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COMPARING RIEMANNIAN FOLIATIONS WITH TRANSVERSALLY SYMMETRIC FOLIATIONS

FRANZ W. KAMBER,
ERNST A. RUH & PHILIPPE TONDEUR

1. Introduction and result

In this paper we compare Riemannian foliations with transversally homogeneous foliations, where the model transverse structure is of the type of a compact symmetric space G/K. The datum needed for this comparison is a connection in the normal bundle, having similar properties as the canonical connection in the case of a transversally symmetric foliation. This similarity is most conveniently formulated in terms of the corresponding Cartan connections. For the symmetric model case the curvature of the Cartan connection vanishes. An almost transversally symmetric foliation is one where this curvature is small in an appropriate norm. In the spirit of Rauch's comparison theorem [15], and more specifically, the comparison theorem of Min-Oo and Ruh [13], we wish to conclude that this assumption already implies the existence of a transversally symmetric structure of type G/K. We succeed in doing so for tense Riemannian foliations with small mean curvature. Here the tenseness means that the mean curvature form of the Riemannian foliation is a basic 1-form. A weaker form of this result was announced in [6], where the mean curvature form was assumed to vanish.

The definitions required to formulate the precise result are in §2. The norms are defined in §5. In the following theorem we let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K respectively.

Theorem. *Let $\mathscr{F}$ be a transversally oriented Riemannian foliation of codimension $q \geqslant 2$ and basic mean curvature form κ on the compact oriented Riemannian manifold (M, g_M). Let G/K be an irreducible compact symmetric space of dimension q and semisimple $\mathfrak{g}$. There exists a constant $A > 0$ depending only on the Lie algebra $\mathfrak{g}$ and curvature bounds on M with the following property.*

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If ω: $TP \to \mathfrak{g}$ is a basic Cartan connection form on the foliated K-reduction P of the normal frame bundle of $\mathcal{F}$ with Cartan curvature Ω and basic mean curvature form κ, then $\|\kappa\|_{1,\infty} + \|\Omega\|_{1,\infty} < A$ in appropriate Sobolev norms implies that $\mathcal{F}$ is transversally symmetric of type G/K.

The idea of the proof is to construct a Cartan connection $\bar{\omega}$ with vanishing curvature. This yields a developing map Φ: $\tilde{P} \to G$ on the universal covering $\tilde{P}$, equivariant with respect to a homomorphism $\pi_1(P) \to \Gamma \subset G$ (holonomy of $\bar{\omega}$). It induces in turn a map φ: $\tilde{M} \to G/K$, possibly after an averaging process. This map defines the transversally symmetric structure of type G/K for the foliation $\mathcal{F}$ as asserted in the theorem, via its lift to the universal covering $\tilde{M}$.

This implies by [3] that the cohomology $H_B(\mathcal{F})$ of basic forms (see (2.2) below) is isomorphic to the De Rham cohomology of G/K. Now it is well known that in the circumstances above $H^1_{\mathrm{DR}}(G/K) \cong H^1(\mathfrak{g}, \mathfrak{k}) \cong m^K = 0$, where $\mathfrak{g} = \mathfrak{k} \oplus m$. It follows that the basic mean curvature κ (see Definition 2.1 below) represents the trivial cohomology class. By the procedure in [10, (4.6)] the metric g_M can be modified by changing it only along the leaves so as to make all leaves minimal, i.e. $\mathcal{F}$ is harmonic with respect to the modified metric. If moreover the holonomy group Γ is dense in G, then $\kappa \in \Omega^1_B \cong H^1_B(\mathcal{F}) \cong H^1_{\mathrm{DR}}(G/K) = 0$, so the form κ must necessarily itself vanish, and $\mathcal{F}$ is already harmonic with respect to the given metric. These arguments show in any case that under the hypothesis of our theorem $\mathcal{F}$ is necessarily a taut foliation.

2. Background material

Let $\mathcal{F}$ be a foliation on M, given by an exact sequence of vector bundles

$$0 \to L \to TM \to Q \to 0.$$

Here L denotes the bundle of vectors tangent to $\mathcal{F}$, and $Q = TM/L$ is the normal bundle of dimension q, the codimension of $\mathcal{F}$. Let G/K be a Riemannian symmetric space of compact type with G and K connected, and $q = \dim G/K$. The foliation is said to be transversally homogeneous of type G/K if $\mathcal{F}$ is given on an atlas of distinguished charts $\mathbf{U} = \{U_\alpha\}$ by local submersions f_α: $U_\alpha \to G/K$, related by transition functions given by the left action of an element $\gamma_{\alpha\beta} \in G$: $f_\alpha = \gamma_{\alpha\beta} f_\beta$ (see e.g. [2]).

This transversal homogeneity can be expressed in terms of the orthonormal frame bundle $F(Q)$ of Q as follows. The isotropy representation of G/K shows that $K \subset \mathrm{SO}(q)$. Therefore, the transversal symmetric structure provides a K-reduction $K \to P \xrightarrow{\pi} M$ of $F(Q)$ with a foliated bundle structure [7].

This means that there is a K-invariant involutive subbundle $\tilde{L} \subset TP$, transversal to the fibers of P. The quotient bundle of $\tilde{L}$ divided by G on the base space M is the given $L \subset TM$. A connection on P is adapted to the foliated bundle structure if the horizontal subspace contains $\tilde{L}$. Starting with a connection on P, the subspace $\tilde{L}_u$ is the horizontal lift of $L_{\pi(u)}$.

