J. DIFFERENTIAL GEOMETRY
29 (1989) 105-112

ON THE SINGULARITIES OF THE SURFACE RECIPROCAL TO A GENERIC SURFACE IN PROJECTIVE SPACE

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1. Introduction

Let $S = S_f = \{[z_0, z_1, z_2, z_3] \in P^3 | f(z_0, z_1, z_2, z_3) = 0\}$ be a smooth surface in the complex projective space, where f is a homogeneous polynomial of degree n. Let P'^3 denote the space of hyperplanes in P^3, and $X_f = \{(a, h) \in S_f \times P'^3 | a \in h\}$, and define $p = p_f : X_f \to P'^3$ to be the natural projection. Denote by $\Sigma(p)$ the points of X_f where the derivative of p is not surjective. Among all the planes through $x \in S$ those tangent to S are special, so there should be no surprise that $\Sigma(p) = \{(a, h) | h = TS_a\}$, where TS_a denotes the tangent plane to S at a, and therefore that $p(\Sigma(p))$ is the surface reciprocal (or dual) to S.

Let A_n denote the vector space of homogeneous polynomials in three variables with complex coefficients, and P_n the projective space associated to A_n. Our purpose is to prove that for f in a nonempty Zariski open subset U_n of A_n the corresponding map p_f is excellent, which means that it has all the transversality properties required for these dimensions (Corollary 2.6). As a consequence, one has a complete description of all possible singularities of the surface reciprocal to S. Also, the fact that p is excellent provides global informations on the various singular loci, which have been exploited in [5] in order to justify some formulas of enumerative geometry found by G. Salmon [6] (the main proofs missed in [5] are provided here). Some work in the same direction was already done in [2], [3] and [4]. I am indebted to Clint McCrory for pointing out to me several mistakes in the first version of this paper.

We shall adopt the notation of [5]. In particular, given a smooth map $F: X \to Y$ and singularity types $\Sigma_1, \cdots, \Sigma_k$ applied to F, we set $M_k(\Sigma_1, \cdots, \Sigma_k) = \{x_1 \in X |$ there are $x_2, \cdots, x_k \in X$, $x_i \neq x_j$ for $i \neq j$ and $f(x_i) = f(x_j)\}$, and $N_k(\Sigma_1, \cdots, \Sigma_k) = f(M(\Sigma_1, \cdots, \Sigma_k))$. We shall denote by $J_0^k(\mathbb{C}^m, \mathbb{C}^n)$ the space of jets of order k of maps sending the origin to the origin.

Received July 20, 1987.

2. The map F is excellent

Given a germ of a map $f\colon(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$, recall that an unfolding of f is a map $F\colon(\mathbb{C}^{n+h},0)\to(\mathbb{C}^{p+h},0)$ such that if $x\in\mathbb{C}^n$ and $t\in\mathbb{C}^p$, then $F(x,t)=(F_1(x,t),t)$ and $F_1(x,0)=f$.

Definition. Let r be an integer and $\Sigma=\{\Sigma_h\}_{h\geq 0}$ be a sequence of singularity types of order k, with $\Sigma_h\subset J_0^k(\mathbb{C}^h;\mathbb{C}^{r+h})$. We shall say that Σ is a u-stable singularity type (for unfolding-stable) if for every germ of unfolding $F\colon(\mathbb{C}^{n+h},0)\to(\mathbb{C}^{n+r+h},0)$ of $f\colon(\mathbb{C}^n,0)\to(\mathbb{C}^{n+r},0)$ the following hold:

(i) $j_0^k(f)\in\Sigma_n$ if and only if $j_0^k(F)\in\Sigma_{n+h}$,

(ii) $j_0^k(f)$ is a transversal to Σ_n if and only if $j_0^k(F)$ is transversal to Σ_{n+h}.

It is usual to write Σ instead of Σ_h for some unspecified h. It follows easily from [1, Theorem 7.15] that all Thom-Boardman singularity types are u-unstable; in their case r, as well as h, is unspecified.

