J. DIFFERENTIAL GEOMETRY
29 (1989) 1–8

ON THE MEAN EXIT TIME FROM A MINIMAL SUBMANIFOLD

STEEN MARKVORSEN

Abstract

Let M^m be an immersed minimal submanifold of a Riemannian manifold N^n, and consider the Brownian motion on a regular ball $\Omega_R \subset M$ with exterior radius R. The mean exit time for the motion from a point $x \in \Omega_R$ is called $E_R(x)$. In this paper we find sharp support functions for $E_R(x)$ in the following distinct cases. The support is from below if the sectional curvatures of N are bounded from below by a nonnegative constant and the support is from above if the sectional curvatures of N are bounded from above by a nonpositive constant. It follows that the minimal submanifolds of $\mathbf{R}^n$ all have the same mean exit time function and we show that this function actually characterizes the minimal hypersurfaces in the set of all hypersurfaces of $\mathbf{R}^n$.

1. Introduction

Let M^m be an immersed (not necessarily minimal) submanifold of a complete Riemannian manifold N^n. The distance function on N is denoted by r so that $\mathrm{dist}_N(p, x) = r_p(x)$ for all p, x in N. We now fix $p \in M \subset N$ and define a *regular domain* $\Omega_R(p) \subset M$ to be a smooth connected component of $B_R(p) \cap M^m$ which contains p. Here $B_R(p)$ is the geodesic R-ball around p in N subject to the usual restriction that $R < \min\{i_N(p), \pi\sqrt{\kappa}\}$, where κ is the supremum of the sectional curvatures of N, and $i_N(p)$ is the injectivity radius of N from p.

We now consider the Brownian motion on the domain Ω_R (cf. [4]) and denote by $E_R(x)$ the mean time of first exit from Ω_R for a Brownian particle starting at $x \in \Omega_R$. We want to compare $E_R(x)$ with the mean exit time function $\tilde{E}_R^b(\tilde{x})$ on the space form R-ball $\tilde{B}_R^m(\tilde{p})$ of dimension $m = \dim M$ and constant curvature $b \in \mathbf{R}$. Since $\tilde{B}_R$ has maximal isotropy at the center $\tilde{p}$, we have that $\tilde{E}_R$ only depends on the distance of $\tilde{x}$ from $\tilde{p}$. Hence we may, and do, write $\tilde{E}_R^b(\tilde{x}) = \tilde{\mathscr{E}}_R^b(r_{\tilde{p}}(\tilde{x})) = \tilde{\mathscr{E}}_R^b(r)$. In order to compare E and $\tilde{E}$ we transplant $\tilde{E}$ to Ω_R by the following definition:

$$\tilde{E}_R^b \colon \Omega_R \to \mathbf{R}; \quad \tilde{E}_R^b(x) = (\tilde{\mathscr{E}}_R^b \circ r_p)(x).$$

Our main result can now be formulated as follows.

Theorem 1. *Let M^m be a minimally immersed submanifold of N^n and let $\Omega_R(p)$ be a regular domain on M.*

(i) *If the sectional curvatures K_N of N satisfy $K_N \geq b \geq 0$, then $E_R(x) \geq \tilde{E}_R^b(x)$ for all $x \in \Omega_R$.*

(ii) *If $K_N \leq b \leq 0$, then $E_R(x) \leq \tilde{E}_R^b(x)$ for all $x \in \Omega_R$.*

Remark. This type of theorem was first proved by Debiard, Gaveau and Mazet in [3] where they used techniques different from ours to obtain bounds on the mean exit times from codimension-0 domains of N. The setting in Theorem 1 is more general in the sense that we have no a priori control on the sectional curvatures of the domains Ω_R in question.

A fundamental observation of Dynkin (cf. [4, vol. 2, p. 51]) states that the function $E_R(x)$ satisfies the Poisson equation on Ω_R with Dirichlet boundary data, i.e.,

$$\Delta_\Omega E_R = -1, \qquad E_R|_{\partial\Omega_R} \equiv 0, \tag{1.1}$$

where Δ_Ω is the induced Laplacian on Ω_R defined by $\Delta_\Omega = \mathrm{div}_\Omega \circ \mathrm{grad}_\Omega = \mathrm{trace}_\Omega \circ \mathrm{Hess}_\Omega$.

