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ON THE DIFFEOMORPHISM TYPES OF CERTAIN ALGEBRAIC SURFACES. I

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Introduction

Using the moduli space of anti-self-dual connections on $SU(2)$-bundles, Donaldson has introduced new invariants for closed, smooth 4-manifolds. The invariant of interest to us here is defined for simply connected, oriented 4-manifolds M of type $(1, n)$ for any $n \geqslant 1$ (type $(1, n)$ meaning that the self-intersection form q_M: $H^2(M; \mathbf{Z}) \to \mathbf{Z}$ defined by $q_M(x) = \int_M x \cup x$ is

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isomorphic to the form $x_1^2 - y_1^2 - \cdots - y_n^2$ on $\mathbf{Z}^{n+1}$). Any two simply connected 4-manifolds of type $(1, n)$ are homotopy equivalent, and consequently h-cobordant [22]. By Freedman's result [13], they are also homeomorphic. As a first application of his invariant, Donaldson showed [8] that not all such manifolds are diffeomorphic. This provides the first example of simply connected h-cobordant manifolds which are not diffeomorphic. More explicitly, he showed that two well-known simply connected algebraic surfaces of type $(1, 9)$ are not diffeomorphic.

We describe briefly these examples. Begin with the complex projective plane, $\mathbf{P}^2$. Blow up the 9 points of intersection of two generic cubics. Call the resulting algebraic surface X. It is elliptic in the sense that there is an algebraic map $\pi\colon X \to \mathbf{P}^1$ with generic fiber an elliptic curve. Being the result of blowing $\mathbf{P}^2$ up at 9 points, X is diffeomorphic to $\mathbf{P}^2$ connected sum with 9 copies of $\overline{\mathbf{P}}^2$, the complex projective plane with the orientation reversed. Thus, X is simply connected and of type $(1, 9)$.

Now suppose we have a pair of natural numbers (p, q). We form a new algebraic surface $S(p, q)$. It is obtained from X by performing log transforms at two of the smooth fibers of π, one of order p and the other of order q. There is an algebraic map $\pi\colon S(p, q) \to \mathbf{P}^1$ with generic fiber an elliptic curve and with two multiple fibers of multiplicities p and q. The diffeomorphism type of $S(p, q)$ depends only on the unordered pair (p, q). Dolgachev [6] showed that if g.c.d.$(p, q) = 1$, then $S(p, q)$ is simply connected and of type $(1, 9)$. Also, $S(1, q)$ is diffeomorphic to X for $q \geqslant 1$. Thus, we shall use the term *Dolgachev surface* to mean an algebraic surface of the form $S(p, q)$ for p and q relatively prime and $p, q > 1$.

Donaldson's example is then the following: $S(2, 3)$ and X are not diffeomorphic.

Our first result is a strengthening of Donaldson's. It is proved using the same invariant.

Theorem 1. (a) *There is a function $n(p, q)$ from unordered pairs of relatively prime natural numbers greater than* 1 *to the natural numbers satisfying*

(i) $n(p, q) \geqslant pq - p - q$.

(ii) $n(2, q) = q^2 - 2$.

(iii) *If $S(p, q)$ and $S(p', q')$ are diffeomorphic Dolgachev surfaces, then* $n(p, q) = n(p', q')$.

(b) *No Dolgachev surface $S(p, q)$ is diffeomorphic to the rational surface X.*

Corollary 2. (a) *The function from unordered pairs of relatively prime integers greater than* 1 *to diffeomorphism classes of* 4*-manifolds given by $(p, q) \mapsto \{$class of $S(p, q)\}$ is finite-to-one. In particular, there are infinitely many algebraic surfaces all homotopy equivalent to X, no two of which are diffeomorphic.*

(b) *If the Dolgachev surfaces $S(2, q)$ and $S(2, q')$ are diffeomorphic, then $q = q'$.*

Corollary 2(b) has been obtained independently by Okonek and Van de Ven [25]. They calculate the appropriate moduli spaces of stable bundles by an argument which is very similar to our Part Two, §§3–4. However, they bypass the analysis of the chamber structure which we give in Chapter II, essentially by deducing the necessary information about chambers from the existence of the Donaldson invariant.

N.B. It is natural to conjecture that if the Dolgachev surfaces $S(p, q)$ and $S(p', q')$ are diffeomorphic, then $(p, q) = (p', q')$ as unordered pairs.

Corollary 2 contrasts markedly with a result in dimensions $\geqslant 5$ which says that specifying the homotopy type and the Pontrjagin classes of a smooth, simply connected manifold determines its diffeomorphism type up to a finite number of possibilities [29]. (Recall that for a 4-manifold M^4, $p_1(M^4) = 3(\text{signature}(M))$, so that specifying the homotopy type of M^4 specifies its Pontrjagin class.) Thus, according to Corollary 2, differentiable classification of simply connected smooth 4-manifolds differs qualitatively from the classification of higher dimensional manifolds.

These results are stable under blowing up. Recall that if Y is a complex surface and if $\{p_1, \cdots, p_r\}$ are distinct points in Y, then there is a complex surface $\tilde{Y}$ and an analytic map $\rho\colon \tilde{Y} \to Y$ which induces an isomorphism $\tilde{Y} - \rho^{-1}(\bigcup_i p_i) \to Y - \bigcup_i p_i$ and for which $\rho^{-1}(p_i)$ is isomorphic to $\mathbf{P}^1$, $1 \leqslant i \leqslant r$. These $\mathbf{P}^1$-fibers are called the *exceptional fibers* of ρ. They have self-intersection equal to -1. This process, which is unique up to isomorphism, is called "blowing Y up at $\{p_1, \cdots, p_r\}$," or less precisely, "blowing Y up r times." The surface $\tilde{Y}$ is diffeomorphic to $Y \# r\overline{\mathbf{P}}^2$. In particular, the diffeomorphism type of $\tilde{Y}$ depends only on that of Y and on r. Furthermore, if Y is of type $(1, n)$ then $\tilde{Y}$ is of type $(1, n + r)$.

Here is the result which says that Theorem 1 is stable under blowing up.

Theorem 3. *Let $r > 0$. Let $\tilde{S}$ and $\tilde{S}'$ be blow ups at r points of Dolgachev surfaces $S = S(p, q)$ and $S' = S(p', q')$. If $\tilde{S}$ and $\tilde{S}'$ are diffeomorphic, then $n(p, q) = n(p', q')$. Furthermore, $\tilde{S}$ is not diffeomorphic to a rational surface.*

As an immediate corollary we have

Corollary 4. *If $\tilde{S}(2, q)$ is diffeomorphic to $\tilde{S}(2, q')$, then $q = q'$.*

Since $\tilde{S}(p, q)$ is of type $(1, 9 + r)$, Theorem 3 implies that for every $n \geqslant 9$, there are infinitely many distinct diffeomorphism classes of simply connected manifolds of type $(1, n)$ (i.e. of manifolds homotopy equivalent to $\mathbf{P}^2 \# n\overline{\mathbf{P}}^2$). In view of Freedman's theorem [13], we can formulate this as follows: for every $n \geqslant 9$, the topological manifold $\mathbf{P}^2 \# n\overline{\mathbf{P}}^2$ admits countably many distinct smooth structures.

Theorem 3 contrasts markedly with a stabilization result due independently to Mandelbaum and Moishezon: if Y is a complex surface, denote by $\hat{Y}$ the anti-complex blow up of Y, so that $\hat{Y}$ is diffeomorphic to $Y\#\mathbf{P}^2$. Then for every Dolgachev surface $S(p,q)$, $\hat{S}(p,q)$ is diffeomorphic to $\hat{X}$, see [20] and [23]. Thus, while any number of complex blow ups preserve the C^∞ distinction between the $S(p,q)$ (roughly speaking), a single anti-complex blow up destroys all such differences. (There is an earlier, more general result of Wall [33], which says that if M and M' are homotopy equivalent simply connected 4-manifolds then after enough connected sums with $\mathbf{P}^2$ and $\overline{\mathbf{P}}^2$ they become diffeomorphic.)

One consequence of Theorem 3 is a finiteness result for moduli spaces of certain algebraic surfaces which we may state very loosely as follows.

Corollary 5. *Fix a simply connected 4-manifold M of type $(1,n)$ for any $n \geqslant 0$. The moduli space of all algebraic surfaces diffeomorphic to M has only finitely many components.*

The techniques that we use to rule out the existence of diffeomorphisms between different $S(p,q)$'s apply equally well to limit the self-diffeomorphisms of a single $S(p,q)$. For any simply connected 4-manifolds M, let $A(M)$ be the group of automorphisms of $H^2(M;\mathbf{Z})$ that preserve the self-intersection form. Let $\mathrm{Diff}^+(M)$ denote the group of orientation-preserving diffeomorphisms of M. Note that if M is of type $(1,n)$, $n \neq 1$, then $\mathrm{Diff}^+(M) = \mathrm{Diff}(M)$. There is a natural homomorphism $\mathrm{Diff}^+(M) \to A(M)$. We denote the image by $D(M)$. By a recent result of Quinn, diffeomorphisms have the same image in $A(M)$ if and only if they are homotopic, or equivalently pseudo-isotopic. The relation of these notions to C^∞ isotopy is not understood in dimension 4.

Theorem 6. *For any Dolgachev surface $S(p,q)$, let*

$$A_f(S(p,q)) \subset A(S(p,q))$$

be the subgroup of elements leaving invariant the subset $\{\pm[f]\}$ in $H^2(S(p,q);\mathbf{Z})$, where $[f]$ is the cohomology class Poincaré dual to a generic fiber of π: $S(p,q) \to \mathbf{P}^1$. Then $D(S(p,q))$ is a subgroup of finite index in $A_f(S(p,q))$ which itself is of infinite index in $A(S(p,q))$.

N.B. We do not determine $D(S(p,q))$ completely, but Theorem 6′ of Chapter III is a refinement of this result and gives an explicit subgroup of finite index in $A_f(S(p,q))$ that is contained in $D(S(p,q))$.

Once again this result is qualitatively different from results in higher dimensions which say that for a simply connected n-manifold, $n \geqslant 5$, the group of homotopy classes, or isotopy classes, of diffeomorphisms is commensurate via the obvious map with the group of automorphisms of the homotopy type preserving the Pontrjagin classes, see [29]. (The group of automorphisms of the homotopy type of a simply connected 4-manifold M maps onto $A(M)$ with finite kernel. The Pontrjagin class is automatically preserved by any element of $A(M)$.)

This result also contrasts markedly with Wall's result [32]: $D(X) = A(X)$.

These results on the image of the diffeomorphism group in the automorphism group of cohomology are also stable under blowing up. Let $\rho\colon \tilde{S} \to S$ be a blow up of a Dolgachev surface at r points. The map $\rho^*\colon H^2(S;\mathbf{Z}) \to H^2(\tilde{S};\mathbf{Z})$ is an injection preserving the quadratic form. If we let $B \subset H^2(\tilde{S};\mathbf{Z})$ be the subgroup generated by the classes dual to the exceptional fibers of ρ then B is the orthogonal complement with respect to $q_{\tilde{S}}$ of $\rho^*(H^2(S;\mathbf{Z}))$. Identifying $H^2(S;\mathbf{Z})$ with its image under ρ^* gives an orthogonal decomposition

$$\left(H^2(\tilde{S};\mathbf{Z}),\, q_{\tilde{S}}\right) \cong \left(H^2(S;\mathbf{Z}), q_S\right) \oplus (B, q_{\tilde{S}} \mid B).$$

Let $A(B)$ denote the automorphism group of $(B, q_{\tilde{S}} \mid B)$. Then $A(B)$ is a finite group. In fact, $A(B) \cong (\mathbf{Z}/2\mathbf{Z})^r \rtimes \mathfrak{S}_r$ and acts on B preserving the r pairs of elements $\{\pm e_i\}$ of square -1 (the e_i being dual to the exceptional fibers of ρ).

Theorem 7. *Let $\tilde{S}$ be the blow up of $S = S(p,q)$ at r points. Then $D(\tilde{S}) \subset A(\tilde{S})$ preserves the decomposition $H^2(\tilde{S};\mathbf{Z}) \cong H^2(S;\mathbf{Z}) \oplus B$. In fact, we have $D(S) \times A(B) \subset D(\tilde{S}) \subset A_f(S) \times A(B)$ so that $D(\tilde{S})$ has finite index in $A_f(S) \times A(B)$.*

One consequence of this result is that $D(S)$ sits inside $D(\tilde{S})$ as a subgroup of finite index.

If $x \in H^2(M;\mathbf{Z})$ has square -1 and has Poincaré dual $PD(x) \in H_2(M;\mathbf{Z})$ represented by a differentiably embedded 2-sphere $S^2 \subset M$, then the element in $A(M)$ defined by reflection in the subspace orthogonal to x is realized by a diffeomorphism, i.e. is an element of $D(M) \subset A(M)$. Using this remark, one derives the following as a corollary to Theorem 7.

Corollary 8. *Let $S = S(p,q)$ be a Dolgachev surface, and let $\tilde{S}$ be the blow up of S at r points. Let $e_1, \cdots, e_r \in H^2(\tilde{S};\mathbf{Z})$ be the classes Poincaré dual to the exceptional fibers of $\rho\colon \tilde{S} \to S$. If $\alpha \in H^2(\tilde{S};\mathbf{Z})$ is dual to a class represented by a differentiably embedded 2-sphere and if $q_{\tilde{S}}(\alpha) = -1$ then $\alpha = \pm e_i$ for some i, $1 \leqslant i \leqslant r$. In particular, no such class exists in $H^2(S;\mathbf{Z})$.*

This leads easily to the following generalization. If $i\colon S^2 \looparrowright M$ is a generic immersion of the 2-sphere into a 4-manifold M, let $d_+(i) =$ the number of double points where the sheets meet with local intersection number $+1$.

Corollary 9. *Let S, $\tilde{S}$, $e_1, \cdots, e_r$ be as in Corollary 8. For any generic immersion $i\colon S^2 \looparrowright \tilde{S}$ representing the Poincaré dual of a class $x \in H^2(\tilde{S};\mathbf{Z})$,*

$x \neq 0$, *we have*

$$d_+(i) \geqslant \frac{q_{\bar{S}}(x)+1}{4}$$

with equality only if $x = \pm e_i$ *for some* i, $1 \leqslant i \leqslant r$.

Actually, the Donaldson invariant also gives information about the diffeomorphism group for $\mathbf{P}^2$ blown up at more than 9 points. It shows that Wall's result [32] fails to generalize.

Theorem 10. *Let* $\tilde{X}$ *be* $\mathbf{P}^2 \# n\overline{\mathbf{P}}^2$, *for some* $n \geqslant 10$.

Then $D(\tilde{X}) \subset A(\tilde{X})$ *is of infinite index.*

In fact, we give a precise description of $D(\tilde{X})$ and, as a by-product, obtain a new proof that $D(X) = A(X)$.

Finally, Wall's theorem holds for no simply connected 4-manifold of type $(1, n)$, $n > 9$.

Theorem 11. *Let* M *be a smooth, simply connected* 4-*manifold of type* $(1, n)$ *with* $n > 9$. *Then* $D(M) \subset A(M)$ *is a proper subgroup.*

Some of these results have been discussed in the note [14].

This paper consists of two parts. The first part consists of the first three chapters and the second part is Chapter IV. The first chapter sets the stage for the remainder of the paper. §1 is an exposition of the fundamental properties of the Donaldson invariant. We review the language of anti-self-dual connections and the Yang-Mills equation on 4-manifolds and describe the work of Donaldson, Taubes and Uhlenbeck which defines the Donaldson invariant and yields its formal properties. §1 concludes with a brief discussion of the relation between anti-self-dual connections and stable holomorphic vector bundles on an algebraic surface. This is the only case where one can actually calculate directly the Donaldson invariant at present. §2 describes the geometry of certain rational surfaces. The purpose of this section is two-fold. First, we describe the rational elliptic surfaces X which are used in the construction of the Dolgachev surfaces. Secondly, we collect various lemmas which will be of use in Chapter II. The literature on rational surfaces is vast and the material in §2 is well known; we have not attempted to give precise references. §3 contains the definition of Dolgachev surfaces, a discussion of their elementary properties, and an irreducibility result for their moduli spaces.

Chapter II is concerned with the arithmetic of forms of type $(1, n)$. Quite generally, let $q\colon \Lambda \to \mathbf{Z}$ be a quadratic function on the free $\mathbf{Z}$-module Λ, of finite rank. We denote the associated bilinear pairing $\Lambda \otimes \Lambda \to \mathbf{Z}$ by

$$(x \cdot y) = \tfrac{1}{2}(q(x+y) - q(x) - q(y)).$$

Definition 0.1. We say that (Λ, q) is *of type* (p, n) (resp., of *signature* (p, n)) if it is isomorphic to the form $q_{(p,n)}(x_1, \cdots, x_p, y_1, \cdots, y_n) = \sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{n} y_j^2$ (resp., if the extensions of q and $q_{(p,n)}$ obtained by tensoring with $\mathbf{R}$ are isomorphic). We will call a basis of Λ *a standard basis* if, via the associated identification of Λ with $\mathbf{Z}^r$, q is identified with $q_{(p,n)}$ for some p and n with $p + n = r$.

If q: $\Lambda \to \mathbf{Z}$ is of type $(1, n)$, then inside $\Lambda_{\mathbf{R}} = \Lambda \otimes \mathbf{R}$ we have

$$\mathbf{H}(q) = \{x \in \Lambda_{\mathbf{R}} | \bar{q}(x) = 1\},$$

where $\bar{q}$: $\Lambda_{\mathbf{R}} \to \mathbf{R}$ is the natural extension of q. Classes $\alpha \in \Lambda$ with $q(\alpha) = -1$ determine walls in $\mathbf{H}(q)$. The collection of all such walls is locally finite. This set of walls divides $\mathbf{H}(q)$ up into chambers. When $\Lambda = H^2(M; \mathbf{Z})$ and q is the self-intersection form q_M the set of chambers $\mathscr{C}(M)$ is the domain of the Donaldson invariant. In fact the Donaldson invariant is a function

$$\Gamma_M\colon \mathscr{C}(M) \to H^2(M; \mathbf{Z}).$$

§1 deals with the basics of this chamber structure. It rapidly becomes apparent that there is a fundamental distinction between forms of type $(1, n)$ for $n \leqslant 8$, $n = 9$, and $n \geqslant 10$. The case $n \leqslant 8$, and to some extent $n = 9$, is rather classical. To deal with the remaining case, it becomes necessary to introduce additional walls corresponding to what we call canonical classes. The cells of this finer decomposition, P-cells, are introduced in §2. There is an interesting twist here: the P-cells are purely arithmetically defined, but we were led to them by algebraic geometry (whence the name canonical class). We prove the basic facts about them using algebraic geometry. This is done in §§3 and 4. §5 deals with some of the deeper properties of this cell structure. Finally, in §6, we reinterpret the Donaldson invariant as a function on the chambers $\mathscr{C}(M)$, and define a modified invariant which is adapted to the finer cell structure. We conclude with a technical result which, when coupled with the appropriate stable bundle calculations, will enable us to distinguish special cells in the hyperbolic spaces contained in the cohomology groups of blown up Dolgachev surfaces purely from the differential topology of these surfaces.

After the detailed analysis of Chapter II, and assuming the stable vector bundle calculations, the proofs of the theorems stated in this introduction are surprisingly easy. These proofs occupy Chapter III. The statements about the distinct diffeomorphism types, Theorem 1 and 3, are given in §1. We also show that the formal properties of the Donaldson invariant give strong restrictions on the self-diffeomorphisms of the algebraic surfaces under study. In §2, we complete the proof of Theorems 6, 7 and 10 by giving constructions for self-diffeomorphism of blown-up Dolgachev surfaces and rational surfaces. Finally, Corollaries 5, 8 and 9 are proved in §3.

It remains to pay the piper and prove the necessary statements on stable vector bundles. The proofs of these statements are largely independent of the rest of the paper, and are deferred to the second part (Chapter IV).

It is a pleasure to acknowledge the help of many mathematicians during the course of this work. Neither of us was an expert in gauge theory or 4-manifolds, and we benefited greatly from the generosity and patience, in particular, of: Cliff Taubes, who guided us through the perilous shoals of *PDE*; Karen Uhlenbeck, for much technical assistance and a flood of ideas, which we are still struggling to comprehend; and finally Simon Donaldson, who in large measure created the mathematical framework in which this paper rests, and who shared with us many of his insights into past, present, and future theorems. Much of the research for this paper was done at MSRI at Berkeley; we found its stimulating intellectual atmosphere and its physical setting ideal. Our thanks to one and all.

Notation and conventions. If Y is a smooth algebraic variety, $\mathcal{O}_Y$ is the sheaf of germs of holomorphic functions. If $\mathcal{S}$ is a sheaf on an algebraic variety Z, then we denote by $H^i(Z; \mathcal{S})$ the sheaf cohomology groups. If Z is clear from context, we abbreviate this $H^i(\mathcal{S})$. If $\mathcal{S}$ is a coherent sheaf, we denote by $\chi(Z; \mathcal{S}) = \chi(\mathcal{S})$ the Euler characteristic of $\mathcal{S}$ and by $\chi(Z)$ the topological Euler characteristic of the space Z. If D is a divisor in Z, we denote by $\mathcal{O}_Z(D)$ the sheaf of germs of sections of the associated line bundle, and we write $H^i(D)$ for $H^i(Z, \mathcal{O}_Z(D))$. K_Z denotes the canonical divisor class of Z. We tacitly identify the group of holomorphic line bundles on an algebraic surface with the group of divisors modulo equivalence. We also identify a vector bundle with its locally free sheaf of sections. If V is a vector bundle, then $\mathbf{V}$ denotes its sheaf of sections. Despite our best efforts to be consistent here, we confess to lapses of notation. Following the standard practice in algebraic geometry, intersection of divisors C and D on a surface Y is denoted by $C \cdot D$; also $C \cdot C$ is denoted C^2. If $[C]$ and $[D]$ denote the cohomology classes Poincaré dual to the fundamental cycles of C and D, then $C \cdot D = [C] \cdot [D]$ where the product on the right-hand-side is the usual cohomological one. If Y is an algebraic surface and $H^1(\mathcal{O}_Y) = 0$, then we can identify a line bundle L over Y with its Chern class $c_1(L) \in H^2(Y; \mathbf{Z})$. If Y is of type $(1, n)$, then $p_g(Y) = \dim H^2(Y; \mathcal{O}_Y) = 0$ and every class in $H^2(Y; \mathbf{Z})$ is $c_1(L)$ for some line bundle L. Finally, a divisor D is *effective* if D can be written in the form $\sum_i n_i D_i$, where the D_i are irreducible hypersurfaces and the n_i are nonnegative integers.

List of Symbols

CHAPTER I

1. A review of the Donaldson invariant

In this section we review the analytic version of Donaldson's invariant for a smooth, simply connected, oriented 4-manifold M of type $(1, n)$. In §II.6 we shall re-interpret these results, as Donaldson did, in terms of a chamber structure on the hyperboloid inside $H^2(M; \mathbf{R})$. But in this section we stick to individual metrics and paths of metrics. All the material in this section can be found in [8], [10].

Before we can define the invariant we need to review some of the basics of the theory of connections.

There is a unique principal $SU(2)$-bundle $P \to M$ with Chern class $c_2(P) = 1$. Let $V \to M$ be the associated complex vector bundle. Let $\mathscr{A}$ be the affine space of $SU(2)$-connections on V (or equivalently connections on P). Let $\mathscr{A}' \subset \mathscr{A}$ be the open subset of irreducible connections. Let $\mathscr{G}$ be the gauge group; that is to say $\mathscr{G}$ is the group of C^∞-automorphisms of P covering the identity on M. Then $\mathscr{G}$ acts on $\mathscr{A}$ and $\mathscr{A}'$. The center of $\mathscr{G}$ is $\mathbf{Z}/2\mathbf{Z}$ and it acts trivially on $\mathscr{A}$. The quotient of $\mathscr{G}$ by its center acts freely on $\mathscr{A}'$. We denote by $\mathscr{X}$ the quotient $\mathscr{A}'/\mathscr{G}$. It has completions with respect to various Sobolev norms which are Banach manifolds. (For a more detailed discussion of all of this see [12, pp. 52–60].)

Over $M \times \mathscr{X}$ there is a universal $U(2)$-bundle $\mathscr{P}$ whose restriction to any slice $M \times \{x\}$ "is" P.

There is a so-called Taubes map (see [30])

$$T\colon M \times [0,\infty) \to \mathscr{X}$$

which associates to (x, λ) a connection on V whose curvature is concentrated near x, with the strength of concentration being a function of λ. For any $\lambda \geqslant 0$ let $T_\lambda\colon M \to \mathscr{X}$ be the restriction of T to the slice $M \times \{\lambda\}$. Here is a basic property, established by Donaldson [9]. The composition

$$H_2(M; \mathbf{Z}) \xrightarrow{(T_\lambda)_*} H_2(\mathscr{X}; \mathbf{Z}) \xrightarrow{c_2(\mathscr{P})/(\ )} H^2(M, \mathbf{Z}) \tag{1.1}$$

(where $/$ is the slant product), is the inverse of Poincaré duality.

Definition 1.2. Let g be a Riemannian metric on M. The subspace $\mathfrak{M}(g) \subset \mathscr{X}$ denotes the space of irreducible connections anti-self-dual with respect to g, modulo gauge equivalence (i.e., modulo the action of $\mathscr{G}$).

There are natural defining equations for $\mathfrak{M}(g)$ inside $\mathscr{X}$. According to [12, Theorem 3.17] there is a dense G_δ-subset G of the space of all metrics such that for each $g \in G$ the differentials of the defining equations for $\mathfrak{M}(g)$ in $\mathscr{X}$ have maximal rank. In this case $\mathfrak{M}(g)$ is a smooth orientable submanifold of $\mathscr{X}$. As a corollary of the Atiyah-Singer Index Theorem, one calculates its dimension to be 2 [12, p. 49].

Associated to g is the Hodge $*$-operator. It splits the vector bundle of 2-forms over M, Ω^2, into eigenspaces $\Omega^2 = \Omega^2_+ \oplus \Omega^2_-$. Since M is of type $(1, n)$, the space of self-dual harmonic 2-forms for g is one dimensional. We view any such form ω as a section of Ω^2_+. As long as $\omega \neq 0$, the zero set of ω depends only on g, not on the choice of ω. We call this zero set $Z_g \subset M$.

Now we are ready to describe the Donaldson invariant. It is an invariant $\tilde{\Gamma}_M(g, \omega) \in H^2(M; \mathbf{Z})$ associated to a "generic" metric g and a nonzero self-dual harmonic 2-form ω for g. Here are the conditions that the metric is required to satisfy in order to define the invariant

(1.3) $g \in G$.

(1.4) ω, considered as a section of Ω^2_+, is transverse to the 0-section.

(1.5) There is no reducible connection on V anti-self-dual with respect to g.

As we have seen, Condition (1.3) defines a dense G_δ-subset of the space of all metrics. Conditions (1.4) and (1.5) define an open dense subset of metrics. Thus, the metrics satisfying all three conditions form a dense G_δ-subspace in the space of all metrics.

Let us examine Condition (1.5) in more detail. If V splits, then, since $c_1(V) = 0$, we have $V \cong L \oplus L^{-1}$. Consequently, $1 = c_2(V) = -q_M(c_1(L))$. For V to admit a connection anti-self-dual with respect to g is for L to admit such a connection. This can happen if and only if all self-dual harmonic forms for q are orthogonal to $c_1(L)$. Thus, Condition (1.5) is simply the statement that the line of self-dual harmonic 2-forms for g, thought of as a line in $H^2(M; \mathbf{R})$, is not orthogonal to any class $\alpha \in H^2(M; \mathbf{Z})$ with $q_M(\alpha) = -1$.

Fix a metric g satisfying (1.3), (1.4), and (1.5) and fix a nonzero self-dual harmonic 2-form ω for g. Because $g \in G$, $\mathfrak{M}(g)$ is a smooth 2-manifold. The choice of ω determines an orientation for $\mathfrak{M}(g)$. Since g satisfies (1.4) $Z_g \subset M$ is a smooth 1-manifold. Let ν be a regular neighborhood of Z_g in M. The form ω determines a nowhere zero section of $\Omega^2_+ \,|(M - \operatorname{int} \nu)$. The orthogonal 2-plane bundle $(\omega^\perp)$ is naturally oriented. Let $K(g, \omega) \in H^2(M - \operatorname{int} \nu)$ be its Euler class, and let $k(g, \omega) \in H_2(M - \operatorname{int} \nu, \partial\nu; \mathbf{Z})$ be the Poincaré dual. This class is represented by the zeros of a generic section of $(\omega^\perp)$. A local computation near Z_g shows that $\partial k(g, \omega) \in H_1(\partial\nu; \mathbf{Z})$ has intersection 2 with any 2-sphere fiber of the projection $\partial\nu \to Z_g$.

It turns out that $\mathfrak{M}(g)$ is noncompact if and only if $Z_g \neq \varnothing$. Furthermore, $\mathfrak{M}(g)$ is asymptotic in $\mathscr{X}$ to $T(Z_g \times [0, \infty))$, see [30] and [31].

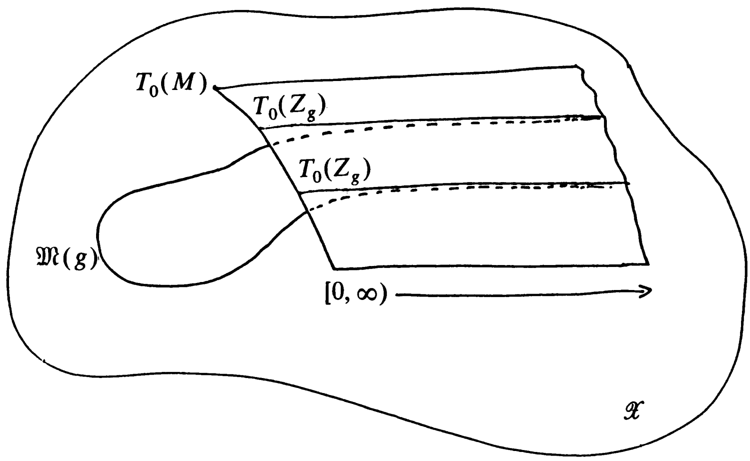

FIGURE 1

Thus, if we truncate $\mathfrak{M}(g)$ near $T_\lambda(M)$ we have a compact 2-chain $m^\lambda(g, \omega)$ whose boundary is nearly $T_\lambda(Z_g)$, provided that λ is sufficiently large. Because of this we can "glue" 2 copies of $m^\lambda(g, \omega)$ to $T_\lambda(k(g, \omega))$ to form a 2-cycle $\gamma(g, \omega)$ in $\mathscr{X}$. The homology class of $\gamma(g, \omega)$ is independent of the choice of λ and of the small deformation of $m^\lambda(g, \omega)$ required to make twice its boundary cancel $\partial T_\lambda(k(g, \omega))$. (For more details on all of this see [10].)

Definiton 1.6. Let g satisfy (1.3), (1.4), and (1.5), and let ω be a nonzero self-dual harmonic 2-form for g. We define the *Donaldson invariant*

$$\tilde{\Gamma}_M(g, \omega) = c_2(\mathscr{P})/\gamma(g, \omega).$$

If we fix $g \in G$ but replace ω by $t\omega$, for some $t \in \mathbf{R}^*$, then $k(g, \omega)$ and $m^\lambda(g, \omega)$ change by sign (t). Hence

$$\tilde{\Gamma}_M(g, t\omega) = \operatorname{sign}(t) \cdot \tilde{\Gamma}_M(g, \omega) \tag{1.7}$$

for any $t \in \mathbf{R}^*$.

