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THE RELATIONS OF PLÜCKER COORDINATES TO SCHUBERT CALCULUS

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To the memory of our mother Nahide Çalişkan

Abstract

We study the relation between the nilpotent and classical descriptions of the cohomology ring of the Grassmann manifold $G_{k,n}$. Our main result is that the Plücker coordinates form a basis for the nilpotent description of the cohomology ring of $G_{k,n}$, which are dual to the Schubert cycles. We also prove that the cohomology ring of any Schubert subvariety of $G_{k,n}$ admits a nilpotent description.

0. Introduction

Let X be a nonsingular complex projective variety having an SL_2 action with the property that any maximal unipotent subgroup of SL_2 has only isolated fixed points. The cohomology ring $H^*(X, \mathbb{C})$ of such an X has been studied in [3], where the authors proved that $H^*(X, \mathbb{C})$ admits the so-called nilpotent and semi-simple descriptions. We start with summarizing these results. Let $\mathbb{B}$ denote the group of upper triangular matrices in SL_2, and suppose V and V_s are respectively the holomorphic vector fields generated by the maximal unipotent subgroup and maximal torus in $\mathbb{B}$. The nilpotent description of $H^*(X, \mathbb{C})$ says that the coordinate ring $A(Z)$ of the zero scheme Z of V has a canonical grading making it isomorphic in the sense of graded rings with $H^*(X, \mathbb{C})$. In the semi-simple case, however, even though the variety Z_s of the zeros of V_s contains only isolated points, the coordinate ring $A(Z_s)$ of Z_s is not graded. But, $A(Z_s)$ admits a filtration $F_0 \subset F_1 \subset \cdots$ such that $F_p F_q \subseteq F_{p+q}$ and

$$\mathrm{Gr}(A(Z_s)) = \bigoplus F_p/F_{p-1} \xrightarrow{\sim} \bigoplus H^{2p}(X, \mathbb{C}) = H^*(X, \mathbb{C}).$$

For any parabolic subgroup P of a complex reductive linear algebraic group G, the space G/P admits such an SL_2 action [3]. Thus, the cohomology ring of

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G/P has the semi-simple and nilpotent descriptions. On the other hand, there is the classical description of $H^*(G/P, \mathbb{C})$, which goes back to Schubert. It is based on the calculation of the homology with the aid of the partition of G/P into the so-called Schubert cells. The relation between the semi-simple and the classical descriptions of the cohomology ring of G/P has been studied by Gel′fand et al. in [5], where the authors constructed a basis in the semi-simple description dual to the Schubert cycles. In this paper, we study the similar problem for the nilpotent and classical descriptions of the cohomology ring of the Grassmann manifold $G_{k,n}$. In Theorem 3.1 we prove that the Plücker coordinates form a basis for the nilpotent description $A(Z)$ of $H^*(G_{k,n}; \mathbb{C})$, which are dual to the Schubert cycles. We also prove in Theorem 3.2 that for any Schubert subvariety Y of $G_{k,n}$ the coordinate ring $A(Y \cap Z)$ of the scheme theoretic intersection $Y \cap Z$ of Y and Z is isomorphic to the cohomology ring $H^*(Y, \mathbb{C})$ of Y. This gives an affirmative answer to the conjecture in [3] for the Grassmann manifolds.

The paper is organized as follows. In §1 we state the known results on the cohomology ring of a complex projective variety X with an SL_2 action. In §2, we compute the ideal $I(Z)$ defining the closed subscheme Z in the full flag manifold. In §3, we prove our main results.

1. Preliminaries and the nilpotent description $A(Z)$

In this section we explain the grading of the nilpotent description $A(Z)$ of $H^*(X, \mathbb{C})$ and review the generalizations of the nilpotent and semi-simple description of $H^*(X, \mathbb{C})$ to the singular subvarieties of X.

We start with the grading of $A(Z)$. We will assume that V has only one zero x_0. The general case is similar. Since the point x_0 is also fixed by the maximal torus $H \cong \mathbb{C}^*$ in $\mathbb{B}$, $\mathbb{C}^*$ acts on the tangent space $T_{x_0}X$ of X at x_0 [3]. Thus $\mathbb{C}^*$ acts on the symmetric algebra $A = \mathrm{Sym}(T^*_{x_0}X)$ of the cotangent space of X at x_0. The weight decomposition of this action makes A into a graded algebra. In the following theorem, A will be a regarded as a graded algebra with this gradation.