If we let $G(x,y)$ denote the Green's functions with Dirichlet boundary conditions on Ω_R, then every smooth function f on Ω_R satisfies

$$f(x) = -\int_{\Omega_R} G(x,y)\Delta f(y)\, dy - \int_{\partial\Omega_R} \frac{\partial G}{\partial \xi}(x,y) f(y)\, dy.$$

In particular, if we set $f = E_R$ we get by (1.1)

$$E_R(x) = \int_{\Omega_R} G(x,y)\, dy.$$

On the other hand, the Green's functions are given by

$$G(x,y) = \int_0^\infty \mathscr{H}(x,y,t)\, dt,$$

where $\mathscr{H}$ is the Dirichlet heat kernel on Ω_R. Thus

$$E_R(x) = \int_{\Omega_R} \int_0^\infty \mathscr{H}(x,y,t)\, dt\, dy, \tag{1.2}$$

and similarly

$$\tilde{E}_R^b(\tilde{x}) = \int_{\tilde{B}_R} \int_0^\infty \tilde{\mathscr{H}}_R^b(\tilde{x},\tilde{y},t)\, dt\, d\tilde{y}.$$

In [8] we showed that under the assumptions of Theorem 1(ii) we have

$$\mathscr{H}(p,y,t) \leq \tilde{\mathscr{H}}_R^b(r_p(y),t), \tag{1.3}$$

and also

$$\mathscr{K}(p,y,t) \leq \tilde{\mathscr{K}}_R^b(r_p(y),t), \tag{1.4}$$

where $\mathscr{K}$ and $\tilde{\mathscr{K}}$ are the Neumann heat kernels on $\Omega_R(p)$ and $\tilde{B}_R(\tilde{p})$ respectively. From these latter kernels one may extract the volumes of the respective domains (cf. [2]) and obtain

$$\int_{\Omega_R} dy \geq \int_{\tilde{B}_R} d\tilde{y}. \tag{1.5}$$

It is now natural to ask for situations where inequalities of type (1.3) and (1.5) balance each other so as to give

$$E_R(x) = \tilde{E}^b_R(x) \quad \text{for all } x \in \Omega_R. \tag{$*$}$$

With this in mind we show the following gap phenomenon for minimal submanifolds of constant curvature space forms: The condition $(*)$ is a Bernstein type condition in every nonflat space form in the sense that Ω_R (and in fact all of M^m) must be totally geodesic if it is minimal and satisfies $(*)$. In contrast, $(*)$ is always satisfied in $\mathbf{R}^n$, where it even *characterizes* minimal hypersurfaces in the set of all hypersurfaces. To be precise we have the following theorems.

Theorem 2. *Under the assumptions of Theorem* 1, *suppose further that* $(*)$ *is fulfilled and that* $b \neq 0$. *Then* Ω_R *is a minimal cone in* N^n *and hence, if* N^n *is actually a space form of constant curvature* b, *then* M^m *is a totally geodesic submanifold of* N^n.

Theorem 3. *Let* Ω_R *be a regular domain of a minimal submanifold of* $\mathbf{R}^n$. *Then* $(*)$ *is satisfied. Conversely, if* $(*)$ *is satisfied for all* x *in a regular domain* Ω_R *of a hypersurface in* $\mathbf{R}^n$, *then* Ω_R *is minimal.*

We note here that Theorem 3 may be viewed as a semiglobal generalization of the following local result which is due to L. Karp and M. Pinsky (cf. [7]).

Theorem A. *Let* M^{n-1} *be a hypersurface of* $\mathbf{R}^n$, *and consider the domains* $\Omega_\epsilon(p) \subset M$ *for some fixed* $p \in M$. *If* $E_\epsilon(p) = \tilde{E}^0_\epsilon(p)$ *for all sufficiently small* $\epsilon > 0$, *then the mean curvature of* M *at* p *is zero.*

Remark. In order to construct the functional dependence of $E_\epsilon(p)$ on ϵ in this theorem it is in principle necessary to solve an infinite number of Poisson equations (one for every ϵ), whereas it is only necessary to solve a single Poisson equation (i.e., (1.1)) to check condition $(*)$ for Theorem 3.