Now suppose that ω is nowhere zero, i.e. that $Z_g = \varnothing$. Then $\mathfrak{M}(g)$ is compact, and $m^\lambda(g, \omega) = \mathfrak{M}(g)$ for all λ sufficiently large. Also, $K(g, \omega) \in H^2(M; \mathbf{Z})$ and $k(g, \omega) \in H_2(M; \mathbf{Z})$ are Poincaré dual.

Claim 1.8. In this case

$$\tilde{\Gamma}_M(g, \omega) = K(g, \omega) + 2\mu(g, \omega),$$

where $\mu(g, \omega) = c_2(\mathscr{P})/[\mathfrak{M}(g)]$. (*N.B.* The orientation of $\mathfrak{M}(g)$ depends on ω.)

Proof. Clearly, $\gamma(g, \omega) = T_\lambda(k(g, \omega)) + 2\mathfrak{M}(g)$ (as cycles).

Thus, to establish the claim we need only see that

$$K(g, \omega) = c_2(\mathscr{P})/T_\lambda(k(g, \omega)).$$

Since $K(g, \omega)$ is Poincaré dual to $k(g, \omega)$, this is exactly the formula given in (1.1). □

Corollary 1.9. *With assumptions as above*

$$\tilde{\Gamma}_M(g, \omega) = K(g, \omega) + 2(p_1)_*(c_2(\mathscr{P}|(M \times \mathfrak{M}(g)))),$$

where p_1: $M \times \mathfrak{M}(g) \to M$ is projection onto M.

Proof. In light of (1.8) we need only see that

$$\mu(g, \omega) = (p_1)_*(c_2(\mathscr{P}|(M \times \mathfrak{M}(g)))).$$

This is clear from the definition of the slant product. □

Let us turn now to the question of how $\tilde{\Gamma}_M$ varies as we vary the metric g and the form ω along a path. Fix a smooth path (g_t, ω_t), $0 \leqslant t \leqslant 1$, where ω_t is a nonzero self-dual harmonic form for g_t. (To say we have a smooth path means that both g_t and ω_t vary smoothly with t.) Suppose that g_0 and g_1 satisfy conditions (1.3), (1.4), and (1.5), so that $\tilde{\Gamma}_M(g_0, \omega_0)$ and $\tilde{\Gamma}_M(g_1, \omega_1)$ are both defined. By deforming the path (g_t, ω_t) slightly, relative to its endpoints, we can assume that for all but finitely many values of t, say $t_1, \cdots, t_r$, Condition (1.5) holds for g_t, and furthermore that for each exceptional t_i the following two properties hold:

(1.10)
(a) There is a unique class $\alpha_i \in H^2(M; \mathbf{Z})$, up to sign, with $q_M(\alpha_i) = -1$ and with $\omega_{t_i} \cdot \alpha_i = 0$.
(b) If $\psi_i(t) = \omega_t \cdot \alpha_i$, then $d\psi_i(t_i)/dt \neq 0$.

By choosing the sign of α_i appropriately we arrange that the derivatives in (b) are all negative. With this normalization of the α_i, according to [10] we have

$$\tilde{\Gamma}_M(g_1, \omega_1) - \tilde{\Gamma}_M(g_0, \omega_0) = -2 \sum_{i=1}^{r} \alpha_i. \tag{1.11}$$

This formula has the following consequence.

(1.12) If the cohomology class of ω_1 is a positive multiple of the cohomology class of ω_0, then $\tilde{\Gamma}_M(g_0, \omega_0) = \tilde{\Gamma}_M(g_1, \omega_1)$.

Actually, there is a more general version of (1.12). Suppose that we have a path (g_t, ω_t), $0 \leqslant t \leqslant 1$, and suppose that both g_0 and g_1 satisfy (1.3), (1.4), and (1.5). Suppose that for each class $\alpha \in H^2(M; \mathbf{Z})$ with $q_M(\alpha) = -1$ the signs of $\omega_0 \cdot \alpha$ and $\omega_1 \cdot \alpha$ are the same.

Proposition 1.13. *Under the above assumptions*

$$\tilde{\Gamma}_M(g_0, \omega_0) = \tilde{\Gamma}_M(g_1, \omega_1).$$

Proof. Deform the path slightly until (1.10) holds. Because of the assumption on ω_0 and ω_1, the exceptional values t_i must pair up in such a fashion that the classes α_i in (1.10) in each pair are negatives of each other. Thus, Formula

(1.11) collapses to

$$\tilde{\Gamma}_M(g_1, \omega_1) = \tilde{\Gamma}_M(g_0, \omega_0). \quad \square$$

Proposition 1.13 allows us to extend the definition of $\tilde{\Gamma}_M$ to all pairs (g, ω) where g satisfies (1.5) and ω is a nonzero self-dual harmonic 2-form for g. Namely given such a (g, ω), we approximate it by (g', ω') where g' satisfies (1.3), (1.4), and (1.5). By (1.13) the value $\tilde{\Gamma}_M(g', \omega')$ will be independent of the approximation, provided only that the approximation is close enough. We define $\tilde{\Gamma}_M(g, \omega)$ to be the value $\tilde{\Gamma}_M(g', \omega')$ for any sufficiently close approximation.

Here is a summary of the basic properties of this extended function

(1.14)
(a) $\tilde{\Gamma}_M(g, \omega)$ is defined for all pairs (g, ω) for which g satisfies (1.5) and ω is a nonzero self-dual harmonic 2-form for g.
(b) $\tilde{\Gamma}_M(g, \omega)$ is locally constant.
(c) $\tilde{\Gamma}_M(g, \omega)$ depends only on the "period point" $[\omega] \in (H^2(M; \mathbf{R}) - \{0\})/\mathbf{R}^+$.

Properties (1.14) allow us to view $\tilde{\Gamma}_M$ as a function Γ_M on the set of chambers cut out of the hyperbolic space $\{x \in H^2(M; \mathbf{R}) \mid q_M(x) = 1\}$ by the walls orthogonal to classes $\alpha \in H^2(M; \mathbf{Z})$ with $q_M(\alpha) = -1$. We give this re-interpretation in §II.6 after we have developed the theory of the associated chamber structure.

We have described in detail the formal properties of the Donaldson invariant $\tilde{\Gamma}_M$, but we have given no indication of how to actually evaluate it. In fact the only examples where $\tilde{\Gamma}_M(g, \omega)$ may be computed directly are when M is an algebraic surface, g is a Hodge metric and ω is its Kähler form. In this case the computation reduces to a computation in algebraic geometry. Enough machinery has been developed to allow one to carry out such computations, at least some of the time. Here, we give the description of $\tilde{\Gamma}_M(g, \omega)$ in the Kähler case in terms of purely algebro-geometric computations.

Let Y be a complex algebraic surface, g a Hodge metric on Y associated to an ample line bundle L over Y, and ω the Kähler form for g.

We begin with some notational conventions. If V is a holomorphic vector bundle, we shall always denote by $\mathbf{V}$ its locally free sheaf of sections.

Definition 1.15. A *subline bundle* of V is given by a line bundle F, together with a nonzero map of $\mathcal{O}_Y$-modules

$$\varphi\colon \mathbf{F} \to \mathbf{V}.$$

Giving φ is the same as giving an algebraic map $F \to V$ which commutes with the projection to Y and is linear on the fibers and nonzero on the generic fiber.

Definition 1.16. Let L be an ample divisor on Y. We say that V is *L-stable* (resp. *L-semistable*) if, for all torsion-free subsheaves $\mathbf{F}$ of $\mathbf{V}$, with $0 < \operatorname{rank}\mathbf{F} < \operatorname{rank}\mathbf{V}$, we have

$$c_1(\mathbf{F}) \cdot c_1(V) < \frac{1}{k} c_1(L) \cdot c_1(V)$$

$$\left(\text{resp.} \leqslant \frac{1}{k} c_1(L) \cdot c_1(V)\right),$$

where $k = \operatorname{rank} V$. If $k = 2$, it is well known that one need only check these conditions for subline bundles F.

We now specialize to the case of interest here, i.e. V of rank 2 with $c_1(V) = 0$ and $c_2(V) = 1$.

Proposition 1.17 (*Maruyama* [21]). *If* $\chi(\mathcal{O}_Y) = 1$, *then there is a fine moduli space* $\mathfrak{M}$ *for L-stable rank* 2 *bundles over* Y *with* $c_1 = 0$ *and* $c_2 = 1$. *It is naturally a scheme whose points are in one-to-one correspondence with such bundles. There is a universal bundle*

$$\mathscr{V} \to Y \times \mathfrak{M}. \quad \square$$

N.B. Just as there are natural defining equations for $\mathfrak{M}(g) \subset \mathscr{X}$, there are (locally) natural defining equations for $\mathfrak{M}$ which give it a scheme structure.

There is no reason to expect $\mathfrak{M}$ to be reduced. We denote by $\mathfrak{M}_{\text{red}}$ its reduction. For each component $\mathfrak{M}_i$ of $\mathfrak{M}$ we have the reduction $(\mathfrak{M}_i)_{\text{red}}$. We denote by n_i the length of the generic point of $\mathfrak{M}_i$. Of course, $n_i \geqslant 1$ and $n_i = 1$ if and only if $\mathfrak{M}_i$ is generically reduced.

Definition 1.18. For each component $\mathfrak{M}_i$ which is compact and of complex dimension 1 we define

$$\mu_i = (p_1)_* c_2(\mathscr{V}_i) \in H^2(Y; \mathbf{Z}),$$

where p_1: $Y \times (\mathfrak{M}_i)_{\text{red}} \to Y$ is the projection and $\mathscr{V}_i$ is the restriction of $\mathscr{V}$ to $Y \times (\mathfrak{M}_i)_{\text{red}}$.

Theorem 1.19 (*Donaldson* [10]). *Suppose that* Y *is a simply connected algebraic surface with* $\chi(\mathcal{O}_Y) = 1$. *Suppose that g is the Hodge metric associated to an ample line bundle L. Let* ω *be the Kähler form for g. Suppose* $[\omega] \in H^2(Y; \mathbf{R})$ *is not perpendicular to any class* $\alpha \in H^2(Y; \mathbf{Z})$ *with* $q_Y(\alpha) = -1$. *Then* $\mathfrak{M}$ *is compact. If each component* $(\mathfrak{M}_i)_{\text{red}}$ *of* $\mathfrak{M}$ *has complex dimension* 1, *then*

$$\tilde{\Gamma}_Y(g, \omega) = K_Y + 2\sum_i n_i \mu_i,$$

where K_Y *is the canonical class of* Y. $\square$

This theorem results from Donaldson's identification of anti-self-dual connections on V for g with L-stable algebraic structures on V, see [7]. (This last result has recently been generalized by Uhlenbeck-Yau.)

2. On the geometry of certain rational surfaces

In this section, we review the construction of rational elliptic surfaces. We also describe certain other rational surfaces whose geometry will be useful in understanding the fundamental domains of various reflection groups.

Definition 2.1. Let X be a simply connected algebraic surface. Call X *good* if we have an equality of divisor classes

$$K_X = -F,$$

where F is a smooth elliptic curve on X. Call X *generic* if, in addition, there does not exist a smooth rational curve $C \subseteq X$ with $C^2 = -2$.

Lemma 2.2. *Let X be a good generic surface. If C is an irreducible curve on X, $C \neq F$, and $C^2 < 0$, then C is an exceptional curve. Hence if C is an irreducible curve on X and $C^2 < -1$, then $C = F$.*

Proof. Let C be an irreducible curve distinct from F. By the adjunction formula for the arithmetic genus $p_a(C)$ [15, p. 471],

$$-2 \leqslant 2p_a(C) - 2 = C^2 + C \cdot K_X = C^2 - C \cdot F.$$

So, if $C^2 < 0$, as $C \cdot F \geqslant 0$ and $2p_a(C) - 2$ is even, either $C \cdot F = 0$, $C^2 = -2$ and $p_a(C) = 0$, or $C^2 = -1$, $C \cdot F = +1$, and $p_a(C) = 0$. In the first case, C is smooth rational, which is excluded, and in the second case C is exceptional. $\square$

Lemma 2.3. *Let X be a good generic surface and ρ: $X \to Y$ the contraction of an exceptional curve E to a point. Then Y is a good generic surface.*

Proof. By the adjunction formula $E \cdot F = 1$. Thus, $\rho(F) = \overline{F}$ is again a smooth elliptic curve. According to [15, p. 187], we have $K_X = \rho^*K_Y + E = -F = -\rho^*\overline{F} + E$. (The last equality follows since $E \cdot F = 1$.) Since ρ^* is injective on divisor classes, $-\overline{F} = K_Y$, and Y is good. If Y fails to be generic, there exists a smooth rational curve C in Y with $C^2 = -2$. If C' is the proper transform of C on X, then $(C')^2 \leqslant C^2 = -2$, contradicting (2.2). $\square$

Proposition 2.4. *Let X be a good generic surface. Then either $X \cong \mathbf{P}^1 \times \mathbf{P}^1$ or there exists a birational map ρ: $X \to \mathbf{P}^2$ such that ρ is obtained by contracting n disjoint exceptional curves on X.*

Proof. By the Castelnuouvo-Enriques theorem [15, p. 536], X is a rational surface. Hence, by the general theory of rational surfaces (e.g. [2, p. 191]) there exists a birational map ρ: $X \to \mathbf{P}^2$ or ρ: $X \to \mathbf{F}_n$, $n \neq 1$ and $n \geqslant 0$, where $\mathbf{F}_n$ is a minimal rational ruled surface over $\mathbf{P}^1$ characterized by the existence of a section σ with $(\sigma)^2 = -n$. Furthermore, ρ can be factored as a sequence of maps each of which contracts an exceptional curve.

If X dominates $\mathbf{F}_n$, then the proper transform on X of the section $\sigma \subset \mathbf{F}_n$ has square $\leqslant -n$. If X is good and generic, this forces $n \leqslant 1$, hence $n = 0$.

The surface $\mathbf{F}_0$ is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. If Y is obtained from $\mathbf{P}^1 \times \mathbf{P}^1$ by blowing up a point, then Y dominates $\mathbf{P}^2$. Hence, if X is not isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ and if X dominates $\mathbf{P}^1 \times \mathbf{P}^1$, then X dominates $\mathbf{P}^2$. Thus, if X is a good generic surface, then either X dominates $\mathbf{P}^2$ or $X \cong \mathbf{P}^1 \times \mathbf{P}^1$.

Finally, we show that if X is generic and ρ: $X \to \mathbf{P}^2$ is a birational morphism, then the positive dimensional fibers of ρ are disjoint smooth rational curves of self-intersection -1. We may factor ρ:

$$X = X_n \xrightarrow{\rho_n} X_{n-1} \xrightarrow{\rho_{n-1}} X_{n-2} \to \cdots \xrightarrow{\rho_1} X_0 = \mathbf{P}^2,$$

where ρ_i: $X_i \to X_{i-1}$ realizes X_i as the blow up of a point $p_i \in X_{i-1}$, $E_i = \rho_i^{-1}(p_i)$ is the exceptional divisor, and $\rho = \rho_1 \circ \cdots \circ \rho_n$. If, for some $j < i$, p_i lies on the proper transform of E_j, then the proper transform of E_j in X has self-intersection $\leqslant -2$, violating (2.2). $\square$

Remark 2.5. (a) By the adjunction formula, the requirement that F be an elliptic curve in (2.1) is implied by the smoothness of F.

(b) The only $\mathbf{F}_n$, $n \geqslant 0$ and $n \neq 1$, which are good are $\mathbf{F}_0$ and $\mathbf{F}_2$. This is a straightforward calculation with the canonical bundle formula for $\mathbf{F}_n$ (see [15, p. 519]).

Proposition 2.6. *Let X_n denote the blow up of $\mathbf{P}^2$ at n points $p_1, \cdots, p_n$. For a suitable choice of the p_i, X_n is a good generic surface.*

Proof. We begin by recalling the following well-known result.

Lemma 2.7. *Let H be the class of a line on $\mathbf{P}^2$, and ρ: $X_n \to \mathbf{P}^2$ the blow up of $\mathbf{P}^2$ at points $p_1, \cdots, p_n$. If $E_i = \rho^{-1}(p_i)$ is the exceptional curve over p_i, then*:

(a) *The classes $\{\rho^*H, E_1, \cdots, E_n\}$ form a standard basis for $H^2(X; \mathbf{Z})$ and exhibit X as a surface of type $(1, n)$ in the notation of the introduction.*

(b) *In the above basis*

$$K_X = -3\rho^*H + E_1 + \cdots + E_n. \quad \text{q.e.d}$$

Returning to the proof of (2.6), first choose all points p_i on a smooth cubic $F \subseteq \mathbf{P}^2$. By abuse of language, we will also denote by F the proper transform of F on X. Then

$$F = 3\rho^*H - E_1 - \cdots - E_n = -K_X,$$

so that X is good. Let C be any irreducible curve on X. Then, for appropriate integers k and a_i, we have an equality of divisor classes

$$C = k\rho^*H + \sum_i a_iE_i.$$

Since C is not homologous to 0, not all of the integers k, a_i are 0.

Let $\operatorname{Pic} F$ denote the group of line bundles on F. As a real Lie group, it is isomorphic to $(\mathbf{R}^2/\mathbf{Z}^2) \times \mathbf{Z}$, where projection onto the second factor is given by the degree of the line bundle. Choosing a point $p_0 \in F$ as origin, there is a natural identification of F with the subgroup of line bundles of degree 0, given by

$$p \in F \to \mathcal{O}_F(p - p_0).$$

Now $\mathcal{O}_X(C)|_F \cong \mathcal{O}_{\mathbf{P}^2}(kH)|_F \otimes \mathcal{O}_F(\Sigma a_ip_i)$, where we identify F with its image in $\mathbf{P}^2$ under ρ. Hence, if $p_1, \cdots, p_n$ are chosen generically, then $\mathcal{O}_X(C)|_F$ is not the trivial line bundle.

On the other hand, suppose that C is smooth rational and $C^2 = -2$. It follows from the adjunction formula that $C \cap F = \varnothing$. Hence

$$\mathcal{O}_X(C)|_F = \mathcal{O}_F \text{ is the trivial bundle,}$$

contrary to the choice of $p_1, \cdots, p_n$. □

To sum up, then

Corollary 2.8. *For every $n \geqslant 0$, there exists a good generic surface of type $(1, n)$. The only good generic surface which is not of type $(1, n)$ (i.e., whose form is not diagonalizable) is $\mathbf{P}^1 \times \mathbf{P}^1$, whose form is even (a hyperbolic plane).* □

Remark 2.9. If the rank of $H^2(X; \mathbf{Z})$ is $\leqslant 8$, it is easy to show that a good generic X is the same as a *del Pezzo surface*, i.e. a surface X such that $-K_X$ is ample.

We now introduce a special kind of rational surface.

Definition 2.10. A surface S is an *elliptic surface* if there exists a holomorphic map

$$\pi: S \to C,$$

with C a smooth curve, such that the general fiber $\pi^{-1}(z)$ is a smooth elliptic curve, and such that there does not exist an exceptional curve $E \subseteq S$ with $E \subseteq \pi^{-1}(x)$ for some $x \in C$. (Some authors term such an S a minimal elliptic surface.)

Definition 2.11. A rational surface X is a *generic rational elliptic surface* if X is an elliptic surface in the sense of (2.10) which is also a good generic surface with $K_X = -F$, F a general fiber of π.

Remark 2.12. If X is a rational elliptic surface with $K_X = -F$, then the base of the elliptic fibration, C in the notation of (2.10), is $\mathbf{P}^1$. All such arise from blowing $\mathbf{P}^2$ up along the nine points of intersection of two cubics. Finally, it is easy to see from the adjunction formula that X is generic if and only if all its fibers are irreducible.

Lemma 2.13. *Generic rational elliptic surfaces exist.*

Proof. Let F_0 and F_∞ be the equations of two smooth cubics in $\mathbf{P}^2$ and consider the pencil of cubics defined by F_0 and F_∞. By definition this is the set of cubics $\{C_t\colon t \in \mathbf{P}^1\}$ defined by $C_t = \{F_0 + tF_\infty = 0\}$. Suppose that C_0 and C_∞ meet transversally. Then, after blowing up $\mathbf{P}^2$ at the set $\{F_0 = F_\infty = 0\}$, we obtain an elliptic surface, $\pi\colon X \to \mathbf{P}^1$. X is clearly good since the points lie on C_0 so that $K_X = -C_0$, say. To insure that X is also generic, it suffices to arrange that all fibers of π are reduced and irreducible. This follows if, for example, $\{C_t\}$ is a Lefschetz pencil [**1**]. □

3. Dolgachev surfaces

We describe the construction of the Dolgachev surfaces, list some elementary properties, and prove an irreducibility result. First, we give a fundamental operation in the theory of elliptic surfaces, due to Kodaira.

(3.1) Let $\pi\colon Y \to C$ be an elliptic surface (2.10). If $x \in C$ is a point with $\pi^{-1}(x) = F$ a smooth (reduced) elliptic curve, it is possible to perform a complex analytic surgery on Y to obtain a new elliptic surface $\pi'\colon Y' \to C$, which is the same as Y away from $\pi^{-1}(x)$. Y' is called a *logarithmic transform* of Y. The construction depends on the choice of a point ξ of order p in Pic F. For details we refer to [15, p. 564 ff], [2, p. 164ff], and to Kodaira's paper [18]. Suffice it to say here that one chooses a small analytic disk Δ_x centered at x, and considers the restriction of π to $U = \pi^{-1}(\Delta_x) \to \Delta_x$. We form the pull back

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow \\ \Delta'_x & \longrightarrow & \Delta_x \\ & & \\ s & \longmapsto & s^p = t. \end{array}$$

The point ξ of order p in Pic$F$, extends uniquely to an analytic family of p-torsion points in PicF_t for $t \in \Delta_x$. These allow us to define a free $(\mathbf{Z}/p\mathbf{Z})$-action on U' covering the standard action of $\mathbf{Z}/p\mathbf{Z}$ on Δ_x. Let $\hat{U}$ be the quotient of this action. It fibers over $\Delta'_x/(\mathbf{Z}/p\mathbf{Z}) \cong \Delta_x$. Let $\pi'\colon \hat{U} \to \Delta_x$ denote this map. The fiber over the origin is a multiple fiber of multiplicity p. Using the logarithm one defines a complex analytic isomorphism $\hat{U} - \pi'^{-1}(0) \cong U - \pi^{-1}(0)$ commuting with the projections to $\Delta_x - \{0\}$. Using this isomorphism, one can glue $\hat{U}$ to $Y - \pi^{-1}(0)$ along $\hat{U} - (\pi')^{-1}(0) \cong U - \pi^{-1}(0)$. We have the important normal bundle formula

(3.2) *If F_p is the reduced fiber of π' over x, $\mathcal{O}_{Y'}(F_p) \mid F_p$ is a line bundle on F_p of order exactly p* [15, *p*. 567].

Definition 3.3. Let $\pi\colon X \to \mathbf{P}^1$ be a rational elliptic surface with $K_X = -F$, F a fiber of π, and let (p, q) be a pair of relatively prime integers greater than 1. A *Dolgachev surface* $S = S(p,q)$ is the surface obtained by applying logarithmic transforms of orders p and q respectively to two smooth fibers $\pi^{-1}(x)$, $\pi^{-1}(y)$, $x, y \in \mathbf{P}^1$. We will call S *generic* if X is, i.e. if all fibers in the elliptic structure $S \to \mathbf{P}^1$ are irreducible.

Remarks 3.4. (a) We drop the primes and write $\pi\colon S \to \mathbf{P}^1$ for the elliptic fibration on S. It is in fact canonically defined by the complex structure of S.

(b) One may allow one of p or q to be 1, i.e., to perform only one logarithmic transform. The resulting surface is rational, and easily seen to be diffeomorphic to X (but does not satisfy (2.11)).

(c) To be completely general we should have allowed logarithmic transforms around certain singular fibers as well. Since such surfaces are limits of those defined in (3.3), for the purposes of topology or irreducibility questions, they may be safely ignored. (For more discussion of this technical point, cf. [23, p. 117].)

Theorem 3.5 (*Dolgachev* [6]; *see also* [23, *p*. 191]). *Dolgachev surfaces are simply connected.* □

This result is false if we perform more than two logarithmic transforms.

Notation 3.6. f is a general fiber on $S = S(p,q)$.

$F_p =$ the fiber of multiplicity p;

$F_q =$ the fiber of multiplicity q.

We will often identify f, F_p, and F_q with their divisor classes or cohomology classes. Hence, as divisor classes,

$$pF_p = qF_q = f.$$

Proposition 3.7. (a) $\chi(S) = 12$ *and* $\chi(\mathcal{O}_S) = 1$.

(b) $K_S = -f + (p-1)F_p + (q-1)F_q = [(pq - p - q)/pq](f)$.

(c) $H^0(K_S) = 0$, *i.e.* $p_g(S) = 0$.

(d) *S is a nonrational algebraic surface with Kodaira dimension* 1.

(e) *The exact order of divisibility of $f \in H^2(S;\mathbf{Z})$ is pq, i.e.*,

$$f = pq \cdot \kappa_S,$$

where $\kappa_S \in H^2(S;\mathbf{Z})$ is a primitive integral class.

(f) *The quadratic form* q_S *is of type* $(1,9)$. *Hence, by Freedman's results* [13], S *is homeomorphic to* $X_9 = X$ *as in* (2.6).

Proof. By a general result on logarithmic transforms [18, p. 773], $\chi(S) = \chi(X) = 12$ so that $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_X) = 1$ by Noether's formula. Part (b) follows from the canonical bundle formulas [18], [15, p. 571], [2, p. 161] and by (a). Part (c) follows from (b) or from (a) and (3.5) as $1 = \chi(\mathcal{O}_S)$ and $H^1(\mathcal{O}_S) = 0$.

Next, from the index theorem [16, p. 86], $b_2^+(S) - b_2^-(S) = -8$ where b_2^+ (resp b_2^-) is the dimension of a maximal subspace of $H^2(Y; \mathbf{R})$ on which q_S is positive (resp. negative) definite. By (a) $b_2^+(S) + b_2^-(S) = 10$. Thus, $b_2^+(S) = 1$ and $b_2^-(S) = 9$. Since $b_2^+(S) > 0$, there exist classes in $H^2(S; \mathbf{Z})$ of positive square. From the exponential sheaf sequence

$$\begin{array}{ccccccc} 0 = H^1(\mathcal{O}_S) & \longrightarrow & H^1(\mathcal{O}_S^*) & \longrightarrow & H^2(S;\mathbf{Z}) & \longrightarrow & H^2(\mathcal{O}_S) \to 0 \\ & & \| & & & & \\ & & \mathrm{Pic}(S) & & & & \end{array}$$

(where we have used duality: $H^2(\mathcal{O}_S)^* = H^0(K_S) = 0$), every class in $H^2(S; \mathbf{Z})$ is the first Chern class of a holomorphic line bundle. There exists a line bundle L with $L^2 > 0$; by a result of Kodaira [18, Theorem 8], S is algebraic. S has Kodaira dimension one since

$$\dim H^0(npqK_S) \sim An, \qquad n > 0,$$

where A is a constant.

We prove (e). Since g.c.d.$(p,q) = 1$, there exists integers r, s such that $rq + sp = 1$. Thus, if we set

$$\kappa_S = \kappa = rF_p + sF_q,$$

$(pq)\kappa = (rq + sp)f = f$, so that f is divisible by pq.

To prove that κ is actually a primitive class, one may easily adapt the argument in Kodaira [19, Lemma 2] to this situation. We shall sketch another, more topological argument. It is sufficient to find a 2-cycle δ in S such that $\delta \cdot f = pq$.

Suppose that S is obtained from X by performing logarithmic transforms over the points $x, y \in \mathbf{P}^1$. Choose a section $s\colon \mathbf{P}^1 \to X$ for the elliptic fibration (e.g., one of the exceptional curves will define such a section). Let Δ_x, Δ_y be small open disks around x and y and set

$$s_0 = s\left(\mathbf{P}^1 - \Delta_x - \Delta_y\right),$$

viewed as a subset of S. Over $\overline{\Delta}_x$ and $\overline{\Delta}_y$, in S, we may find multi-sections s_x and s_y of π, of orders p and q respectively, (i.e. $s_x \cdot f = p$, $s_y \cdot f = q$, for a generic f over $\overline{\Delta}_x$ or $\overline{\Delta}_y$). Let

$$\Sigma = pqs_0 + qs_x + ps_y.$$

Then $\partial\Sigma \subseteq \pi^{-1}(\partial\Delta_x \cup \partial\Delta_y)$. It is clearly sufficient to prove

Claim. $\partial\Sigma$ *bounds a 2-cycle* Σ' *in* $X - \pi^{-1}(x) - \pi^{-1}(y)$.

Proof of the claim. Let $F_x = \pi^{-1}(x)$ and $F_y = \pi^{-1}(y)$. Consider the Gysin sequence

$$H_2(X;\mathbf{Z}) \to H_0(F_x \cup F_y;\mathbf{Z}) \to H_1(X - F_x - F_y;\mathbf{Z}) \to H_1(X;\mathbf{Z}) = 0.$$

Since $[F_x] = [F_y]$ defines a primitive cohomology class in $H^2(X;\mathbf{Z})$, it follows that $H_1(X - F_x - F_y;\mathbf{Z}) \cong \mathbf{Z}$, in fact $H_1(X - F_x - F_y;\mathbf{Z}) \cong H_1(\mathbf{P}^1 - \{x, y\};\mathbf{Z})$ via π_*. One checks easily that, with our choices of sign, $\pi_*[\partial\Sigma] = 0$. This proves that $\partial\Sigma$ bounds in $X - F_x - F_y$; (e) follows.

Finally, we must prove (f). Since κ is primitive, there exists $\delta \in H^2(S;\mathbf{Z})$ with $\delta \cdot \kappa = 1$, so that

$$[K_S] \cdot \delta = pq - p - q.$$

Since at least one of p, q is odd, $pq - p - q$ is odd. From the Wu formula,

$$\delta^2 \equiv K_S \cdot \delta \equiv 1 \mod 2,$$

so that q_S is odd. (Another proof consists in quoting Rokhlin's theorem.) By the classification of integral unimodular forms [28, p. 92], q_S is of type $(1, 9)$. □

Proposition 3.8. *For a given p and q relatively prime the moduli space of all Dolgachev surfaces with multiple fibers of multiplicities p and q is irreducible, i.e. there exists an irreducible complex space T (which may be assumed smooth) and a proper smooth map* $\Phi\colon \mathfrak{X} \to T$ *such that every Dolgachev surface* $S(p, q)$ *is isomorphic to* $\Phi^{-1}(t)$ *for some* $t \in T$.

Proof. Let $\pi\colon X \to \mathbf{P}^1$ be a rational elliptic surface and $x, y \in \mathbf{P}^1$ two points such that $\pi^{-1}(x) = F_x$ and $\pi^{-1}(y) = F_y$ are smooth. The construction of a Dolgachev surface S from X depended on a choice of two line bundles $\eta_x \in \operatorname{Pic} F_x$, $\eta_y \in \operatorname{Pic} F_y$, of orders exactly p and q respectively. The next lemma states that, given X, x, and y, the choice of η_x and η_y is the only choice involved.