2.1 Proposition. *Let Σ be a u-stable sequence of singularity types. Consider the commutative diagram:*

$$\begin{array}{ccc}(X,x_0) & \xrightarrow{F} & (Y,Y_0)\\ {\scriptstyle p_X}\searrow & & \swarrow{\scriptstyle p_Y}\\ & (T,t_0) & \end{array}$$

and assume that p_X and p_Y are germs of submersions. Denote by X_t and Y_t the fibers over t of p_X and p_Y respectively, and let $F_t=F|p_X^{-1}(t_0)\colon X_{t_0}\to Y_{t_0}$. Then the following hold:

(i) *If F_t is Σ-transversal at $x\in X_t$, then so is F itself.*

(ii) *Let F be Σ-transversal at $x\in X_t$. Then F_t is Σ-transversal at x if and only if t is a regular value of the germ of $p_X|\Sigma(F)$ at x. This occurs only when*

$$\dim(T\Sigma_x\cap T(X_t)_x)\leq\dim(T\Sigma_x)-\dim(X)+\dim(X_t),$$

(i.e., codimension is preserved; in fact inequality is equivalent to equality).

(iii) *Let $\Sigma_1,\cdots,\Sigma_m$ be u-stable singularity types, and F be multitransversal to them at $x\in X_t$. Then F_t is multitransversal to the same at x if and only if t is a regular value of the germ at x of the restriction of p_X to $\Sigma_1(F)\cap\cdots\cap\Sigma_m(F)$.*

The proof is straightforward and is left to the reader.

We introduce now some notation. Let $X=\{([x],[\alpha],[f])\in P^3\times P'^3\times P_n \mid\alpha(x)=0,\ f(x)=0\}$, $Y=P'^3\times P_n$, and $F\colon X\to Y$ be the natural projection. We shall write $[x,\alpha,f]$ for an element of X, and $[\alpha,f]$ for an element of Y. Let $[x_0]\in P_3$, H_∞ be a hyperplane in P^3 not containing $[x_0]$, and $V=P^3-H_\infty$. Let E_0 and E_1 be affine subspaces of V of dimension 2 and 1 respectively such that $E_0\cap E_1=\{[x_0]\}$; we will choose $[x_0]$ as the origin in E_0 and E_1, so that they become vector spaces. Choose $x_\infty\in H_\infty$ and L to be a nonzero linear form on $\mathbb{C}^4$ whose kernel is H_∞. Set $B_n = \{f \in A_n | f(x_\infty) = 0\}$; we have an isomorphism $B_n \times \mathbb{C} \to A_n$ sending (f, c) to $f - c \cdot L^n$. Denote by $A_1(E_0, E_1)$ the space of affine maps from E_0 to E_1. Let U be the subset of $E_0 \times A_1(E_0, E_1) \times B_n$ consisting of triples (x, α, f) such that $(f(x+\alpha(x)), f)$ is nonzero in $\mathbb{C} \times B_n$, and let W be the subset of $A_1(E_0, E_1) \times \mathbb{C} \times B_n$ consisting of triples (α, c, f) such that (c, f) is nonzero. Define $\Phi: U \to W$ by

$$\Phi(x, \alpha, f) = (\alpha, f(x + \alpha(x))/L^n(x + \alpha(x)), f).$$

The group $\mathbb{C}^*$ of nonzero complex numbers acts freely on U and W by $z \cdot (x, \alpha, f) = (x, \alpha, z \cdot f)$ and $z \cdot (\alpha, c, f) = (\alpha, z \cdot c, z \cdot f)$, and Φ is equivariant for these actions. We have a commutative diagram:

$$\begin{array}{ccccc} U & \to & U' & = U/\mathbb{C}^* & \xrightarrow{h} & X \\ {\scriptstyle\Phi}\downarrow & & {\scriptstyle F'}\downarrow & & & \downarrow{\scriptstyle F} \\ W & \to & W' & = W/\mathbb{C}^* & \xrightarrow{H} & Y \end{array}$$