2. Some preliminary inequalities

The following Hessian comparison theorem is well known and may be obtained from standard index comparison theory (cf. [1]).

Lemma 1. *If* $K_N \leq b$ *(respectively* $K_N \geq b$*),* $b \in \mathbf{R}$, *then for every unit vector* X *in the tangent bundle of* $B_R(p) - p$ *we have*

$$\operatorname{Hess}_N(r_p)(X, X) \geq \ (\leq)\ h_b(r_p)(1 - \langle \operatorname{grad}_N r_p, X\rangle^2), \tag{2.1}$$

where $h_b(r)$ is the constant mean curvature of any distance sphere of radius r in a space form of constant curvature b.

Now let F be a smooth function on $\mathbf{R}$. Then $F \circ r_p$ is a smooth function on $B_R(p) - p$ with a smooth restriction to $\Omega_R(p) - p$. A calculation along the lines of [6, p. 713] now gives

(2.2)

$$\begin{aligned}\mathrm{Hess}_\Omega(F \circ r)(X,X) &= F''(r)\langle \mathrm{grad}_N(r), X\rangle^2 \\ &\quad + F'(r)\left(\mathrm{Hess}_N(r)(X,X) + \langle \mathrm{grad}_N(r), \alpha(X,X)\rangle\right)\end{aligned}$$

for all $X \in T(\Omega_R(p) - p)$, where $\alpha(\cdot,\cdot)$ is the second fundamental form of Ω in N.

Combining (2.1) and (2.2) we get

Lemma 2. *Let $q \in \Omega_R(p) - p$ and let $\{X_i\}$, $1 \leq i \leq m$, be an orthonormal basis of $T_q\Omega \subset T_qN$. Suppose that $F'(r) \geq 0$ for all $r \in [0,R]$ and that $K_N \leq b$ (respectively $K_N \geq b$), $b \in \mathbf{R}$. Then*

$$\begin{aligned}\Delta_\Omega(F \circ r)|_q \geq\ (\leq)\ &(F''(r) - F'(r)h_b(r)) \cdot \sum_{j=1}^{m}\langle \mathrm{grad}_N(r), X_j\rangle^2 \\ &+ mF'(r)(h_b(r) + \langle \mathrm{grad}_N(r), H(q)\rangle),\end{aligned} \tag{2.3}$$

where $H(q)$ is the mean curvature vector of Ω at q in N.

In what follows we shall only consider functions $F = F_b$ defined by

$$F_b(r) = \begin{cases} \frac{1}{b}(1 - \cos(\sqrt{b}r)) & \text{if } b > 0, \\ r^2/2 & \text{if } b = 0, \\ \frac{1}{b}(1 - \cosh(\sqrt{-b}r)) & \text{if } b < 0. \end{cases} \tag{2.4}$$

Then $F_b \circ r$ is a smooth function on all of $\Omega_R(p)$, and moreover $F_b''(r) = F_b'(r)h_b(r)$ so that Lemma 2 specializes to

Corollary 3. *If $K_N \leq b$ (resp. $K_N \geq b$), then*

$$\Delta_\Omega(F_b \circ r)|_q \geq\ (\leq)\ mF_b'(r)(h_b(r) + pH_\Omega(q)), \tag{2.5}$$

where we have defined $pH_\Omega(q) = \langle \mathrm{grad}_N(r), H(q)\rangle$, $pH_\Omega(p) = 0$.

3. Some properties of $\tilde{E}$

The mean time of first exit from a space form ball of constant curvature b, radius R and dimension m is given explicitly by (cf. [3])

$$\begin{aligned}\tilde{\mathscr{E}}_R^b(r) &= \int_r^R (\theta_b(u)u^{m-1})^{-1}\left(\int_0^u \theta_b(t)t^{m-1}\,dt\right)du \\ &= \tilde{\mathscr{E}}_R^b(0) - \tilde{\mathscr{E}}_r^b(0),\end{aligned} \tag{3.1}$$

where θ_b is the change of volume factor induced by the exponential map, i.e.,

$$\theta_b(r) = \begin{cases} (\sin\sqrt{b}r/\sqrt{b}r)^{m-1} & \text{if } b > 0, \\ 1 & \text{if } b = 0, \\ (\sinh\sqrt{-b}r/\sqrt{-b}r)^{m-1} & \text{if } b < 0. \end{cases} \tag{3.2}$$

Changing the variable from r to $s(r) = F_b(r)$ defined in (2.4) we get $\tilde{\mathscr{E}}^b_R(r) = \tilde{\mathscr{E}}^b_R(r(s)) = \mathbf{E}^b_R(s)$.