A $\mathfrak{k}$-valued adapted connection η on P gives rise to a $\mathfrak{g}$-valued Cartan connection $\omega = \eta + \varphi$, where φ is the canonical $\mathbb{R}^q$-valued (solder) 1-form on P defined by $\varphi(X) = u^{-1}(\pi(X))$ for $X \in T_uP$, and the frame u of Q at $\pi(u)$ is viewed as a linear isomorphism $\mathbb{R}^q \to Q_{\pi(u)}$. The curvature

$$\Omega_\omega = d\omega + \tfrac{1}{2}[\omega, \omega]$$

can be expressed in terms of the curvature $\Omega_\eta = d\eta + \frac{1}{2}[\eta, \eta]$, and the torsion $\Phi_\eta = d\varphi + [\eta, \varphi]$ by

$$\Omega_\omega = \Omega_\eta + \tfrac{1}{2}[\varphi, \varphi] + \Phi_\eta,$$

where the brackets are expressed in terms of the brackets in the Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus m$.

In case η is the unique torsion free connection, the symmetric space structure implies $\Omega_\eta = -\frac{1}{2}[\varphi, \varphi]$ and thus $\Omega_\omega = 0$. The last equation is the integrability condition for a locally symmetric transversal structure, and therefore is equivalent to the definition of a transversally symmetric foliation by local submersions outlined at the beginning of this section.

In our theorem we allow a slightly more general situation. We start with a basic Cartan connection (see [5] and the definition below) ω: $TP \to \mathfrak{g}$ with small curvature. It is not necessary to assume that the 1-form φ in $\omega = \eta + \varphi$ defined by the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus m$ is the canonical 1-form. It suffices to assume that φ is nondegenerate. To simplify notation we write Ω instead of Ω_ω.

The Cartan connection ω is said to be adapted to the foliation $\tilde{\mathscr{F}}$ on P if ω restricted to $\tilde{L}$ vanishes. An adapted Cartan connection is basic if $i(\tilde{X})\Omega = 0$ for all $\tilde{X} \in \Gamma\tilde{L}$ (compare [7]). This implies that Ω itself is a differential form in the basic complex defined later in this section.

A foliation $\mathscr{F}$ on M is said to be almost transversally symmetric of type G/K if there exists a foliated K-reduction P of $F(Q)$ and a basic Cartan $\mathfrak{g}$-connection ω with small curvature Ω. In order to avoid technical difficulties of the kind dealt with in [13], we measure Ω in terms of a norm involving first derivatives (compare §5 for the precise definition). In view of [1] there is no loss in generality.

Next we describe the mean curvature form κ of $\mathscr{F}$ as given in [9], [10]. Since $K \subset \mathrm{SO}(q)$, the foliations considered are necessarily Riemannian. It is therefore no restriction to assume g_M to be a bundle-like metric on M [16]. The choice of a metric defines a splitting $\sigma\colon Q \to TM$, with isometric identification $(Q, g_Q) \overset{\cong}{\to} (L^\perp, g_M \mid L^\perp)$. The (partial) Bott connection $\overset{\circ}{\nabla}$ in Q is given by

$$\overset{\circ}{\nabla}_X s = \pi[X, Y] \quad \text{for } X \in \Gamma L,\ Y \in \Gamma TM \text{ with } \pi(Y) = s \in \Gamma Q.$$

The Riemannian connection ∇^M of g_M gives rise to an adapted connection in Q as follows:

$$\nabla_X^Q s = \begin{cases} \overset{\circ}{\nabla}_X s & \text{for } X \in \Gamma L,\ s \in \Gamma Q, \\ \pi\left(\nabla_X^M Y_s\right) & \text{for } X \in \Gamma\sigma Q,\ s \in \Gamma Q, \text{ and } Y_s = \sigma(s) \in \Gamma\sigma Q. \end{cases}$$

This connection is metric and torsionfree (see [8]). The notations of that paper are also used below.

Consider the Weingarten map $W(s)\colon L \to L$ given for $s \in \Gamma Q$ by $W(s)X = -\pi^\perp(\nabla_X^M Y_s)$ for $X \in \Gamma L$, $Y_s = \sigma(s) \in \Gamma\sigma Q$, and $\pi^\perp$ the orthogonal projection $TM \to L$.

(2.1) Definition. *The* 1*-form* κ *given by* $\kappa(s) = \operatorname{Tr} W(s)$ *is the mean curvature form of* $\mathscr{F}$.

The basic complex of $\mathscr{F}$ is given by

(2.2) $\Omega_B(\mathscr{F}) = \{\beta \in \Omega(M) \mid i(X)\beta = 0,\ \Theta(X)\beta = 0 \text{ for all } X \in \Gamma L\}$,

where $\Theta(X)$ denotes the Lie derivative. In our theorem we assume the mean curvature κ to be a basic 1-form (this is the tenseness property of [9], [10]). As a consequence we have $d\kappa = 0$ (see [10, (4.4)] or the appendix to [11]).

Let $\nu \in \Omega_B^q$ denote the transversal volume form associated to the metric g_Q. Clearly $d\nu = 0$. Then $\chi_{\mathscr{F}} = *\nu$ is the characteristic p-form of $\mathscr{F}$ ($p = \dim \mathscr{F}$), expressed in terms of the star operator associated to g_M. The form $\mu = \nu \wedge \chi_{\mathscr{F}}$ is the Riemannian volume of g_M. The fundamental relationship between $\chi_{\mathscr{F}}$ and κ is given by the congruence [10, 4.3]

(2.3) $$d\chi_{\mathscr{F}} + \kappa \wedge \chi_{\mathscr{F}} \equiv 0 \quad \text{mod } \mathscr{F}\text{-trivial forms.}$$

This means that the $(p+1)$-form on the left-hand side vanishes when evaluated on p tangent vectors in L.