where h sends the class of (x, α, f) in U' to

$$[x + \alpha(x), \mathrm{graph}(\alpha), f - L^n \cdot (f(x + \alpha(x)/L^n(x + \alpha(x))],$$

and H sends the class of (α, c, f) to $[\mathrm{graph}(\alpha), f - c \cdot L^n]$ and $\mathrm{graph}(\alpha) = \{x + \alpha(x) | x \in E_0\}$. It is readily verified that h and H are coordinate charts on X and Y respectively, and that the diagram commutes.

2.2 Proposition. *F is Σ-transversal to all u-stable singularity types of order not exceeding n.*

Proof. The partial map Φ_α sends $(x, f) \in E_0 \times B_n$ to $(f(x + \alpha(x)), f) \in \mathbb{C} \times B_n$, and is therefore obviously transversal to any u-stable singularity type of order not exceeding n. It follows from 2.1(i) that Φ itself is transversal to the same kind of singularities. Since Φ is $\mathbb{C}^*$-equivariant and the action is free, it follows from 2.1(ii) that F' and hence F have the same transversality property.

For the next proposition we shall use a slightly different local description of X, Y and F. Let $\{x_0\} \in P^3$, V, H_∞, E_0 and E_1 be as before, and set

$$S = \{(x, \alpha, f) \in V \times A_1(E_0, E_1) \times (A_n - \{0\}) | f(x + \alpha(x)) = 0\},$$

$$T = A_1(E_0, E_1) \times (A_n - \{0\}).$$

The group $\mathbb{C}^*$ acts on S and T by multiplication on $A_n - \{0\}$. Define $\Phi: S \to T$ by $\Phi(x, \alpha, f) = (\alpha, f)$. We have a commutative diagram

$$\begin{array}{ccccc} S & \to & S/\mathbb{C}^* & \xrightarrow{h} & X \subset P^3 \times P'^3 \times P_n \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle F'} & & \downarrow{\scriptstyle F} \\ T & \to & T/\mathbb{C}^* & \xrightarrow{H} & Y = P'^3 \times P_n \end{array}$$

where h sends $(x, \alpha, [f])$ to $([x], \text{graph}(\alpha), [f])$, which we shall denote by $[x, \alpha, f]$, and H sends $(\alpha, [f])$ to $[\alpha, f] = (\text{graph}(\alpha, [f])$. H and h are diffeomorphisms onto open subsets of Y and X respectively, and we shall write $[\underline{x}, \underline{\alpha}, \underline{f}]$ and $[\underline{\alpha}, \underline{f}]$ for elements of the tangent spaces of X and Y respectively. For example, taking the derivative of the equation defining S, we obtain:

$$(1) \qquad TX_a = \{[\underline{x}, \underline{\alpha}, \underline{f}] | df_x(\underline{x} + \alpha(\underline{x}) - \alpha(x_0)) + df_x(\underline{\alpha}(x)) \\ + \underline{f}(x + \alpha(x)) = 0\},$$

where $a = [x, \alpha, f]$, from which it follows that

$$\ker(dF_a) = \{[\underline{x}, 0, 0] | df_x(\underline{x} + \alpha(\underline{x}) - \alpha(x)) = 0\},$$

and hence that $[x, \alpha, f] \in \Sigma^2(F)$ if and only if $\ker(df_x) = \text{graph}(\alpha)$.

2.3 Proposition. *Let $a = [x, \alpha, f] \in X$ and assume that $a \in \Sigma^2(F)$). Then the following hold:*

(i) $\text{Im}(dF_a) = \{[\underline{\alpha}, \underline{f}] | df_x(\underline{\alpha}(x)) + \underline{f}(x + \alpha(x)) = 0\}$.