Lemma 3.9. *Given* η_x, η_y *as above, there is a unique elliptic surface* $S \to \mathbf{P}^1$ *with multiple fibers at x and y and associated invariants* η_x *and* η_y *and which is isomorphic to X over* $\mathbf{P}^1 - \{x, y\}$.

Proof. Any two such surfaces differ by an element in the Brauer group $\operatorname{Br}(X)$, by [6, pp. 123–126]. Since X is a rational surface, $\operatorname{Br}(X) = 0$.

In a slightly different language, by results of Kodaira ([17], or (11.1)(c) on p. 160 of [2]), the set of such surfaces up to isomorphism is a principal homogeneous space over $H^1(\mathbf{P}^1, \mathcal{J})$, where $\mathcal{J}$ is the sheaf on $\mathbf{P}^1$ defined by

$$0 \to R^1\pi_*\mathbf{Z} \to R^1\pi_*\mathcal{O}_X \to \mathcal{J} \to 0.$$

An argument with the Leray spectral sequence and standard facts on rational surfaces shows that $H^1(R^1\pi_*\mathcal{O}_X)$ and $H^2(R^1\pi_*\mathbf{Z})$ are zero; hence so is $H^1(\mathbf{P}^1; \mathscr{J})$. This completes the proof of (3.9). □

Returning to the proof of (3.8), let T denote the space of quintuples:

$$T = \{(X, x, y, \eta_x, \eta_y)\}$$

subject to the conditions that X is a rational, elliptic surface π: $X \to \mathbf{P}^1$ is the projection, $x \neq y$ are points of $\mathbf{P}^1$, F_x and F_y are smooth fibers of X, $\eta_x \in \operatorname{Pic} F_x$ is a p-torsion point and $\eta_y \in \operatorname{Pic} F_y$ is a q-torsion point. The space T has a natural analytic structure, and by (3.9) there is a family $\mathfrak{X}$ over T containing every Dolgachev surface of type (p, q). Hence, it suffices to show that T is irreducible. The family R of all rational elliptic surfaces is identified with the family of all cubic pencils in $\mathbf{P}^2$ whose generic member is smooth. This is a Zariski open subset of the space of lines in the projective space of cubic polynomials on $\mathbf{P}^3$. Hence R is irreducible.

To show that T is irreducible, it suffices to show that the fibers of $T \to R$ are irreducible. Thus we fix a rational elliptic surface π: $X \to \mathbf{P}^1$ and we consider all quadruples

$$F = \{(x, y, \eta_x, \eta_y) \mid x \neq y \in \mathbf{P}^1,\ F_x \text{ and } F_y \text{ are smooth};$$
$$\eta_x \in \operatorname{Pic} F_x \text{ is a } p\text{-torsion point};\ \eta_y \in \operatorname{Pic} F_y \text{ is a } q\text{-torsion point}\}.$$

Of course the pairs $\bar{F} = \{(x, y)\}$ as above form an open subset of $\mathbf{P}^1 \times \mathbf{P}^1$ and hence the space of them is irreducible. The map $F \to \bar{F}$ is finite to one. It suffices to show that $\pi_1(\bar{F})$ acts transitively on a fiber of $F \to \bar{F}$ in order to show that F is irreducible.

Pick a point $p_0 \in \mathbf{P}^1$ distinct from x and y so that F_{p_0} is smooth. Choose paths in $\mathbf{P}^1$ from x to p_0 and y to p_0 avoiding the images of the singular fibers of X. Using these paths we identify η_x and η_y respectively with p- and q-torsion points in $\operatorname{Pic} F_{p_0}$. What we are reduced to showing is that the fundamental group $\pi_1(\bar{F}, p_0)$ acts transitively on pairs $\alpha, \beta \in \operatorname{Pic}(F_{p_0})$, α being a p-torsion point and β being a q-torsion point. Choosing a basis for $H_1(F_{p_0}; \mathbf{Z})$, we identify α with an element in $\frac{1}{p}(\mathbf{Z}^2)/\mathbf{Z}^2$, and β with an element in $\frac{1}{q}(\mathbf{Z}^2)/\mathbf{Z}^2$. We claim that the action of $\pi_1(\bar{F}, p_0)$ on such pairs $\{\alpha, \beta\}$ is transitive. Of course the action of $\pi_1(\bar{F}, p_0)$ on such pairs is via a representation μ: $\pi_1(\bar{F}, p_0) \to SL_2(\mathbf{Z})$. According to [23, proof of Theorem 9, p. 175], $\operatorname{Im}\mu = SL_2(\mathbf{Z})$ if X is generic. Thus, it suffices to prove

Lemma 3.10. *$SL_2(\mathbf{Z})$ acts transitively on pairs (α, β) where $\alpha \in \frac{1}{p}(\mathbf{Z}^2)/\mathbf{Z}^2$ is of order p and $\beta \in \frac{1}{q}(\mathbf{Z}^2)/\mathbf{Z}^2$ is of order q.*

Proof. Since p and q are relatively prime, there is a unique $\gamma \in \frac{1}{pq}(\mathbf{Z}^2)/\mathbf{Z}^2$ such that $q\gamma = \alpha$ and $p\gamma = \beta$. We need to see that $SL_2(\mathbf{Z})$ acts transitively on such γ, i.e. on elements of order pq in $\frac{1}{pq}(\mathbf{Z}^2)/\mathbf{Z}^2$. This is a standard fact. □

Corollary 3.11. *Let S_1 and S_2 be Dolgachev surfaces with multiple fibers of orders p and q. Let $[f_i] \in H^2(S_i; \mathbf{Z})$ be the class dual to a generic fiber of the elliptic fibration on S_i. Then there is a diffeomorphism φ: $S_1 \to S_2$ such that $\varphi^*[f_2] = [f_1]$.*

Proof. By (3.8) there exist a family Φ: $\mathfrak{X} \to T$ with T irreducible and points $t_1, t_2 \in T$ such that $\Phi^{-1}(t_i) \cong S_i$. Choose a real curve γ in T joining t_1 to t_2. If we pull the bundle $\mathfrak{X} \to T$ back to the domain of γ, it has a C^∞-trivialization. Hence, there is a diffeomorphism φ: $S_1 \to S_2$. The relative canonical class $K_{\mathfrak{X}/T}$ restricts to any fiber to give the canonical class of that fiber. As $\gamma^* K_{\mathfrak{X}/T}$ is a class in the pulled-back bundle which restricts to the ends to give K_{S_1} and K_{S_2}, $\varphi^* K_{S_2} = K_{S_1}$. Since $[f_i] = (pq - p - q)K_{S_i}$, we have $\varphi^*[f_2] = [f_1]$. □

CHAPTER II

1. Generalities on forms of type $(1, n)$

In this section we develop some of the basic theory for quadratic forms of signature $(1, n)$. We concentrate on the hyperbolic space inside the vector space supporting this form. We are particularly interested in the chamber structure associated to the walls perpendicular to integral classes of square -1.

We fix a nonsingular quadratic form q: $\Lambda \to \mathbf{Z}$ of signature $(1, n)$, $n \geqslant 1$. Set $\Lambda_{\mathbf{R}} = \Lambda \otimes \mathbf{R}$ and let $\bar{q}$: $\Lambda_{\mathbf{R}} \to \mathbf{R}$ be the extension of q to a real-valued quadratic form on $\Lambda_{\mathbf{R}}$. We denote by $(\cdot)$ the symmetric bilinear form on $\Lambda_{\mathbf{R}}$ associated to $\bar{q}$. Let $\mathscr{X} = \mathscr{X}(q)$ be the level set $\{x \in \Lambda_{\mathbf{R}} \mid \bar{q}(x) = 0\}$. It is a double cone. Let $\mathscr{P} = \mathscr{P}(q)$ be $\{x \in \Lambda_{\mathbf{R}} \mid \bar{q}(x) > 0\}$. It is the interior of the cone. Let $\mathscr{Q} = \mathscr{Q}(q)$ be $\mathscr{P} \cup \mathscr{X} - \{0\}$. The level set $\mathbf{H} = \mathbf{H}(q) = \{x \in \Lambda_{\mathbf{R}} \mid \bar{q}(x) = 1\}$ is a hyperboloid of two sheets. We give $\mathbf{H}(q)$ the Riemannian metric induced by $(-\bar{q})$ restricted to its tangent spaces. It is isometric to two copies of n-dimensional hyperbolic space. (See Figure 2.) There is the obvious identification of $\mathscr{P}/\mathbf{R}^+$ with $\mathbf{H}$. If we form $\mathscr{Q}/\mathbf{R}^+$, then this gives a compactification of $\mathscr{P}/\mathbf{R}^+$ and hence of $\mathbf{H}$. We denote it by $\overline{\mathbf{H}}$. Clearly $\overline{\mathbf{H}}$ is just two copies of the unit disk in $\mathbf{R}^n$.

Lemma 1.1. *$\mathscr{Q}(q)$ has two components, as does $\mathscr{P}(q)$. If $x \in \mathscr{P}(q)$ and $y \in \mathscr{Q}(q)$, then $x \cdot y \neq 0$ and $x \cdot y > 0$ if and only if x and y lie in the same component of $\mathscr{Q}(q)$.*

Proof. Fix $x \in \mathscr{P}(q)$. Suppose the function i_x: $\mathscr{Q}(q) \to \mathbf{R}$ defined by $i_x(y) = x \cdot y$ never vanishes. Then since $x \cdot x > 0$, i_x must be positive on the

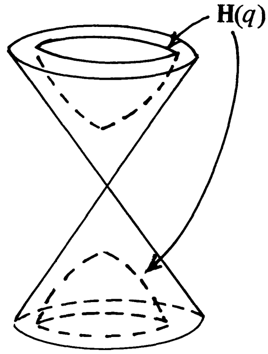

FIGURE 2

component of $\mathcal{Q}(q)$ containing x and negative on the other component (which contains $-x$). But if $x \cdot y = 0$ for some $y \in \mathcal{Q}(q)$, then y is not a multiple of x, so that $\langle x, y\rangle$ has dimension 2, where $\langle x, y\rangle$ is the **R**-linear span of x and y. The form $\bar{q}\,|\,\langle x, y\rangle$ is positive semidefinite. This is a contradiction. □

The point of this section is to give some of the basic facts about certain sets of walls in hyperbolic space and their associated chamber structures. Included in these is a general condition for a set of walls to be locally finite. We are most interested in the set $\mathscr{W}_1$ of walls defined by classes $\alpha \in \Lambda$ with $q(\alpha) = -1$. But first we do some very general preliminary work.

For any $x \in \Lambda_{\mathbf{R}}$ we define $(x^{\perp}) \subset \Lambda_{\mathbf{R}}$ to be the orthogonal subspace to x: $(x^{\perp}) = \{y \in \Lambda_{\mathbf{R}} \,|\, x \cdot y = 0\}$. When $x \neq 0$, this is a codimension-1 linear subspace. There is a very simple condition on x for this subspace to meet **H**.

Lemma 1.2. *If* $x \neq 0$, *then* $(x^{\perp}) \cap \mathbf{H} \neq \varnothing$ *if and only if* $\bar{q}(x) < 0$.

Proof. Suppose that $x \neq 0$ and that $(x^{\perp}) \cap \mathbf{H} \neq \varnothing$. By Lemma 1.1 this implies that $x \notin \mathcal{Q}(q)$, and hence that $\bar{q}(x) < 0$.

Conversely, suppose that $\bar{q}(x) < 0$, or equivalently $x \cdot x < 0$. Since $x \cdot x \neq 0$, we can decompose $\Lambda_{\mathbf{R}}$ as $\langle x\rangle \oplus (x^{\perp})$. Since $x \cdot x < 0$, the signature of $\bar{q}\,|(x^{\perp})$ is $(1, n-1)$. Hence, inside $(x^{\perp})$ there is a vector of positive square. An appropriate multiple of this vector lies in **H**, and hence $(x^{\perp}) \cap \mathbf{H} \neq \varnothing$. □

Definition 1.3. If $\bar{q}(x) < 0$, then $(x^{\perp}) \cap \mathbf{H}$ is of the form $W \amalg -W$ where W is a totally geodesic codimension-1 subspace. We call $W \amalg -W$ the *wall* of **H** determined by x, and we denote it W^x. If W is a totally geodesic, codimension-1 subspace of one of the components of **H**, then there is a class $x \in \Lambda_{\mathbf{R}}$ such that $W^x = W \amalg -W$. This class x is unique up to nonzero scalar multiples.

Lemma 1.4. *Let $B \subset \mathbf{H}$ be a compact set, and let $K > 0$ be a constant. Then the set*

$$D = \left\{ x \in \Lambda_{\mathbf{R}} \,|\, (x^{\perp}) \cap B \neq \varnothing \text{ and } |\bar{q}(x)| \leqslant K \right\}$$

is compact.

Before giving the proof we introduce another definition.

Definition 1.5. A *diagonal basis* for $\Lambda_{\mathbf{R}}$ is an $\mathbf{R}$-basis for $\Lambda_{\mathbf{R}}$ $(e_0, e_1, \cdots, e_n)$ such that $\bar{q}(e_0) = 1$, $\bar{q}(e_i) = -1$ for $1 \leqslant i \leqslant n$, and $e_i \cdot e_j = 0$ for $i \neq j$. Notice that $\Lambda_{\mathbf{R}}$ always has a diagonal basis since q is of signature $(1, n)$.

Proof of 1.4. Clearly $D \subset \Lambda_{\mathbf{R}}$ is closed. We show that it is bounded. To do this choose a diagonal basis $(e_0, \cdots, e_n)$ for $\Lambda_{\mathbf{R}}$ and write all elements of $\Lambda_{\mathbf{R}}$ in terms of this basis. Possibly after expanding B, we can assume that it is of the form $B = \{(b_0, \cdots, b_n) | b_0^2 - \sum_{i=1}^n b_i^2 = 1 \text{ and } \sum_{i=1}^n b_i^2 \leqslant R\}$ for some $R > 0$.

For any $x = (x_0, \cdots, x_n) \in \Lambda_{\mathbf{R}}$ we denote by $\|x\|_-$ the quantity $(\sum_{i=1}^n x_i^2)^{1/2}$. Suppose $x \in D$. Then there exists $(b_0, \cdots, b_n) \in B$ with $x \cdot b = x_0 b_0 - \sum_{i=1}^n x_i b_i = 0$. Thus,

$$|x_0 b_0| = \left| \sum_{i=1}^n x_i b_i \right| \leqslant \|x\|_- \|b\|_- .$$

Since $|b_0| \leqslant \sqrt{\|b\|_-^2 + 1}$, this yields

$$|x_0| \leqslant \|x\|_- \cdot \left(\frac{\|b\|_-}{\sqrt{\|b\|_-^2 + 1}} \right).$$

Thus, $x_0^2 \leqslant \|x\|_-^2 (\|b\|_- / (\|b\|_-^2 + 1))$ or

$$\bar{q}(x) = x_0^2 - \|x\|_-^2 \leqslant (\|x\|_-)^2 \left(\frac{-1}{(\|b\|_-)^2 + 1} \right).$$

Since $x \in D$, Lemma 1.2 says $\bar{q}(x) \leqslant 0$. Since $-K \leqslant \bar{q}(x) \leqslant 0$, we have,

$$-K \leqslant \left(\frac{-1}{(\|b\|_-)^2 + 1} \right) \|x\|_-^2$$

or

$$(\|x\|_-)^2 \leqslant K(R + 1).$$

This shows that x_i for $1 \leqslant i \leqslant n$ is bounded in absolute value by $\sqrt{K(R+1)}$. Since $\bar{q}(x) \leqslant 0$, $x_0^2 \leqslant \|x\|_-^2 \leqslant K\,(R + 1)$ also. □

Corollary 1.6. *Let $B \subset \mathbf{H}$ be compact and fix $N > 0$. The set of walls $\{W^\alpha \,|\, \alpha \in \Lambda - \{0\}, |q(\alpha)| \leqslant N\}$ which meet B is a finite set.*

Proof. It suffices to prove that

$$\left\{ \alpha \in \Lambda \,|\, (\alpha^{\perp}) \cap B \neq \varnothing \text{ and } |q(\alpha)| \leqslant N \right\}$$

is finite. But this set is the intersection of a compact subset of $\Lambda_{\mathbf{R}}$ with the lattice Λ. □

Definition 1.7. For any $N \in \mathbf{Z}$, $N > 0$, we define $\mathscr{W}_N$ to be the set of walls

$$\{W^\alpha \mid \alpha \in \Lambda \text{ and } -N \leqslant q(\alpha) < 0\}.$$

Corollary 1.8. *For any $N > 0$, $\mathscr{W}_N$ is a locally finite set of walls in* $\mathbf{H}$. □

Definition 1.9. Let $\mathscr{W}$ be a locally finite set of walls in $\mathbf{H}$. A *chamber* (for $\mathscr{W}$) is the closure in $\mathbf{H}$ of a connected component of $\mathbf{H} - \bigcup_{W \in \mathscr{W}} W$. A wall $W \in \mathscr{W}$ is a *wall of the chamber* C if C is not a chamber for the set $\mathscr{W} - \{W\}$ of walls. Equivalently, W is a wall of C if $C \cap W$ contains a nonempty open subset of W. In this case $C \cap W$ is a *face* of C. The frontier of C in $\mathbf{H}$ is the union of its faces. A point in the frontier of C lies in the interior of a face of C if and only if it lies in a unique wall $W \in \mathscr{W}$.

If F is a face of C, then there is a unique chamber $C' \neq C$ which also has F as a face. The chambers C and C' lie on opposite sides of the wall containing the common face.

Definition 1.10. We say that a class $\alpha \in \Lambda_{\mathbf{R}}$ determines an *oriented wall* of C if:

(i) W^α is a wall of C, and

(ii) $\alpha \cdot C \geqslant 0$, i.e. for all $x \in C$, $\alpha \cdot x \geqslant 0$.

Given the chamber C and one of its walls W, the above two conditions determine α up to a positive scalar multiple. If, as will be the case, the line $\mathbf{R}^+ \cdot \alpha \in \Lambda_{\mathbf{R}}$ passes through Λ, then we make the extra implicit assumption that α is primitive in Λ. This will determine α uniquely.

We say that a wall $W \in \mathscr{W}$ *separates* chambers C_0 and C_1 if C_0 and C_1 lie in opposite closed half-spaces bounded by W, or equivalently if $W = W^\alpha$ for some $\alpha \in \Lambda_{\mathbf{R}}$ with $\alpha \cdot C_0 \geqslant 0$ and $\alpha \cdot C_1 \leqslant 0$.

There is one locally finite set of walls that is of particular interest to us. This is the set $\mathscr{W}_1 = \{W^\alpha \mid \alpha \in \Lambda \text{ and } q(\alpha) = -1\}$. For the remainder of this paper, we will reserve the term chamber for a chamber for the set $\mathscr{W}_1$ unless we explicitly say otherwise. We denote by $\mathscr{F} = \{\alpha \in \Lambda \mid q(\alpha) = -1\}$ and by $\mathscr{C} = \mathscr{C}(q)$ the set of chambers for $\mathscr{W}_1$. Here is the first special fact about $\mathscr{W}_1$: the walls meet orthogonally.

Lemma 1.11. *Let W^α and W^β be distinct walls in the set $\mathscr{W}_1$. If $W^\alpha \cap W^\beta \neq \varnothing$, then $\alpha \cdot \beta = 0$, or equivalently, W^α and W^β are perpendicular.*

Proof. We have $\alpha, \beta \in \mathscr{F}$. If $W^\alpha \cap \mathscr{W}^\beta \neq \varnothing$, then there is $x \in \mathbf{H} \cap (\alpha^\perp) \cap (\beta^\perp)$. Thus, $x \in (\alpha \pm \beta)^\perp \cap \mathbf{H}$. By (1.2) this means $q(\alpha \pm \beta) < 0$ ($\alpha \neq \pm\beta$ since $W^\alpha \neq W^\beta$). But

$$q(\alpha + \beta) = -1 \pm 2(\alpha \cdot \beta) - 1 = -2(1 \mp (\alpha \cdot \beta)).$$

Since $\alpha \cdot \beta \in \mathbf{Z}$, for both these numbers to be negative, $\alpha \cdot \beta = 0$. □

Definition 1.12. Suppose $\alpha \in \mathscr{F}$. Set $\Lambda_\alpha = (\alpha^\perp) \cap \Lambda$. The quadratic form $q \mid \Lambda_\alpha$ is of signature $(1, n-1)$. If it is of type $(1, n-1)$, then we say that W^α is an *ordinary* wall in $\mathscr{W}_1$. Otherwise, it is an *extraordinary wall.* In either case we identify W^α with $\mathbf{H}(q \mid \Lambda_\alpha)$.

Proposition 1.13. *Let $\alpha \in \mathscr{F}$. Suppose that C is a chamber which has W^α as a wall. Then*

(a) *If W^α is an ordinary wall of $\mathscr{W}_1(q)$, then $C \cap W^\alpha$ is a chamber for $\mathscr{W}_1(q|\Lambda_\alpha)$.*

(b) *If W^α is an extraordinary wall of $\mathscr{W}_1(q)$, then $C \cap W^\alpha$ is a component of $\mathbf{H}(q \mid \Lambda_\alpha)$.*

Proof. Let W be a wall in the set $\mathscr{W}_1(q \mid \Lambda_\alpha)$. It is defined by a class $\beta \in \Lambda_\alpha$ with $q(\beta) = -1$. Considering β as an element in Λ, this gives a wall $\overline{W} \in \mathscr{W}_1(q)$ which meets W^α perpendicularly in W. Clearly this sets up a bijection between the walls of $\mathscr{W}_1(q \mid \Lambda_\alpha)$ and the walls of $\mathscr{W}_1(q)$ meeting W^α perpendicularly. According to Lemma 1.11, the latter set is the set of walls of $\mathscr{W}_1(q)$ meeting W^α. From this the result is immediate. □

There are two discontinuous groups associated with all this structure.

Definition 1.14. Let $A(q)$ denote the full automorphism group of (Λ, q). It acts on $\Lambda_{\mathbf{R}}$ preserving $\mathscr{X}(q)$ and $\mathbf{H}(q)$, and the action of $A(q)$ on $\mathbf{H}(q)$ is properly discontinuous. Notice that $A(q)$ preserves the set of walls $\mathscr{W}_1$ and hence acts on the chamber structure associated to $\mathscr{W}_1$.

There is another group closely related to $\mathscr{W}_1$—the reflection group in the walls of $\mathscr{W}_1$.

Definition 1.15. For each $\alpha \in \mathscr{F}$ we define *the reflection* $R_\alpha\colon \Lambda \to \Lambda$ by

$$R_\alpha(\gamma) = \gamma + 2(\alpha \cdot \gamma)\alpha.$$

One checks easily that $R_\alpha \in A(q)$ and is of order 2. The induced action of R_α on $\mathbf{H}(q)$ is geometrically reflection in the wall W^α. (Notice that R_α leaves invariant the components of $\mathbf{H}(q)$.)

Let $\mathscr{R} \subset A(q)$ be the group generated by $\{R_\alpha \mid \alpha \in \mathscr{F}\}$. Actually, $\mathscr{R}$ is generated by the reflections in the walls of any single chamber C. Since $\mathscr{R} \subseteq A(q)$, the action of $\mathscr{R}$ on $\mathbf{H}$ is properly discontinuous. If we restrict to a connected component $\mathbf{H}_0$ of $\mathbf{H}$, then the quotient space of the action and also a fundamental domain for the action is a single chamber C in $\mathbf{H}_0$. In particular, $\mathscr{R}$ acts simply transitively on the chambers contained in $\mathbf{H}_0$.

Inside $A(q)$ there is the group $\tilde{\mathscr{R}} = \mathscr{R} \times \{\pm \mathrm{Id}\}$. This group acts simply transitively on all chambers. Consequently, $A(q)$ acts transitively on the chambers. However, the isotropy group of a chamber is nontrivial, and in fact is infinite for $n \geqslant 9$.

Definition 1.16. If C is a chamber, then let $\mathscr{F}_C = \{\alpha \in \mathscr{F} \mid \alpha \text{ defines an oriented wall of } C\}$.

Lemma 1.17. *Let $n \geqslant 2$, and let q: $\Lambda \to \mathbf{Z}$ be a quadratic form of type $(1, n)$. If $C \subset \mathbf{H}(q)$ is a chamber, then the classes $\alpha \in \mathscr{F}_C$ span Λ over $\mathbf{Z}$.*

Proof. Since $A(q)$ acts transitively on the chambers, it suffices to prove this for a single chamber. Since q is of type $(1, n)$ there is a standard basis $(x, e_1, \cdots, e_n)$ for Λ. Let C be the chamber with $x \in C$ and $e_1 \cdot C \geqslant 0$ for $i = 1, \cdots, n$. Clearly, $e_1, \cdots, e_n$ define oriented walls for C. We claim that $x - e_1 - e_2$ also defines an oriented wall for C. To see this restrict to $(e_3^\perp) \cap \cdots \cap (e_n^\perp)$. A straightforward computation shows that $C \cap (e_3^\perp) \cap \cdots \cap (e_n^\perp) = C'$ is as pictured below

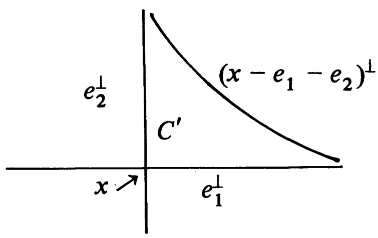

FIGURE 3

That is to say, the walls of C' are defined by e_1, e_2, and $x - e_1 - e_2$. Consequently, $x - e_1 - e_2$ defines a wall for C. Since $x - e_1 - e_2$, $e_1, \cdots, e_n$ span Λ, the result is established. $\square$

Here is another transitivity result.

Lemma 1.18. *Consider the set $\mathscr{S}$ of all ordered pairs (C, W) where C is a chamber and W is an ordinary wall of C. Clearly $A(q)$ acts on $\mathscr{S}$. This action is transitive.*

Proof. Let (C, W) and (C', W') be elements of $\mathscr{S}$ with, say, $W = W^\alpha$ and $W' = W^\beta$. We choose $\alpha \in \mathscr{F}_C$ and $\beta \in \mathscr{F}_{C'}$. There are decompositions:

$$(\Lambda, q) = (\langle\alpha\rangle, q \mid \langle\alpha\rangle) \oplus (\Lambda_\alpha, q \mid \Lambda_\alpha),$$
$$(\Lambda, q) = (\langle\beta\rangle, q \mid \langle\beta\rangle) \oplus (\Lambda_\beta, q \mid \Lambda_\beta).$$

Since W^α and W^β are ordinary walls, $q \mid \Lambda_\alpha$ and $q \mid \Lambda_\beta$ are both of type $(1, n-1)$. Thus, there is $\psi_0 \in A(q)$ with $\psi_0(\alpha) = \beta$. Clearly, $\psi_0(W^\alpha) = W^\beta$.

The chambers $\psi_0(C)$ and C' both lie in the same side of W^β and both meet it in a face. By Proposition 1.13, $\psi_0(C) \cap W^\beta$ and $C' \cap W^\beta$ are both chambers for $\mathscr{W}_1(q \mid \Lambda_\beta)$. Thus, there is $\mu \in A(q \mid \Lambda_\beta)$ sending $\psi_0(C) \cap W^\beta$ to $C' \cap W^\beta$. Extend μ to an element $\tilde{\mu} \in A(q)$ by setting $\tilde{\mu}(\beta) = \beta$. Clearly, $\tilde{\mu} \circ \psi_0(C) \cap W^\beta = C' \cap W^\beta$ is a face of C', and $\tilde{\mu} \circ \psi_0(C)$ and C' lie in the same side of W^β. This means that $\tilde{\mu} \circ \psi_0(C) = C'$. Since $\tilde{\mu} \circ \psi_0(W^\alpha) = \tilde{\mu}(W^\beta) = W^\beta$, this proves the lemma. $\square$

So far we have been working inside $\mathbf{H}$ where local finiteness holds for the set $\mathscr{W}_1$ of walls. We show now that we lose local finiteness even at the boundary of hyperbolic space, on the double cone $\mathscr{X}$.

Proposition 1.19. *Suppose that (Λ, q) is of type $(1, n)$ for some $n \geqslant 2$. Then*

$$\mathscr{X}(q) \subset \overline{\left(\bigcup_{\alpha \in \mathscr{F}} ((\alpha^{\perp}) \cap \mathscr{P})\right)}.$$

Proof. Since the points with rational coordinates form a dense subset of the unit sphere in $\mathbf{R}^n$, lines in $\mathscr{X}(q)$ through points of $\Lambda - \{0\}$ form a dense subset of $\mathscr{X}(q)$. Clearly, $\overline{(\bigcup(\alpha^{\perp}) \cap \mathscr{P})}$ is invariant under the $\mathbf{R}^+$ action. Hence, the proposition will be established when we show that for each $k \in \Lambda \cap \mathscr{X}(q)$, we have $k \in \overline{\bigcup((\alpha^{\perp}) \cap \mathscr{P})}$. Actually it suffices to prove this result for all $k \in \Lambda \cap \mathscr{X}(q)$ which are primitive (i.e., indivisible) in Λ.

Let $k \in \Lambda \cap \mathscr{X}(q)$ be primitive in Λ. Since $k \in \mathscr{X}(q)$, $q(k) = 0$. Since k is primitive, there is $l \in \Lambda$ with $k \cdot l = 1$. The form $q | \langle k, l\rangle$ is unimodular. Thus, there is a decomposition

$$\Lambda = (\langle k, l\rangle) \oplus (\langle k, l\rangle^{\perp}).$$

Since q is not even, either $q(l) = 1(2)$ or there is $x \in (\langle k, l\rangle^{\perp})$ with $q(x) \equiv 1(2)$. In the latter case if $q(l) \equiv 0(2)$, then we can replace l by $l + x$. Thus, we can always arrange that $q(l) \equiv 1(2)$. Once we have this, replacing l by $l - ak$ varies $q(l)$ by $2a$. Hence, we can arrange that $q(l) = -1$. Since $q | (\langle k, l\rangle)$ has type $(1,1)$, $q | (\langle k, l\rangle^{\perp})$ is negative definite. Thus, for any $N > 0$, there is a class $d_N \in (\langle k, l\rangle^{\perp})$ with $q(d_N) \equiv 0(2)$ and $q(d_N) < -N$. Define $\alpha_N = (q(d_N)/2)k - l + d_N$. Obviously, $\alpha_N \in \mathscr{F}$. The element $x_N = k + (1 + q(d_N)/2)^{-1} \cdot l$ is contained in $\mathscr{P} \cap (\alpha_N^{\perp})$. As $N \to \infty$, $x_N \to k$. Hence $k \in \overline{\bigcup_{\alpha \in \mathscr{F}}(\alpha^{\perp}) \cap \mathscr{P})}$. □

This density result has two useful corollaries.