Now let E be a vector bundle over M with covariant derivative D. The associated covariant exterior derivative d^E on the E-valued form $\Omega(M, E)$ is given by

(2.4) $$(d^E\beta)(X_0, \cdots, X_p) = \sum_{i=0}^{p} (-1)^i (\nabla_{X_i}\beta)(X_0, \cdots, \hat{X}_i, \cdots, X_p),$$

where ∇ is the connection defined by D on E and ∇^M on M. The corresponding Lie derivative is given $\Theta^E(X) = d^E i(X) + i(X) d^E$. The complex of basic E-valued forms is then given by

(2.5) $$\Omega_\beta(\mathscr{F}, E) = \{\beta \in \Omega(M, E) \mid i(X)\beta = 0,\ \Theta^E(X)\beta = 0 \quad \text{for all } x \in \Gamma L\}.$$

The Bianchi identity $d^E R^D = 0$ shows that the condition $i(X)R^D = 0$ for all $X \in \Gamma L$ implies that the curvature tensor R^D is a basic 2-form. As a consequence of this condition the exterior derivative d^E on $\Omega(M, E)$ restricts to

$$d_B^E : \Omega_B^r(\mathcal{F}, E) \to \Omega_B^{r+1}(\mathcal{F}, E). \tag{2.6}$$

Next we assume $E \to M$ to be equipped with a bundle metric g_E and D to be a metric covariant derivative. We need the star operator $\bar{*}$: $\Omega_B^r(\mathcal{F}, E) \to \Omega_B^{q-r}(\mathcal{F}, E)$ associated to g_Q, and characterized by $g_E(\beta \wedge \bar{*}\beta) = \nu$ (transversal volume). It is related to the ordinary star operator $*$: $\Omega^r(M, E) \to \Omega^{n-r}(M, E)$ ($n = \dim M$) associated to g_M by the formula $*\beta = \bar{*}\beta \wedge \chi_{\mathcal{F}}$ for $\beta \in \Omega_B^r(\mathcal{F}, E)$ (see [10, (4.12)] for the case of the trivial line bundle E). The global scalar product $\langle\ ,\ \rangle$ on $\Omega(M, E)$ restricted to basic forms is then

$$\langle \beta, \beta' \rangle_B = \int_M g_E(\beta \wedge \bar{*}\beta') \wedge \chi_{\mathcal{F}}.$$

With respect to this scalar product the adjoint δ_B^E: $\Omega_B^r(\mathcal{F}, E) \to \Omega_B^{r-1}(\mathcal{F}, E)$ of d_B^E is given by

$$\delta_B^E \beta = (-1)^{q(r+1)+1} \bar{*}(d_B^E - \kappa \wedge)\bar{*}\beta \tag{2.7}$$

for $\beta \in \Omega_B^r(\mathcal{F}, E)$ (see [10, (4.14)] for the case of the trivial line bundle E).

For the computations of §4 it will be convenient to use a special orthonormal moving frame on M. For $x \in M$ let $\{e_A\}_{A=1}^n \subset T_x M$ be an oriented orthonormal basis with $\{e_i\}_{i=1}^p \subset L_x$ and $\{e_\alpha\}_{\alpha=p+1}^n \subset Q_x \cong L_x^\perp$ ($p + q = n$). Let U be a distinguished (flat) neighborhood of x for $\mathcal{F}$ with local Riemannian submersion f: $U \to B$. For $\alpha = p + 1, \cdots, n$ let $E_\alpha \in \Gamma(U, Q)$ be the pull back of the extension $f_* e_\alpha$ to a vector field on B by parallel transport along geodesic rays emanating from $f(x)$ (use [16, Proposition 4.2]). We complete $\{E_\alpha\}_{\alpha=p+1}^n$ by the Gram-Schmidt process to a moving frame $\{E_A\}_{A=1}^n$ by adding $\{E_i\}_{i=1}^p \subset \Gamma(U, L)$ with $(E_i)_x = e_i$. We then have

$$\nabla_{e_\alpha}^Q E_{\alpha'} = \left(\nabla_{E_\alpha}^Q E_{\alpha'}\right)_x = 0 \quad \text{for } \alpha, \alpha' = p + 1, \cdots, n.$$

As a consequence of the torsionfreeness of ∇^Q (see [8, (1.5)]), we have $[E_\alpha, E_{\alpha'}]_x \in L_x$. Furthermore, since the E_α are infinitesimal automorphisms of $\mathcal{F}$, we have $\nabla_X^Q E_\alpha = \pi[X, E_\alpha] = 0$ for $X \in \Gamma(U, L)$. In terms of this special frame, δ_B^E takes the following form:

$$\begin{aligned}(\delta_B^E \beta)(X_2, \cdots, X_p) = &- \sum_{\alpha=p+1}^n (\nabla_{E_\alpha} \beta)(E_\alpha, X_2, \cdots, X_p) \\ &+ \sum_{\alpha=p+1}^n \kappa(E_\alpha) \cdot \beta(E_\alpha, X_2, \cdots, X_p).\end{aligned} \tag{2.8}$$

As usual the Laplacian is defined by

$$\Delta_B^E = d_B^E \delta_B^E + \delta_B^E d_B^E. \tag{2.9}$$

3. The proof

In this section we deal with the formal aspects of the proof. The main work is done in the remaining sections. For a motivation of the construction below we refer to [13] where the nonfoliated version of the theorem was proved and discussed.

Let $\mathscr{F}$ denote a Riemannian foliation on the compact Riemannian manifold (M, g_M) with basic mean curvature κ. Let $\tilde{\mathscr{F}}$ denote the lift of $\mathscr{F}$ to a K-invariant foliation on the K-principal bundle of orthonormal frames P associated to the normal bundle Q of the foliation $\mathscr{F}$. The data of the Theorem specifies an adapted Cartan connection ω: $TP \to \mathfrak{g}$ with small basic Cartan curvature $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$. The Cartan curvature is defined relative to the model G/K, which means that the bracket in the definition of Ω is the Lie bracket of the Lie algebra $\mathfrak{g}$ of G, and that the structure group K of P coincides with the isotropy group K of the symmetric space G/K.