Theorem 1.1 ([3], *the nilpotent description*). *There exists a $\mathbb{C}^*$-invariant open affine neighborhood U of x_0 such that U is $\mathbb{C}^*$-equivariantly isomorphic to* Spec(A)*, and consequently, the ring of regular functions $A(U)$ on U admits a graded algebra structure. The ideal $I(Z)$ of the zero scheme Z of V is homogeneous in $A(U)$, and moreover $A(Z) = A(U)/I(Z)$ is isomorphic to $H^*(X, \mathbb{C})$.*

The generalizations of the semi-simple and nilpotent descriptions of $H^*(X, \mathbb{C})$ to the singular subvarieties of X have been studied in [4], where the following results were obtained as particular cases: Let Y be a $\mathbb{B}$-invariant subvariety of X such that $H^*(X, \mathbb{C})$ surjects into $H^*(Y, \mathbb{C})$. In the semi-simple case, the coordinate ring $A(Y \cap Z_s)$ of the intersection $Y \cap Z_s$ of Y and Z_s has a filtration such that the associated graded algebra $\mathrm{Gr}(A(Y \cap Z_s))$ admits a homomorphism into $H^*(Y, \mathbb{C})$ making the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Gr}(A(Z_s)) & \xrightarrow[\sim]{\phi} & H^*(X, \mathbb{C}) \\ \downarrow & & \downarrow \\ \mathrm{Gr}(A(Y \cap Z_s)) & \xrightarrow{\overline{\phi}} & H^*(Y, \mathbb{C}). \end{array}$$

In this case the main result is that the map

$$\overline{\phi}\colon \mathrm{Gr}(A(Y \cap Z_s)) \to H^*(Y, \mathbb{C})$$

is an isomorphism, i.e., $H^*(Y, \mathbb{C})$ admits a semi-simple description. On the other hand, in the nilpotent case $A(Z)$, there is a canonical grading of the coordinate ring $A(Y \cap Z)$ of the scheme theoretic intersection $Y \cap Z$ of Y and Z such that the natural map $A(Z) \to A(Y \cap Z)$ is a graded algebra homomorphism. The main theorem is that $A(Y \cap Z)$ admits a homomorphism into $H^*(Y, \mathbb{C})$, which is compatible with the isomorphism $\psi\colon A(Z) \xrightarrow{\sim} H^*(X, \mathbb{C})$ (cf. [3], [4]). Thus, the map ψ induces a surjective graded algebra homomorphism

$$\overline{\psi}\colon A(Y \cap Z) \to H^*(Y, \mathbb{C}).$$

While $\overline{\phi}$ is an isomorphism in the semi-simple case, it is not known whether this is true for $\overline{\psi}$ in the nilpotent case. But, when X is the algebraic homogeneous space G/P and Y is a Schubert subvariety of G/P, it has been conjectured in [3] that $\overline{\psi}$ is an isomorphism. This would imply that the cohomology ring of a Schubert variety Y in G/P admits a nilpotent description.

2. Graded algebra $A(Z)$ when $X = \mathrm{GL}_n/B$

In this section we give the complete description of $A(Z)$ when $X = \mathrm{GL}_n/B$ is the full flag manifold or the Grassmann manifold $G_{k,n}$ of k-planes in $\mathbb{C}^n$.

Let $G = \mathrm{GL}_n$, and let B be the group of upper triangular matrices in G, P the parabolic subgroup of all matrices in G of the form $\left(\begin{smallmatrix} A & * \\ 0 & C \end{smallmatrix}\right)$, where 0 is the $(n-k) \times k$ zero matrix, $\pi\colon G/B \to B/P$ the natural projection map, e_{ij} the $n \times n$ matrix having 1 in the (i,j)th entry and zero everywhere else, $n = \sum_{i=1}^{n-1} e_{i,i+1}$, and x_0 (resp. $\pi(x_0)$) the element B (resp. P) in G/B (resp. G/P). By the Jacobson-Morosov Lemma, associated with n there exists an SL_2 action on G/B (resp. G/P) such that the vector field $\tilde{V}$ (resp. V) generated by the maximal unipotent subgroup in $\mathbb{B}$ is induced from the one parameter subgroup $\exp(tn)$ of G, and has exactly one zero x_0 (resp. $\pi(x_0)$). The algebraic homogeneous space G/B is the full flag manifold, and $G/P = G_{k,n}$ is the Grassmann manifold of k-planes in $\mathbb{C}^n$. Let z_{ij} be the functions on G defined by $z_{ij}(x) = x_{ij}$, where $x = (x_{ij}) \in G$. It follows from [3] that $A(U)$ for G/B (resp. $G_{k,n}$) is isomorphic to the graded algebra