Corollary 1.20. *Suppose that (Λ, q) is of type $(1, n)$ for some $n \geqslant 2$ and that C is a chamber. Then $\overline{(\mathbf{R}^+ \cdot C)} \cap \mathscr{X}(q)$ has no interior in $\mathscr{X}(q)$.*

Proof. Suppose $U \subset \overline{\mathbf{R}^+ \cdot C} \cap \mathscr{X}(q)$ is a nonempty open subset of $\mathscr{X}(q)$. Let $x \in \mathbf{R}^+ \cdot (\operatorname{int} C)$. Denote by $(x * U)$ the cone in $\Lambda_{\mathbf{R}}$ with base U and vertex x. Clearly $(x * U) \cap \mathscr{P} \subset \mathbf{R}^+ \cdot \operatorname{int} C$. Thus, if $W \in \mathscr{W}_1$ then $W \cap (x * U) = \varnothing$. Hence, $\overline{(\bigcup_{W \in \mathscr{W}_1} W)} \cap (x * U) = \varnothing$, and hence $\overline{(\bigcup_{W \in \mathscr{W}_1} W)} \cap U = \varnothing$. This contradicts Proposition 1.19. □

Corollary 1.21. *Suppose that (Λ, q) is of type $(1, n)$ for some $n \geqslant 2$. Suppose that C is a chamber. There is no class $x \in \Lambda_{\mathbf{R}} - \mathscr{Q}(q)$ such that $x \cdot \alpha > 0$ for all $\alpha \in \mathscr{F}_C$.*

Proof. Let $\tilde{C} = \bigcap_{\alpha \in \mathscr{F}_C} \{x \in \Lambda_{\mathbf{R}} | \alpha \cdot x \geqslant 0\}$ This is a convex subset of $\Lambda_{\mathbf{R}}$ which meets $\mathscr{P}$ exactly in $\mathbf{R}^+ \cdot \bar{C}$. Suppose $y \in \tilde{C} \cap (\Lambda_{\mathbf{R}} - \mathscr{Q})$. For each $x \in \mathbf{R}^+ \cdot \operatorname{int} C$ we form the line $l(x, y)$ in $\Lambda_{\mathbf{R}}$ joining x to y. This line must pierce $\mathscr{X}(q)$ at a unique point $z(x)$. Clearly, the segment $[x, z(x)) \subset l(x, y)$ lies in $\mathbf{R}^+ \cdot C$. Since $\mathbf{R}^+ \cdot C$ contains an open subset of $\mathscr{P}$, the $\{z(x)\}$ sweep out a subset of $\mathscr{X}(q)$ with nonempty interior in $\mathscr{X}(q)$. This means that $\overline{(\mathbf{R}^+ \cdot C)} \cap \mathscr{X}(q)$ contains a nonempty open subset of $\mathscr{X}(q)$, contradicting Corollary 1.20. $\square$

2. The canonical class of a chamber and a corner

In this section we define the canonical class associated with a chamber and a corner. We state two basic results—the first describes the convex subset of the chamber cut out by the wall perpendicular to the canonical class, the second gives a disjointness result for these convex subsets. Both are purely arithmetic results, and presumably both can be proved directly. We find it easier to use some of the theory of Kähler metrics on good generic rational surfaces to prove these results. The study of these metrics is carried out in §3. The proofs of the results stated in this section are given in §4. We keep the notation of the previous section.

Definition 2.1. A *corner* is a point $x \in \mathbf{H}$ at which n mutually perpendicular walls of $\mathscr{W}_1$ meet, where $n =$ dimension $\mathbf{H}$. If $(x, \alpha_1, \cdots, \alpha_n)$ and $(x', \alpha'_1, \cdots, \alpha'_n)$ are standard bases for Λ, then they are *equivalent* if they are the same as unordered sets. Notice that this forces $x = x'$.

Lemma 2.2. *Suppose q: $\Lambda \to \mathbf{Z}$ is a quadratic form of type* $(1, n)$.

(a) *$x \in \mathbf{H}$ is a corner if and only if $x \in \Lambda \cap \mathbf{H}$ and $q \,|((x^\perp) \cap \Lambda)$ is of type* $(0, n)$.

Now let us assume that $x \in \mathbf{H}$ is a corner.

(b) *There are 2^n chambers meeting at x.*

(c) *There are 2^n equivalence classes of standard bases for Λ whose first element is x.*

(d) *There is a bijection between the set of chambers having x as a corner and equivalence classes of standard bases for Λ whose first element is x. This bijection is defined by*

$$C \leftrightarrow \{x, \alpha_1, \cdots, \alpha_n\} \text{ if and only if } \alpha_1, \cdots, \alpha_n \in \mathscr{F}_C.$$

Proof. (a) Suppose $x \in \mathbf{H}$ is a corner. There are classes $\alpha_1, \cdots, \alpha_n \in \mathscr{F}$ with $\alpha_i \cdot \alpha_j = 0$ for $i \neq j$ and $x \cdot \alpha_i = 0$ for $i = 1, \cdots, n$. (These classes determine the n mutually perpendicular walls meeting at x.) Clearly, $q \,|\langle \alpha_1, \cdots, \alpha_n \rangle$ is nonsingular and of type $(0, n)$. Hence $q \,|((\langle \alpha_1, \cdots, \alpha_n \rangle^\perp) \cap \Lambda)$ is also nonsingular and of signature $(1, 0)$. The generators of this sublattice lie in $\mathbf{H}$. Thus, $x \in ((\langle \alpha_1, \cdots, \alpha_n \rangle^\perp)$ and generates it. Clearly $(x^\perp) = \langle \alpha_1, \cdots, \alpha_n \rangle$.

Conversely, if $x \in \Lambda \cap \mathbf{H}$, then $q(x) = 1$ and we have a decomposition

$$(\Lambda, q) \cong (\langle x \rangle, q | \langle x \rangle) \oplus (\Lambda \cap (x^{\perp}), q | \Lambda \cap (x^{\perp})).$$

The second form is of signature $(0, n)$. If it is of type $(0, n)$, then there are vectors $\alpha_1, \cdots, \alpha_n \in (x^{\perp}) \cap \mathscr{F}$, for $1 \leqslant i \leqslant n$, with $\alpha_i \cdot \alpha_j = 0$ for $i \neq j$. These classes determine the n mutually perpendicular walls meeting at $x \in \mathbf{H}$.

Parts (b), (c) and (d) are immediate given (a). $\square$

Definition 2.3. If $x \in \mathbf{H}$ is a corner and C is a chamber with $x \in C$, then any standard basis $(x, \alpha_1, \cdots, \alpha_n)$ for Λ with $\alpha_i \cdot C \geqslant 0$ for $1 \leqslant i \leqslant n$ is called a standard basis *adapted to* C. We define the *canonical class* $\kappa(x, C)$ to be $(3x - \sum_{i=1}^{n} \alpha_i) \in \Lambda$.

Notice that if we permute the α_i, then $\kappa(x, C)$ remains unchanged. Hence, $\kappa(x, C)$ depends only on x and C and not on the choice of basis adapted to C.

Lemma 2.4. *Let C be a chamber and x a corner for C. Then*

(a) $q(\kappa(x, C)) = 9 - n$.

(b) $\kappa(x, C) \in \Lambda$ *is indivisible.*

(c) $q | (\kappa(x, C)^{\perp}) \cap \Lambda$ *is even.*

(d) *If $n = 9$ and if $\kappa \in \Lambda$ satisfies* (a), (b), *and* (c) *then there is a chamber C' and a corner x' with $\kappa(x', C') = \kappa$.*

Proof. $\kappa(x, C) = 3x - \sum_{i=1}^{n} \alpha_i$ where $(x, \alpha_1, \cdots, \alpha_n)$ is a standard basis. Clearly then (a), (b), and (c) hold, where (c) follows by reducing mod 2. Suppose $n = 9$ and $\kappa \in \Lambda$ satisfies (a), (b), and (c). Since κ is indivisible, there is $\delta \in \Lambda$ with $\kappa \cdot \delta = 1$. Since $q(\kappa) = 0$, $q | \langle \kappa, \delta \rangle$ is nonsingular and of signature (1,1). Hence, $\Lambda \cong \langle \kappa, \delta \rangle \oplus (\langle \kappa, \delta \rangle^{\perp})$. By (c), $q | (\langle \kappa, \delta \rangle^{\perp})$ is even. It is of signature (0,8). Thus, $q | (\langle \kappa, \delta \rangle^{\perp})$ must be isomorphic to $-E_8$. Since q is of type (1,9) it is odd. This means that $q(\delta) \equiv 1(2)$. By subtracting a multiple of κ from δ we can arrange that $q(\delta) = 1$. This proves that for any class $\kappa \in \Lambda$ satisfying (a), (b), and (c) of the lemma there is a decomposition

$$\Lambda \cong \langle \kappa, \delta \rangle \oplus (\langle \kappa, \delta \rangle^{\perp})$$

with

$$q(\delta) = 1, \qquad \kappa \cdot \delta = 1, \qquad q | (\langle \kappa, \delta \rangle^{\perp}) \cong -E_8.$$

Thus, if κ and κ' are two such classes, then there is an automorphism of (Λ, q) carrying κ to κ'. In particular, for any such κ there is a standard basis for Λ, $(x', \alpha_1', \cdots, \alpha_9')$ with $\kappa = 3x' - \sum_{i=1}^{9} \alpha_i'$. Let C' be the chamber with corner x' and with $\alpha_i' \cdot C' \geqslant 0$ for $1 \leqslant i \leqslant 9$. Then $\kappa(x', C') = \kappa$. $\square$

Since $q(\kappa(x, C)) = 9 - n$, $\kappa(x, C) \in \mathbf{R}^{+} \cdot \mathbf{H}$ for $n < 9$, $\kappa(x, C)$ determines an ideal point in $\overline{\mathbf{H}}$ for $n = 9$ and $(\kappa(x, C)^{\perp}) \cap \mathbf{H}$ is a wall for $n \geqslant 10$.

Definition 2.5. If x is a corner of a chamber C, then we define

$$P(x,C) = C \cap \{ y \in \mathbf{H} \mid \kappa(x,C) \cdot y \geqslant 0 \}.$$

(Clearly, if $n \leqslant 9$, then $P(x,C) = C$.) Any subset of $\mathbf{H}$ of the form $P(x,C)$ is called a *P-cell.*

Now we state the basic properties of P-cells. Let us begin with some obvious one

(a) If φ: $(\Lambda, q) \to (\Lambda', q')$ is an isomorphism of forms of type $(1, n)$, then the extension $\bar{\varphi}$: $\Lambda_{\mathbf{R}} \xrightarrow{\cong} \Lambda'_{\mathbf{R}}$ sends P-cells for q to P-cells for q'.

(b) $P(x,C)$ contains a neighborhood in C of x. In particular, $P(x,C) \neq \varnothing$.

(c) $P(x,C)$ is a chamber for the set of walls

(2.6) $\mathscr{W}_1 \cup \{(\kappa(x,C)^{\perp}) \cap \mathbf{H}\}$.

(d) If $P(x,C) \neq C$, then $\kappa(x,C)^{\perp} \cap \mathbf{H}$ defines a wall of $P(x,C)$.

(e) If W^{α} is a wall of $\mathscr{W}_1$ passing through x, and if we identify W^{α} with $\mathbf{H}(q \mid \Lambda_{\alpha})$, then $P(x,C) \cap W^{\alpha} = P(x, C \cap W^{\alpha})$.

The main results we need about P-cells are contained in the following two propositions whose proofs are deferred to §4.

Proposition 2.7. *Let q: $\Lambda \to \mathbf{Z}$ be a quadratic form of type $(1,n)$. Let $C \subset \mathbf{H}(q)$ be a chamber, and let x be a corner for C.*

(a) *If $n < 9$, then $P(x,C) = C$, $\kappa(x,C) \in \mathbf{R}^{+}\cdot(\operatorname{int} C)$, and for every $\alpha \in \mathscr{F}_C$, we have $\alpha \cdot \kappa(x,C) = 1$.*

(b) *If $n = 9$, then $P(x,C) = C$, $\kappa(x,C) \in \overline{(\mathbf{R}^{+}\cdot C)}$, and for every $\alpha \in \mathscr{F}_C$, we have $\alpha \cdot \kappa(x,C) = 1$.*

(c) *If $n = 10$, then $P(x,C) = C$, $\kappa(x,C) \in \mathscr{F}_C$, and for any $\alpha \in \mathscr{F}_C$, distinct from $\kappa(x,C)$, we have $\alpha \cdot \kappa(x,C) = 1$.*

(d) *If $n \geqslant 11$, then $P(x,C) \subsetneqq C$, $\kappa(x,C)$ defines an oriented wall of $P(x,C)$, and if $\alpha \in \mathscr{F}_C$ defines any other oriented wall of $P(x,C)$, then $\alpha \cdot \kappa(x,C) = 1$.*

Proposition 2.8. (a) *If $P(x,C) = P(x',C')$, then $\kappa(x,C) = \kappa(x',C')$.*

(b) *If $(\operatorname{int} P(x,C)) \cap (\operatorname{int} P(x',C')) \neq \varnothing$, then $P(x,C) = P(x',C')$.*

Definition 2.9. Notice from (2.8)(a) that $\kappa(x,C)$ is an invariant of the P-cell itself, not its description as $P(x,C)$. Thus for $n \geqslant 10$, the wall $\kappa(x,C)^{\perp} \cap \mathbf{H}$ of $P(x,C)$ is a distinguished wall. We call it the *exceptional* wall of the P-cell.

Propositions 2.7 and 2.8 are purely arithmetic in nature and presumably can be established by direct arithmetic arguments. We prefer to argue using the algebraic geometry of good generic rational surfaces. This involves a detour to first establish the requisite results in algebraic geometry. That is the purpose of the next section.

3. The Kähler cone of a good generic surface

Recall from §I.2 that a good generic surface X is one for which (a) $K_X = -F$ for F a smooth elliptic curve, and (b) if C is a smooth rational curve on X, then $C^2 \geqslant -1$. All such surfaces are rational and have self-intersection forms of signature $(1, n)$. If $n \neq 1$, then the form is actually of type $(1, n)$. According to I(2.6), for each $n \geqslant 0$, there is a good generic surface X whose form is of type $(1, n)$. We fix a good generic surface X of type $(1, n)$. We denote by $\Lambda(X)$ the lattice $H^2(X; \mathbf{Z})$, by q_X the self-intersection quadratic form, by $\mathscr{P}(X)$ the set $\{x \in \Lambda_{\mathbf{R}}(X)\colon q_X(x) > 0\}$, by $\mathscr{W}_1(X)$ the set of walls in $\mathbf{H}(X)$ defined by classes $\alpha \in \mathscr{F}(X)$, i.e. by classes $\alpha \in \Lambda(X)$ satisfying $q_X(\alpha) = -1$, and by $\mathscr{C}(X)$ the set of chambers in $\mathbf{H}(X)$ associated to $\mathscr{W}_1(X)$.

The surface X has many Kähler metrics. Associated to each such metric g, there is its Kähler form ω and associated cohomology class $[\omega] \in H^2(X; \mathbf{R})$. Changing g by a positive factor changes $[\omega]$ by the same factor.

Definition 3.1. Let $\mathring{\mathscr{K}}(X) \subset \mathbf{H}(X)$ be the set of all Kähler cohomology classes contained in $\mathbf{H}(X)$. Let $\mathscr{K}(X) \subset \mathbf{H}(X)$ be the closure of $\mathring{\mathscr{K}}(X)$ in $\mathbf{H}(X)$. (Notice that the set of all Kähler cohomology classes is $\mathbf{R}^+ \cdot \mathring{\mathscr{K}}(X)$.)

Let $H \subset X$ be a hyperplane section for a projective embedding of X. Then $H^2 > 0$, so that $H \in \mathscr{P}(X)$. Let $\mathscr{P}_+(X)$ denote the component of $\mathscr{P}(X)$ containing H, let $\mathbf{H}_+(X)$ be the corresponding component of $\mathbf{H}(X)$, and let $\mathscr{Q}_+(X)$ be the corresponding component of $\mathscr{Q}(X) = \mathscr{Q}(q_X)$.

Claim 3.2. (a) $\mathscr{K}(X) \subset \mathbf{H}_+(X)$.

(b) If $C \subset X$ is any irreducible curve with $C^2 \geqslant 0$ then $C \in \mathscr{Q}_+(X)$.

(c) If L is an ample line bundle over X, then $c_1(L) \subset \mathscr{P}_+(X)$.

Proof. If $C \subset X$ is any irreducible curve and g is a Kähler metric for X with associated Kähler form ω, then $[\omega] \cdot C = \int_C \omega = \operatorname{vol}(C) > 0$. Thus, if $C^2 \geqslant 0$, C and $[\omega]$ lie in the same component of $\mathscr{Q}(X)$. Applying this remark to $C = H$, we see that $\mathring{\mathscr{K}}(X) \subset \mathbf{H}_+(X)$; hence so is its closure $\mathscr{K}(X)$. Part (b) is immediate from this argument. Lastly, if L is ample then NL is very ample for some $N > 0$ so that $Nc_1(L)$ is the class of a hyperplane section of a projective embedding. Hence $Nc_1(L) \in \mathscr{P}_+(X)$, and consequently $c_1(L) \in \mathscr{P}_+(X)$. $\square$

Definition 3.3. Let $\mathscr{I}(X)$ be the set of irreducible curves in X, and let $\mathscr{E}(X)$ be the set of exceptional curves.

Proposition 3.4. (a) $\mathscr{K}(X) = \{y \in \mathbf{H}_+(X) \mid y \cdot F \geqslant 0$ *and* $y \cdot E \geqslant 0$ *for all* $E \in \mathscr{E}(X)\}$.

(b) $\mathring{\mathscr{K}}(X) =$ *interior of* $\mathscr{K}(X)$.

Proof. General theory ([2, p. 238]) tells us that $\mathring{\mathscr{K}}(X)$ is equal to the interior of $\mathscr{K}(X)$. Also by [2, p. 127], we have

$$\mathscr{K}(X) = \{y \in \mathbf{H}_+(X) \mid y \cdot C \geqslant 0 \text{ for all } C \in \mathscr{I}(X)\}.$$

As we have already seen, if $C \in \mathscr{I}(X)$ and $C^2 \geqslant 0$, then $C \in \mathscr{P}_+(X)$. By Lemma 1.1, $y \cdot C > 0$ for any $y \in \mathbf{H}_+(X)$. Thus $\mathscr{K}(X)$ is also given as

$$\mathscr{K}(X) = \{y \in \mathbf{H}_+(X) \mid y \in C \geqslant 0 \text{ for all } C \in \mathscr{I}(X) \text{ with } C^2 < 0\}.$$

By Lemma I(2.2), except possibly for $F \subset X$, all $C \in \mathscr{I}(X)$ with $C^2 < 0$ are exceptional curves. □

Let $E \subset X$ be an exceptional curve and let ρ: $X \to Y$ be the result of blowing down E. Then by I(2.3) Y is a good generic surface. We can identify $(\Lambda(Y), q_Y)$ with $(\rho^*\Lambda(Y), q_X \mid \rho^*\Lambda(Y))$ inside $(\Lambda(X), q_X)$. Furthermore,

$$\rho^*\Lambda(Y) = ([E]^\perp) \cap \Lambda(X).$$

Proposition 3.5. $\mathscr{K}(X) \cap ([E]^\perp) = \rho^*\mathscr{K}(Y)$. *In particular, all exceptional curves on X define walls of $\mathscr{K}(X)$.*

Proof. Let $H \subset Y$ be a hyperplane section missing the point $\rho(E)$. Then $H^2 > 0$. Clearly, $[H]$ lies in $\mathscr{P}_+(Y)$. Since $\rho^*[H] = [\rho^{-1}H]$ is also algebraic, $\rho^*[H]$ lies in $\mathscr{P}_+(X)$. This proves that $\rho^*\mathbf{H}_+(Y) \subset \mathbf{H}_+(X)$.

Now if $y \in \mathbf{H}_+(Y)$ and $y \cdot C \geqslant 0$ for all irreducible curves $C \subset Y$, then $\rho^*y \in \mathbf{H}_+(X)$ and $(\rho^*y) \cdot C' = y \cdot \rho_*(C') \geqslant 0$ for all irreducible curves $C' \subset X$. Thus, $\rho^*y \in \mathscr{K}(X)$. This proves $\rho^*\mathscr{K}(Y) \subset \mathscr{K}(X) \cap ([E]^\perp)$.

Conversely, suppose $x \in \mathscr{K}(X) \cap ([E]^\perp)$. Then $x = \rho^*y$ for some $y \in \mathbf{H}_+(Y)$. Clearly, if $C \subset Y$ is irreducible, then $y \cdot C = (\rho^*y) \cdot (\rho^*C) = x \cdot \rho^*C$. But ρ^*C is effective in X. (It is the proper transform of C plus some nonnegative multiple of E.) Since $x \in \mathscr{K}(X)$, $x \cdot \rho^*C \geqslant 0$. □

Now we are ready to show that $\mathscr{K}(X)$ is a P-cell in $\mathbf{H}(X)$. Let ρ: $X \to \mathbf{P}^2$ be a holomorphic map with exceptional fibers $E_1, \cdots, E_n$ where each E_i is an exceptional curve. (Such a map exists by Proposition I(2.4).) Let $H \subset \mathbf{P}^2$ be a hyperplane section and set $x_0 = \rho^*(H)$ and $e_i = [E_i]$ for $1 \leqslant i \leqslant n$. Then $(x_0, e_1, \cdots, e_n)$ is a standard basis for $\Lambda(X)$. There is a unique chamber $C_0(X)$ which has x_0 as a corner and for which the given basis is adapted (i.e., for which $e_i \cdot C_0(X) \geqslant 0$ for $i = 1, \cdots, n$). Notice that $\kappa(x_0, C_0(X)) = 3x_0 - \sum_{i=1}^n e_i = 3\rho^*H - \sum_{i=1}^n [E_i] = -K_X$.

Proposition 3.6. (a) $\mathscr{K}(X) = P(x_0, C_0(X))$.

(b) *If $\kappa(x_0, C_0(x))$ does not define a wall of $\mathscr{K}(X)$, then $\mathscr{K}(X) = C_0(X)$.*

(c) *If* $\alpha \in \mathcal{F}(X)$ *is not equal to* $\kappa(x_0, C_0(X))$ *and if* α *defines an oriented wall for* $\mathcal{K}(X)$, *then* $\alpha \cdot \kappa(x_0, C_0(X)) = 1$.

(d) $\kappa(x_0, C_0(X)) = -K_X$.

Proof. We have already established (d) by direct computation. We consider (a), (b) and (c).

Claim I. $\mathcal{K}(X) \subseteq C_0(X)$.

Proof of Claim I. Since $x_0 = \rho^* H$, $x_0 \cdot C \geqslant 0$ for any irreducible curve $C \subset X$. Hence, $x_0 \in \mathcal{K}(X)$. Clearly $\mathcal{K}(X) \cdot e_i \geqslant 0$ for all $i = 1, \cdots, n$. Since $\mathring{\mathcal{K}}(X) = \operatorname{int} \mathcal{K}(X)$ is dense in $\mathcal{K}(X)$, it follows that $\mathcal{K}(X) \cap \operatorname{int} C_0(X) \neq \varnothing$.

To complete the proof of Claim I, we show that $\mathcal{K}(X)$ is contained in a single chamber. To do this it suffices to show that $\mathring{\mathcal{K}}(X)$ meets no wall of $\mathcal{W}_1(X)$. Suppose, by contradiction, that $W \in \mathcal{W}_1(X)$ contains a Kähler class $[\omega]$. Let $W = W^\alpha$ for some $\alpha \in \mathcal{F}(X)$. By changing the sign of α, if necessary, we arrange that $\alpha \cdot K_X \leqslant 0$. Let L_α be the holomorphic line bundle with Chern class α. Then by Riemann-Roch

$$\chi(X; L_\alpha) = 1 + \left(c_1(L_\alpha)^2 - L_\alpha \cdot K_X\right)/2 = 1 + \left(\alpha^2 - \alpha \cdot K_X\right)/2 \geqslant 1/2.$$

Hence $\chi(X; L_\alpha) \geqslant 1$.

By Serre duality $H^2(X; L_\alpha) \cong H^0(X; K_X \otimes L_\alpha^{-1})$. Since $\dim H^0(X; L_\alpha) + \dim H^0(K_X \otimes L_\alpha^{-1}) \geqslant \chi(X; L_\alpha) \geqslant 1$ we see that either L_α or $K_X \otimes L_\alpha^{-1}$ has a global holomorphic section. This means that either α or $K_X - \alpha$ is an effective divisor. Since $[\omega] \cdot \alpha = 0$, α cannot be effective. Since $[\omega] \cdot (K_X - \alpha) = [\omega] \cdot K_X = -[\omega] \cdot F < 0$, neither can $K_X - \alpha$. This is a contradiction.

Claim II. The walls of $\mathcal{K}(X)$ are contained in

$$\mathcal{W}_1(X) \cup \left\{\left(\kappa(x_0, C_0(X))^{\perp}\right) \cap \mathbf{H}(X)\right\}.$$

Proof. All walls of $\mathcal{K}(X)$, except possibly one, are defined by exceptional curves in X. These, being of square -1, define walls of $\mathcal{W}_1(X)$. The only other possible wall of $\mathcal{K}(X)$ is defined by $[F] = 3x_0 - \sum_{i=1}^n e_i$. This class is the canonical class $\kappa(x_0, C_0(X))$.

Proof of (a) *and* (b). We have established that $\mathcal{K}(X) \subseteq C_0(X)$, that $\mathcal{K}(X) \cdot \kappa(x_0, C_0(X)) \geqslant 0$, and that all walls of $\mathcal{K}(X)$ are perpendicular either to $\kappa(x_0, C_0(X))$ or to classes in $\mathcal{F}(X)$. Hence, all walls of $\mathcal{K}(X)$, except at most 1, are walls of $\mathcal{W}_1(X)$. The exceptional wall, if it exists, is $(\kappa(x_0, C_0(X))^{\perp}) \cap \mathbf{H}(X)$. This shows that

$$\mathcal{K}(X) = C_0(X) \cap \{ y \in \mathbf{H}(X) \mid y \cdot \kappa(x_0, C_0(X)) \geqslant 0\},$$

which is exactly the definition of $P(x_0, C_0(X))$. This proves (a), and (b) is now clear.

Proof of (c). All othe walls of $\mathcal{K}(X)$ are defined by $[E]$ for $E \subset X$ exceptional curves. But for any exceptional curve E we have $K_X \cdot E = -1$. Since $K_X = -\kappa(x_0, C_0(X))$, this gives $E \cdot \kappa(x_0, C_0(X)) = +1$. □

Proposition (3.7). *If X is a good generic surface and $n \leqslant 8$, or if X is a generic rational elliptic surface and $n = 9$, then α is the cohomology class dual to an exceptional curve E if and only if $q_X(\alpha) = -1$ and $\alpha \cdot \kappa(x_0, C_0(X)) = 1$.*

Proof. Clearly, if $\alpha = [E]$ where E is an exceptional curve, then

$$q_X(\alpha) = \alpha \cdot [K_X] = -1.$$

Conversely, suppose that $q_X(\alpha) = \alpha \cdot [K_X] = -1$. We let L_α be the holomorphic line bundle corresponding to α. Suppose that we are in the case $n = 9$ and X is a generic rational elliptic surface (the case $n \leqslant 8$ is similar and simpler). By Riemann-Roch as in the proof of Claim I of (3.6), since $\chi(L_\alpha) = 1$, either L_α or $K_X \otimes L_\alpha^{-1}$ is of the form $\mathcal{O}_X(D)$, where D is an effective divisor. Since $K_X = \mathcal{O}_X(-F)$, where $|F|$ is a basepoint-free linear series, $F \cdot D \geqslant 0$ for every effective divisor D. Moreover $F \cdot L_\alpha = 1$ and $F \cdot K_X = 0$. Thus, if $K_X \otimes L_\alpha^{-1} = \mathcal{O}_X(D)$, $F \cdot D = -1$, which is a contradiction. It follows that $L_\alpha = \mathcal{O}_X(D)$ for an effective divisor D. Since $F \cdot D = 1$, and F is a fiber of the elliptic fibration, D is linearly equivalent to $E + nF$, where E is a section of the fibration and hence an exceptional curve, and $n \geqslant 0$. Since

$$D^2 = (E + nF)^2 = -1 + 2n = -1,$$

$n = 0$, and D is an exceptional curve. □

Remark 3.8. It is easy to see that if X is a good generic surface and $n \geqslant 10$, then the conclusion of (3.7) is no longer true.

4. A study of P-cells

In the last section we showed that for X a good generic surface of type $(1, n)$ $\mathcal{K}(X) = P(x_0, C_0(X))$ for an appropriate choice of x_0 and $C_0(X)$. Actually, as the next lemma shows, we can use $\mathcal{K}(X)$ as a model for any P-cell.

Lemma 4.1. *Let q: $\Lambda \to \mathbf{Z}$ be a quadratic form of type $(1, n)$. Let $P(x, C) \subset \mathbf{H}(q)$ be a P-cell. Then there is an isometry of quadratic forms φ: $(\Lambda, q) \xrightarrow{\cong} (\Lambda(X), q_X)$ with $\varphi(P(x, C)) = P(x_0, C_0(X))$.*

Proof. Let $(x, e_1, \cdots, e_n)$ be a standard basis for Λ adapted to C. Let $(x_0, e_1', \cdots, e_n')$ be a standard basis for $\Lambda(X)$ as in Proposition 3.6. The isomorphism φ: $\Lambda \to \Lambda(X)$ defined by $\varphi(x) = x_0$ and $\varphi(e_i) = e_i'$ for $1 \leqslant i \leqslant n$, is as required. □

Now we are ready to begin the proofs of (2.7) and (2.8).

Proof of (2.7). By the previous lemma it suffices to prove (2.7) for the P-cell $\mathcal{K}(X) = P(x_0, C_0(X))$, where X is a good generic surface of type $(1, n)$.

Proof of (a) *and* (b). If $n \leqslant 9$, then $q_X(\kappa(x_0, C_0(X))) \geqslant 0$. Thus, $\kappa(x_0, C_0(X))$ does not define a wall in $\mathbf{H}(X)$. Hence, by Proposition 3.6(b), $\mathscr{K}(X) = C_0(X)$.

If $n < 9$, then $q_X(\kappa(x_0, C_0(X))) > 0$. By (3.6)(c), $\alpha \cdot \kappa(x_0, C_0(X)) > 0$ for all $\alpha \in \mathscr{F}_{C_0(X)}$. Therefore, $\kappa(x_0, C_0(X)) \in \mathbf{R}^+ \cdot (\operatorname{int} C_0(X))$. If $n = 9$, then $q_X(\kappa(x_0, C_0(X))) = 0$ and $\kappa(x_0, C_0(X)) \in \mathscr{Z}(q_X)$. As before, it has positive intersection with all classes defining oriented walls of $C_0(X)$. Thus, $\kappa(x_0, C_0(X)) \in \overline{\mathbf{R}^+ \cdot C_0(X)}$.

Proof of (c) *and* (d). If $n \geqslant 10$, then $q_X(\kappa(x_0, C_0(X))) < 0$. Thus by Corollary 1.21, it cannot be the case that $\kappa(x_0, C_0(X))$ has positive intersection with every class defining an oriented wall of $C_0(X)$. By Proposition 3.6(b) and (c), it follows that $\kappa(x_0, C_0(X))$ defines a wall of $\mathscr{K}(X)$. Also, by Proposition 3.6(b) for every $\alpha \in \mathscr{F}(X)$ defining an oriented wall of $\mathscr{K}(X)$ with $\alpha \neq \kappa(x_0, C_0(X))$, we have $\alpha \cdot \kappa(x_0, C_0(X)) = 1$.

If $n = 10$, then $q_X(\kappa(x_0, C_0(X))) = -1$, and $\kappa(x_0, C_0(X))$ defines a wall in $\mathscr{W}_1(X)$. Thus, all walls of $\mathscr{K}(X)$ are walls in $\mathscr{W}_1(X)$. Hence, $\mathscr{K}(X)$ is a union of chambers. Since $\mathscr{K}(X) \subset \overline{C_0(X)}$, it follows that $\mathscr{K}(X) = C_0(X)$.