It is well known (compare [13]) that a Cartan connection $\bar{\omega}$ with vanishing curvature $\bar{\Omega}$ gives rise to a developing map ϕ: $\tilde{P} \to G$, and a representation of the fundamental group $\pi_1(P) \to \Gamma \subset G$. The map ϕ is equivariant with respect to the actions of the fundamental group $\pi_1(P)$ on $\tilde{P}$ and left translation by Γ on G respectively. If the Cartan connection $\bar{\omega}$: $TP \to \mathfrak{g}$ respects the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus m$, i.e., if $\bar{\omega}$ restricted to the tangent space of the fibers in the K-principal bundle P takes values in the Lie algebra $\mathfrak{k}$ of K, then ϕ factors to φ: $\tilde{M} \to G/K$. The submersion φ defines the transversal symmetric structure of the foliation $\mathscr{F}$.

The main work in the proof is to establish the existence of an adapted Cartan connection $\bar{\omega}$ on P with vanishing Cartan curvature $\bar{\Omega}$. It turns out that the flat Cartan connection constructed in the present proof respects the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus m$ only up to an error controlled by the constant A of the Theorem. This does not interfere with the construction of a developing map ϕ: $\tilde{P} \to G$ but φ: $\tilde{M} \to G/K$ has to be constructed as in [13, p. 343] by an averaging process.

We will obtain $\bar{\omega}$: $TP \to \mathfrak{g}$ as the limit of a sequence of Cartan connections. The sequence starts with $\omega^0 = \omega$, the Cartan connection of the Theorem. To define the iteration step let $E = P \times \mathfrak{g}$ denote the trivial vector bundle over P whose fiber is the Lie algebra $\mathfrak{g}$. On E we define the linear connection

$$D_X s = Xs + [\omega(X), s], \tag{3.1}$$

where s is a section in E, Xs is the derivative of s in direction X, $\omega = \omega^0$ is the original Cartan connection, and $[\ ,\ ]$ is the Lie bracket of $\mathfrak{g}$. The curvature R^D of D is

$$R^D(X,Y)s = [\Omega(X,Y), s]. \tag{3.2}$$

In particular, R^D is a basic 2-form and we can consider the basic complex of E-valued differential forms on P as in §2. Obviously, the norms of R^D and Ω are identical.

Now we apply the DeRham-Hodge decomposition for Δ_B^E on $\Omega_B(\tilde{\mathscr{F}}, E)$ as established in [11]. We define

$$\omega^{i+1} = \omega^i + \delta_B^E \beta^{i+1}. \tag{3.3}$$

Here β^{i+1} is the unique solution of

$$\Delta_B^E \beta^{i+1} = -\Omega^i, \tag{3.4}$$

where Ω^i is the curvature of ω^i, and the operators δ_B^E and Δ_B^E are defined by (2.7) and (2.9) respectively. We note that these operators are defined relative to the foliation $\tilde{\mathscr{F}}$ on P. Proposition (4.8) guarantees that the solution of (3.4) is unique.

Because the initial Cartan connection $\omega = \omega^0$ as well as the mean curvature form $\tilde{\kappa}$ of $\tilde{\mathscr{F}}$ is in the basic complex $\Omega_B(\tilde{F}, E)$ of differential forms of P, the connection forms ω^i and the curvature forms Ω^i are in this complex as well.

The requirements for the convergence of $\{\omega^i\}$ to a flat Cartan connection $\overline{\omega}$ are formulated as the following result.

(3.5) Main Lemma. *Let ω: $TP \to \mathfrak{g}$ denote the basic Cartan connection form with basic curvature form Ω, and let κ denote the mean curvature form of the foliation $\mathscr{F}$ assumed in the Theorem of §1. There exists a constant $A' > 0$ depending only on $\mathfrak{g}$ and curvature bounds for the metric g_M on the basis M of P such that $(\|\kappa\|_{1,\infty} + \|\Omega\|_{1,\infty}) < A'$ implies that ω^{i+1} of (3.3) satisfies*

(i) $\|\Omega^{i+1}\|_{1,m} < c(\|\kappa\|_{1,\infty} + \|\Omega\|_{1,\infty})\|\Omega^i\|_{1,m}$,

(ii) $\|\omega^{i+1} - \omega^i\|_{2,m} < c\|\Omega^i\|_{1,m}$,

where c is a constant depending only on $\mathfrak{g}$ and curvature bounds on g_M.

The assertion (i) of (3.5) shows that $\{\|\Omega^i\|_{1,m}\}$ is a geometric sequence whose ratio can be made arbitrarily small by choosing A' suitably. This implies by assertion (ii) of (3.5) that $\Sigma_{i=0}^{\infty}\|\omega^{i+1} - \omega^i\|_{2,m}$ can be made arbitrarily small by choosing A' suitably. Therefore $\{\omega^i\}$ converges to an element $\overline{\omega}$ in the Sobolev space $W_{2,m}$. Since $\|\omega - \overline{\omega}\|_{2,m}$ is small, $\overline{\omega}$ is nondegenerate and hence a Cartan connection form. By (3.5)(i), $\overline{\Omega} = d\overline{\omega} + \frac{1}{2}[\overline{\omega}, \overline{\omega}] = 0$. The regularity theorem for this differential equation implies that $\overline{\omega}$ is a smooth differential form. This completes the formal part of the proof.