(iia) *a is an $\Sigma^{2,1}(F)$ if and only if there exists a line l_0 in E_0 such that $d^2 f_x | E_0 x l_0 = 0$. In this case we have*

(iib) $\text{Im}(d(F|\Sigma^2(F))_a) = \{[\underline{\alpha}, \underline{f}] \in \text{Im}(dF_a) | d\underline{f}_x | l_0 + df_x(\underline{\alpha}|l_0) = 0\}$.

(iiia) *$a \in \Sigma^{2,1,1}(F)$ if and only if $a \in \Sigma^{2,1}(F)$ and $d^2 f_x | E_0 \times l_0 \times l_0 = 0$. In this case we have*

(iiib)

$$\text{Im}(d(F|\Sigma^{2,1}(F))_a) \\ = \{[\underline{\alpha}, \underline{f}] \in \text{Im}(d(F|\Sigma^2(F))_a) | d^2 \underline{f}_x | l_0 \times l_0 + 2d^2 f_x(\underline{\alpha}, \ ) | l_0 \times l_0 = 0\}.$$

The following corollary is a consequence of the very definition of Thom-Boardman's singularities.

2.4 Corollary. *If $a \in \Sigma^{2,0}(F)$, $\Sigma^{2,1,0}(F)$ or $\Sigma^{2,1,1,0}(F)$, then the images by dF_a of the tangent spaces to these strata are described by* (i), (iib) *and* (iiib) *above respectively, that is:*

(i) $T(F(\Sigma^{2,1,0}))_b = \{[\underline{\alpha}, \underline{f}] | df_x(\underline{\alpha}(x)) + \underline{f}(x + \alpha(x)) = 0\}$.

(ii) $T(F(\Sigma^{2,1,0}))_b = \{[\underline{\alpha}, \underline{f}] \in \text{Im}(dF_a) | d\underline{f}_x | l_0 + df_x(\underline{\alpha}|l_0) = 0\}$.

(iii)

$$T(F(\Sigma^{2,1,1,0}))_b \\ = \{[\underline{\alpha}, \underline{f}] \in \text{Im}(d(F|\Sigma^2(F))_a) | d^2 \underline{f}_x | l_0 \times l_0 + 2d^2 f_x(\underline{\alpha}, \ ) | l_0 \times l_0 = 0\},$$

where $a = [x, \alpha, f]$ and $b = [\alpha, f]$.

Proof of 2.3. We shall work near the point a and will assume for simplicity that $E_0 = \text{graph}(\alpha)$. If $a \in \Sigma^2(F)$, then $\ker(df_x) = E_0$ and (i) follows from equality (1). As we have already seen in the proof of Proposition 2.2, the map

F can be seen as an unfolding of the function $f|E_0$. Since Thom-Boardman singularities are u-stable, (iia) and (iiia) follow at once.

We have that $\Sigma^2(F) = \{[x,\alpha,f] | df_x(1_{E_0}+\alpha) = 0\}$, where $1_{E_0}: E_0 \to V$ denotes the inclusion, and the equation takes place in $L(E_0, \mathbb{C})$. Taking derivatives we get

$$(2) \qquad T\Sigma^2(F)_a = \{[\underline{x}, \underline{\alpha}, \underline{f}] \in TX_a | d^2 f_x(1_{E_0}, \underline{x}) + d\underline{f}_x(1_{E_0}) + df_x(\underline{\alpha}) = 0\}.$$

The presence of the term $d\underline{f}_x$ in (2) shows that the equation is of maximal rank. If $a \in \Sigma^{2,1}(F)$, then $d^2 f_x: E_0 \to L(E_0, \mathbb{C})$ vanishes exactly on l_0. Therefore, if $\underline{\alpha}$ and $\underline{f}$ are such that $d\underline{f}_x|l_0 + df_x(\underline{\alpha}|l_0) = 0$, then there exists exactly one $\underline{x}$ such that (2) is satisfied, (iib) is proved.