If $n > 10$, then $q_X(\kappa(x_0, C_0(X))) < -1$. Since this class is indivisible in $\Lambda(X)$, the wall it defines is not a wall in $\mathscr{W}_1(q)$. Since this wall is a wall of $\mathscr{K}(X)$, it must be the case that $\mathscr{K}(X) \subsetneqq C_0(X)$. This completes the proof of Proposition 2.7. □

Before beginning the proof of Proposition 2.8, we need a couple of lemmas.

Lemma 4.2. *Suppose $n \geqslant 3$, and let q: $\Lambda \to \mathbf{Z}$ be a form of type $(1, n)$. Let $P(x, C)$ be a P-cell for q. Let $\alpha \in \Lambda$ define a wall of $P(x, C)$ distinct from $(\kappa(x, C)^\perp) \cap \mathbf{H}$ (if the latter is a wall of $P(x, C)$). Then W^α is an ordinary wall in $\mathscr{W}_1(q)$.*

Proof. By Lemma 4.1, it suffices to prove this result for the P-cell $P(x_0, C_0(X))$ where X is a good generic surface of type $(1, n)$. By Proposition 3.4, $\alpha = [E]$ where $E \subset X$ is an exceptional curve. Thus, $q_X(\alpha) = -1$ and $W^\alpha \in \mathscr{W}_1$. Furthermore, in the notation of (1.12), $\Lambda_\alpha(X) = \rho^*\Lambda(Y)$ where ρ: $X \to Y$ is the result of contracting E. To prove that W^α is an ordinary wall is to prove that $(\Lambda(Y), q_Y)$ is of type $(1, n-1)$. Since, by Lemma I(2.3), Y is a good generic surface and since rank $\Lambda(Y) = n \neq 2$, by Proposition I(2.4), Y is of type $(1, n-1)$. □

Lemma 4.3. *Let $n \geqslant 10$, and let q: $\Lambda \to \mathbf{Z}$ be a form of type $(1, n)$. Suppose $P(x, C)$ is a P-cell for q. The wall $(\kappa(x, C)^\perp) \cap \mathbf{H}(q)$ for $P(x, C)$ is distinguished among all walls of $P(x, C)$ by being the only one which is not an ordinary wall in $\mathscr{W}_1(q)$.*

Proof. Since $n \geqslant 10$, we know that $(\kappa(x, C)^\perp) \cap \mathbf{H}(q)$ is a wall of $P(x, C)$. By Lemma 4.2 all other walls of $P(x, C)$ are ordinary walls in $\mathscr{W}_1(q)$. Since

$\kappa(x, C)$ is indivisible in Λ and $q(\kappa(x, C)) = 9 - n$, the wall $(\kappa(x, C)^{\perp}) \cap \mathbf{H}$ is not in $\mathscr{W}_1(q)$ unless $n = 10$. In this case, $q | ((\kappa(x, C)^{\perp}) \cap \Lambda$ is even, by (2.4)(c), so the wall is extraordinary. □

Corollary 4.4. *Let q: $\Lambda \to \mathbf{Z}$ and q': $\Lambda' \to \mathbf{Z}$ be forms of type $(1, n)$, for some $n \geqslant 2$. Let $P \subset \mathbf{H}(q)$ and $P' \subset \mathbf{H}(q')$ be P-cells, say $P = P(x, C)$ and $P' = P(x', C')$. Suppose φ: $\Lambda \to \Lambda'$ is an isomorphism of forms and that $\bar{\varphi}(P) = P'$ where $\bar{\varphi}$ is the natural extension of φ to $\Lambda_{\mathbf{R}}$. Then $\varphi(\kappa(x, C)) = \kappa(x', C')$.*

N.B. We assume that $\varphi(P) = P'$ but not that $\varphi(x) = x'$.

Proof. If $2 \leqslant n \leqslant 9$, then $P = C$ and $P' = C'$. Thus, our assumption becomes $\bar{\varphi}(C) = C'$ in this case. By Proposition 3.6(c), if $\alpha \in \mathscr{F}_C$, then $\alpha \cdot \kappa(x, C) = 1$. Likewise, if $\alpha' \in \mathscr{F}_{C'}$, then $\alpha' \cdot \kappa(x', C') = 1$. Clearly then, $\varphi(\kappa(x, C))$ and $\kappa(x', C')$ both have the property that their intersection with any $\alpha \in \mathscr{F}_{C'}$ is 1. By Lemma 1.17 $\mathscr{F}_{C'}$ spans Λ'. Thus, $\varphi(\kappa(x, C)) = \kappa(x', C')$ in this case.

Now suppose that $n \geqslant 10$. By Lemma 4.3 the fact that $\bar{\varphi}(P) = P'$ implies that $\bar{\varphi}((\kappa(x, C)^{\perp}) \cap \mathbf{H}(q)) = (\kappa(x', C')^{\perp}) \cap \mathbf{H}(q')$. Hence $\bar{\varphi}(\kappa(x, C)^{\perp}) = (\kappa(x', C')^{\perp})$. Thus, $\varphi(\kappa(x, C)) = m\kappa(x', C')$ for some $m \in \mathbf{Z}$. Since $q(\kappa(x, C)) = q'(\kappa(x', C'))$, $m = \pm 1$. Since $P \cdot \kappa(x, C) \geqslant 0$, $P' \cdot \kappa(x, C') \geqslant 0$ and $\varphi(P) = P'$, it follows that $m \geqslant 0$. Thus $m = 1$, and $\varphi(\kappa(x, C)) = \kappa(x', C')$. □

Definition 4.5. Let P be a P-cell. We define the *canonical class* $\kappa(P)$ to be $\kappa(x, C)$ where $P = P(x, C)$.

Corollary 4.4 tells us that $\kappa(P)$ is well defined, i.e. independent of the representation of P as $P(x, C)$, provided $n \geqslant 2$. This is obvious for $n = 1$.

Lemma 4.6. (a) *Suppose that q: $\Lambda \to \mathbf{Z}$ is of type $(1, n)$ for some $n \geqslant 11$. Let P be a P-cell for q, and let $\alpha \in \mathscr{F}$ define an oriented, ordinary wall of P. Then $P \cap W^{\alpha} = P'$ is a P-cell for $q | \Lambda_{\alpha}$. Let $\kappa(P) \in \Lambda$ and $\kappa(P') \in \Lambda_{\alpha}$ be the canonical classes for these P-cells. We have*

$$\kappa(P) = \kappa(P') - \alpha.$$

(b) *$\{\alpha \in \mathscr{F} \mid \alpha$ defines an oriented ordinary wall for $P\}$ spans Λ.*

Proof. As before we identify (Λ, q) with $(\Lambda(X), q_X)$ for some good generic surface X of type $(1, n)$. We do this in such a way that P is identified with $\mathscr{K}(X)$. Thus $\kappa(P)$ becomes $-K_X$.

Since α defines an ordinary wall for P, it is identified with $[E]$ for some exceptional curve $E \subset X$. Let ρ: $X \to Y$ be the result of contracting E. By Proposition 3.5, $\mathscr{K}(X) \cap ([E]^{\perp}) = \rho^*\mathscr{K}(Y)$. Since Y is a good generic surface and $n \neq 2$, Y is of type $(1, n-1)$. Hence, $\mathscr{K}(Y)$ is a P-cell in $\mathbf{H}(Y) = \mathbf{H}(X) \cap ([E]^{\perp})$. Furthermore, the canonical class of this P-cell is $-K_Y$. We know that $K_X = \rho^*K_Y + [E]$. Hence, $-K_X = \rho^*(-K_Y) - [E]$. Translating back to Λ gives the claimed formula in (a).

If $n \leqslant 9$, then Part (b) is immediate from (1.17) and (2.7).

To prove (b) for $n > 9$ let x be a corner of P and $(x, e_0, \cdots, e_n)$ a standard basis adapted to the chamber containing P. Then by Part (a), $P \cap (e_{10}^{\perp}) \cap \cdots \cap (e_n^{\perp})$ is a chamber C for the form $q | (\langle e_{10}, \cdots, e_n \rangle^{\perp})$. Hence, the oriented ordinary walls of P include $(e_{10}^{\perp}), \cdots, (e_n^{\perp})$, and the oriented ordinary walls of C. Thus, the classes defining them span Λ. □

Proof of Proposition 2.8. Part (a) of (2.8) is immediate from Corollary 4.4. Part (b) is obvious from (2.7) for $n \leqslant 10$. We suppose that $n \geqslant 11$, and that, by induction, we have established Part (b) of (2.8) for $n - 1$.

If P_1 and P_2 are P-cells and if $(\operatorname{int} P_1) \cap (\operatorname{int} P_2) \neq \varnothing$, then clearly P_1 and P_2 lie in the same chamber, say C. Furthermore, $I = P_1 \cap P_2$ is a chamber for the locally finite set of walls

$$\mathscr{W}_1 \cup \left\{ \left(\kappa(P_1)^{\perp} \right) \cap \mathbf{H}, \left(\kappa(P_2)^{\perp} \right) \cap \mathbf{H} \right\}.$$

Thus, either I has at most two faces or a face of I lies in a wall in $\mathscr{W}_1$. Since $I \subset C$, by Corollary 1.20 $\overline{\mathbf{R}^+ \cdot I} \cap \mathscr{X}$ has no interior in $\mathscr{X}$. Hence, I must have more than two faces. Let F be a face of I contained in $W^{\alpha} \in \mathscr{W}_1$. Clearly, both P_1 and P_2 have faces, say P_1' and P_2' respectively, contained in W^{α}. Furthermore $P_1' \cap P_2' = F$ contains a nonempty open subset of W^{α}. By Lemma 4.6, P_1' and P_2' are P-cells for the form $q | \Lambda_{\alpha}$. By induction, it follows that $P_1' = P_2'$. In particular, $\kappa(P_1') = \kappa(P_2')$. But by Lemma 4.6 again, we have

$$\kappa(P_1) = \kappa(P_1') - \alpha \quad \text{and} \quad \kappa(P_2) = \kappa(P_2') - \alpha.$$

Consequently, $\kappa(P_1) = \kappa(P_2)$. It follows immediately that $P_1 = P_2$. This completes the proof of Proposition 2.8. □

Note. It happens that $P(x, C) = P(x', C')$ without $x = x'$.

Having established Propositions 2.7 and 2.8, we continue now with a further description of P-cells.

Let $q \colon \Lambda \to \mathbf{Z}$ be of type $(1, n)$, let C be a chamber and let $\{P_i\}_{i \in I_C}$ be the set of P-cells contained in C.

Lemma 4.7. *The $\{P_i\}_{i \in I_C}$ form a locally finite set of convex subspaces of C.*

Proof. This is immediate from the definition of a P-cell and the local finiteness result, Corollary 1.8. □

Definition 4.8. Let $Q(C) = \overline{\left(C - \bigcup_{i \in I_C} P_i\right)}$. We call this the *core* of C.

Lemma 4.9. *$Q(C)$ is convex. It is nonempty if and only if $n \geqslant 12$.*

Proof. Since $P_i = \{x \in C \,|\, x \cdot \kappa(P_i) \geqslant 0\}$ and since the walls $\{(\kappa(P_i)^{\perp}) \cap \mathbf{H}(q)\}_{i \in I_C}$ form a locally finite family, either $Q(C) = \varnothing$ or $Q(C)$ contains a nonempty open subset of C and is given by

$$Q(C) = \left\{ x \in C \,|\, x \cdot \kappa(P_i) \leqslant 0 \text{ for all } i \in I_C \right\}.$$

Clearly in both cases $Q(C)$ is convex.

If $n \leqslant 10$, then each P-cell is a chamber, and $Q(C) = \varnothing$. We consider $n \geqslant 11$. Let P_0 be a P-cell in C. Then the intersection $I_0 = (\kappa(P_0)^{\perp}) \cap C$ contains a nonempty open subset of $(\kappa(P_0)^{\perp}) \cap \mathbf{H}(q)$. If $Q(C) = \varnothing$, then every point of I_0 lies in a wall $(\kappa(P_i)^{\perp}) \cap \mathbf{H}(q)$, for some $P_i \neq P_0$. Since the walls are locally finite, there is another P-cell, $P_1 \neq P_0$ in C such that $(\kappa(P_1)^{\perp}) = (\kappa(P_0)^{\perp})$. Since $P_1 \neq P_0$, by Proposition 2.8(b), P_1 and P_0 lie on opposite sides of $(\kappa(P_0)^{\perp}) \cap C$. Now consider $P_0 \cup P_1 \subset C$. It is a closed subset of C without frontier in C. Hence $P_0 \cup P_1 = C$. We have just shown that for any $n \geqslant 11$, $Q(C) = \varnothing$ if and only if C is a union of two P-cells.

Suppose $n = 11$. Then $q(\kappa(P_0)) = -2$. Thus, there is a reflection in $(\kappa(P_0)^{\perp})$, $R \in A(q)$, defined by

$$R(x) = x + (x \cdot \kappa(P_0))\kappa(P_0).$$

Clearly, $R(C) = C$ and $R(P_0) = P_1$ is another P-cell in C with $\kappa(P_1) = -\kappa(P_0)$. Thus, $C = P_0 \cup P_1$ and $Q(C) = \varnothing$.

Conversely, suppose that $C = P_0 \cup P_1$. We shall show that $n = 11$. Let $\alpha \in \mathscr{F}$ define an ordinary wall for P_0. We consider $P_0 \cap W^{\alpha} = P_0'$. We claim that P_0' is a chamber for $\mathscr{W}_1(q \mid \Lambda_{\alpha})$. If so, then $(n-1) \leqslant 10$ so that $n \leqslant 11$. Since we are assuming $n \geqslant 11$, this forces $n = 11$.

First notice that since P_0 and P_1 meet along $(\kappa(P_0)^{\perp}) \cap \mathbf{H}$, $\kappa(P_0) = -\kappa(P_1)$. Thus, by Proposition 3.6(c) there is no wall W in $\mathscr{W}_1(q)$ which is a wall for both P_0 and P_1. Set $C' = C \cap W^{\alpha}$. It is our purpose to show that $P_0' = C'$. Suppose not. Then $(\kappa(P_0')^{\perp}) \cap \operatorname{int} C' \neq \varnothing$. Since $(\kappa(P_0')^{\perp}) \cap C' = (\kappa(P_0)^{\perp}) \cap C'$, this means

$$\left(\kappa'(P_0)^{\perp}\right) \cap (\operatorname{int} C') \neq \varnothing .$$

Consequently, C' meets both sides of $(\kappa(P_0)^{\perp})$. Thus, W^{α} contains a face of both P_0 and $P_1 = \overline{C - P_0}$. This is a contradiction. □

We need one last fact about P-cells:

Lemma 4.10. *If P is a P-cell and if $W \in \mathscr{W}_1(q)$ is a wall for P, then there is a P-cell P' meeting P along its face $P \cap W$.*

Proof. Let $W = W^{\alpha}$ and let R_{α} be reflection in W^{α}. Clearly $P' = R_{\alpha} \cdot P$ is as required. □

This completes our description of P-cells. The following pictures summarize what we have shown.

For $n \leqslant 8$, the chamber structure is well known (and classical). In particular, all chambers have finite volume, and the stabilizer of a chamber is essentially the Weyl group of an appropriate root system of type A, D, or E; moreover, the Weyl group arises as a finite subgroup of the Cremona group.

5. Super P-cells

Throughout this section we fix a quadratic form $q\colon \Lambda \to \mathbf{Z}$ of type $(1, n)$ and a P-cell $P \subset \mathbf{H}(q)$. We denote by $\mathscr{F}_P$ the set $\{\alpha \in \mathscr{F} \mid \alpha$ defines an oriented, ordinary wall of $P\}$. Let C be the chamber containing P.

Definition 5.1. $\mathscr{R}(P) \subset A(q)$ is the group generated by reflections in the $\{W^\alpha \mid \alpha \in \mathscr{F}_P\}$.

If $3 \leqslant n \leqslant 9$, then P is a chamber and all walls of P are ordinary. Hence, $\mathscr{R}(P)$ is the full group $\mathscr{R}$ generated by reflections in all walls in $\mathscr{W}_1(q)$. For $n \geqslant 10$ this is not true, and, as we shall see, $\mathscr{R}(P)$ is of infinite index in $\mathscr{R}$.

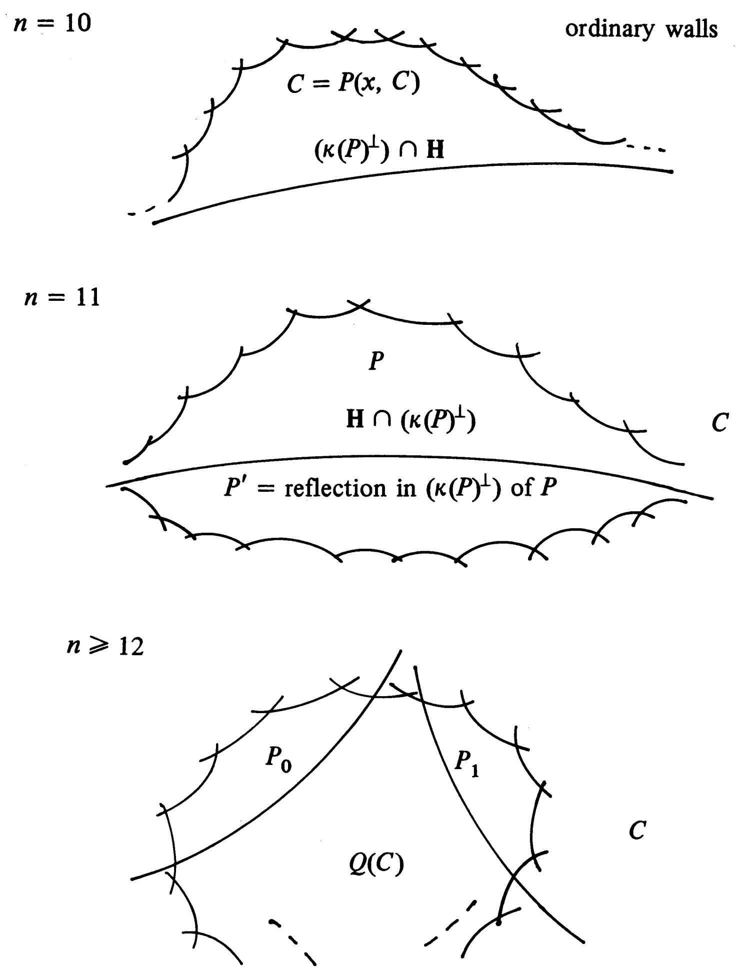

FIGURE 4

Definition 5.2. The *super* P-cell $\mathbf{S}(P)$ is defined by

$$\mathbf{S}(P) = \bigcup_{\varphi \in \mathscr{R}(P)} \varphi \cdot P.$$

Lemma 5.3. (a) *$\mathbf{S}(P)$ is a union of P-cells with disjoint interiors.*

(b) *If C' is a chamber, then $\mathbf{S}(P) \cap C' = \varnothing$ or $\mathbf{S}(P) \cap C'$ is a single P-cell.*

(c) *$\mathscr{R}(P)$ acts simply transitively on the P-cells in $\mathbf{S}(P)$.*

(d) *A P-cell P' is contained in $\mathbf{S}(P)$ if and only if there is a sequence $P = P_0, P_1, \cdots, P_t = P'$ of P-cells with P_i and P_{i+1} sharing a face in an ordinary wall in $\mathscr{W}_1$.*

(e) *If* $(\operatorname{int}\mathbf{S}(P)) \cap (\operatorname{int}\mathbf{S}(P')) \neq \varnothing$, *then* $\mathbf{S}(P) = \mathbf{S}(P')$ *and* $\mathscr{R}(P) = \mathscr{R}(P')$.

Proof of (a). This is clear from the definition and the fact that distinct P-cells have disjoint interiors.

Proof of (b) *and* (c). By definition $\mathscr{R}(P)$ acts transitively on the P-cells in $\mathbf{S}(P)$. Since $\mathscr{R}$ acts simply transitively on the set of chambers, $\mathscr{R}(P)$ acts freely on the set of chambers. Thus, if $\varphi \cdot P$, and $\varphi' \cdot P$ are contained in the same chamber C', then $\varphi \cdot C = \varphi' \cdot C$. Hence, $\varphi = \varphi'$. This proves that $\mathbf{S}(P) \cap C'$ is at most a single P-cell and that $\mathscr{R}(P)$ acts freely on the P-cells in $\mathbf{S}(P)$.

Proof of (d). Suppose $P = P_0, P_1, \cdots, P_t = P'$ is a path as described in (d). We prove by induction on i that $P_i \subset \mathbf{S}(P)$. This is clear for $i = 0$. Suppose we know the result for $(i - 1)$. Then the ordinary walls of P_{i-1} are images under an element of $\mathscr{R}(P)$ of the ordinary walls of P. Hence, reflections in them are conjugate by elements of $\mathscr{R}(P)$ to generates of $\mathscr{R}(P)$. Thus, reflections in all ordinary walls of P_{i-1} are elements of $\mathscr{R}(P)$. One of these reflections carries P_{i-1} to P_i. Since $P_{i-1} \subset \mathbf{S}(P)$, so is P_i.

Conversely, suppose $P' = (r_1 \circ \cdots \circ r_t) \cdot P$ where the r_i are reflections in the ordinary walls of P. Let $P_0 = P$, and $P_i = (r_1 \circ \cdots \circ r_i) \cdot P$ for $1 \leqslant i \leqslant t$. Clearly, $P_t = P'$. We claim that for each $0 \leqslant i \leqslant t - 1$, P_i and P_{i+1} share a face in an ordinary wall. To see this let $w = r_1 \circ \cdots \circ r_i$. Then $P_i = w \cdot P$ and $P_{i+1} = w \cdot (r_{i+1} \cdot P)$. Since P and $r_{i+1} \cdot P$ share a face in the fixed wall of r_{i+1}, which is ordinary, $w \cdot P$ and $w \cdot (r_{i+1} \cdot P)$ share a face in an ordinary wall.

Proof of (e). Suppose $(\operatorname{int}\mathbf{S}(P)) \cap (\operatorname{int}\mathbf{S}(P')) \neq \varnothing$. Then there are P-cells $P_0 \subset \mathbf{S}(P)$ and $P_0' \subset \mathbf{S}(P')$ with $(\operatorname{int} P_0) \cap (\operatorname{int} P_0') \neq \varnothing$. By Proposition 2.8(b), it follows that $P_0 = P_0'$. Clearly, by (d) $P' \subset \mathbf{S}(P)$, and thus $\mathbf{S}(P') \subset \mathbf{S}(P)$. By symmetry, $\mathbf{S}(P') = \mathbf{S}(P)$. Since $\mathscr{R}(P)$ is the group generated by the reflections in any P-cell contained in $\mathbf{S}(P)$, it also follows that $\mathscr{R}(P') = \mathscr{R}(P)$. □

Remark 5.4. If $3 \leqslant n \leqslant 9$, then the P-cells are chambers and all walls are ordinary. Thus a super P-cell is simply a component of $\mathbf{H}$. □

Corollary 5.5. $\mathbf{S}(P)$ *is connected.* $\square$

Now we state the main result of this section.

Proposition 5.6. $\mathbf{S}(P) \subset \mathbf{H}(q)$ *is convex.*

Proof. In light of Corollary 5.5, it suffices to show that $\mathbf{S}(P)$ is locally convex. Fix $x \in \mathbf{S}(P)$, say $x \in P_0$ a P-cell of $\mathbf{S}(P)$. Let C_0 be the chamber containing P_0. Let $N(x)$ be the union of all chambers containing x. Then $N(x)$ is a neighborhood in $\mathbf{H}(q)$ of x. We prove the local convexity of $\mathbf{S}(P)$ at x by showing that $\mathbf{S}(P) \cap N(x)$ is convex inside $N(x)$.

Suppose that the walls in $\mathscr{W}_1(q)$ passing through x are $W^{\alpha_1}, \cdots, W^{\alpha_t}$. We can choose the $\alpha_i \in \mathscr{F}_{C_0}$. The reflections R_{α_i} commute by Lemma 1.11. The group $\mathscr{R}_x$ that they generate is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^t$. It acts simply transitively on the 2^t chambers making up $N(x)$. We have two cases to consider.

Case 1. One of the W^{α_i} is an exceptional wall in $\mathscr{W}_1(q)$.

Case 2. All the W^{α_i} are ordinary walls in $\mathscr{W}_1(q)$.

In Case 1, $q \mid \Lambda_{\alpha_i}$ is even so that there is no other wall in $\mathscr{W}_1(q)$ meeting W^{α_i}. Hence, in this case $i = t = 1$ and $N(x)$ is the union of two chambers. By Lemma 4.2, all walls of P_0 except $(\kappa(P_0)^{\perp}) \cap \mathbf{H}(q)$ are ordinary in $\mathscr{W}_1(q)$. Thus, if W^{α_1} is extraordinary, it must be the case that $W^{\alpha_1} = (\kappa(P_0)^{\perp}) \cap \mathbf{H}(q)$. Hence, $q(\kappa(P_0)) = -1$, and $n = 10$. Thus, all P-cells are chambers in this case, and $\mathbf{S}(P) \cap N(x)$ is either a single chamber or all of $N(x)$. In either case the intersection is convex inside $N(x)$. (In fact, the intersection is a single chamber.)

We consider Case 2. The images of P_0 under all elements of $\mathscr{R}_x$ are contained in $\mathbf{S}(P)$. By Part (b) of Lemma 5.3, it follows that $\mathbf{S}(P) \cap N(x) = \bigcup_{w \in \mathscr{R}_x} w \cdot P_0$. The following claim establishes the convexity of $\mathbf{S}(P) \cap N(x)$ inside $N(x)$.

Claim 5.7. $\mathbf{S}(P) \cap N(x) = \{ y \in N(x) \mid y \cdot \kappa(w \cdot P_0) \geqslant 0$ *for all* $w \in \mathscr{R}_x \}$.

Proof of Claim. Let $T(x)$ denote the right-hand side. Since $\mathbf{S}(P) \cap N(x) = \bigcup_{w \in \mathscr{R}_x} w \cdot P_0$ and since $w \cdot P_0 = w \cdot C_0 \cap \{ y \mid y \cdot \kappa(w \cdot P_0) > 0\}$, we see that $T(x) \cap w \cdot C_0 \subset w \cdot P_0$ for all $w \in \mathscr{R}_x$. Hence, $T(x) \subset \mathbf{S}(P) \cap N(x)$.

We must establish the opposite inclusion. To do this, it suffices to show that for all $w, w' \in \mathscr{R}_x$,

$$\kappa(w \cdot P_0) \cdot (w' \cdot P_0) \geqslant 0,$$

or equivalently to show that for all $w \in \mathscr{R}_x$, $\kappa(w \cdot P_0) \cdot P_0 \geqslant 0$. Any $w \in \mathscr{R}_x$ can be written as $R_{\alpha_{i_k}} \circ \cdots \circ R_{\alpha_{i_1}}$, for some $i_1, \cdots, i_k$ distinct indices between 1 and t. Since $\alpha_i \cdot \alpha_j = 0$ for $i \neq j$, we see that

$$\kappa(w \cdot P_0) = w \cdot \kappa(P_0) = \kappa(P_0) + 2 \sum_{j=1}^{k} \left(\kappa(P_0) \cdot \alpha_{i_j}\right) \alpha_{i_j}.$$

Since each α_{i_j} defines an ordinary oriented wall for P_0, Proposition 2.7 tells us that $\kappa(P_0)\cdot\alpha_{i_j}=1$. Hence

$$\kappa(w\cdot P_0)=\kappa(P_0)+2\sum_{j=1}^{k}\alpha_{i_j}.$$

Since $\alpha_{i_j}\cdot C_0\geqslant 0$, for any $y\in C_0$, we have

$$\kappa(w\cdot P_0)\cdot y=\kappa(P_0)\cdot y+2\sum_{j=1}^{k}\alpha_{i_j}\cdot y\geqslant\kappa(P_0)\cdot y.$$

Thus, $\kappa(w\cdot P_0)\cdot P_0\geqslant 0$. $\square$

We finish the section by proving a result about the way a super P-cell meets infinity. Let us introduce some notation. Suppose that P is a P-cell contained in the chamber C. Suppose that $(x,e_1,\cdots,e_n)$ is a standard basis for Λ adapted to C and that $x\in P$. Then $\kappa(P)=3x-\sum_{i=1}^{n}e_i$. Set $k=\kappa(P)+\sum_{i=10}^{n}e_i=3x-\sum_{i=1}^{9}e_i$. Clearly, $P'=P\cap W^{e_{10}}\cap\cdots\cap W^{e_n}$ is a P-cell for the form $q\,|(\Lambda\cap(\langle e_{10},\cdots,e_n\rangle^{\perp})$ and $k=\kappa(P')$.

Proposition 5.8. (a) $k\cdot k=0$ *so that* k *determines a point in* $\overline{\mathbf{H}(q)}$.

(b) $k\in\overline{\mathbf{R}^{+}\cdot P'}\subset\overline{\mathbf{R}^{+}\cdot P}$.

(c) *If* $\mathbf{S}(P)$ *denotes the super P-cell in* $\mathbf{H}(q)$ *containing* P, *then the only walls in* $\mathscr{W}_1(q)$ *which pass through* $\operatorname{int}\mathbf{S}(P)$ *and contain* k *in their closures are* $W^{e_{10}},\cdots,W^{e_n}$.

Proof. It is clear that $k\cdot k=0$, that $k\cdot e_i=0$ for $10\leqslant i\leqslant n$, and that $k=\kappa(P')$. It is also clear by (2.7)(b) that the point determined by k in $\overline{\mathbf{H}(q)}$ is contained in $\overline{P'}$ and *a fortiori* in $\overline{P}$.

Now suppose W^{α} is a wall of $\mathscr{W}_1(q)$ and k is contained in the closure of W^{α}. This means $k\cdot\alpha=0$. We write $\alpha=\alpha_0+\sum_{i=10}^{n}r_ie_i$. Since $\alpha\cdot k=0$, we see $\alpha_0\cdot k=0$, and hence $q(\alpha_0)\leqslant 0$, by (1.1).

We have

$$-1=q(\alpha)=q(\alpha_0)-\sum_{i=10}^{n}r_i^2.$$

Hence $q(\alpha_0)=0$ or -1. Since by Lemma 2.4(c) $q\,|((\langle k,e_{10},\cdots,e_n\rangle^{\perp})\cap\Lambda$ is even, $q(\alpha_0)=0$. Thus r_i is nonzero for exactly one value of i, say $i=i_0$, and $\alpha_0=mk$ for some $m\in\mathbf{Z}$. It follows that $\alpha=mk\pm e_{i_0}$. By symmetry we can assume that $i_0=10$.

We shall show that, if $m\neq 0$, then W^{α} does not pass through $\mathbf{S}(P)$, which will complete the proof of the proposition. Since $(mk\pm e_{10})$ and $\mathbf{S}(P)$ are both invariant under the reflections R_{e_j} for $11\leqslant j\leqslant n$ and since $\mathbf{S}(P)$ is convex, if $W^{(mk\pm e_{10})}\cap(\operatorname{int}\mathbf{S}(P)))\neq\varnothing$, then $W^{(mk\pm e_{10})}\bigcap_{11\leqslant j\leqslant n}(e_j^{\perp})\cap(\operatorname{int}\mathbf{S}(P))\neq\varnothing$. Clearly, $\mathbf{S}(P)\cap(e_{11}^{\perp})\cap\cdots\cap(e_n^{\perp})$ is a super P-cell $\mathbf{S}(P'')$

for the form $q\,|\,\langle x, e_1, \cdots, e_{10}\rangle$. Thus, we have reduced the problem to showing that the result is true for $n = 10$.