4. A Bochner-Lichnerowicz formula for basic E-valued 2-forms

In the course of the proof (see the previous section) we need to establish the strict positivity of the Laplace operator Δ_B^E of (2.9) on basic 2-forms. We expect this operator to be positive because the transversal structure of $\tilde{\mathscr{F}}$ on P is close to the structure of a semisimple Lie group of compact type. For such groups the Laplace operator on real valued 2-forms is well known to be positive. Our computation follows closely the standard derivation of the Bochner-Lichnerowicz formula (compare [4]). The additional complications here are due to the curvature R^D in the vector bundle E and the mean curvature form $\tilde{\kappa}$ of the foliation $\tilde{\mathscr{F}}$ on P. Since both tensors are small their contribution will not interfere with the positivity of Δ_B^E.

We refer to §2 for the basic concepts. Here we deal with the special case of a foliation $\tilde{\mathscr{F}}$ on P induced by a foliation $\mathscr{F}$ on M. To simplify the computations we work with a special moving frame adapted to the foliation $\tilde{\mathscr{F}}$ on P as defined at the end of §2. Since the norms of $\tilde{\kappa}$ and κ agree up to the volume of K, we will write κ also for the mean curvature form of $\tilde{\mathscr{F}}$.

To formulate the result, we introduce the following operators on the space of basic 2-forms $\Omega_B(\tilde{\mathscr{F}}, E)$. The first operator is related to the contribution of the curvature of the Riemannian manifold P to the Bochner formula. For $\beta \in \Omega_B(\tilde{\mathscr{F}}, E)$ we define

$$\begin{aligned} \mathscr{R}^P(\beta)(X,Y) = -\sum_{\alpha=p+1}^{N} \{ &\beta(R^P(E_\alpha, X)E_\alpha, Y) - \beta(R^P(E_\alpha, Y)E_\alpha, X) \\ &+ \beta(E_\alpha, R^P(E_\alpha, X)Y) - \beta(E_\alpha, R^P(E_\alpha, Y)X) \}, \end{aligned} \tag{4.1}$$

where R^P is the Riemann curvature tensor of the Riemannian metric induced on P by ω: $TP \to \mathfrak{g}$ and the Cartan-Killing form on $\mathfrak{g}$, and $N = p + \dim \mathfrak{g}$.

The second operator is related to the contribution of the curvature of the connection D on the vector bundle E to the Bochner formula. We define

$$\begin{aligned} &\mathscr{R}^D(\beta)(X,Y) \\ &= \sum_{\alpha=p+1}^{N} \{ R^D(E_\alpha, X)\beta(E_\alpha, Y) - R^D(E_\alpha, Y)\beta(E_\alpha, X) \}. \end{aligned} \tag{4.2}$$

The third operator is related to the contribution of the mean curvature form κ to the Bochner formula. We define

$$\begin{aligned}
&\mathscr{K}(\beta)(X,Y)\\
(4.3)\quad &= \sum_{\alpha=p+1}^{N} \left\{\nabla_X(\kappa(E_\alpha)\beta(E_\alpha,Y)) - \nabla_Y(\kappa(E_\alpha)\beta(E_\alpha,X))\right.\\
&\qquad \left. +\kappa(E_\alpha)\left((\nabla_{E_\alpha}\beta)(X,Y) + (\nabla_X\beta)(Y,E_\alpha) + (\nabla_Y\beta)(E_\alpha,X)\right)\right\}.
\end{aligned}$$

where ∇ is the product connection of D on E and the extension of the Levi-Civitá connection ∇^P to real valued differential forms on P.

In terms of the operators defined above, the Bochner formula is given in the following proposition. ∇ is the product connection defined above and $\nabla^*\nabla = -\operatorname{trace}\nabla\nabla$.

(4.4) Proposition. *If Δ_B^E is the Laplace operator on basic* 2*-forms with values in E defined in* (2.9), *then*

$$\Delta_B^E\beta = \nabla^*\nabla\beta + \mathscr{R}^D(\beta) + \mathscr{R}^P(\beta) + \mathscr{K}(\beta).$$

Proof.

$$\begin{aligned}
(d_B^E\delta_B^E\beta)(X,Y) = -\sum_{\alpha=p+1}^{N} &\left\{(\nabla_X\nabla_{E_\alpha}\beta)(E_\alpha,Y) - (\nabla_Y\nabla_{E_\alpha}\beta)(E_\alpha,X)\right.\\
&\left. -\nabla_X(\kappa(E_\alpha)\beta(E_\alpha,Y)) + \nabla_Y(\kappa(E_\alpha)\beta(E_\alpha,X))\right\},
\end{aligned}$$

$$\begin{aligned}
&(\delta_B^E d_B^E\beta)(X,Y)\\
&= -\sum_{\alpha=p+1}^{N} \left\{(\nabla_{E_\alpha}\nabla_{E_\alpha}\beta)(X,Y) + (\nabla_{E_\alpha}\nabla_X\beta)(Y,E_\alpha) + (\nabla_{E_\alpha}\nabla_Y\beta)(E_\alpha X)\right\}\\
&\quad + \sum_{\alpha=p+1}^{N} \kappa(E_\alpha)\left\{(\nabla_{E_\alpha}\beta)(X,Y) + (\nabla_X\beta)(Y,E_\alpha) + (\nabla_Y\beta)(E_\alpha,X)\right\},
\end{aligned}$$

$$\begin{aligned}
(\Delta_B^E\beta)(X,Y) = &-\sum_{\alpha=p+1}^{N} (\nabla_{E_\alpha}\nabla_{E_\alpha}\beta)(X,Y)\\
&+ \sum_{\alpha=p+1}^{N} \left\{-(R(X,E_\alpha)\beta)(E_\alpha,Y) + (R(Y,E_\alpha)\beta)(E_\alpha,X)\right\}\\
&+\mathscr{K}(\beta)(X,Y).
\end{aligned}$$

Since R is the curvature tensor of ∇, the product connection of D on E and the extension of ∇^P to differential forms on P, the second line on the right-hand side for $\Delta_B^E\beta$ above is equal to

$$\sum_{\alpha=p+1}^{N}\left\{-R^D(X,E_\alpha)\beta(E_\alpha,Y)+R^D(Y,E_\alpha)\beta(E_\alpha,X)\right\}$$

$$+\sum_{\alpha=p+1}^{N}\left\{\beta\left(R^P(X,E_\alpha)E_\alpha,Y\right)+\beta\left(E_\alpha,R^P(X,E_\alpha)Y\right)\right.$$

$$\left.-\beta\left(R^P(Y,E_\alpha)E_\alpha,X\right)-\beta\left(E_\alpha,R^P(Y,E_\alpha)X\right)\right\},$$

The first line above equals $(\mathscr{R}^D\beta)(X,Y)$, the second line equals $(R^P\beta)(X,Y)$, and Proposition (4.4) is proved.