We need now the equation of $\Sigma^{2,1}$ in Σ^2; let l_1 be a supplementary subspace of l_0 in E_0. Consider the space $X' = \Sigma^2(F) \times L(E_0, E_1) \times L(l_0, l_1)$ and the natural projection $P: X' \to \Sigma^2(F)$, and define $\Sigma' \subset X'$ by the equation

$$(3) \qquad d^2 f_x(\theta_1, \theta_2) = 0,$$

where $\theta_1 = (1_{E_0} + \alpha) \cdot (1_{l_0} + \beta)$, $\theta_2 = (1_{E_0} + \alpha) \cdot (\beta \cdot \pi_0 + \pi_1)$, π_0, π_1 are the projections of E_0 onto l_0 and l_1 parallel to l_1 and l_0 respectively, $\alpha \in L(E_0, E_1)$, $\beta \in L(l_0, l_1)$ and $[x', \alpha, f] \in \Sigma^2(F)$. Let $l = \mathrm{Im}(\theta_1)$, $E = \mathrm{Im}(\theta_2)$; note that l is included in E. The product $\theta_1 \times \theta_2$ induces an isomorphism from $l_0 \times E_1$ to $l \times E$, which is symmetric on $l_1 \times l_1$, thanks to the complicated expression for θ_2. If $([x,\alpha,f],\beta,l)$ satisfies (3), then $d^2 f_x | l \times E = 0$ and hence $[x,\alpha,f] \in \Sigma^{2,1}(F)$. Taking the derivative of (3) at $[x,0,0]$ we get

$$(4) \qquad \begin{aligned} &d^3 f_x(\underline{x}, 1_{E_0}) + d^2 \underline{f}_x(1_{E_0}, 1_{l_0}) + d^2 f_x(\underline{\alpha} \cdot 1_{l_0}, 1_{E_0}) \\ &\quad + d^2 f_x(\underline{\beta}, 1_{E_0}) + d^2 f_x(1_{l_0}, \underline{\alpha}) = 0. \end{aligned}$$

Because of the term $d^2\underline{f}_x$, (4) is of maximal rank; also, if $d^2 f_x(\underline{\beta}, 1_{E_0}) = 0$ then $\underline{\beta} = 0$ since we are on $\Sigma^{2,1}$ and not on $\Sigma^{2,2}$. It follows that $P|\Sigma'$ is a diffeomorphism on some open set of $\Sigma^{2,1}(F)$, that

$$(5) \qquad T\Sigma^{2,1}(F)_a = \{[\underline{x}, \underline{\alpha}, \underline{f}] \in T\Sigma^2(F) | \text{ there is } \beta \text{ satisfying (4)}\},$$

and that if $\underline{\alpha}$ and $\underline{f}$ are such that

$$d^2 f_x(1_{l_0}, 1_{l_0}) + d^2 f_x(\underline{\alpha}|l_0, 1_{l_0}) + d^2 f_x(1_{l_0}, \underline{\alpha}|l_0) = 0,$$

then there exist unique $\underline{\beta}$ and $\underline{x}$ such that (3) is satisfied, and (iiib) is proved.

2.5 Proposition. *Let $n \geq 3$; then the map $F: X \to Y$ is multitransversal to the following singularity types:*

(i) $M_k(\Sigma^{2,0}, \cdots, \Sigma^{2,0})$, *provided $n \geq k$.*

(ii) $M_2(\Sigma^{2,0}, \Sigma^{2,1,0})$.

(iii) $M_3(\Sigma^{2,0}, \Sigma^{2,0}, \Sigma^{2,1,0})$.

(iv) $M_2(\Sigma^{2,1,0}, \Sigma^{2,1,0})$.

(v) $M_2(\Sigma^{2,0}, \Sigma^{2,1,1,0})$.