Now suppose $n = 10$. We have $\kappa(P) = k - e_{10}$. Since $n = 10$, $(\kappa(P)^{\perp}) = ((k - e_{10})^{\perp})$ is an exceptional wall for P and hence for $\mathbf{S}(P)$. Likewise, $((k + e_{10})^{\perp})$ is an exceptional wall for the P-cell $R_{e_{10}} \cdot P$, and hence $((k + e_{10})^{\perp})$ is also a wall for $\mathbf{S}(P)$. The other walls $(mk \pm e_{10})^{\perp})$ are separated from $(e_{10}^{\perp})$ by either $((k + e_{10})^{\perp})$ or $((k - e_{10})^{\perp})$. Since $(e_{10}^{\perp})$ cuts $\mathbf{S}(P)$, none of the others does, by the convexity of $\mathbf{S}(P)$. □

6. The Donaldson invariant revisited

We begin by re-interpreting the results of §I.1 in terms of the chamber structure. Let M be a closed, smooth, simply connected, oriented 4-manifold of type $(1, n)$.

Proposition 6.1. *Suppose g_0 and g_1 are metrics satisfying condition* I(1.5). *Suppose ω_0 and ω_1 are self-dual harmonic 2-forms with respect to these metrics with $[\omega_0]$ and $[\omega_1] \in \mathbf{H}(M)$. Then $[\omega_0]$ and $[\omega_1]$ are both in the interiors of chambers. If they are in the interior of the same chamber, then $\tilde{\Gamma}_M(g_0, \omega_0) = \tilde{\Gamma}_M(g_1, \omega_1)$.*

Proof. This is immediate from I(1.14). □

If $\Lambda(M)$ is the lattice $H^2(M, \mathbf{Z})$ with self-intersection form q_M, we shall let $(\Lambda_{\mathbf{R}}(M), \bar{q}_M)$ denote $\Lambda(M) \otimes \mathbf{R}$, equipped with the natural extension of q_M. We may then define $\mathbf{H}(M)$, $\mathscr{C}(M)$, etc.

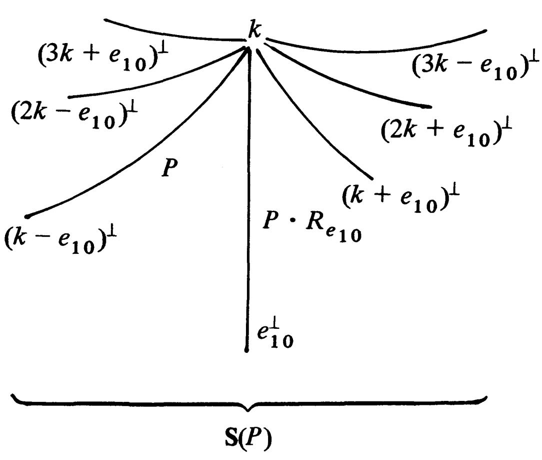

FIGURE 5

Let $\mathscr{C}'(M) \subset \mathscr{C}(M)$ be the subset of all chambers whose interiors contain the cohomology class of a self-dual 2-form for some metric on M.

Definition 6.2. There is a unique function

$$\Gamma_M\colon \mathscr{C}'(M) \to \Lambda(M)$$

defined by $\Gamma_M(C) = \tilde{\Gamma}_M(g, \omega)$ if $[\omega] \in C$. It satisfies:

(a) If $C \in \mathscr{C}'(M)$, then so is $-C$ and $\Gamma_M(-C) = -\Gamma_M(C)$.
(b) If C and $C' \in \mathscr{C}'(M)$ both lie in the same component of $\mathbf{H}(M)$, then

$$\Gamma_M(C') = \Gamma_M(C) - 2\sum_{i=1}^{r} \alpha_i \tag{6.2}$$

where the classes $\alpha_i \in \Lambda(M)$ run over all classes satisfying $q_M(\alpha_i) = -1$ and $\alpha_i \cdot C \geqslant 0 \geqslant \alpha_i \cdot C'$.

(That is to say, the α_i run over all classes defining walls separating C and C', oriented so that C is in the positive side.)

Clearly, $\bigcup_{C\in\mathscr{C}'(M)} C$ meets both components of $\mathbf{H}(M)$. Since the set of metrics on M is connected $\bigcup_{C\in\mathscr{C}'(M)} C$ meets each component of $\mathbf{H}(M)$ in a connected set. Thus, it is easy to see that there is a unique extension of Γ_M to a function

$$\Gamma_M\colon \mathscr{C}(M) \to \Lambda(M)$$

satisfying (6.2)(b). This extension also satisfies (6.2)(a). This then is the *Donaldson invariant*. Notice that Γ_M is determined by its value on any chamber via formulae (6.2)(a) and (b). Suppose that M' is also a closed, smooth, simply connected, oriented 4-manifold of type $(1, n)$, and that $f\colon M \to M'$ is an orientation-preserving diffeomorphism. Then f induces an isometry $f^*\colon \mathbf{H}(M') \to \mathbf{H}(M)$ and consequently a function $f^*\colon \mathscr{C}(M') \to \mathscr{C}(M)$. Clearly, $f^*\mathscr{C}'(M') = \mathscr{C}'(M)$. Furthermore, if $C' \in \mathscr{C}'(M')$ then $f^*\Gamma_{M'}(C') = \Gamma_M(f^*C')$. It follows that

(6.2)(c) For any $C' \in \mathscr{C}(M')$ we have $f^*\Gamma_{M'}(C') = \Gamma_M(f^*C')$.

We find it convenient to work with another invariant, Δ_M, derived from the Donaldson invariant. The domain of Δ_M is a convex cell decomposition of $\mathbf{H}(M)$ finer than $\mathscr{C}(M)$.

Definition 6.3. Let $\mathfrak{D}(M)$ be the union over $C \in \mathscr{C}(M)$ of $\{Q(C), P(x, C) \mid x \text{ is a corner of } C\}$. (Thus, each $D \in \mathfrak{D}(M)$ is either a P-cell or a core of a chamber.)

If $D \in \mathfrak{D}(M)$ then D is a closed convex subset of $\mathbf{H}(M)$, and its frontier in $\mathbf{H}(M)$ is a union of a locally finite set of faces with each face contained in a wall in $\mathscr{W}_1 \cup \mathscr{W}_{|n-9|}$. Furthermore, each $D \in \mathfrak{D}(M)$ is a subset of some

$C \in \mathscr{C}(M)$. If $n \leqslant 10$, then $\mathfrak{D}(M) = \mathscr{C}(M)$. If $n \geqslant 11$, then each $D \in \mathfrak{D}(M)$ is a union of chambers cut out by $\mathscr{W}_{n-9}(q_M)$. By Proposition 2.8(b), and the definition of $Q(C)$, the interiors of the $D \in \mathfrak{D}(M)$ are disjoint.

The invariant Δ_M which we shall now define is a function $\Delta_M\colon \mathfrak{D}(M) \to \Lambda(M)$.

Definition 6.4. (a) If P is a P-cell in $\mathbf{H}(M)$ contained in the chamber C, then

$$\Delta_M(P) = \Gamma_M(C) + \kappa(P).$$

(b) If Q is the core of a chamber C, then

$$\Delta_M(Q) = \Gamma_M(C).$$

Lemma 6.5. (a) *If $D \in \mathfrak{D}(M)$, then $-D \in \mathfrak{D}(M)$ and $\Delta_M(-D) = -\Delta_M(D)$.*

(b) *If $f\colon M \to M'$ is an orientation-preserving diffeomorphism, then it induces a map $f^*\colon \mathfrak{D}(M') \to \mathfrak{D}(M)$. We have*

$$f^*\Delta_{M'}(D') = \Delta_M(f^*D')$$

for any $D' \in \mathfrak{D}(M')$.

Proof of (a). Clearly, if $D \in \mathfrak{D}(M)$ and D is contained in the chamber C, then $-D \in \mathfrak{D}(M)$ and $(-D) \subseteq (-C)$. Furthermore, D is a P-cell if and only if $-D$ is. Since $\kappa(-P) = -\kappa(P)$ for any P-cell, (a) is immediate from (6.2)(a).

Proof of (b). If $f\colon M \to M'$ is a diffeomorphism, then clearly it induces a map $f^*\colon \mathfrak{D}(M') \to \mathfrak{D}(M)$. For any P-cell P' for $q_{M'}$, f^*P' is a P-cell for q_M and $\kappa(f^*P') = f^*\kappa(P)$. Given this result, (b) is immediate from (6.2)(c). □

We also need a formula for the effect on Δ_M of crossing a face of the decomposition $\mathfrak{D}(M)$.

Let us list the possibilities.

Lemma 6.6. *Suppose that D_0 and D_1 are distinct elements of $\mathfrak{D}(M)$ and that D_0 and D_1 share a face. Then one of the following holds.*

(a) *One of D_0 and D_1 is a P-cell P contained in a chamber C and the other is $Q(C)$. Furthermore $D_0 \cap D_1 = (\kappa(P)^{\perp}) \cap C$.*

(b) *D_0 and D_1 are both P-cells and $D_0 \cap D_1$ is a face of each which is contained in an ordinary wall in $\mathscr{W}_1$.*

(c) *D_0 and D_1 are both Q-cells, cores of chambers C_0 and C_1 respectively, and $D_0 \cap D_1 \subset C_0 \cap C_1 \subset W$ for some wall $W \in \mathscr{W}_1$.*

(d) *D_0 and D_1 are both P-cells, $D_0 \cap D_1 \subset (\kappa(D_0)^{\perp})$ and $n = 10$ or 11.*

Proof. That these are the only possibilities is immediate from Lemma 4.9 and Lemma 4.10. □

Having listed the possibilities, we now give the formula in each case.

Lemma 6.7. *Let M be a closed smooth, simply connected, oriented 4-manifold of type $(1, n)$.*

(a) *Suppose P is a P-cell for q_M, and P is contained in a chamber C. If $n \geqslant 12$, then*

$$\Delta_M(Q(C)) = \Delta_M(P) - \kappa(P).$$

(b) *Suppose that P and P' are P-cells sharing a face in an ordinary wall W^α in $\mathscr{W}_1(M)$. Then*

$$\Delta_M(P') = \Delta_M(P).$$

(c) *Suppose that $Q(C_0)$ and $Q(C_1)$ are Q-cells sharing a face in a wall W^α where $\alpha \in \mathscr{F}(M)$ and $\alpha \cdot C_0 \geqslant 0$.*

$$\Delta_M(Q(C_1)) = \Delta_M(Q(C_0)) - 2\alpha.$$

(d) *Suppose that P and P' are P-cells sharing a face in $(\kappa(P)^\perp)$. Then*

$$\Delta_M(P') = \Delta_M(P) + \frac{4 \cdot \kappa(P)}{q_M(\kappa(P))}.$$

Since $n = 10$ or 11 in this case (d) breaks up into:

$$(\mathrm{d}_1)\colon n = 10, \qquad \Delta_M(P') = \Delta_M(P) - 4\kappa(P),$$
$$(\mathrm{d}_2)\colon n = 11, \qquad \Delta_M(P') = \Delta_M(P) - 2\kappa(P).$$

Proof. (a) $\Delta_M(P) = \Gamma_M(C) + \kappa(P)$ and $\Delta_M(Q(C)) = \Gamma_M(C)$. From this (a) is clear.

(b) Suppose $P \subset C$ and $P' \subset C'$ and that $C \cap C' \subset W^\alpha$ for $\alpha \in \mathscr{F}(M)$ with $\alpha \cdot C \geqslant 0$. Then by Proposition 2.8, $P' = R_\alpha \cdot P$ where R_α is reflection in W^α. Thus $\kappa(P') = R_\alpha \cdot \kappa(P) = \kappa(P) + 2(\alpha \cdot \kappa(P))\alpha$. Since α defines an oriented ordinary wall of P, by Proposition 2.7 $\alpha \cdot \kappa(P) = 1$. Hence, we have $\kappa(P') = \kappa(P) + 2\alpha$. By (6.2)(b), $\Gamma_M(C') = \Gamma_M(C) - 2\alpha$. Hence

$$\begin{aligned}\Delta_M(P') &= \Gamma_M(C') + \kappa(P') = (\Gamma_M(C) - 2\alpha) + (\kappa(P) + 2\alpha)\\ &= \Gamma_M(C) + \kappa(P) = \Delta_M(P).\end{aligned}$$

(c)
$$\begin{aligned}\Delta_M(Q(C_1)) &= \Gamma_M(C_1) = \Gamma_M(C_0) - 2\alpha\\ &= \Delta_M(Q(C_0)) - 2\alpha.\end{aligned}$$

(The second equality is a consequence of (6.2)(b).)

(d_1): Here P is a chamber, say C, and P' is another chamber say C' and $C \cap C' \subset (\kappa(P)^\perp)$, the latter being an exceptional wall in $\mathscr{W}_1(M)$. Thus P' is the image of P under reflection in $(\kappa(P)^\perp)$. Hence, $\kappa(P') = -\kappa(P)$, so that

$$\begin{aligned}\Delta_M(P') &= \Gamma_M(C') + \kappa(P') = (\Gamma_M(C) - 2\kappa(P)) - \kappa(P)\\ &= \Gamma_M(C) - 3\kappa(P) = \Delta_M(C) - 4\kappa(P).\end{aligned}$$

(d_2): Here, P and P' both lie in the same chamber, say C. Again $\kappa(P') = -\kappa(P)$. Thus

$$\Delta_M(P') = \Gamma_M(C) + \kappa(P') = \Gamma_M(C) - \kappa(P)$$
$$= \Delta_M(C) - 2\kappa(P). \quad \square$$

Remark 6.8. If D and D' are elements of $\mathfrak{D}(M)$ which share a face in the wall W^α, with α chosen to be indivisible on $\Lambda(M)$ and to satisfy

$$\alpha \cdot D \geqslant 0, \quad \text{then } \Delta_M(D') = \Delta_M(D) - m\alpha$$

where $m = 0, 1, 2$, or 4.

Corollary 6.9. *Δ_M is constant on super P-cells in* $\mathbf{H}(M)$.

Proof. This is immediate from (6.7)(b) and (5.3)(d). $\square$

Now we come to our main technical result.

Theorem 6.10. *Suppose that for some P-cell, P, we have $\Delta_M(P) \in \overline{\mathbf{R}^+ \cdot P}$. Then for any $D \in \mathfrak{D}(M)$ we have $q_M(\Delta_M(D)) \leqslant q_M(\Delta_M(P))$ with equality if and only if D is a P-cell contained in one of the super P-cells* $\pm\, \mathbf{S}(P)$.

Proof. By (6.5)(a) it suffices to consider those D contained in the same component of $\mathbf{H}(M)$ as P. Let $D \in \mathfrak{D}(M)$ be contained in this component. Let γ be a geodesic arc in $\mathbf{H}(M)$ from a point $x \in \operatorname{int} P$ to a point $y \in \operatorname{int} D$. We choose x and y so that γ is generic with respect to the locally finite cell decomposition $\{D \in \mathfrak{D}(M)\}$ of $\mathbf{H}(M)$. Let $P = D_0, D_1, \cdots, D_s = D$ be the cells (in order) that γ crosses. For each i, D_{i-1} is separated from D_i by a wall W^{α_i} defined by a unique primitive class $\alpha_i \in \Lambda$ satisfying $\alpha_i \cdot D_{i-1} \geqslant 0$.

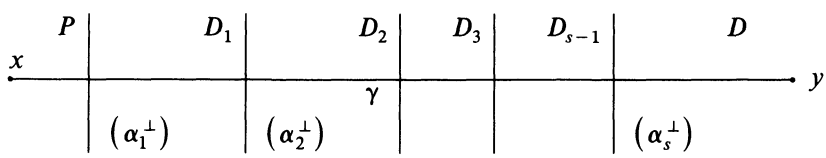

FIGURE 6

According to Remark 6.8, $\Delta_M(D) = \Delta_M(P) - \sum_{i=1}^s m_i\alpha_i$ where each m_i is $0, 1, 2$, or 4.

Claim 6.11. For all i, $1 \leqslant i \leqslant s$, $\alpha_i \cdot x > 0 > \alpha_i \cdot y$.

Proof. Parameterize γ so that $\gamma(0) = x$ and $\gamma(1) = y$. For any i the function $\alpha_i \cdot \gamma(t)$ is strictly monotone with respect to t. It vanishes at $t_i = \gamma^{-1}(D_{i-1} \cap D_i)$ and is positive for $t < t_i$. This proves (6.11).

Since $(\alpha_i^\perp) \cap (\operatorname{int} P) = \varnothing$ and $(\alpha_i^\perp) \cap \operatorname{int} D = \varnothing$, it follows immediately from (6.11) that

$$\alpha_i \cdot P \geqslant 0 \geqslant \alpha_i \cdot D \quad \text{for } 1 \leqslant i \leqslant s. \tag{6.12}$$

Now we are ready to compute:

$$\Delta_M(D) = \Delta_M(P) - \sum_{i=1}^{s} m_i\alpha_i,$$

hence

$$q_M(\Delta_M(D)) = q_M(\Delta_M(P)) - 2\Delta_M(P)\cdot\left(\sum_{i=1}^{s} m_i\alpha_i q\right) + q_M\left(\sum_{i=1}^{s} m_i\alpha_i\right).$$

Since $\Delta_M(P) \in \overline{\mathbf{R}^+\cdot P}$, (6.12) implies that

$$\Delta_M(P)\cdot\left(\sum_{i=1}^{s} m_i\alpha_i\right) \geqslant 0.$$

Thus

$$q_M(\Delta_M(D)) \leqslant q_M(\Delta_M(P)) + q_M\left(\sum_{i=1}^{s} m_i\alpha_i\right).$$

Since $m_i \geqslant 0$ for all i, and $\alpha_i \cdot x > 0 > \alpha_i \cdot y$, either

$$\left(\sum_{i=1}^{s} m_i\alpha_i\right)\cdot x > 0 > \left(\sum_{i=1}^{s} m_i\alpha_i\right)\cdot y, \quad \text{or} \quad m_i = 0 \text{ for all } i,\ 1 \leqslant i \leqslant s.$$

Thus, by (1.1) either $q_M(\sum_{i=1}^{s} m_i\alpha_i) < 0$ or $m_i = 0$ for all i, $1 \leqslant i \leqslant s$. Consequently, either

$$q_M(\Delta_M(D)) < q_M(\Delta_M(P))$$

or

$$q_M(\Delta_M(D)) = q_M(\Delta_M(P)) \quad \text{and} \quad m_i = 0 \quad \text{for } i = 1,\cdots,s.$$

But if $m_i = 0$ then D_{i-1} and D_i are both P-cells and their intersection is a face contained in an ordinary wall of each. Thus, if $m_i = 0$ for all i, $1 \leqslant i \leqslant s$ then D is a P-cell contained in the super P-cell $\mathbf{S}(P)$. □

CHAPTER III

1. Proofs of the main theorems

In this section we prove Theorems 1, 3 and 11 stated in the Introduction, and we prove one inclusion in each of Theorems 6, 7, and 10. All the arguments use Theorem I (1.19) and are based on computations of the moduli spaces of stable rank-2 vector bundles over Dolgachev surfaces and their blow ups. These computations are stated as needed, but the proofs are postponed to Part Two (Chapter IV). We shall refer to the necessary results from Part Two by IV.n.m. for appropriate n and m.

Throughout this section we denote by $S = S(p, q)$ a Dolgachev surface. We shall calculate Γ_S by calculating it on a natural chamber $C_0(S)$ associated to the elliptic fibration structure on S.

Definition 1.1. Let $[f]$ be the cohomology class dual to a generic fiber in S. By I(3.7)(e), there is an indivisible class $\kappa_S \in \Lambda(S)$ with $[f] = pq\kappa_S$. Since $f \cdot f = 0$, we have $q_S(\kappa_S) = 0$.

Lemma 1.2. *$q_S|((\kappa_S^{\perp}) \cap \Lambda(S))$ is even.*

Proof. Let $w_2(S)$ denote the second Stiefel-Whitney class of S. Then the reduction of $[K_S] \bmod 2$ is $w_2(S)$. Suppose $x \in \Lambda(S)$ and $x \cdot \kappa_S = 0$. Then $x \cdot [K_S] = 0$ as well, by I(3.7)(b). Hence $q_S(x) \equiv 0 \pmod 2$. □

Corollary 1.3. *There is a unique chamber $C_0(S)$ in $\mathbf{H}(S)$ with $\kappa(C_0(S)) = \kappa_S$.*

Proof. This is immediate from (1.1), (1.2), and II(2.4)(d). □

Now we are ready to calculate $\Gamma_S(C_0(S))$. This we do with the aid of the following definition and theorem.

Definition 1.4. A line bundle L on S will be called *suitable* if there exists an ample line bundle L_0 on S and an integer

$$t \geqslant \left(\frac{1}{pq - p - q}\right)(L_0 \cdot K_S),$$

such that $L = L_0 \otimes K_S^t$.

It follows from the Nakai-Moishezon criterion [2, p. 127], that a suitable line bundle is ample. We shall use the following result from Chapter IV to prove Theorem 1.

Theorem IV 4.4. *Let $S_0 = S_0(p, q)$ be a generic Dolgachev surface, and let L be a suitable line bundle over S_0. The moduli space $\mathfrak{M}$ of L-stable rank-2 vector bundles V with $c_1(V) = 0$ and $c_2(V) = 1$ is compact. Each component of $\mathfrak{M}_{\mathrm{red}}$ is identified with either F_p or F_q. If a component $(\mathfrak{M}_i)_{\mathrm{red}}$ of $\mathfrak{M}_{\mathrm{red}}$ is identified with F_p (resp. F_q) and $\mathscr{V}_i$ is the restriction of the universal bundle over $S_0 \times \mathscr{M}$ to $S_0 \times (\mathscr{M}_i)_{\mathrm{red}}$, then $\mu_i = (p_i)_* c_2(\mathscr{V}_i)$ is Poincaré dual to $[F_p]$ (resp. F_q). Lastly, if $p = 2$, then $\mathfrak{M}$ is reduced and consists of $(q-1)/2$ components, each identified with F_2.*

Corollary 1.5. *Suppose that S_0 is a generic Dolgachev surface with multiple fibers of orders p and q. Let L be a suitable line bundle over S_0. Let g be a Kähler metric associated to L, and let ω be the Kähler form of g. Then*

$$\tilde{\Gamma}_{S_0}(g, \omega) = (pq - p - q)\kappa_{S_0} + 2\left(\sum_{i \in I_p} n_i q + \sum_{i \in I_q} n_i p\right)\kappa_{S_0},$$

where I_p and I_q are the sets of components of $\mathfrak{M}_{\mathrm{red}}$ identified with F_p and F_q respectively and n_i is the length of the generic point of $\mathfrak{M}_i$. If $p = 2$, then

$$\tilde{\Gamma}_{S_0}(g, \omega) = (q^2 - 2)\kappa_{S_0}.$$

Proof. By I(3.7)(a), $\chi(\mathcal{O}_{S_0}) = 1$. By Proposition IV(4.3) all components of $\mathfrak{M}_{\text{red}}$ are compact and of dimension 1. Thus, by Theorem I(1.19)

$$\tilde{\Gamma}_{S_0}(g, \omega) = K_{S_0} + 2 \sum_{i \in I_p \cup I_q} n_i \mu_i.$$

By I(3.6) and I(3.7)(b), $K_{S_0} = (pq - p - q)\kappa_{S_0}$. If $i \in I_p$, then $\mu_i = [F_p] = q\kappa_{S_0}$ and if $i \in I_q$ then $\mu_i = [F_q] = p\kappa_{S_0}$. Thus,

$$\tilde{\Gamma}_{S_0}(g, \omega) = \Big\{(pq - p - q) + 2 \sum_{i \in I_p} n_i q + 2 \sum_{i \in I_q} n_i p\Big\}\kappa_{S_0}.$$

If $p = 2$, then $I_q = \varnothing$, I_2 has $(q-1)/2$ elements, and $n_i = 1$ for all $i \in I_2$. Thus, in this case, we have

$$\tilde{\Gamma}_{S_0}(g, \omega) = \{(2q - 2 - q) + 2q(q-1)/2\}\kappa_{S_0}$$
$$= (q^2 - 2)\kappa_{S_0}. \quad \square$$

Definition 1.6. For all pairs (p, q) with g.c.d.$(p, q) = 1$, we define $n(p, q) = (pq - p - q) + 2\Sigma_{i \in I_p} n_i q + 2\Sigma_{i \in I_q} n_i p$. Clearly $n(p, q) > pq - p - q$ and $n(2, q) = q^2 - 2$.

Note. The proof of Theorem IV(3.9) shows, in fact, that $n(p, q)$ is independent of the choice of generic Dolgachev surface S_0. A slightly better lower bound for $n(p, q)$ is given in IV(4.9).

Thus, we can reformulate Corollary 1.5 as follows

Corollary 1.7. *With S_0, g and ω as in* 1.5, *we have*

$$\tilde{\Gamma}_{S_0}(g, \omega) = n(p, q)\kappa_{S_0}. \quad \square$$

Now we identify the chamber containing $c_1(L)$.

Proposition 1.8. *Let $S = S(p, q)$ be an arbitrary Dolgachev surface, and let L be a suitable bundle over S. Then*

$$c_1(L) \in \mathbf{R}^+ \cdot (\operatorname{int} C_0(S)).$$

Proof. We write $L = L_0 + (pq - p - q)(L_0 \cdot \kappa) \cdot \kappa$ where

$$q_S(c_1(L_0)) > 0 \quad \text{and} \quad c_1(L_0) \cdot \kappa > 0,$$

and we set $\kappa = \kappa_S$. Hence, (1.8) is an immediate consequence of the following lemma.

Lemma 1.9. *Let $x \in \Lambda_{\mathbf{R}}(S)$. Suppose that $\bar{q}_S(x) > 0$ and that $x \cdot \kappa > 0$ (i.e. that x and κ are in the same component of $\mathcal{P}(q_S)$). Then, for all $t \geqslant \frac{1}{2}$, $x + t(x \cdot \kappa)\kappa \in \mathbf{R}^+ \cdot (\operatorname{int} C_0(S))$.*

Proof. After rescaling x, we may assume that $x \cdot \kappa = 1$. Let $\alpha \in \mathcal{F}_{C_0(S)}$. Then $\alpha \cdot \kappa = 1$, so that $(x - \alpha) \cdot \kappa = 0$. Hence $\bar{q}_S(x - \alpha) \leqslant 0$, i.e., $q_S(x) - 2(x \cdot \alpha) - 1 \leqslant 0$. It follows that

$$(x \cdot \alpha) \geqslant -\frac{1}{2} + \frac{\bar{q}_S(x)}{2} > -\frac{1}{2}.$$

If $t \geqslant \frac{1}{2}$, we have $((x + t(x \cdot \kappa)\kappa) \cdot \alpha) = (x \cdot \alpha) + t(x \cdot \kappa) = (x \cdot \alpha) + t > -\frac{1}{2} + t \geqslant 0$. As this holds for all $\alpha \in \mathscr{F}_{C_0(S)}$, $x + t(x \cdot \kappa)\kappa \in \mathbf{R}^+ \cdot (\operatorname{int} C_0(S))$ for all $t \geqslant \frac{1}{2}$. □

Corollary 1.10. *Let S_0 be a generic Dolgachev surface with multiple fibers of orders p and q. Then*

$$\Gamma_{S_0}(C_0(S_0)) = n(p,q)\kappa_{S_0} = n(p,q)\kappa(C_0(S_0)).$$

Proof. This is immediate from (1.7), (1.8), and the definition of Γ_{S_0}. □

We now pass from a generic Dolgachev surface S_0 to an arbitrary one.

Corollary 1.11. *Let $S = S(p,q)$ be a Dolgachev surface. Then $\Gamma_S(C_0(S)) = n(p,q)\kappa_S = n(p,q)\kappa(C_0(S))$.*

Proof. Let S_0 be a generic Dolgachev surface with multiple fibers of orders p and q. By I(3.11) there is a diffeomorphism φ: $S \to S_0$ (automatically orientation-preserving, since S is of type $(1,9)$) with $\varphi^* K_{S_0} = K_S$. Thus, $\varphi^*\kappa_{S_0} = \kappa_S$, and hence $\varphi^*(C_0(S_0)) = C_0(S)$. By II(6.2)(c) and (1.10), we have

$$\begin{aligned}\Gamma_S(C_0(S)) &= \varphi^*\Gamma_{S_0}(C_0(S)) = \varphi^* n(p,q)\kappa_{S_0} \\ &= n(p,q)\kappa_S = n(p,q)\kappa(C_0(S)).\ \square\end{aligned}$$

Corollary 1.12. $\Delta_S(C_0(S)) = (n(p,q)+1)\kappa(C_0(S))$. □

Now we are ready to prove Theorem 1.

Theorem 1. (a) *Let $n(p,q)$ be the function from pairs of relatively prime integers greater than* 1 *to* $\mathbf{N}$ *defined in* (1.6). *Then*

(i) $n(p,q) \geqslant pq - p - q$;

(ii) $n(2,q) = q^2 - 2$; *and*

(iii) *if $S(p,q)$ and $S(p',q')$ are diffeomorphic then $n(p,q) = n(p',q')$.*

(b) *No Dolgachev surface $S(p,q)$ is diffeomorphic to a rational surface.*

Proof. We have already seen that $n(p,q)$ satisfies (i) and (ii). Let $S = S(p,q)$ be a Dolgachev surface. By Corollary 1.12

$$\Delta_S(C_0(S)) = (n(p,q)+1)\kappa(C_0(S)) = (n(p,q)+1)\kappa,$$

where we let $\kappa = \kappa(C_0(S)) = \kappa_S$. Since $n = 9$ the super P-cell containing $C_0(S)$ is exactly the component of $\mathbf{H}(S)$ containing $C_0(S)$. Since Δ_S is constant on super P-cells (by II(6.9)), we see that $\Delta_S(C) = (n(p,q)+1) \cdot \kappa$ for all C in the same component of $\mathbf{H}(S)$ as $C_0(S)$. Hence, by II(6.5)(a), $\Delta_S(C) = \pm(n(p,q)+1) \cdot \kappa$ for all chambers C in $\mathbf{H}(S)$. Thus for all chambers C in $\mathbf{H}(S)$, $\Delta_S(C)$ is divisible in $H^2(S;\mathbf{Z})$ by exactly $(n(p,q)+1)$.

