to $\tilde{l}_\infty$ and preserves the real structures, it therefore follows that ν carries the family $\tilde{\mathscr{F}}^1$ to the family $\tilde{\mathscr{F}}^2$. Near l_∞, the twistor lines in $\overline{Z}^s$ belonging to the families $\mathscr{F}^1$ and $\mathscr{F}^2$ are just the images of the families $\tilde{\mathscr{F}}^1$ and $\tilde{\mathscr{F}}^2$; so these coincide also.

It now follows that in the hyper-Kähler manifolds X^1 and X^2 there will be open neighborhoods of infinity, say V^1 and V^2, on which there is an isometry $\eta\colon V^1 \to V^2$. If we choose an $a \in \mathbf{C}P^1$ for which Z^s_a is nonsingular, then the maps χ^i give isomorphisms on the fibers $Z^i_a \cong Z^s_a$ $(i = 1, 2)$; and since Z^1_a and Z^2_a are just the manifolds X^1 and X^2 equipped with the complex structure I_a, we conclude that η extends to a global diffeomorphism $\eta_a\colon X^1 \to X^2$ which is holomorphic with respect to I_a. Since this map is an isometry on the open set V^1, it is an isometry everywhere by analytic continuation and will be holomorphic with respect to all the complex structures.

References

[1] M. F. Atiyah, *Geometry of Yang-Mills fields*, Fermi Lectures, Scuola Normale, Pisa, 1979.

[2] M. F. Atiyah, N. J. Hitchin & I. M. Singer, *Self-duality in four dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A **362** (1978) 425–461.

[3] W. L. Bailey, *The decomposition theorem for V-manifolds*, Amer. J. Math. **78** (1956) 862–888.

[4] E. Brieskorn, *Die Auflösung der rationalen Singularitäten holomorpher Abbildungen*, Math. Ann. **178** (1968) 255–270.

[5] ——, *Singular elements of semisimple algebraic groups*, Actes Congr. Internat. Math., No. 2, 1970, 279–284.

[6] N. J. Hitchin, *Polygons and gravitons*, Math. Proc. Cambridge Philos. Soc. **83** (1979) 465–476.

[7] ——, *Linear field equations on self-dual spaces*, Proc. Roy. Soc. London Ser. A **370** (1980) 173–191.

[8] N. J. Hitchin, A. Karlhede, U. Lindström & M. Roček, *Hyper-Kähler metrics and supersymmetry*, Comm. Math. Phys. **108** (1987) 535–589.

[9] F. Klein, *Vorlesgungen über das Ikosaeder und die Auflösung der Gleichung vom funften Grade*, Leipzig, 1884.

[10] P. B. Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients*, J. Differential Geometry **29** (1989), 665–683.

[11] E. J. N. Looijenga, *Isolated singular points on complete intersections*, London Math. Soc. Lecture Note Series 77, Cambridge University Press, Cambridge, 1984.

[12] I. Satake, *On a generalization of the notion of manifolds*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956) 359–363.

[13] P. Slodowy, *Simple singularities and simple algebraic groups*, Lecture Notes in Math., Vol. 815, Springer, Berlin, 1980.

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