We shall need the following observations.

Proposition 4. *$\mathbf{E}^b_R(s)$ is a smooth function of s for $0 \le s \le s(R)$, and in this interval we have*

$$\frac{d}{ds}\mathbf{E}^b_R(s) < 0 \quad \textit{for every } b, \tag{3.3}$$

$$\frac{d^2}{ds^2}\mathbf{E}^b_R(s) \begin{cases} < 0 & \textit{if } b > 0, \\ = 0 & \textit{if } b = 0, \\ > 0 & \textit{if } b < 0. \end{cases} \tag{3.4}$$

Proof. Since $r(s)$ and $\tilde{\mathscr{E}}^b_R(r)$ (in (3.1)) are smooth when the respective arguments are nonzero, we see that $\mathbf{E}(s)$ is a smooth function of s for $0 < s \le s(R)$. The fact that $\tilde{E}^b_R(x)$ solves the Poisson equation (1.1) on $\tilde{B}_R(\tilde{p})$ can therefore be written as

$$-1 = \tilde{\Delta}\tilde{E} = \mathbf{E}''(s) \cdot \|\operatorname{grad}_{\tilde{B}} s\|^2 + \mathbf{E}'(s) \cdot \tilde{\Delta}s, \tag{3.5}$$

and so from (2.5),

$$-1 = \mathbf{E}''(s) \cdot (s'(r))^2 + \mathbf{E}'(s) \cdot ms'(r)h_b(r),$$

or equivalently

$$-1 = \mathbf{E}''(s) \cdot s(2 - bs) + \mathbf{E}'(s) \cdot m(1 - bs). \tag{3.6}$$

The unique solution $\mathbf{E}(s)$ to (3.6) with $\mathbf{E}(0) = c_0$ has a power series expansion $\mathbf{E}(s) = \sum_{k=0}^{\infty} c_k s^k$ whose radius of absolute convergence is $2/|b|$. Indeed, a substitution of the series into (3.6) leads to the recurrence relation

$$c_1 = -1/m, \quad c_{k+1} = c_k \cdot b(k^2 + (m-1)k)/(2k^2 + (m+2)k + m), \quad k \ge 1,$$

and the claim now follows from the ratio test together with

$$|c_{k+1}|/|c_k| \to |b|/2 \quad \text{for } k \to \infty.$$

In particular $\mathbf{E}^b_R(s)$ is therefore smooth in the closed interval $0 \le s \le s(R)$, and

$$\begin{aligned} &\mathbf{E}^b_R(s) - \mathbf{E}^b_R(0) \\ &= -\frac{1}{m}s - \frac{b}{2(m+2)}s^2 - \frac{b^2 \cdot (m+1)}{3(m+2)(m+4)}s^3 - \cdots, \qquad s < \frac{2}{|b|}. \end{aligned} \tag{3.7}$$

We see that $\mathbf{E}'(0) = -1/m < 0$, and the inequality (3.3) follows for all other values of s from $r'(s) > 0$ and $\tilde{\mathscr{E}}'(r) < 0$ (from (3.1)). Furthermore, the series (3.7) also gives directly the statement (3.4) for $b > 0$ and $b = 0$ respectively. In case $b < 0$ we first observe from (3.7) that $\mathbf{E}''(0) = -b/(m+2) > 0$. A differentiation of (3.6) with respect to s gives

$$0 = (\mathbf{E}''(s))' \cdot s(2 - bs) + (\mathbf{E}''(s))(m+2)(1 - bs) - mb\mathbf{E}'(s).$$

Here $mb\mathbf{E}'(s)$ is positive so that $\mathbf{E}''(s) = \eta(s)$ satisfies

$$0 < \eta'(s) \cdot s(2 - bs) + \eta(s) \cdot (m+2)(1 - bs).$$

If a first zero s_0 exists for η, then $\eta'(s_0) > 0$, but this is clearly ruled out by $\eta(0) > 0$. Hence $E''(s)$ is positive for all s when $b < 0$.