Now suppose $S(p,q)$ and $S(p',q')$ are diffeomorphic. By II(6.5)(b) and the above computation, it follows that

$$n(p,q) + 1 = n(p',q') + 1, \quad \text{so } n(p,q) = n(p',q').$$

Lastly, we show that no $S(p,q)$ is diffeomorphic to a good generic rational surface X. From this it will follow from the proof of Theorem 3.4 in this chapter that no $S(p,q)$ is diffeomorphic to any rational surface. To do this we compute Δ_X. Let L be an ample line bundle over X. By [8], or by IV(5.12), there are no L-stable rank-2 bundles over L with $c_2 = 1$. Thus, if g is a Kähler metric for L and ω is its Kähler form, we have $\mathfrak{M}(g) = \varnothing$ and

$$\tilde{\Gamma}_X(g,\omega) = K_X.$$

Let $C_0(X)$ be the chamber containing all Kähler forms (up to scalar multiple). Then

$$\Gamma_X(C_0(X)) = K_X.$$

Since by II(3.6)(d), $\kappa(C_0(X)) = -K_X$, we see that

$$\Delta_X(C_0(X)) = 0.$$

Again using II(6.9) and II(6.5)(a), we have $\Delta_X(C) = 0$ for all chambers C in $\mathbf{H}(X)$. Since $n(p,q) \geqslant pq - p - q \geqslant 1$ if g.c.d.$(p,q) = 1$ and $p,q > 1$, we see immediately from II(6.5)(b), that $S(p,q)$ and X are not diffeomorphic. This completes the proof of Theorem 1 (modulo Theorem IV(4.4) and IV(5.10)). □

Corollary 1.13. *Let C be a chamber in $\mathbf{H}(S)$. Then $q_S(\Gamma_S(C)) = 0$ if and only if $C = \pm C_0(S)$.*

Proof. $\Gamma_S(C) = \Delta_S(C) - \kappa(C) = \pm(n(p,q)+1)\kappa(C_0(S)) - \kappa(C)$. Since $q_S(\kappa(C)) = q_S(\kappa(C_0)) = 0$, $q_S(\Gamma_S(C)) = 0$ if and only if $\kappa(C_0(S)) \cdot \kappa(C) = 0$. But $\kappa(C_0(S)) \cdot \kappa(C) = 0$ if and only if $\kappa(C_0(S))$ and $\kappa(C)$ are multiples of each other. Since $\kappa(C_0(S))$ and $\kappa(C)$ are indivisible in $\Lambda(S)$, this happens only when $\kappa(C) = \pm\kappa(C_0(S))$, i.e. only when $C = \pm C_0(S)$. □

We turn next to the proof of Theorem 3. Let $\tilde{S} = \tilde{S}_r(p,q)$ denote the blow up of $S = S(p,q)$ at r distinct points, and let $\rho\colon \tilde{S} \to S$ be the natural map. Here is another result from Chapter IV which we shall use to prove Theorem 3.

Corollary IV (5.9). *Let L be a suitable line bundle on S, and let $E_1, \cdots, E_r$ denote the exceptional fibers of ρ. Then there exist positive integers N, m_i, $i = 1, \cdots, r$, with m_i/N arbitrarily small, such that, if we set*

$$\tilde{L} = N\rho^*L - \sum_1^r m_i E_i,$$

then:

(a) *$\tilde{L}$ is ample on $\tilde{S}$.*

(b) *The moduli space $\tilde{\mathfrak{M}}$ of $\tilde{L}$-stable rank 2 vector bundles on $\tilde{S}$ with $c_1 = 0$ and $c_2 = 1$ is isomorphic as a scheme to $\mathfrak{M}$, the moduli space of L-stable rank-2 vector bundles on S with $c_1 = 0$ and $c_2 = 1$. This isomorphism is induced by ρ^*.*

(c) *The universal bundle* $\tilde{\mathcal{V}}$ *over* $\tilde{S} \times \tilde{\mathfrak{M}}$ *is the pullback via* $(\rho \times \mathrm{Id})^*$ *of the universal bundle* $\mathcal{V}$ *over* $S \times \mathfrak{M}$.

Lemma 1.14. *There is a unique chamber* $C_0(\tilde{S})$ *in* $\mathbf{H}(\tilde{S})$ *with the following properties*

(a) $[E_i] \cdot C_0(\tilde{S}) \geqslant 0$ *for* $1 \leqslant i \leqslant r$.

(b) $C_0(\tilde{S}) \cap ([E_1]^\perp) \cap \cdots \cap ([E_r]^\perp) = \rho^* C_0(S)$.

There is a unique P-cell $P_0(\tilde{S})$ *in* $\mathbf{H}(\tilde{S})$ *such that*

$$P_0(\tilde{S}) \subset C_0(\tilde{S}) \quad \textit{and} \quad P_0(\tilde{S}) \cap \left([E_1]^\perp\right) \cap \cdots \cap \left([E_r]^\perp\right) = \rho^* C_0(S).$$

Proof. That there is a unique chamber $C_0(\tilde{S})$ satisfying (a) and (b) is clear. By Proposition II(2.8) there is at most one P-cell as required. Furthermore, if $x \in \mathbf{H}(S)$ is a corner for $C_0(S)$, then $\rho^*(x) \in \mathbf{H}(\tilde{S})$ is a corner for $C_0(\tilde{S})$. The P-cell $P_0(\tilde{S})$ is $P(\rho^* x, C_0(\tilde{S}))$. □

Lemma 1.15. *If* $y \in \operatorname{int} C_0(S)$, N *and* m_i *are positive integers*, $1 \leqslant i \leqslant r$, *and if* m_i/N *is sufficiently small, then* $N\rho^* y - \sum_1^r m_i [E_i] \in \mathbf{R}^+ \cdot C_0(\tilde{S})$.

Proof. As $m_i/N \to 0$, $\frac{1}{N}(\rho^* N y - \sum_1^r m_i [E_i]) \to \rho^* y \in \operatorname{int} \rho^* C_0(S)$. Since $(N\rho^* y - \sum_1^r m_i [E_i]) \cdot E_i = m_i > 0$ for all i, $1 \leqslant i \leqslant r$, it follows that if the m_i/N are sufficiently small, then $N\rho^* y - \sum_1^r m_i [E_i] \in C_0(\tilde{S})$. □

Corollary 1.16. $\Gamma_{\tilde{S}}(C_0(\tilde{S})) = \rho^* \Gamma_S(C_0(S)) + \sum_{i=1}^r [E_i]$.

Proof. First let us assume that S is a generic Dolgachev surface. Let L be a suitable line bundle over S. Then by Proposition 1.8, $c_1(L) \in \mathbf{R}^+ \cdot (\operatorname{int} C_0(S))$. Let $\tilde{L}$ over $\tilde{S}$ be the bundle $N\rho^* L - \sum_{i=1}^r m_i E_i$, where N and m are chosen to satisfy the conclusions of IV(5.9) and (1.15). Then $\tilde{L}$ is ample and $c_1(\tilde{L}) \in C_0(\tilde{S})$. Thus

$$\Gamma_S\left(C_0(S)\right) = K_S + 2\mu,$$
$$\Gamma_{\tilde{S}}\left(C_0(\tilde{S})\right) = K_{\tilde{S}} + 2\tilde{\mu},$$

where μ and $\tilde{\mu}$ are calculated from $\mathfrak{M}$ and $\tilde{\mathfrak{M}}$ as in Theorem I(1.19). By Corollary IV(5.9), $\rho^*\mu = \tilde{\mu}$. Of course, $\rho^* K_S + \sum_{i=1}^r [E_i] = K_{\tilde{S}}$. Thus,

$$\Gamma_{\tilde{S}}\left(C_0(\tilde{S})\right) = \rho^* \Gamma_S\left(C_0(S)\right) + \sum_{i=1}^r [E_i].$$

This proves the result when S is generic. To pass to a general Dolgachev surface S, we use I(3.11) to construct a diffeomorphism from S to a generic Dolgachev surface S_0. This lifts to a diffeomorphism between the blow ups which makes the exceptional curves correspond. The result for $\tilde{S}$ then follows from that for $\tilde{S}_0$ by the naturality property II(6.2)(b). □

Corollary 1.17. $\Delta_{\tilde{S}}(P_0(\tilde{S})) = \rho^* \Delta_S(C_0(S)) = (n(p,q) + 1)\rho^* \kappa_S$.

Proof. By II(4.6), we have $\kappa(P_0(\tilde{S})) = \rho^* \kappa(C_0(S)) - \sum_{i=1}^r [E_i]$. The result is immediate from this and (1.16). □

We recall the statement of Theorem 3.

Theorem 3. *Let $r \geqslant 0$. Let $\tilde{S}$ and $\tilde{S}'$ be blow ups at r points of Dolgachev surfaces $S = S(p,q)$ and $S' = S(p',q')$. If $\tilde{S}$ and $\tilde{S}'$ are diffeomorphic, then $n(p,q) = n(p',q')$. Furthermore $\tilde{S}$ is not diffeomorphic to a rational surface.*

Proof. Since by II(2.7)(b) $\kappa_S = \kappa(C_0(S)) \in \overline{C_0(S)}$, $\rho^*\kappa_S$ is in the closure of $P_0(\tilde{S})$. By Theorem II(6.10), for any $D \in \mathfrak{D}(\tilde{S})$, we have

$$q_{\tilde{S}}(\Delta_{\tilde{S}}(D)) \leqslant q_{\tilde{S}}(\Delta_{\tilde{S}}(P_0(\tilde{S})) = 0,$$

with equality only if D is in one of the super P-cells $\pm\mathbf{S}(P_0(\tilde{S}))$. Now suppose $\tilde{S}'$ is a blow up of $S(p',q')$, and suppose $\varphi\colon \tilde{S} \to \tilde{S}'$ is a diffeomorphism. Again, φ will be automatically orientation-preserving, since S is of type $(1, 9+r)$. Let $P_0(S')$ be the P-cell in $\tilde{S}'$ as in Lemma 1.14. Then by II(6.5)(b) we have

$$q_{\tilde{S}}\big(\Delta_{\tilde{S}}\big(\varphi^*P_0(\tilde{S})\big)\big) = q_{\tilde{S}'}\big(\Delta_{\tilde{S}'}\big(P_0(\tilde{S}')\big)\big) = 0.$$

Thus, $\varphi^*P_0(\tilde{S}')$ is in one of the super P-cells $\pm\,\mathbf{S}(P_0(S))$. Since $\Delta_{\tilde{S}}$ is constant on super P-cells (II(6.9)), by II(6.5)(a) and Corollary 1.17, we have

$$\Delta_{\tilde{S}}\big(\varphi^*P_0(\tilde{S}')\big) = \pm(n(p,q)+1)\varphi^*\kappa_{S'}.$$

Of course, by Corollary 1.17, $\Delta_{\tilde{S}'}(P_0(\tilde{S}')) = (n(p',q')+1)(\varphi')^*\kappa_{S'}$. By II(6.5)(b), the divisibilities of these classes must be the same. Hence, $n(p,q) + 1 = n(p',q') + 1$, and $n(p,q) = n(p',q')$.

Lastly, we show that $\tilde{S}$ is not diffeomorphic to a rational surface of type $(1, 9+r)$. To do this we can again work with $\tilde{X}$ a good generic surface. Let $C_0(\tilde{X})$ be the chamber in $\mathbf{H}(\tilde{X})$ containing all Kähler forms associated to Kähler metrics. By IV(5.10) there are no L-stable bundles on $\tilde{X}$ for any ample line bundle L. Hence,

$$\Gamma_{\tilde{X}}(C_0(\tilde{X})) = K_{\tilde{X}} = -\kappa(C_0(\tilde{X})). \tag{1.19}$$

Hence $\Delta_{\tilde{X}}(C_0(\tilde{X})) = 0$. Thus, $\Delta_{\tilde{X}}$ takes on the value 0 whereas $\Delta_{\tilde{S}}$ never does. Thus, by II(6.5)(b) this implies that $\tilde{S}$ and $\tilde{X}$ are not diffeomorphic. This completes the proof of Theorem 3, modulo Theorem IV(4.4) and Corollary IV(5.11). □

Now we turn to Theorem 6. At this point we shall prove part of Theorem 6. Recall that if $S = S(p,q)$ is a Dolgachev surface, then we define

$$A_f(S) = \{\psi \in A(S) \mid \psi([f]) = \pm[f]\},$$

where $[f]$ is the cohomology class dual to a generic fiber of S.

Theorem 6A. *For any Dolgachev surface $S = S(p,q)$ we have $D(S) \subseteq A_f(S) \subset A(S)$. Furthermore, the subgroup $A_f(S)$ is of infinite index in $A(S)$.*

Proof. Suppose that $\varphi\colon S \to S$ is a diffeomorphism. We must show that $\varphi^*[f] = \pm[f]$: We know that Δ_S takes on only two values, $\pm(n(p,q)+1)\kappa_S$. Thus, $\varphi^*\kappa_S = \pm\kappa_S$. Since $[f]$ is a multiple of κ_S, $\varphi^*[f] = \pm[f]$.

To show that $A_f(S)$ is of infinite index in $A(S)$, we note that $A_f(S) = \{\psi \in A(S) \mid \psi(\kappa) = \pm\kappa\}$. Since $\kappa = \kappa(C_0(S))$ we have $A_f(S) = \{\psi \in A(S) \mid \psi(C_0(S)) = \pm C_0(S)\mid$. Since there are infinitely many chambers in $\mathbf{H}(S)$ and since $A(S)$ acts transitively on the set of chambers, it follows that $A_f(S) \subset A(S)$ has infinite index. $\square$

Next we turn to Theorem 7. At this point we prove a partial result along these lines.

Theorem 7A. *Let $\tilde{S}$ be the blow up of $S = S(p,q)$ at r points. Let $B \subset H^2(\tilde{S};\mathbf{Z})$ be the subgroup generated by the classes dual to the exceptional fibers of $\rho\colon \tilde{S} \to S$. If $\varphi\colon \tilde{S} \to \tilde{S}$ is a diffeomorphism, then φ preserves the orthogonal splitting*

$$H^2(\tilde{S};\mathbf{Z}) = H^2(S;\mathbf{Z}) \oplus B.$$

Furthermore, the restriction of φ^* to $H^2(S;\mathbf{Z})$ is contained in $A_f(S)$.

Proof. Let $P_0(\tilde{S})$ be the P-cell in $\mathbf{H}(\tilde{S})$ described in Lemma 1.14. Since $q_{\tilde{S}}(\Delta_{\tilde{S}}(\varphi^*P_0(\tilde{S}))) = 0$, we see that $\varphi^*P_0(\tilde{S})$ is contained in one of the super P-cells $\pm\mathbf{S}(P_0(\tilde{S}))$. Since $\Delta_{\tilde{S}}$ is constant on super P-cells, it follows that $\varphi^*\Delta_{\tilde{S}}(P_0(\tilde{S})) = \pm\Delta_{\tilde{S}}(P_0(\tilde{S}))$. But $\Delta_{\tilde{S}}(P_0(\tilde{S})) = \varphi^*\kappa_S$. Thus

$$\varphi^*\left(\rho^*\kappa_S\right) = \pm\rho^*\kappa_S.$$

Let us consider first the case when $\varphi^*\rho^*\kappa_S = \rho^*\kappa_S$. In this case $\varphi^*\mathbf{S}(P_0(\tilde{S})) = \mathbf{S}(P_0(\tilde{S}))$. Let $(x, e_1, \cdots, e_9)$ be a standard basis for $\Lambda(S)$ adapted to $C_0(S)$. Let $e_{10}, \cdots, e_n$ be the classes dual to the exceptional fibers of $\rho\colon \tilde{S} \to S$. Then $(\rho^*x, \rho^*e_1, \cdots, \rho^*e_9, e_{10}, \cdots, e_n)$ is a standard basis for $\Lambda(\tilde{S})$ adapted to $C_0(\tilde{S})$, and $\rho^*x \in P_0(\tilde{S})$. Of course, $\rho^*\kappa_S = \kappa(C_0(\tilde{S})) - \Sigma_{i=10}^n e_i$. Thus, by Proposition II(5.8), the only walls in $\mathscr{W}_1(q_{\tilde{S}})$ which pass through $\mathbf{S}(P_0(\tilde{S}))$ and contain $\rho^*\kappa_S$ in their closures are $W^{e_{10}}, \cdots, W^{e_n}$. The automorphism φ^* must then preserve this set of walls. Thus for each i, $10 \leqslant i \leqslant n$, there is a $j = j(i)$, $10 \leqslant j \leqslant n$ with $\varphi^*(e_i) = \pm e_j$. Hence, φ^* leaves invariant $\langle e_{10}, \cdots, e_n\rangle \subset \Lambda(\tilde{S})$. Consequently, it also leaves invariant $(\langle e_{10}, \cdots, e_n\rangle^\perp) \cap \Lambda(\tilde{S}) = \rho^*\Lambda(S)$.

If $\varphi^*\rho^*\kappa_S = -\rho^*\kappa_S$, then $\varphi^*\mathbf{S}(P_0(\tilde{S})) = -\mathbf{S}(P_0(\tilde{S}))$. We apply the above argument to $\psi = (-\mathrm{Id}) \circ \varphi^*$. We conclude that ψ preserves the given decomposition. Since $(-\mathrm{Id})$ also preserves this decomposition, so does φ^* in this case as well. Since $\varphi^*\rho^*\kappa_S = \pm\rho^*\kappa_S$, it follows that $\varphi^*\rho^*[f] = \pm\rho^*[f]$, so that $\varphi^* \mid H^2(S;\mathbf{Z}) \in A_f(S)$. $\square$

We shall establish the following refinement of Theorem 10.

Theorem 10′. *Let $\tilde{X}$ be a rational surface of type $(1, n)$ for some $n > 9$. There is a P-cell $P_0(\tilde{X}) \subset \mathbf{H}(\tilde{X})$ such that $D(\tilde{X}) = \{\psi \in A(\tilde{X}) \mid \psi(\mathbf{S}(P_0(\tilde{X}))) = \pm\mathbf{S}(P_0(\tilde{X}))$ where $\mathbf{S}(P_0(\tilde{X}))$ is the super P-cell containing $P_0(\tilde{X})$. The group $D(\tilde{X})$ has infinite index in $A(\tilde{X})$.*

In this section we prove

Theorem 10A. *If $\tilde{X}$ is a good generic surface, then setting $P_0(\tilde{X})$ equal to the closure of the Kähler cone $\mathscr{K}(\tilde{X})$ we have*

$$D(\tilde{X}) \subset \{\psi \in A(\tilde{X}) \mid \psi(\mathbf{S}(P_0(\tilde{X}))) = \pm\mathbf{S}(P_0(\tilde{X}))\}.$$

The group $D(\tilde{X})$ has infinite index in $A(\tilde{X})$.

Proof. Let $C_0(\tilde{X})$ be the chamber containing $P_0(\tilde{X})$. By (1.19), $\Gamma_{\tilde{X}}(C_0(\tilde{X})) = K_{\tilde{X}} = -\kappa(P_0(\tilde{X}))$. Hence $\Delta_{\tilde{X}}(P_0(\tilde{X})) = 0$. Applying Theorem II(6.10), we see that $q_{\tilde{X}}(\Delta_{\tilde{X}}(D)) \leqslant 0$ with equality if and only if D is in one of the super P-cells $\pm\,\mathbf{S}(P_0(\tilde{X}))$. Thus, if $\varphi\colon\ \tilde{X} \to \tilde{X}$ is a diffeomorphism then $\varphi^*(\mathbf{S}(P_0(\tilde{X}))) = \pm\mathbf{S}(P_0(\tilde{X}))$.

Finally, we must show that $D(\tilde{X})$ has infinite index in $A(\tilde{X})$. Since $D(\tilde{X})$ acting on the P-cells preserves the value of $q_{\tilde{X}} \circ \Delta_{\tilde{X}}$, and since $A(\tilde{X})$ acts transitively on the P-cells, it suffices to show that $q_{\tilde{X}} \circ \Delta_{\tilde{X}}$ takes infinitely many distinct values on P-cells.

It suffices to prove that $q_{\tilde{X}} \circ \Delta_{\tilde{X}}$ takes on infinitely many values when $n = 10$, since the cohomology of a good generic rational surface of type $(1, 10)$ injects into that for a rational surface of type $(1, n)$ for any $n \geqslant 10$ preserving the values of $q_{\tilde{X}} \circ \Delta_{\tilde{X}}$. In the case $n = 10$, the super P-cells are exactly chambers for the subset of exceptional walls in $\mathscr{W}_1$. If $\mathbf{S}_0$ is the super P-cell containing the Kähler cone $P_0(\tilde{X})$ then by II(6.7)(d) for any other super P-cell $\mathbf{S}_1$ in the same component of $\mathbf{H}$ as $\mathbf{S}_0$ we have

$$q_{\tilde{X}}(\Delta_{\tilde{X}}(S_1)) = q_X\left(-4\sum \alpha_i\right),$$

where the α_i are the classes defining the exceptional walls which separate $\mathbf{S}_0$ and $\mathbf{S}_1$. Of course, $q_{\tilde{X}}(\alpha_i) = -1$. Since the walls W^{α_i} and W^{α_j} do not meet, and since there is a geodesic crossing the oriented walls W^{α_i} and W^{α_j} both from the positive side to the negative side, $\alpha_i\alpha_j \leqslant 0$ for all i and j. Hence $q_{\tilde{X}}(\Delta_{\tilde{X}}(\mathbf{S}_1)) \leqslant -16r$, where r is the number of exceptional walls separating $\mathbf{S}_0$ and $\mathbf{S}_1$.

To complete the proof we need only see that given $r > 0$ there is a super P-cell $\mathbf{S}_1(r)$ separated from $\mathbf{S}_0$ by at least r exceptional walls in $\mathscr{W}_1$. Since each super P-cell has at least 2 exceptional walls and they meet each other, if at all, along one wall, this is obvious. □

Finally, we shall prove Theorem 11. It is an easy corollary of the formal properties of Γ and the discussion of Chapter II.

Theorem 11. *Let M be a smooth simply connected oriented 4-manifold of type $(1, n)$, with $n \geqslant 10$. Then $D(M)$ is a proper subgroup of $A(M)$.*

Proof. Let $C \in \mathscr{C}(M)$, and let $\alpha \in \mathscr{F}_C$. Then

$$\Gamma_M(R_\alpha C) = \Gamma_M(C) - 2\alpha.$$

Suppose that $R_\alpha \in D(M)$, say $R_\alpha = \varphi^*$. Then

$$\begin{aligned}\Gamma_M(R_\alpha C) &= \Gamma_M(\varphi^* C) = \varphi^*\Gamma_M(C) = R_\alpha\Gamma_M(C)\\ &= \Gamma_M(C) + 2(\Gamma_M(C)\cdot\alpha)\alpha.\end{aligned}$$

Thus, if $R_\alpha \in D(M)$ for all α, we must have $\Gamma_M(C)\cdot\alpha = -1$ for every chamber C and every $\alpha \in \mathscr{F}_C$. By II(4.6)(b), this is only possible if

$$\Gamma_M(C) = -\kappa(x, C)$$

for every corner x of C. By II(1.21), $-\Gamma_M(C)$ lies in $\mathscr{Q} = \{x \in \Lambda_{\mathbf{R}}(M)\colon \bar{q}_M(x) \geqslant 0\}$. Since $\bar{q}_M(-\Gamma_M(C)) = q_M(\kappa(x, C)) = 9 - n$, $n \leqslant 9$. $\square$

2. Completion of the proofs of Theorems 6, 7, and 10′

We begin by stating a refinement of Theorem 6 of the Introduction. Let $S = S(p, q)$ be a Dolgachev surface. The complement of tubular neighborhoods of the multiple fibers is naturally identified with the complement X_0 in X of tubular neighborhoods of two smooth fibers. Let $A_0(S) \subset A(S)$ be the stabilizer of the subspace in $H^2(S; \mathbf{Z})$ Poincaré dual to the image of $H_2(X_0; \mathbf{Z})$ in $H_2(S; \mathbf{Z})$.

Theorem 6′. *We have inclusions $A_0(S) \subset D(S) \subset A_f(S) \subset A(S)$. The index of $A_0(S)$ in $A_f(S)$ is $(pq)^8$; the index of $A_f(S)$ in $A(S)$ is infinite.*

Theorem 6A established that $D(S) \subset A_f(S) \subset A(S)$ and that $A_f(S)$ has infinite index in $A(S)$. Thus, to prove Theorem 6′, we need to show

(2.1)(a) $A_0(S)$ is a subgroup of $A_f(S)$ of index $(pq)^8$.

(2.1)(b) $$A_0(S) \subset D(S).$$

In order to prove these results we introduce some notation. We set $\tilde{L} \subset H^2(S; \mathbf{Z})$ equal to the subspace Poincaré dual to the image of $H_2(X_0; \mathbf{Z})$ in $H_2(S; \mathbf{Z})$; and we set $L \subset H^2(S; \mathbf{Z})$ equal to $([f]^\perp)$. The Dynkin diagram for $-\tilde{E}_8$ is

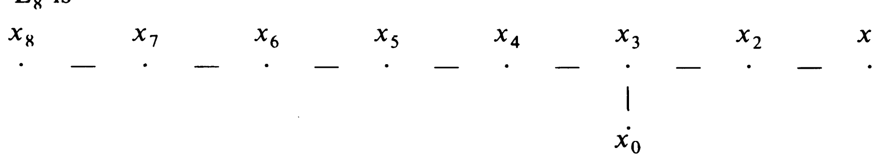

FIGURE 7

The associated symmetric bilinear form on the free abelian group with basis $\{x_0, x_1, \cdots, x_8\}$ is defined by

$$x_i \cdot x_j = \begin{cases} -2 & \text{if } i = j, \\ 1 & \text{if } i \neq j \text{ and } x_i \text{ and } x_j \text{ are connected} \\ & \text{by an edge in the diagram,} \\ 0 & \text{otherwise.} \end{cases}$$

We denote this form by $-\tilde{E}_8$. It is negative semidefinite. It has a one-dimensional radical. The quotient by the radical is $-E_8$.

Claim 2.2. (a) The form $q_S|L$ is isomorphic to $-\tilde{E}_8$. Its radical is $\langle \kappa_S \rangle$

(b) $\tilde{L} \subset L$, and the quotient $\tilde{L}/L$ is isomorphic to $\mathbf{Z}/pq\mathbf{Z}$ and is generated by κ_S. The form $q_S \mid \tilde{L}$ is also isomorphic to $-\tilde{E}_8$ and its radical is $\langle [f] \rangle$

Proof. Using the notation in the proof of II(2.4), we have $H^2(S; \mathbf{Z}) \cong \langle \kappa_S, \delta \rangle \oplus (\langle \kappa_S, \delta \rangle^{\perp})$ with $q_S|(\langle \kappa_S, \delta \rangle^{\perp})$ isomorphic to $-E_8$. Clearly, $L = \langle \kappa_S \rangle \oplus (\langle \kappa_S, \delta \rangle^{\perp})$. Part (a) follows immediately.

Since $X_0 \subset S$ misses the multiple fibers, any $a \in H_2(S; \mathbf{Z})$ in the image of $H_2(X_0; \mathbf{Z})$ has zero homological intersection with the classes of the multiple fibers and hence with the classes of the ordinary fibers (these classes being rational multiples of those of the multiple fibers). Dualizing gives $\tilde{L} \subset L$.

To compute $L/\tilde{L}$, let T_p and T_q be tubular neighborhoods in S of the multiple fibers F_p and F_q. A simple computation shows that $H_2(T_p, \partial T_p; \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}$ with the class of F_p generating the finite cyclic factor (and similarly for F_q). From the exact sequence

$$\begin{array}{ccccccc} H_2(X_0) \to H_2(S) & \longrightarrow & H_2(S, X_0) & \longrightarrow & H_1(X_0) & \longrightarrow & H_1(S) \\ & & \cong & & & & \\ & & H_2(T_p, \partial T_p) \oplus H_2(T_q, \partial T_q) & & \cong & & \| \\ & & \cong & & & & \\ & & (\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}) \oplus (\mathbf{Z} \oplus \mathbf{Z}/q\mathbf{Z}) & \to & \mathbf{Z} & \longrightarrow & 0 \end{array}$$

we see that $H_2(S)/\mathrm{Im}\, H_2(X_0) \cong \mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/q\mathbf{Z} \cong \mathbf{Z} \oplus \mathbf{Z}/pq\mathbf{Z}$ and that the class dual to κ_S generates the finite cyclic factor. Dualizing, we have $H^2(S; \mathbf{Z})/\tilde{L} \cong \mathbf{Z} \oplus \mathbf{Z}/pq\mathbf{Z}$ with κ_S generating the finite cyclic factor. Since $\tilde{L} \subset L$ and since $H^2(S; \mathbf{Z}) \cong \mathbf{Z} \oplus L$, it follows that $L/\tilde{L} \cong \mathbf{Z}/pq\mathbf{Z}$ and that κ_S generates the quotient. Part (b) is now an easy consequence of Part (a).

Now we show that $A_0(S) \subset A_f(S)$. The first step is to identify $A_f(S)$ with the stabilizer of L. This is immediate, for $\alpha \in A(S)$ leaves $\{\pm[f]\}$ invariant if and only if it leaves $\langle \kappa_S \rangle$ invariant if and only if it leaves $(\langle \kappa_S \rangle^{\perp}) = L$ invariant. Since $L = \mathbf{Q}\tilde{L} \cap H^2(S; \mathbf{Z})$ any $\alpha \in A(S)$ leaving $\tilde{L}$ invariant also leaves L invariant. This proves that $A_0(S) \subset A_f(S)$.

Now for any $\alpha \in A_f(S)$, consider the subgroup $\alpha(\tilde{L}) \subset L$. It is a subgroup with cyclic quotient isomorphic to $\mathbf{Z}/pq\mathbf{Z}$ generated by κ_S. It is straightforward to verify that there is one such subgroup for each element of $\mathrm{Hom}(\tilde{L}/\langle[f]\rangle, \langle\kappa_S\rangle/\langle[f]\rangle)$. Since $\tilde{L}/\langle[f]\rangle \cong \mathbf{Z}^8$ and since $[f] = pq\kappa_S$, there are $(pq)^8$ such subgroups. It is also easy to check that each such arises as $\alpha(\tilde{L})$ for some $\alpha \in A_f(S)$. To complete the proof that $A_0(S) \subset A_f(S)$ has index $(pq)^8$, note that $\alpha(\tilde{L}) = \beta(\tilde{L})$ if and only $\alpha^{-1}\beta(\tilde{L}) = L$, if and only if the right cosets $\alpha A_0(S)$ and $\beta A_0(S)$ are equal. This completes the proof of (2.1)(a).

Now we turn to (2.1)(b). We begin with a lemma.

Lemma 2.3. *If α and β are elements of $A(S)$ and if $\alpha | \tilde{L} = \beta | \tilde{L}$, then $\alpha = \beta$.*

Proof. It suffices to suppose that $\alpha \in A(S)$ and that $\alpha | \tilde{L} = \mathrm{Id}_{\tilde{L}}$, and to prove that $\alpha = \mathrm{Id}$. Using the notation established in the proof of Claim 2.2(a), we have $\alpha | (\langle\kappa_S, \delta\rangle^\perp) = \mathrm{Id}$ and $\alpha([f]) = [f]$, so that $\alpha(\kappa_S) = \kappa_S$. We need to show that $\alpha(\delta) = \delta$. But $\alpha(\delta) \in ((\langle\kappa_S, \delta\rangle^\perp)^\perp) = \langle\kappa_S, \delta\rangle$ so that $\alpha(\delta) = a\kappa_S + b\delta$. On the other hand, $\alpha(\delta) \cdot \kappa_S = 1$, implying that $b = 1$. Lastly, $\delta \cdot \delta = \alpha(\delta) \cdot \alpha(\delta) = 2a + \delta \cdot \delta$, so that $a = 0$, and $\alpha(\delta) = \delta$. □

In light of (2.3) to prove that $A_0(S) \subset D(S)$ it suffices to prove that any automorphism of the indefinite form $q_S | \tilde{L}$ is realized by a diffeomorphism. As a first step to doing this, recall that, if $q\colon \Lambda \to \mathbf{Z}$ is a quadratic form on the lattice Λ, and if $\alpha \in \Lambda$, $q(\alpha) = -1$, then

$$R_\alpha(x) = x + 2(x \cdot \alpha)\alpha$$

is an integral isometry of Λ. Similarly, if $q(\alpha) = -2$, we define

$$R_\alpha(x) = x + (x \cdot \alpha)\alpha.$$

In both cases, $R_\alpha(\alpha) = -\alpha$, $R_\alpha^2 = \mathrm{Id}$, and R_α fixes $(\alpha^\perp)$.