$$\tilde{R} = \mathbb{C}[z_{ij}: 1 \leq j < i \leq n] \quad (\text{resp. } R = \mathbb{C}[z_{k+i\,j}: 1 \leq i \leq n-k,\ 1 \leq j \leq k]),$$

where the grading is determined by taking degree $(z_{pq}) = p - q$. In the rest of the paper $\tilde{Z}$ (resp. Z) denotes, as before, the zero scheme of $\tilde{V}$ (resp. V), and we take $z_{ij} = 0$ if either $i > n$ or $j < 1$, or $j > i$, and $z_{ii} = 1$ for $1 \leq i \leq n$. The following is the key proposition for the rest of the paper.

Proposition 2.1. (i) *The graded algebra $A(\tilde{Z})$ is isomorphic to $\tilde{R}/I(\tilde{Z})$, where $I(\tilde{Z})$ is the homogeneous ideal generated by*

$$a_{ij}(z) = z_{i+1\,j} - z_{i\,j-1} + z_{ij}(z_{j\,j-1} - z_{j+1\,j}).$$

(ii) *Let $x_1 = z_{21}, x_2 = z_{32} - z_{21}, \cdots, x_j = z_{j+1\,j} - z_{j\,j-1}, \cdots, x_n = -z_{n\,n-1}$, and let $h_m(y_1, \cdots, y_s)$ be the mth complete symmetric homogeneous function in $y_1, \cdots, y_s$. For any i, j the following identity holds in $\tilde{R}/I(\tilde{Z})$:*

$$z_{ij} = h_{i-j}(x_1, x_2, \cdots, x_j).$$

(iii) *Under the isomorphism $\tilde{\psi}: \tilde{R}/I(\tilde{Z}) \cong A(\tilde{Z}) \xrightarrow{\sim} H^*(G/B, \mathbb{C})$, $\tilde{\psi}(z_{ij} \bmod I(\tilde{Z})) = c_{i-j}(Q_j)$, $(i-j)$th Chern class of the universal quotient bundle Q_j of rank $n - j$ on G/B.*

Proof. To prove (i) we need to compute the local expression of $\tilde{V}$ in the local coordinates z_{ij}, $1 \leq j < i \leq n$. Let $M = (z_{ij})$ be the $n \times n$ lower triangular unipotent matrix having z_{ij} as its entries. The change of the local coordinates z_{ij} by the action of $\exp(tn)$ around x_0 is given by the functions $z_{ij}(t)$, $1 \leq j < i \leq n$, which satisfy the following matrix identity for some $n \times n$ upper triangular matrix $B(t)$:

$$\exp(tn) M B(t) = (z_{ij}(t)).$$

Here $(z_{ij}(t))$ represents the $n \times n$ lower triangular unipotent matrix. The point is that one can compute these $z_{ij}(t)$ explicitly. Once this is done it is not hard to see that

$$\tilde{V}(z_{ij}) = \frac{d}{dt}(z_{ij}(t))\big|_{t=0} = z_{i+1\,j} - z_{i\,j-1} + z_{ij}(z_{j\,j-1} - z_{j+1\,j}).$$

We leave these calculations to the reader.

Part (ii) follows from the defining relations $a_{ij}(z) = 0$ in $\tilde{R}/I(\tilde{Z})$. Part (iii) follows from [3], part (ii) and the well-known formula for $c_k(Q_j)$ in $H^*(G/B, \mathbb{C})$.

For the Grassmann manifold $G_{k,n}$ similar results can be found in [6]. In this case the homogeneous ideal $I(Z)$ of Z in R is generated by

$$z_{k+1+ij} - z_{k+ij-1} - z_{k+ik}z_{k+1j}, \qquad 1 \leq j \leq k,\ 1 \leq i \leq n-k.$$

In the rest of the paper we shall take $A(\tilde{Z}) = \tilde{R}/I(\tilde{Z})$, $A(Z) = R/I(Z)$, and keep the notations as before.