4. Proof of Theorems 1 and 2

The comparison of $E_R(x)$ with $\mathbf{E}^b_R(s(r_p(x)))$ will follow from the identity

$$\Delta_\Omega \mathbf{E}^b_R(s(r(x))) = \mathbf{E}''(s) \cdot \|\operatorname{grad}_\Omega s\|^2 + \mathbf{E}'(s) \cdot \Delta_\Omega s. \tag{4.1}$$

In fact, suppose that $K_N \geq b > 0$. Then by Corollary 3 (with $H_\Omega = 0$), Proposition 4 and the fact that $\|\operatorname{grad}_\Omega s\|^2 \leq \|\operatorname{grad}_N s\|^2 = \|\operatorname{grad}_{\tilde{B}_R} s\|^2$ we get

$$\begin{aligned} \Delta_\Omega \tilde{E}^b_R(x) &\geq \mathbf{E}''(s) \cdot \|\operatorname{grad}_N s\|^2 + \mathbf{E}'(s) \cdot ms'(r)h_b(r) \\ &\geq \mathbf{E}''(s) \cdot \|\operatorname{grad}_{\tilde{B}_R} s\|^2 + \mathbf{E}'(s) \cdot \tilde{\Delta}_{\tilde{B}_R} s \\ &= \tilde{\Delta}_{\tilde{B}_R} \tilde{E}^b_R = -1 = \Delta_\Omega E_R(x). \end{aligned}$$

Therefore $\tilde{E}^b_R(x) - E_R(x)$ is a subharmonic function on Ω_R vanishing on $\partial\Omega_R$ so that the maximum principle applies and gives $\tilde{E}^b_R(x) \leq E_R(x)$ for all $x \in \Omega_R$.

In case $K_N \leq b < 0$ we get similarly

$$\Delta_\Omega \tilde{E}^b_R(x) \leq \tilde{\Delta}_{\tilde{B}_R} \tilde{E}^b_R = -1 = \Delta_\Omega E_R(x).$$

Thus $\tilde{E}^b_R(x) - E_R(x)$ is a superharmonic function vanishing on the boundary so that $\tilde{E}^b_R(x) \geq E_R(x)$ for all $x \in \Omega_R$. This proves Theorem 1.

If $K_N \equiv b = 0$, then by continuity $\tilde{E}^b_R(x) = E_R(x)$ for all $x \in \Omega_R$. Conversely, if $\tilde{E}^b_R(x) = E_R(x)$ for all $x \in \Omega_R$, and we are in one of the two cases of Theorem 1 with $b \neq 0$, then we conclude from $(\mathbf{E}^b_R(s))'' \neq 0$ that $\|\operatorname{grad}_\Omega s\| = \|\operatorname{grad}_N s\|$. Hence Ω_R is a minimal cone in N. If N has constant curvature $K_N \equiv b \neq 0$, then by analytic continuation from $\Omega_R = \tilde{B}_R$ we finally get that all of M^m is a totally geodesic submanifold of N^n. q.e.d.

We note here that if $\Omega_R(p)$ is a codimension-0 domain of N we get $\|\operatorname{grad}_\Omega s\| = \|\operatorname{grad}_N s\|$ for free, so that the assumption of $E_R(x) = \tilde{E}^b_R(x)$ for all $x \in \Omega_R(p)$ now gives (from (4.1))

$$\Delta_\Omega s = \tilde{\Delta}_{\tilde{B}_R} s \quad \text{for all } s. \tag{4.2}$$

Under any of the curvature assumptions $\mathrm{Ric}_N \geq (n-1)b$ or $K_N \leq b$, (4.2) is only possible if $K_N \equiv b$ along every 'radial' plane in $\Omega_R(p)$, which therefore must be isometric to the space form ball $\tilde{B}_R$ of constant curvature b (cf. [1]). Thus we get the following rigidity theorem which completes the Corollaire in [3, p. 796].