In order to prepare for the estimate of the right-hand side in Proposition (4.4), we compute $\mathscr{R}^G$, where $P = G$ is a simple compact Lie group foliated by points and E is the trivial line bundle over G with the product connection. We obtain the following result.

(4.5) Proposition. *If G is a compact simple Lie group with the Riemannian metric provided by the (negative) Cartan-Killing form, then*

$$\langle\mathscr{R}^G\beta,\beta\rangle \geqslant \tfrac{1}{4}|\beta|^2.$$

To prove this proposition we need a lemma. Let $\mathfrak{g}$ denote the Lie algebra of a compact simple Lie group G. Let $\mathfrak{g}\to\mathfrak{g}\wedge\mathfrak{g}$ denote the injection defined by the adjoint representation and the Killing form identifying $\mathfrak{g}\cong\mathfrak{g}^*$, and let Proj: $\mathfrak{g}\wedge\mathfrak{g}\to\mathfrak{g}$ denote the corresponding orthogonal projection. (See [12] for a version of the following fact.)

(4.6) Lemma. *For any compact simple Lie algebra, the exterior product and Lie bracket are related by the equation*

$$\operatorname{Proj}(x\wedge y)=[x,y].$$

Proof. (We thank Bob Stanton for this proof.) Let $\{x_i\}$ denote an orthonormal basis with respect to the Killing form. Then

$$\sum_i\langle x\wedge y,\operatorname{ad}x_i\rangle_{\operatorname{End}\mathfrak{g}}\operatorname{ad}x_i=\sum_i\operatorname{tr}(x\wedge y\circ\operatorname{ad}x_i)\operatorname{ad}x_i.$$

Further

$$\begin{aligned}\operatorname{tr}(x\wedge y\circ\operatorname{ad}x_i)&=\sum_j\langle\operatorname{ad}x_i\circ x\wedge y(x_j),x_j\rangle\\&=\frac{1}{2}\sum_j\langle\operatorname{ad}x_i(\langle y,x_j\rangle x-\langle x,x_j\rangle y),x_j\rangle\\&=\frac{1}{2}(\langle\operatorname{ad}x_i(x),y\rangle-\langle\operatorname{ad}x_i(y),x\rangle.\end{aligned}$$

Substituting this in the first line yields

$$\sum_i \langle x \wedge y, \operatorname{ad} x_i \rangle_{\operatorname{End}\mathfrak{g}} \operatorname{ad} x_i = \frac{1}{2} \sum_i (\langle \operatorname{ad} x_i(x), y\rangle - \langle \operatorname{ad} x_i(y), x\rangle) \operatorname{ad} x_i$$
$$= \frac{1}{2} \cdot 2 \sum_i \langle x_i, [x, y]\rangle \operatorname{ad} x_i = \sum_i \langle [x, y], x_i \rangle \operatorname{ad} x_i,$$

and $\operatorname{Proj}(x \wedge y) = [x, y]$.

Proof of Proposition (4.5). By definition (4.1) and the Bianchi identity applied to the last two terms of (4.1), we have

$$\begin{aligned}(\mathscr{R}^G(\beta))(X, Y) = &\sum_{\alpha=p+1}^{N} \{\beta(R^G(X, E_\alpha)E_\alpha, Y) + \beta(X, R^G(Y, E_\alpha)E_\alpha)\} \\ &- \sum_{\alpha=p+1}^{N} \beta(E_\alpha, R^G(X, Y)E_\alpha).\end{aligned} \tag{4.7}$$

Since $R^G(X, Y) = \frac{1}{4}[X, Y]$ the above equation shows that $\mathscr{R}^G$ respects the splitting $\mathfrak{g} \wedge \mathfrak{g} \cong \mathfrak{g} \oplus \mathfrak{g}^\perp$ defined by the adjoint representation. In fact, the first line in (4.7) is given by scalar multiplication because G is an Einstein manifold. The second line, by Lemma (4.6), vanishes for $\beta \in \mathfrak{g}^\perp$. For $\beta \in \mathfrak{g}^\perp$ we obtain

$$\begin{aligned}\langle \mathscr{R}^G\beta, \beta\rangle = &\sum_{\alpha=p+1}^{\dim\mathfrak{g}} \sum_{\gamma<\delta=p+1}^{\dim\mathfrak{g}} \\ &\cdot \{\beta(R^G(E_\alpha, E_\gamma)E_\alpha, E_\delta) - \beta(R^G(E_\alpha, E_\delta)E_\alpha, E_\gamma)\} \cdot \beta(E_\gamma, E_\delta) \\ = &\sum_{\alpha=p+1}^{\dim\mathfrak{g}} 2 \cdot \sum_{\gamma<\delta=p+1}^{\dim\mathfrak{g}} \frac{1}{4}\beta([[E_\alpha, E_\gamma], E_\alpha], E_\delta)\beta(E_\gamma, E_\delta) \\ = &\frac{1}{2} \sum_{\gamma<\delta=1}^{\dim\mathfrak{g}} \beta(E_\gamma, E_\delta)\beta(E_\gamma, E_\delta) = \frac{1}{2}|\beta|^2.\end{aligned}$$