Proof. If $a_1 = [x_1, \alpha, f] \in \Sigma_1(F), \cdots, a_k = [x_k, \alpha, f] \in \Sigma_k(F)$ (and so $F(a_1) = \cdots = F(a_k) = [\alpha, f] = b$), we shall write $T_i = \operatorname{Im}(d(F|\Sigma_i)_{a_i}) \subset TY_b$. With no further mention, the sequences $\Sigma_1, \cdots, \Sigma_k$ and $a_1, \cdots, a_k$ will be those appearing in the case under consideration. The symbols h_* will denote elements of $A_1 - \{0\}$, i.e., nonzero linear forms on $\mathbf{C}^4$.

(i) Let H_i, $i = 1, \cdots, k$, be such that $h_i(x_j) = 0$ if and only if $i = j$. Set $\underline{f}_i = h_1 \cdot \cdots \cdot h_{i-1} \cdot h_i^{(n-i+1)}$. It follows from 2.3(i) that $[0, \underline{f}_i] \in T_1 \cap \cdots \cap T_i$ but $\underline{f}_i \notin T_{i+1}$.

(ii) Here we have a line $l \subset E = \ker(df_{x_1})$ such that $d^2 f_{x_1}|l \times E = 0$. Let h_1, h_2 be such that $h_1|l = 0$ and $h_2(x_1) = 0$, $h_2(x_2) \neq 0$. Set $\underline{f} = h_1 \cdot h_2^{n-1}$; then by 2.3 $[0, \underline{f}] \in T_2 - T_1$.

(iii) We already know by (ii) that T_2 and T_3 meet transversally, and hence it suffices to prove that $T_2 \cap T_3 \neq T_1$. Let $l \subset E$ be for x_3 be what it was under (ii) for x_2, and let h_1 and h_2 be such that $h_1(x_1) \neq 0$, $h_1(x_2) = 0$, $h_2(x_1) \neq 0$, $h_2(x_3) = 0$, and set $\underline{f} = h_1 \cdot h_2^{n-1}$. Then $d\underline{f}_{x_3}|l = 0$ since $n \geq 3$, and so $[0, \underline{f}] \in T_2 \cap T_3 - T_1$.

(iv) Let h_1 and h_2 satisfy $h_1(x_1) = 0$, $h_1(x_2) \neq 0$, $h_2(x_1) \neq 0$, $h_2(x_2) = 0$. Then $[0, h_1 \cdot h_2^{n-1}]$, $[0, h_2^n] \in T_2 - T_1$ (since $n \geq 3$) and are linearly independent.

(v) Let h satisfy $h(x_2) = 0$, $h(x_1) \neq 0$; then $[0, h^n] \in T_2 - T_1$.

It is certainly no coincidence that the above proof is based on the geometry of points, lines and planes.

From 2.5 and 2.1(ii) and (iii) it follows:

2.6 Corollary. *For f in some Zariski open dense subset U_n of A_n, the partial map $F_f = p_f = p: X_f \to Y_f = P'^3 \times \{f\}$ is excellent (i.e., transversal and multitransversal to all possible Thom-Boardman singularities—in these dimensions only Σ^2, $\Sigma^{2,1}$, $\Sigma^{2,1,1}$, $M_2(\Sigma^2, \Sigma^2)$, $M_2(\Sigma^2, \Sigma^{2,1})$, $M_3(\Sigma^2, \Sigma^2, \Sigma^2)$).*

We will now recall and justify the geometric description of the set U_n given in [5, Proposition 2.1]. Let $f \in A_n$ define a nonsingular projective surface $S_f = S$.

2.7 Proposition. *The map p_f is excellent if and only if for any $x \in S_f$ the intersection of TS_x with S is a curve with singularities of the following types only:*

(a) *one ordinary double point;*

(b) *two ordinary double points;*

(c) *three ordinary double points, not lying on a same line;*

(d) *one ordinary cusp;*

(e) *one ordinary double point and one double point, not lying on the line tangent to the cusp*;

(f) *one ordinary tacnodal point.*

(See [2, Example 1.5b], or [5, Proposition 2.1] for the interpretation of these singularities in terms of singularities of p_f and $S' = p_f(\Sigma^2(p_f))$ = the surface reciprocal to S.)