Proposition 5. *Let $\Omega^n_R(p)$ be a regular domain of N^n. Suppose that either $\mathrm{Ric}_N \geq (n-1)b$ or $K_N \leq b$ for some $b \in \mathbf{R}$, and that $E_R(x) = \tilde{E}^b_R(x)$ for all $x \in \Omega_R$. Then Ω_R is isometric to $\tilde{B}_R$ with constant curvature b.*

5. Submanifolds of $\mathbf{R}^n$

Now suppose again that Ω_R is a regular domain of any submanifold of $\mathbf{R}^n$. Then

$$\begin{aligned} \Delta_\Omega \tilde{E}^0_R(x) &= \mathbf{E}'(s) \cdot \Delta_\Omega s \\ &= \mathbf{E}'(s) \cdot m s'(r)(h_0(r) + pH_\Omega(x)) \\ &= \tilde{\Delta}_{\tilde{B}_R} \tilde{E}^0_R(x) + \mathbf{E}'(s) \cdot m r(x) p H_\Omega(x) \\ &= \Delta_\Omega E_R(x) + m r(x) \mathbf{E}'(s) \cdot pH_\Omega(x). \end{aligned} \tag{5.1}$$

If we assume that $E_R(x) = \tilde{E}^0_R(x)$ for all $x \in \Omega_R$, then we must have $pH_\Omega(x) \equiv 0$.

To establish Theorem 3 stated in §1 we therefore only have to prove the following

Lemma 6. *Let M^{n-1} be an immersed hypersurface of $\mathbf{R}^n$. If M has everywhere vanishing pH, then M is minimal.*

Proof. Suppose for contradiction that $H(q) \neq 0$ for some $q \in M$. Then $H \neq 0$ in some maximal neighborhood $\mathscr{U}(q) \subset M$. Since

$$pH(x) = \langle \operatorname{grad}_{R^n} r, H \rangle = 0,$$

and H is orthogonal to the tangent space $T_x\mathscr{U}$, we get $\operatorname{grad}_{R^n}(x) \in T_x\mathscr{U}$. Thus $Z(x) = \operatorname{grad}_{R^n} r(x)$ is a unit vector field whose straight line integral curves foliate $\mathscr{U}$. Hence $\mathscr{U}$ is part of a cone with vertex p. But on a cone in $\mathbf{R}^n$ the length of the mean curvature vector grows to ∞ as one approaches the vertex. In particular $\mathscr{U}(q)$ must contain p where $\|H\| = \infty$. This contradicts the smoothness of M and proves the lemma.

References

[1] J. Cheeger & D. G. Ebin, *Comparison theorems in Riemannian geometry*, North-Holland Mathematical Library, Vol. 9, North-Holland, Amsterdam, 1975.

[2] S.-Y. Cheng, P. Li & S.-T. Yau, *Heat equations on minimal submanifolds and their applications*, Amer. J. Math. **106** (1984) 1033–1065.

[3] A. Debiard, B. Gaveau & E. Mazet, *Temps de sortie des boules normales et minoration locale de* λ_1, C. R. Acad. Sci. Paris. Sér. A **278** (1974) 795–798.

[4] E. B. Dynkin, *Markov processes*, Vols. 1 and 2, Grundlehren Math. Wiss. Vols. 121 and 122, Springer, Berlin, (1965).

[5] A. Gray & M. A. Pinsky, *The mean exit time from a small geodesic ball in a Riemannian manifold*, Bull. Sci. Math. (2) **107** (1983) 345–370.

[6] K. Jorge & D. Koutroufiotis, *An estimate for the curvature of bounded submanifolds*, Amer. J. Math. **103** (1981) 711–725.

[7] L. Karp & M. Pinsky, *Mean exit time from an extrinsic ball of a hypersurface* (Proc. Warwick Sympos. 1985, K. Elworthy, ed.), Pitman Research Notes in Math., No. 150, London, 1986, 179–186.

[8] S. Markvorsen, *On the heat kernel comparison theorems for minimal submanifolds*, Proc. Amer. Math. Soc. **97** (1986) 479–482.

THE TECHNICAL UNIVERSITY OF DENMARK