For $\beta \in \mathfrak{g} \subset \mathfrak{g} \wedge \mathfrak{g}$ we have to take the additional term

$$-\sum_{\gamma<\delta=1}^{\dim\mathfrak{g}} \sum_{\alpha=1}^{\dim\mathfrak{g}} \beta(E_\alpha, R^G(E_\gamma, E_\delta)E_\alpha)\beta(E_\gamma, E_\delta)$$

into account. Using Lemma (4.6) we obtain for the above expression

$$-\sum_{\gamma<\delta=1}^{\dim\mathfrak{g}} \frac{1}{4} \cdot \sum_{\alpha=1}^{\dim\mathfrak{g}} \beta([E_\alpha, [[E_\gamma, E_\delta], E_\alpha]])\beta([E_\gamma, E_\delta])$$
$$= -\frac{1}{4} \cdot \sum_{\gamma<\delta=1}^{\dim\mathfrak{g}} \beta(E_\gamma, E_\delta)\beta(E_\gamma, E_\delta) = -\frac{1}{4}|\beta|^2.$$

This proves Proposition (4.5) for $\beta \in \mathfrak{g} \subset \mathfrak{g} \wedge \mathfrak{g}$ as well.

We conclude this section with a proposition which summarizes the above results to the extent needed in §§3 and 5. By choosing the constant A of the Theorem of §2 small enough we can render the operators $\mathscr{R}^E$ and $\mathscr{K}$ arbitrarily small. Therefore, the main term in Proposition (4.4) is $\langle \mathscr{R}^G\beta, \beta\rangle$. Proposition (4.5) shows that for $P = G$ this term is strictly positive. Since $\|\Omega\|_{1,\infty}$ measures the difference between R^P and R^G we can render this difference arbitrarily small. We replace the factor $\frac{1}{4}$ in Proposition (4.5) by $\frac{1}{5}$ to absorb the errors caused by Ω and $\mathscr{K}$. Taking into consideration

$$\begin{aligned}\langle \nabla^*\nabla\beta, \beta\rangle &= -\sum_\alpha \langle \nabla_{E_\alpha}\nabla_{E_\alpha}\beta, \beta\rangle = -\sum_\alpha \left\{ E_\alpha\langle \nabla_{E_\alpha}\beta, \beta\rangle + |\nabla E_\alpha \beta|^2 \right\} \\ &= \frac{1}{2}\Delta_B|\beta|^2 + |\nabla\beta|^2.\end{aligned}$$

we obtain the following result.

(4.8) Proposition. *Under the assumptions of the Theorem of* §1 *with the constant A sufficiently small, the following inequality holds for $\beta \in \Omega_B^2(\tilde{\mathscr{F}}, E)$:*

$$\langle \Delta_B^E\beta, \beta\rangle \geqslant \tfrac{1}{2}\Delta_B|\beta|^2 + |\nabla\beta|^2 + \tfrac{1}{5}|\beta|^2.$$

5. The estimates

The purpose of this section is to prove the Main Lemma of §3. To achieve this we derive certain estimates related to the equation $\Delta_B^E\beta = -\Omega$, where the elliptic operator Δ_β^E is defined in (2.9), and Ω is one of the curvatures of the iteration of Cartan connections defined in (3.3). The basic strategy is the same as in [13]. Complications arise because, in contrast to [13], we deal with the transversal geometry of a foliation. We utilize the de Rham-Hodge theory for basic differential forms relative to a foliation as worked out in [11]. In contrast to the main body of [11], where the differential forms are real valued, we have to work here with vector valued differential forms. As a consequence, the exterior derivative may not be exact. The assumption of small curvature is sufficient.

As in [13] we have to make sure that the constants in the estimates can be controlled by curvature assumptions only. In particular, the diameter and the injectivity radius of the manifolds M and P should not enter the estimates. This can be done, as in [13], by localizing the definition of the Sobolev norms. Instead of defining the norms by integrating the appropriate expression over the Riemannian manifold P, we pull back the differential forms via the exponential map $\exp_p$: $T_pP \to P$ to a ball $B_r(0)$ of a suitable radius r and center $0 \in T_pP$, define the Sobolev norms on these balls as in [11, §4], and then take the supremum over $p \in P$:

$$\|\beta\|_{s,m}^{m} = \sup_{p\in P} \sum_{|\mu|\leqslant s} \int_{B_r(0)} |\nabla^{\mu}\beta|^{m}, \tag{5.1}$$

where $m \in \mathbb{N}$ is the exponent and $s \in \mathbb{N}$ is the number of derivatives taken into account in the definition of the Sobolev space $W_{s,m}$. As usual, $|\mu|$ is the degree of the multi-index μ, and $m = \infty$ indicates the essential supremum.

Proposition (4.8) shows that Δ_B^E is positive on E-valued basic 2-forms. Therefore, the equation $\Delta_B^E\beta = -\Omega$ has a unique solution. An application of Proposition (4.3) at the point where the norm $|\beta|$ of the solution β of $\Delta_B^E\beta = -\Omega$ is maximal shows

$$\|\beta\|_{0,\infty} \leqslant 5\|\Omega\|_{0,\infty}. \tag{5.2}$$

Now, with the supremum of $|\beta|$ under control it is sufficient to control the constants in the Sobolev embedding theorem and in the regularity estimates for the solution β of the elliptic system $\Delta_B^E\beta = -\Omega$. Because of the localization of the definition of the Sobolev norms these constants are controlled by curvature bounds. The main result of [1] shows that, without loss of generality, bounded curvature in the C^0-sense can be replaced by the assumption of a C^k-bound for a fixed $k \in \mathbb{N}$. Thus, for convenience we may assume that the assumption of bounded curvature in the Theorem means that we have bounds for a certain number of covariant derivatives of the curvature tensor as well. One way to be in the most favorable case of the Sobolev embedding theorem is to fix the exponent m in $W_{s,m}$ to be larger than the dimension of P.