Proof. Step 1: *Monotransversality.* It is easily checked that the curve $S \cap TS_x$ has ordinary double points, cusp or tacnodal points at x if and only if $x \in \Sigma^{2,0}(p)$, $\Sigma^{2,1,0}(p)$ or $\Sigma^{2,1,1,0}(p)$ respectively. It remains to show that at those points, p is transversal to the corresponding Thom-Boardman strata.

Let us show first transversality to Σ^2 and $\Sigma^{2,1}$. Consider the commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr, "p_X"'] & & Y \arrow[dl, "p_Y"] \\ & P_n & \end{tikzcd}$$

Since p_X and p_Y are submersions and F is transversal to Thom-Boardman singularities of order $\leq n$, we can apply Proposition 2.1(ii). Let $a = [x, \alpha, f] \in \Sigma^2(F)$. Let T_1 be the tangent space of X_f at $[x, \alpha]$, and T_2 be the tangent space to $\Sigma^2(F)$ at a. Consider the natural projection $P: T_1 \cap T_2 \to TP^3$ sending $[\underline{x}, \underline{\alpha}]$ to $[\underline{x}]$. From (1), (2) and the fact that $\Sigma^{2,2}(F) = \varnothing$ it follows that P is injective, so that $\dim(T_1 \cap T_2) \leq \dim(E_0) = 2$, and hence p is Σ^2-transversal by 2.1(ii). Assume now that $a \in \Sigma^{2,1}(F)$, and let T_3 be the tangent space to $\Sigma^{2,1}(F)$ at a. From (5) and the fact that $\Sigma^{2,2}(F) = \varnothing$ it follows that $P(T_1 \cap T_3) = l_0$, and so by 2.3(ii) we are done again. From [1, Theorem 7.15] follows easily a general fact, about Thom-Boardman singularities, that a Σ^2-transversal map $g: X^{n+1} \to Y^n$ is automatically $\Sigma^{2,1}$-transversal at points of $\Sigma^{2,1,0}(g)$, $\Sigma^{2,1,1}$-transversal at points of $\Sigma^{2,1,1,0}(g)$, and so on. Therefore Step 1 is complete.

Step 2: *Multitransversality.* It follows from 2.4 that:

$$T(p(\Sigma^2(p)))_{[\alpha]} = \{[\underline{\alpha}] \in TP'^3_{[\alpha]} | \underline{\alpha}(x) = 0\},$$
$$T(p(\Sigma^{2,1}(p)))_{[\alpha]} = \{[\underline{\alpha}] \in TP'^3_{[\alpha]} | \underline{\alpha} \cdot 1_{l_0} = 0\},$$
$$T(p(\Sigma^{2,1,1}(p)))_{[\alpha]} = \{[\underline{\alpha}] | \underline{\alpha} \cdot 1_{E_0} = 0\} = \{0\}.$$

In other words, these three tangent spaces can be intrepreted respectively as the two planes in projective space passing through x, the two planes containing l_0, and the two planes tangent to S_f at x (note that our description of TP'^3 at x depends on the choice of E_0 and E_1, but the above description does not). From this the end of the proof follows immediately.

Remark. As C. McCrory pointed out to me, the methods used in this paper can be used to handle the case of hypersurfaces contacting planes of any dimension (rather than hyperplanes only) in projective spaces of any dimension. However, the degree of the hypersurface might put some limits on the excellency of the analogues of the map F. For example, in the case of hypersurfaces of degree n and hyperplanes in P^k, the singularity type $\Sigma^{k-1,1,\cdots,1}$, with 1 occurring $k-1$ times, appears generically; this is a singularity of order k, and therefore probably one should require that $n \geq k$ in order to make sure that F is excellent (a similar but erroneous—as C. McCrory noticed—statement was made in [5, Remark 2.3(ii))].

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