3. Cohomology of Schubert varieties in $G_{k,n}$

In this section, we first give the explicit description of the isomorphism $\psi: A(Z) \xrightarrow{\sim} H^*(G_{k,n}, \mathbb{C})$ by providing the representatives of Schubert cycles in $A(Z)$, and then prove that $\overline{\psi}: A(Y \cap Z) \xrightarrow{\sim} H^*(Y, \mathbb{C})$ is an isomorphism for any Schubert variety Y in $G_{k,n}$.

Let W be the symmetric group in $1, 2, \cdots, n$. For any permutation $\tau = (a_1, \cdots, a_n)$ in W, let $\tau(e)$ be the $n \times n$ permutation matrix obtained from the identity matrix e by permuting the rows relative to $(a_1, \cdots, a_n)$. Let $S = \{(i) = (i_1, \cdots, i_k): 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$. For any (i) in S there exists a unique permutation $(i_1, \cdots, i_k, i_{k+1}, \cdots, i_n)$ in W with the property $i_{k+1} < \cdots < i_n$. We denote this permutation by $\sigma(i) = (i_1, \cdots, i_n)$. For $(i) = (i_1, \cdots, i_k)$ in S, let $Y_{(i)} = \overline{B\sigma(i)(e)\pi(x_0)}$ be the Schubert subvariety of $G_{k,n}$ associated with $1 \leq i_1 < \cdots < i_k \leq n$, and let $\Omega(i_1, \cdots, i_k)$ be the Poincaré dual of the cycle class of the Schubert variety $Y_{(n-i_k+1, \cdots, n-i_1+1)}$ in $H^*(G_{k,n}, \mathbb{C})$. Let $\tilde{U} = B^-$ denote the affine space of all $n \times n$ lower triangular unipotent matrices, and let $U = \pi(\tilde{U})$. $\tilde{U}$ is naturally biholomorphic to the open big cell in the Bruhat decomposition of $G/B = \bigcup B\tau(e)x_0$, $\tau \in W$. Thus $\tilde{U}$ (resp. U) is an open affine neighborhood of x_0 (resp. $\pi(x_0)$) in G/B (resp. $G_{k,n}$).

Theorem 3.1. *For any $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, we have*

$$\psi(P_{(i_1, \cdots, i_k)} \bmod I(Z)) = \Omega(i_1, \cdots, i_k),$$

where $P_{(i_1, \cdots, i_k)}$ is the Plücker coordinate of $G_{k,n}$ associated with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.

Proof. Let $j: A(Z) \to A(\tilde{Z})$ be the natural map induced from the $\mathbb{C}$-equivariant map $\pi: G/B \to G_{k,n}$. It follows from [1] that j is a graded algebra homomorphism and the following diagram is commutative:

$$\begin{array}{ccc} \tilde{\psi}: A(\tilde{Z}) & \xrightarrow{\sim} & H^*(G/B, \mathbb{C}) \\ j \big\uparrow & & \big\uparrow \pi^* \\ \psi: A(Z) & \xrightarrow{\sim} & H^*(G_{k,n}, \mathbb{C}) \end{array}$$

where π^* is the cohomology map of π. Since π^* is injective, j is also an injective map [1], [2]. Thus, to prove the theorem, it is enough to show that $\tilde{\psi}(j(P_{(i_1,\cdots,i_k)})) = \pi^*(\Omega_{(i_1,\cdots,i_k)})$. For any $x = (x_{ij})$ in $\tilde{U}$, since

$$j(P_{(i_1,\cdots,i_k)})(x) = \det \begin{bmatrix} x_{i_1 1} & \dots & x_{i_1 k} \\ \vdots & & \vdots \\ x_{i_k 1} & \dots & x_{i_k k} \end{bmatrix} = \begin{vmatrix} x_{i_1 1} & \dots & x_{i_1 k} \\ \vdots & & \vdots \\ x_{i_k 1} & \dots & x_{i_k k} \end{vmatrix} = \begin{vmatrix} z_{i_1 1}(x) & \dots & z_{i_1 k}(x) \\ \vdots & & \vdots \\ z_{i_k 1}(x) & \dots & z_{i_k k}(x) \end{vmatrix},$$

we get

$$j(P_{(i_1,\cdots,i_k)}) = \begin{vmatrix} z_{i_1 1} & \dots & z_{i_1 k} \\ \vdots & & \vdots \\ z_{i_k 1} & \dots & z_{i_k k} \end{vmatrix} \quad \text{on } \tilde{U}.$$