To obtain more precise estimates on β we observe (compare [11, Proposition 3.6] and (4.3), (4.4)) that the Laplace operator restricted to basic forms differs from Δ_B^E by $\mathscr{K}$ only. As (4.3) shows, $\mathscr{K}$ involves first derivatives only and the usual elliptic estimates apply. We have

$$\|\beta\|_{3,m} \leqslant c\left(\|\beta\|_{0,m} + \left\|\Delta_B^E\beta\right\|_{1,m}\right),$$

and (5.2) implies

$$\|\beta\|_{3,m} \leqslant c\|\Omega\|_{1,m}. \tag{5.3}$$

After these preparations we are ready to prove the Main Lemma (3.5). In fact, (5.3) applied to the solution β^{i+1} of $\Delta_B^E\beta^{i+1} = -\Omega^i$ proves assertion (ii) of (3.5). In order to prove the first assertion of (3.5) we compute Ω^{i+1}:

$$\begin{aligned}\Omega^{i+1} &= d\omega^{i+1} + \tfrac{1}{2}[\omega^{i+1}, \omega^{i+1}] \\ &= d\omega^i + \tfrac{1}{2}[\omega^i, \omega^i] + d\delta_B^E\beta^{i+1} + \tfrac{1}{2}\left[\omega^i, \delta_B^E\beta^{i+1}\right] \\ &\quad + \tfrac{1}{2}\left[\delta_B^E\beta^{i+1}, \omega^i\right] + \tfrac{1}{4}\left[\delta_B^E\beta^{i+1}, \delta_B^E\beta^{i+1}\right].\end{aligned}$$

In this formula, d is the standard exterior derivative. We have

$$\Omega^{i+1} = \Omega^i + d\delta_B^E\beta^{i+1} + \tfrac{1}{2}\left[\omega^0, \delta_B^E\beta^{i+1}\right] + \tfrac{1}{2}\left[\delta_B^E\beta^{i+1}, \omega^0\right]$$
$$+ \tfrac{1}{2}\left[\omega^i - \omega^0, \delta_B^E\beta^{i+1}\right] + \tfrac{1}{2}\left[\delta_B^E\beta^{i+1}, \omega^i - \omega^0\right] + \tfrac{1}{4}\left[\delta_B^E\beta^{i+1}, \delta_B^E\beta^{i+1}\right].$$

The purpose of the introduction of δ_B^E is to simplify the first line in the expression for Ω^{i+1} above. Abbreviating the second line by r we obtain

$$\begin{aligned} \Omega^{i+1} &= \Omega^i + d_B^E\delta_B^E\beta^{i+1} + r, \quad \text{or} \\ \Omega^{i+1} &= \Omega^i + \Delta_B^E\beta^{i+1} - \delta_B^E d_B^E\beta^{i+1} + r. \end{aligned} \tag{5.4}$$

The sum of the first two terms of the right-hand side above vanishes by the definition (3.4) of β^{i+1}. It suffices to estimate the last two terms.

Assuming the Main Lemma proved up to index i, we can choose the constant A' small enough such that

$$\|\omega^0 - \omega^i\|_{2,m} \leqslant c\left(\|\kappa\|_{1,\infty} + \|\Omega\|_{1,\infty}\right), \tag{5.5}$$

and (5.3) implies

$$\|r\|_{1,m} < c\left(\|\kappa\|_{1,\infty} + \|\Omega\|_{1,\infty}\right)\|\Omega^i\|_{1,m}. \tag{5.6}$$

To estimate $\gamma = \delta_B^E d_B^E\beta^{i+1}$ we define

$$d_B^E\gamma = \varphi, \qquad \delta_B^E\gamma = \psi, \tag{5.7}$$

and estimate the basic forms φ and ψ. Now

$$\Psi = \delta_B^E\gamma = \delta_B^E\delta_B^E d_B^E\beta^{i+1} = *\left(R^D \wedge *d_B^E\beta^{i+1}\right),$$

where R^D is the curvature of the connection D on E. We are only interested in the norm of this term, for which it follows that

$$\|\psi\|_{1,m} \leqslant c\|\Omega\|_{1,\infty}\|\Omega^i\|_{1,m}. \tag{5.8}$$

To estimate φ we write $\varphi = d_B^E\gamma = -d_B^E\Omega^{i+1} + d_B^E r$. Using (5.5) again together with the Bianchi equation for Ω^{i+1} we obtain

$$\|\Omega\|_{0,m} \leqslant c\left(\|\kappa\|_{1,\infty} + \|\Omega\|_{1,\infty}\right)\|\Omega^i\|_{1,m} + \left\|d_B^E r\right\|_{0,m}. \tag{5.9}$$

Now, (5.6), (5.8), and (5.9) imply

$$\|\varphi\|_{0,m} + \|\psi\|_{0,m} < c\left(\|\kappa\|_{1,\infty} + \|\Omega\|_{1,\infty}\right)\|\Omega^i\|_{1,m}, \tag{5.10}$$

and Proposition (4.8) together with an elliptic regularity estimate implies

$$\|\gamma\|_{1,m} \leqslant c\left(\|\kappa\|_{1,\infty} + \|\Omega\|_{1,\infty}\right)\|\Omega^i\|_{1,m}. \tag{5.11}$$

This proves assertion (i) of the Main Lemma.

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UNIVERSITY OF ILLINOIS, URBANA—CHAMPAIGN
OHIO STATE UNIVERSITY