Thus, by Proposition 2.1, in $A(\tilde{Z})$ we have the identity

$$j(P_{(i_1,\cdots,i_k)}) = \begin{vmatrix} h_{i_1-1}(x_1) & \dots & h_{i_1-k}(x_1,\cdots,x_k) \\ \vdots & & \vdots \\ h_{i_k-1}(x_1) & \dots & h_{i_k-k}(x_1,\cdots,x_k) \end{vmatrix}.$$

In this determinant, by replacing the 1st column by the 1st column $+x_2\cdot$ (the 2nd column), we obtain

$$j(P_{(i_1,\cdots,i_k)}) = \begin{vmatrix} h_{i_1-1}(x_1,x_2) & h_{i_1-2}(x_1,x_2) & \dots & h_{i_1-k}(x_1,\cdots,x_k) \\ \vdots & \vdots & & \vdots \\ h_{i_k-1}(x_1,x_2) & h_{i_k-2}(x_1,x_2) & \dots & h_{i_k-k}(x_1,\cdots,x_k) \end{vmatrix},$$

just because $h_l(x_1,x_2) = h_l(x_1) + x_2 h_{l-1}(x_1,x_2)$. Now, by replacing the 2nd column by the 2nd column $+x_3\cdot$ (the 3rd column) one gets

$$j(P_{(i_1,\cdots,i_k)}) = \begin{vmatrix} h_{i_1-1}(x_1,x_2) & h_{i_1-2}(x_1,x_2,x_3) & \dots & h_{i_1-k}(x_1,\cdots,x_k) \\ \vdots & \vdots & & \vdots \\ h_{i_k-1}(x_1,x_2) & h_{i_k-2}(x_1,x_2,x_3) & \dots & h_{i_k-k}(x_1,\cdots,x_k) \end{vmatrix}.$$

This time, replace the 1st column by 1st column $+x_3\cdot$ (2nd column) to obtain

$$j(P_{(i_1,\cdots,i_k)}) = \begin{vmatrix} h_{i_1-1}(x_1,x_2,x_3) & h_{i_1-2}(x_1,x_2,x_3) & \dots & h_{i_1-k}(x_1,\cdots,x_k) \\ \vdots & \vdots & & \vdots \\ h_{i_k-1}(x_1,x_2,x_3) & h_{i_k-2}(x_1,x_2,x_3) & \dots & h_{i_k-k}(x_1,\cdots,x_k) \end{vmatrix}.$$

By using similar column operations and the (obvious) identity $h_l(x_1, \cdots, x_s) = h_l(x_1, \cdots, x_{s-1}) + x_s h_{l-1}(x_1, \cdots, x_s)$ one obtains in $A(\tilde{Z})$,

$$j(P_{(i_1,\cdots,i_k)}) = \begin{vmatrix} h_{i_1-1}(x_1,\cdots,x_k) & \dots & h_{i_1-k}(x_1,\cdots,x_k) \\ \vdots & & \vdots \\ h_{i_k-1}(x_1,\cdots,x_k) & \dots & h_{i_k-k}(x_1,\cdots,x_k) \end{vmatrix}.$$

Since $z_{i+k\,k} = h_i(x_1, \cdots, x_k)$ in $A(\tilde{Z})$ and $\tilde{\psi}(z_{i+k\,k}) = c_i(Q_k)$, by Proposition 2.1 we get

$$\tilde{\psi}(j(P_{(i_1,\cdots,i_k)})) = \begin{vmatrix} c_{i_1-1}(Q_k) & \dots & c_{i_1-k}(Q_k) \\ \vdots & & \vdots \\ c_{i_k-1}(Q_k) & \dots & c_{i_k-k}(Q_k) \end{vmatrix}.$$

Since the pull back $\pi^*(Q_{k,n})$ of the universal quotient bundle $Q_{k,n}$ on $G_{k,n}$ is isomorphic to Q_k on G/B, we obtain

$$\tilde{\psi}(j(P_{(i_1,\cdots,i_k)})) = \pi^*\left(\begin{vmatrix} c_{i_1-1}(Q_{k,n}) & \dots & c_{i_1-k}(Q_{k,n}) \\ \vdots & & \vdots \\ c_{i_k-1}(Q_{k,n}) & \dots & c_{i_k-k}(Q_{k,n}) \end{vmatrix} \right).$$

Since

$$\Omega(i_1, \cdots, i_k) = \begin{vmatrix} c_{i_1-1}(Q_{k,n}) & \dots & c_{i_1-k}(Q_{k,n}) \\ \vdots & & \vdots \\ c_{i_k-1}(Q_{k,n}) & \dots & c_{i_k-k}(Q_{k,n}) \end{vmatrix},$$

by the determinantal formula in Schubert calculus [2], we get $\tilde{\psi}(j(P_{(i_1,\cdots,i_k)})) = \pi^*(\Omega_{(i_1,\cdots,i_k)})$, and the proof is complete.

We consider the natural partial order on $S = \{(i) = (i_1, \cdots, i_k): 1 \le i_1 < \cdots < i_k \le n\}$ defined by: for (i) and (j) in S, $(i) \le (j)$ if $i_1 \le j_1, \cdots, i_k \le j_k$. It is well known that this partial order on S is compatible with the Bruhat ordering on $G_{k,n} = \bigcup B\sigma(i)(e)\pi(x_0)$, $(i) \in S$. That is, for (i) and (j) in S, $(i) \le (j)$ if and only if $Y_{(i)} \subseteq Y_{(j)}$ [7].

Lemma. *For any (j) in S, we have*:

(i) *the ideal $I(Y_{(j)})$ of the Schubert variety $Y_{(j)}$ in the neighborhood U of $\pi(x_0)$ is generated by the Plücker coordinates $P_{(l)}$, $(l) \not\le (j)$,*

(ii) *the Euler-Poincaré characteristic $\chi(Y_{(j)})$ of $Y_{(j)}$ is equal to the cardinality of the set $\{(l) \in S: (l) \le (j)\}$.*

Proof. This lemma is not new. In fact, part (i) can be found in [7], and part (ii) follows from the cellular decomposition $Y_{(j)} = \bigcup B\sigma(l)(e)\pi(x_0)$, $(l) \le (j)$, of $Y_{(j)}$ [4].

Theorem 3.2. *Let $Y = Y_{(i)}$, $(i) \in S$, be a Schubert subvariety of $G_{k,n}$. The graded algebra isomorphism $\psi: A(Z) \xrightarrow{\sim} H^*(G_{k,n}, \mathbb{C})$ induces an isomorphism $\overline{\psi}: A(Y \cap Z) \xrightarrow{\sim} H^*(Y, \mathbb{C})$ which commutes with the natural maps $\alpha: A(Z) \to A(Y \cap Z)$ and $i^*: H^*(G_{k,n}, \mathbb{C}) \to H^*(Y, \mathbb{C})$.*

Proof. By [4], we know that ψ induces a graded algebra homomorphism $\overline{\psi}: A(Y \cap Z) \to H^*(Y, \mathbb{C})$ which commutes with α and i^*. Since $\overline{\psi}$ is a surjective map, we only need to show that $\dim_{\mathbb{C}} A(Y \cap Z) \le \dim_{\mathbb{C}} H^*(Y, \mathbb{C})$. By the basis theorem of Schubert calculus and Theorem 3.1, we know that the Plücker coordinates $P_{(j)}$, $(j) \in S$, form a basis of $A(Z)$. Thus $\{\alpha(P_{(l)}): (l) \in S\}$ spans the vector space $A(Y \cap Z)$. By the lemma, $P_{(j)}$ is in $I(Y_{(i)})$ when $(j) \not\le (i)$, so $\alpha(P_{(j)}) = 0$ in $A(Y \cap Z)$ for $(j) \not\le (i)$. This implies $I = \{\alpha(P_{(l)}): (l) \le (i)\}$ spans $A(Y \cap Z)$. Therefore, the cardinality of $I = \#\{(l) \in S: (l) \le (i)\} \ge \dim_{\mathbb{C}} A(Y \cap Z)$. By the same Lemma, since $\chi(Y) = \dim_{\mathbb{C}} H^*(Y, \mathbb{C}) = \#\{(l) \in S: (l) \le (i)\}$, we get $\dim_{\mathbb{C}} A(Y \cap Z) \le \dim_{\mathbb{C}} H^*(Y, \mathbb{C})$, and the proof is complete.

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