Proposition 2.4. *Let M be an oriented 4-manifold and $S^2 \subseteq M$ be an embedded sphere with $\alpha \in H^2(M; \mathbf{Z})$ the cohomology class dual to S^2. If $q_M(\alpha) = -1$ or -2, there is an orientation-preserving self-diffeomorphism φ of M such that $\varphi^* = R_\alpha$.*

Proof. We deal with the two cases separately. First suppose that $q_M(\alpha) = -1$. Then there is a neighborhood of S^2 in M which is diffeomorphic to $\mathbf{C}P^2 - \operatorname{int} D^4$, via an orientation-reversing diffeomorphism. We may thus write

$$M = M' \# \overline{\mathbf{C}P}^2$$

with

$$S^2 = \overline{(\mathbf{C}P^1)} \subseteq \overline{\mathbf{C}P}^2 - \operatorname{int} D^4 \subseteq M' \# \overline{\mathbf{C}P}^2,$$

where M' is a smooth oriented 4-manifold. It suffices to construct an orientation-preserving diffeomorphism $\varphi_0\colon \overline{\mathbf{C}P}^2 \to \overline{\mathbf{C}P}^2$ such that

(i) $\varphi_0 \mid D^4$ is the identity on D^4;

(ii) $\varphi_0^*(x) = -x$, where x is a generator of $H^2(\overline{\mathbf{C}P}^2; \mathbf{Z})$.

Indeed, given φ_0, we can extend $\varphi_0 \mid \overline{\mathbf{C}P}^2 - \operatorname{int} D^4$ by the identity on M' and smooth to obtain an diffeomorphism φ of M with the required properties.

To construct φ_0, it is sufficient to replace $\overline{\mathbf{C}P}^2$ by $\mathbf{C}P^2$ and construct a map satisfying the analogues of (i) and (ii). Begin with complex conjugation

$$\sigma\colon \mathbf{C}P^2 \to \mathbf{C}P^2.$$

It is orientation-preserving and satisfies (ii). Let p be a fixed point of σ, i.e. $p \in \mathbf{R}P^2$. Let $D^4 \cong \mathbf{C}^2$ be the affine piece of $\mathbf{C}P^2$ given by $\{z_0 \neq 0\}$, where z_0, z_1, z_2 are homogeneous coordinates on $\mathbf{C}P^2$. Then D^4 is invariant under σ, and $(z_1, z_2) = (x_1, y_1, x_2, y_2)$ are coordinates on U. Moreover, in these coordinates,

$$\sigma(x_1, y_1, x_2, y_2) = (x_1, -y_1, x_2, -y_2).$$

There is an isotopy of $\sigma \mid D^4$ to the identity on D^4 given by rotating the 2-plane spanned by the y_1- and y_2-coordinates at time t by an angle $t\pi$. By the isotopy extension theorem, we can extend this isotopy to an isotopy from σ to a diffeomorphism φ_0, which has the required properties (i) and (ii).

Now suppose that $q_M(\alpha) = -2$. Then a neighborhood of S^2 in M is diffeomorphic (via an orientation-reversing diffeomorphism), to $T_{D^2}(S^2)$, the unit disk bundle of the tangent bundle of S^2. We construct an orientation-preserving diffeomorphism $\varphi_0\colon T_{D^2}(S^2) \to T_{D^2}(S^2)$ such that, if Σ is the zero-section of $T_{D^2}(S^2)$,

(i) $\varphi_0 \mid \partial T_{D^2}(S^2)$ is the identity:

(ii) $\varphi_0(\Sigma) = -\Sigma (= \Sigma$ with the orientation reversed). Extending φ_0 by the identity on the rest of M and smoothing near $\partial T_{D^2}(S^2) \subset M$ then gives the desired diffeomorphism φ. To construct φ_0, view $T_{D^2}(S^2)$ as

$$T_{D^2}(S^2) = \{(v, w) \in \mathbf{R}^3 \times \mathbf{R}^3\colon \|v\| = 1,\ \|w\| \leqslant 1, \text{ and } v \cdot w = 0\},$$

where $\|\cdot\|$ and $(\cdot)$ are the standard norm and inner product on $\mathbf{R}^3$. Define

$$\varphi_0(v, w) = \begin{cases} \left(\cos(t\pi)v + \sin(t\pi)\dfrac{w}{\|w\|},\ -\|w\|\sin(t\pi)\cdot v + \cos(t\pi)w\right), \\ \qquad\qquad\qquad\qquad \text{if } w \neq 0, \\ (-v, 0), \qquad\qquad\quad\ \text{if } w = 0, \end{cases}$$

where $t = 1 - \|w\|$.

One checks easily that φ_0: $T_{D^2}(S^2) \to T_{D^2}(S^2)$ is a diffeomorphism, with $\varphi_0(v, w) = (v, w)$ if $\|w\| = 1$ and $\varphi_0(v, 0) = (-v, 0)$. $\square$

Remark 2.5. If p: $Z \to \Delta$ is a family of algebraic surfaces over the complex disk acquiring an ordinary double point at $t = 0$, and Z is smooth then there is so-called "vanishing cycle" associated to p. It is a smoothly embedded sphere $S^2 \subseteq Z_t$, where Z_t is a general fiber of p, and S^2 has self-intersection -2. According to the Picard-Lefschetz formula, the monodromy diffeomorphism associated to a loop in Δ enclosing 0 is the map φ_0 constructed above, up to isotopy.

We next turn to the construction of embedded spheres in Dolgachev surfaces. To do this, it suffices to consider one very special Dolgachev surface $\bar{S} = \bar{S}(p, q)$; this is a consequence of I(3.11).

We turn to the construction of $\bar{S}$. Let C_0 be a smooth cubic curve in $\mathbf{P}^2$, and l a line of inflection of C_0. (There are exactly 9 such lines.) Thus, l meets C_0 at a unique point p, counted with multiplicity 3. Make 9 "infinitely near" blowups at p. In other words, blow up p, then blow up the point of intersection of the proper transform of C_0 with the newly created exceptional curve, and so on. In the following schematic picture, we indicate the newly created exceptional curves E_i by dotted lines and use the same letter for a curve and its proper transform. Call the resulting surface $\bar{X}$. If we continue to use the same symbol to denote a curve and its proper transform on $\bar{X}$, then $\bar{X}$ has classes $E_1, \cdots, E_8$ and l which are smooth rational curves of square -2, and all indicated intersections are transverse. $\bar{X}$ has the structure of a rational elliptic surface in the sense of I(2.10). In fact, $\bar{X}$ is a resolution of singularities of the pencil defined by C_0 and $3l$. Moreover, $\bar{X}$ may be deformed complex-analytically to a generic rational elliptic surface. (In Kodaira's notation [17], X has a singular fiber of type II*, sometimes called of type $\tilde{E}_8$.)

Let $\bar{S} = \bar{S}(p, q)$ denote the result of performing two logarithmic transforms of orders p and q on two smooth fibers of $\bar{X}$. Thus, $\bar{S}$ is a Dolgachev surface as defined in I(3.3). Let $e_0, e_1, \cdots, e_8$ be the cohomology classes on $\bar{S}$ dual to $l, E_1, \cdots, E_8$. They are orthogonal to $[f]$, the class dual to a general fiber in the elliptic fibration on $\bar{S}$.

Clearly, we can define a map of quadratic forms ψ: $(-\tilde{E}_8) \to (\tilde{L}, q_{\bar{S}} | \tilde{L})$ by sending x_i to e_i. Let $q'_{\bar{S}}$ be the induced form on $\tilde{L}/\langle [f] \rangle$. Then ψ induces a map of quadratic forms $\bar{\psi}$: $(-E_8) \to (\tilde{L}/\langle [f] \rangle, q'_{\bar{S}})$. Since both these forms are nonsingular, $\bar{\psi}$ is automatically an isomorphism. The class $[f]$ is an integral linear combination of $e_0, \cdots, e_8$. Thus, it is contained in the image of ψ. Hence, ψ is onto, and consequently, ψ is an isomorphism.

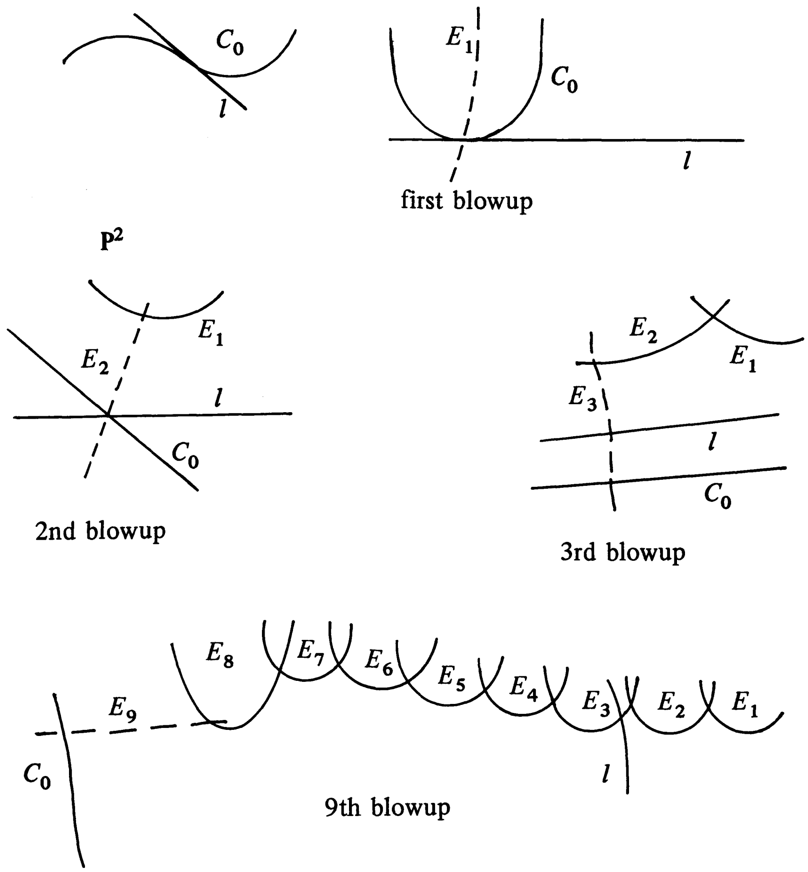

FIGURE 8

Lemma 2.6. *Every automorphism of* $(\tilde{L}, q_{\bar{S}}|\tilde{L})$ *sending* $[f]$ *to* $[f]$ *can be written as a composition of the reflections* R_{e_i}, $0 < i < 8$.

Proof. By the discussion above, it suffices to show that the subgroup G of $\mathrm{Aut}(-\tilde{E}_8)$ generated by $R_{x_0}, \cdots, R_{x_8}$ is the subgroup which is the identity on the radical of $-E_8$. Let $k \in (-\tilde{E}_8)$ generate the radical. Since the reflections in the simple roots generate $\mathrm{Aut}(-E_8)$, it follows that the image of G in $\mathrm{Aut}(-E_8)$ is all of $\mathrm{Aut}(-E_8)$. Next, using [4, p. 198], we see that G contains all the transvections: $T_y(x) = x + (x \cdot y)k$ for $y \in (-\tilde{E}_8)$. Now let $\psi \in \mathrm{Aut}(-\tilde{E}_8)$ satisfy $\psi(k) = k$. After composing with an element of G we can suppose that ψ is the identity on the quotient $-E_8$. It follows that ψ is a transvection and hence an element of G. □

Lemma 2.7. *If $\psi \in A_0(\bar{S})$ satisfies $\psi([f]) = [f]$, then $\psi \in D(\bar{S})$.*

Proof. If $\psi \in A_0(\bar{X})$, then $\psi | \tilde{L} \in \mathrm{Aut}(\tilde{L})$. If in addition, $\psi([f]) = [f]$, then by (2.6), ψ is in the subgroup generated by $R_{e_0}, \cdots, R_{e_8}$. But by construction each e_i is the Poincaré dual of an embedded 2-sphere in $\bar{S}$. By (2.4), each R_{e_i} is induced by a self-diffeomorphism of $\bar{S}$. Hence, so is ψ. □

Finally, we construct a diffeomorphism $\varphi: S \to S$ with $\varphi^*([f]) = -[f]$. We may view a Dolgachev surface S as the zero locus of a finite set $\{p_1, \cdots, p_M\}$ of homogeneous polynomials in $z_0, \cdots, z_N$ which define the embedding $S \subseteq \mathbf{P}^N = \mathbf{P}^N(\mathbf{C})$.

Let $\sigma: \mathbf{C} \to \mathbf{C}$ denote complex conjugation and let $S^\sigma \subseteq \mathbf{P}^N(\mathbf{C})$ be defined by the vanishing of $\{p_1^\sigma, \cdots, p_M^\sigma\}$. (If p is a polynomial with complex coefficients, we denote by p^σ the polynomial obtained by applying σ to all the coefficients.)

By transport of structure, S^σ is a smooth algebraic surface with the same numerical invariants of S, and every purely algebraic statement about S is true for S^σ as well. Hence S^σ is an elliptic surface with $p_g = g = 0$ and with exactly two multiple fibers, of multiplicities p and q. It follows that S^σ is a Dolgachev surface, by [6] or [23, Theorem 10, p. 191].

By I(3.11), there is a diffeomorphism $\varphi_0: S^\sigma \to S$ such that, if $[f^\sigma]$ is the class dual to a general fiber in the elliptic fibration for S^σ, then

$$\varphi_0^*[f] = [f^\sigma].$$

Moreover, $\sigma: \mathbf{P}^N(\mathbf{C}) \to \mathbf{P}^N(\mathbf{C})$ induces a diffeomorphism $S \to S^\sigma$, also denoted σ, and clearly $\sigma^*[f^\sigma] = -[f]$. Hence, if we set $\varphi = \varphi_0 \circ \sigma$,

$$\varphi^*[f] = \sigma^*\varphi_0^*[f] = \sigma^*[f^\sigma] = -[f].$$

This concludes the proof of (2.1)(b), and hence of Theorem 6′. □

We now consider Theorem 7.

Let $\tilde{S}$ be $S(p, q)$ blown up at r points. Of course we have the map $\rho^*: H^2(S) \hookrightarrow H^2(\tilde{S})$. We identify $H^2(S)$ with $\rho^* H^2(S) \subset H^2(\tilde{S})$. By Theorem 7A we have $D(\tilde{S}) \subset \{\psi \in A(\tilde{S}) \mid \psi(H^2(S)) = H^2(S)$ and $\psi | H^2(S) \in A_f(\tilde{S})\}$. That is to say

$$D(\tilde{S}) \subset A_f(S) \times A(B),$$

where B is the subgroup of $H^2(\tilde{S}; \mathbf{Z})$ generated by the classes $e_1, \cdots, e_r$ dual to the exceptional fibers of ρ and $A(B)$ is the automorphism group of the negative definite form $q_{\tilde{S}} | B$. We complete the proof of Theorem 7 by showing

Theorem 7B. $D(S) \times A(B) \subset D(\tilde{S})$.

Proof. First let us show that $D(S) \subset D(\tilde{S})$. Suppose $\psi \in A(S)$. Let $\tilde{\psi} \in A(\tilde{S})$ be the element defined by $\tilde{\psi}(x) = \psi(x)$ if $x \in H^2(S)$ and $\tilde{\psi}(e_i) = e_i$ for $1 \leqslant i \leqslant r$. We wish to show that if $\psi \in D(S)$, then $\tilde{\psi} \in D(\tilde{S})$. Suppose that ψ

is realized by a diffeomorphism φ: $S \to S$. Since φ is orientation-preserving, we can isotope φ until it is the identity in a disk $D^4 \subset S$. We take this disk to include the r points we blow up. Then φ lifts to a diffeomorphism $\tilde{\varphi}$: $\tilde{S} \to \tilde{S}$. Clearly $\tilde{\varphi}$ realizes $\tilde{\psi}$.

Now we show that $A(B) \in D(\tilde{S})$. The group $A(B)$ acts as a permutation group on the vectors of square -1. These vectors are $\{\pm e_1, \cdots, \pm e_r\}$, where $e_1, \cdots, e_r$ are dual to the exceptional fibers of ρ. It follows easily that $A(B)$ is generated by the reflections defined by e_i and by transpositions R_{ij} defined by $e_i \to -e_j$, $e_j \to -e_i$, $e_k \to e_k$ for $k \neq i, j$ for all pairs $i \neq j$. Since the e_i are dual to embedded 2-spheres in $\tilde{S}$, reflections defined by e_i are realized by self-diffeomorphisms of $\tilde{S}$, by (2.4). The transposition R_{ij} is in fact the reflection defined by $e_i + e_j$. Clearly, $q_{\tilde{S}}(e_i + e_j) = -2$. Since e_i and e_j are dual to classes represented by disjointly embedded 2-spheres in $\tilde{S}$, $e_i + e_j$ is dual to a class represented by an embedded 2-spheres. Hence by (2.4), R_{ij} is realized by a diffeomorphism of $\tilde{S}$. □

Now we turn to the completion of Theorem 10′.

Theorem 10B. *Let $\tilde{X}$ be a good generic rational surface of type* $(1, n)$ *for* $n \geqslant 10$. *Let $P_0(\tilde{X})$ be the P-cell in $\mathbf{H}(\tilde{X})$ which is the Kähler cone. Then*

$$\{\psi \in A(\tilde{X}) \mid \psi(\mathbf{S}(P_0(\tilde{X}))) = \pm\mathbf{S}(P_0(\tilde{X}))\} \subset D(\tilde{X}).$$

Proof. Clearly, the group on the left-hand side is generated by (i) $-$ Id, (ii) reflections in the ordinary walls of $P_0(\tilde{X})$, and (iii) $\{\psi \in A(\tilde{X}) \mid \psi(P_0(\tilde{X})) = P_0(\tilde{X})\}$. We show all these elements are realized by self-diffeomorphisms of $\tilde{X}$. Let $(x, e_1, \cdots, e_n)$ be a standard basic for $\tilde{X}$ coming from a representation of $\tilde{X}$ as $\mathbf{P}^2$ with n points blown up. Then x and the e_i are represented by disjointly embedded spheres. By (2.4) R_{e_i} is realized by a self-diffeomorphism of $\tilde{X}$. By the obvious analogue of (2.4) for classes of square $+1$, so is R_x where $R_x(y) = y - 2(x \cdot y)x$. The composition $R_x \circ R_{e_1} \circ \cdots \circ R_{e_n} = -\mathrm{Id}$. Hence $-\mathrm{Id} \in D(\tilde{X})$.

Since the ordinary walls of $P_0(\tilde{X})$ are defined by classes dual to exceptional curves the reflections in (ii) are realized by self-diffeomorphisms of $\tilde{X}$.

Lastly suppose $\psi \in A(\tilde{X})$ and $\psi(P_0(\tilde{X})) = P_0(\tilde{X})$. Let x be a corner of $P_0(\tilde{X})$ and let $(x, e_1, \cdots, e_n)$ be a standard basis for $\Lambda(\tilde{X})$ adapted to the chamber $\tilde{C}_0$ containing $P_0(\tilde{X})$. Since the e_i define oriented, ordinary walls for $P_0(\tilde{X})$, each e_i is dual to an exceptional curve E_i in $\tilde{X}$. Since $e_i \cdot e_j = 0$ for $i \neq j$, $E_i \cap E_j = \varnothing$. Thus, there is a diffeomorphism φ_0: $\tilde{X} \to \mathbf{C}P^2 \# n\overline{\mathbf{C}P}^2$ carrying the obvious standard basis for $\mathbf{C}P^2 \# n\overline{\mathbf{C}P}^2$ to $(x, e_1, \cdots, e_n)$.

Now suppose $\psi \in A(\tilde{X})$ leaves $P_0(\tilde{X})$-invariant. Then

$$(\psi(x), \bar{\psi}(e_1), \cdots, \psi(e_n))$$

is another standard basis adapted to $C_0(\tilde{X})$ with $\psi(x)$ a corner of $P_0(\tilde{X})$. Thus, there is another diffeomorphism φ_0^{ψ}: $\tilde{X} \to \mathbf{C}P^2\#n\overline{\mathbf{C}P}^2$ carrying the obvious standard basis for $\mathbf{C}P^2\#n\overline{\mathbf{C}P}^2$ to $(\psi(x), \psi(e_1), \cdots, \psi(e_n))$. The composition $(\varphi_0)^{-1} \circ \varphi_0^{\psi}$: $\tilde{X} \to \tilde{X}$ is a diffeomorphism realizing ψ. This completes the proof of Theorem 10B. $\square$

Proof that Theorem 10A *and Theorem* 10B $\Rightarrow$ *Theorem* 10$'$. Theorem 10A and 10B taken together say that if $\tilde{X}$ is a good generic rational surface of type $(1, n)$, $n \geqslant 10$, then $D(\tilde{X}) = \{\psi \in A(\tilde{X}) \mid \psi(\mathbf{S}(P_0(\tilde{X}))) = \pm\mathbf{S}(P_0(\tilde{X}))\}$, where $P_0(\tilde{X})$ is $\mathscr{K}(\tilde{X})$, and that $D(\tilde{X})$ has infinite index in $A(\tilde{X})$. Now suppose that $\tilde{X}'$ is any rational surface of type $(1, n)$. We know that there is a diffeomorphism φ: $\tilde{X}' \to \tilde{X}$. Let $P_0(\tilde{X}') = \varphi^* P_0(\tilde{X})$. Then $D(\tilde{X}') = \{\psi \in A(X') \mid \psi(\mathbf{S}(P_0(\tilde{X}')) = \pm\mathbf{S}(P_0(\tilde{X}'))\}$ and this group has infinite index in $A(\tilde{X}')$. $\square$

3. Some corollaries

In this section we deduce some corollaries of the main theorem. We first give lower bounds for the number of double points on immersed 2-spheres in blown up Dolgachev surfaces, in terms of the cohomology class dual to the image of the 2-sphere. As a consequence we find that there are strong restrictions on the possibilities for connected sum decompositions of Dolgachev surfaces and their blow ups. Finally, we give a reformulation of our main theorem in terms of the finiteness of the number of components of moduli spaces for algebraic surfaces of type $(1, n)$.

Our first result characterizes those cohomology classes of square -1 in a blown up Dolgachev surface which are represented by differentiably embedded 2-spheres.

Corollary 8. *Let $S = S(p, q)$ be a Dolgachev surface, and let $\tilde{S}$ be the blow up of S at r points. Let $e_1, \cdots, e_r \in H^2(\tilde{S}; \mathbf{Z})$ be the classes Poincaré dual to the exceptional fibers of ρ: $\tilde{S} \to S$. If $\alpha \in \Lambda(\tilde{S})$ is dual to a class represented by a differentiably embedded 2-sphere and if $q_{\tilde{S}}(\alpha) = -1$ then $\alpha = \pm e_i$ for some i, $1 \leqslant i \leqslant r$. In particular, no such class exists in $H^2(S; \mathbf{Z})$.*

Proof. Suppose that $q_{\tilde{S}}(\alpha) = -1$ and α is Poincaré dual to a class represented by a differentiably embedded 2-sphere. By (2.1) the reflection R_α: $\mathbf{H}(\tilde{S}) \to \mathbf{H}(\tilde{S})$ is realized by a diffeomorphism. Since R_α leaves invariant the components of $\mathbf{H}(\tilde{S})$, Theorem 7 implies that R_α preserves the super P-cell $\mathbf{S}(P_0(\tilde{S}))$ and also the class $k = \rho^*\kappa_S \in H^2(\tilde{S}; \mathbf{Z})$. Since R_α: $\mathbf{H}(\tilde{S}) \to \mathbf{H}(\tilde{S})$ is geometric reflection in the wall W^α in $\mathscr{W}_1(q)$, this implies that $\mathbf{R}^+ \cdot W^\alpha$ contains k in its closure and that $\mathbf{S}(P_0(\tilde{S}))$ is invariant under this reflection.

Since $\mathbf{S}(P_0(\tilde{S}))$ is convex, it must be the case that $W^\alpha \cap \operatorname{int}\mathbf{S}(P_0(\tilde{S})) \neq \varnothing$. Hence by II(5.8) $W^\alpha = W^{e_i}$ for some i, $1 \leqslant i \leqslant r$. $\square$

This result has a corollary concerning the types of immersions that can represent classes in $H^2(S;\mathbf{Z})$ or $H^2(\tilde{S};\mathbf{Z})$. If M^4 is an oriented 4-manifold and if $i\colon S^2 \looparrowright M$ is a generic C^∞-immersion, then i has a finite set of double points. At each double point there is a sign for the self-intersection (± 1). Let $d_+(i)$ (resp. $d_-(i)$) be the number of double points whose self-intersection is $+1$ (resp. -1). If χ denotes the Euler characteristic of the normal bundle of i and if $x \in H^2(M;\mathbf{Z})$ is the class Poincaré dual to $i_*[S^2]$, then

$$q_M(x) = 2d_+(i) - 2d_-(i) + \chi.$$

Corollary 9. *Let S, $\tilde{S}$, $e_1, \cdots, e_r$, and ρ be as in Corollary 8. For any generic C^∞-immersion $i\colon S^2 \looparrowright \tilde{S}$ representing the Poincaré dual of a class $x \in H^2(\tilde{S};\mathbf{Z})$, $x \neq 0$, we have*

$$d_+(i) \geqslant \frac{q_{\tilde{S}}(x) + 1}{4}$$

with equality only if $x = \pm e_i$ for some i, $1 \leqslant i \leqslant r$.

Proof. Suppose $i\colon S^2 \looparrowright \tilde{S}$ is a generic C^∞-immersion representing the Poincaré dual of a class $x \in H^2(\tilde{S};\mathbf{Z})$ and $d_+(i) \leqslant (q_{\tilde{S}}(x) + 1)/4$. Let the double points of i be $a_1, \cdots, a_s$, $b_1, \cdots, b_t$ with the sign of self-intersection at each a_i positive and at each b_j negative. Let $u = q_{\tilde{S}}(x) + 1 - 4d_+(i)$. By hypothesis $u > 0$. Choose smooth points $c_1, \cdots, c_u$ on $i(S^2)$. We can assume that the two sheets of $i(S^2)$ near each a_i are complex analytic and that the single sheet of $i(S^2)$ near each c_i is complex analytic. Near the b_j we can arrange that the sheets are geometrically complex analytic but that one has the opposite orientation. Now let $\lambda\colon \tilde{S}' \to \tilde{S}$ be the result of blowing up $\{a_1, \cdots, a_s, b_1, \cdots, b_t, c_1, \cdots, c_u\}$ in $\tilde{S}$. Let A_i, B_j and C_k denote the fibers over b_i, b_j, and c_k respectively. In $\tilde{S}'$ we have the proper transform $i'\colon S^2 \looparrowright \tilde{S}'$ of $i\colon S^2 \looparrowright \tilde{S}$. It is an embedding, and it represents the class Poincaré dual to $y = \lambda^*(x) - 2\Sigma_{i=1}^s A_i - \Sigma_{j=1}^u C_j$. Thus,

$$q_{\tilde{S}'}(y) = q_{\tilde{S}}(x) - 4s - u = q_{\tilde{S}}(x) - 4d_+(i) - u = -1.$$

By Corollary 8, y is Poincaré dual to an exceptional fiber of $\tilde{S}' \to S$. Clearly, since $x \neq 0$, the only way that this can happen is for $q_{\tilde{S}}(x) = -1$ and $d_+(i) = 0$, and hence by Corollary 8 again $x = \pm e_i$ for some i, $1 \leqslant i \leqslant r$. $\square$

Corollary 3.1. *Let S be a Dolgachev surface and let $i\colon S^2 \looparrowright S$ be a generic C^∞-immersion. Suppose that the cohomology class dual to $i_*[S^2]$ is x and $x \neq 0$. Then,*

$$d_+(i) > \frac{q_S(x) + 1}{4}. \quad \square$$

Corollary 3.2. *Let $\tilde{S}$ be the blow up of a Dolgachev surface S at r points.*

(a) *There does not exist a connected sum decomposition*

$$\tilde{S} = M \# \mathbf{C}P^2 \quad \text{or} \quad \tilde{S} = M \# (S^2 \times S^2),$$

for any smooth 4-manifold M.

(b) *Suppose that there is a connected sum decomposition*

$$\tilde{S} = M \# \overline{\mathbf{C}P}^2.$$

Let $x \in H^2(\tilde{S}; \mathbf{Z})$ be the element corresponding to the image of a generator of $H^2(\overline{\mathbf{C}P}^2; \mathbf{Z})$. Then $x = \pm e_i$ for some i. □

Our final application is, in a certain sense, a much less precise reformulation of the main result on diffeomorphism types of blown up Dolgachev surfaces. We include it as a clue to the very striking relationship between the moduli of algebraic surfaces and 4-manifold theory.

Definition 3.3. Two smooth compact complex surfaces Y_1 and Y_2 are *of the same deformation type* if there exist

(i) connected complex spaces T and Y (which need not be smooth or irreducible);

(ii) a smooth proper map $\Phi\colon Y \to T$; and

(iii) two points $t_1, t_2 \in T$ with

$$\Phi^{-1}(t_i) \cong Y_i, \qquad i = 1, 2.$$

Of course, if Y_1 and Y_2 are of the same deformation type, then they are diffeomorphic. Corollary 5 in the introduction is a paraphrase of the following theorem.

Theorem 3.4. *The forgetful map*

$$\left\{\begin{array}{l}\textit{smooth simply connected}\\ \textit{algebraic surfaces of}\\ \textit{type } (1,n) \textit{ modulo defor-}\\ \textit{mation type}\end{array}\right\} \to \left\{\begin{array}{l}\textit{oriented, simply connected}\\ \textit{4-manifolds of type } (1,n)\\ \textit{modulo diffeomorphism}\end{array}\right\}$$

is finite-to-one.

An equivalent reformulation of (3.4) is

Theorem 3.4′. *Let M be an oriented simply connected 4-manifold of type $(1, n)$. Then the set of all nonrational algebraic surfaces diffeomorphic to M may be parametrized by a union of finitely many quasi-projective varieties.*

Proof of (3.4). First note that if Y_1 and Y_2 are of the same deformation type, then so are $\tilde{Y}_1$ and $\tilde{Y}_2$, where $\tilde{Y}_i$ is the blow up of Y_i at some point. By Kodaira's classification, surfaces fall into 4 types, characterized by Kodaira dimension $-\infty$, 0, 1, or 2. Surfaces of Kodaira dimension $-\infty$ have minimal models which are rational or ruled, and if the surface is simply connected it is rational.

Rational surfaces of type $(1, n)$ fall into one or two equivalence classes under deformation type, depending on whether $n > 1$ or $n = 1$. (A simply connected algebraic surface of type $(1, 0)$ is automatically $\mathbf{P}^2$, which is elementary if the surface is rational and a deep theorem of Yau [34] otherwise.) To see this, first assume that the rational surface is minimal ruled, and so $\mathbf{F}_a$ for some $a \geqslant 0$. It is well known that $\mathbf{F}_a$ and $\mathbf{F}_{a+2}$ are of the same deformation type [5]. Since $\mathbf{F}_0 \cong \mathbf{P}^1 \times \mathbf{P}^1$ has an even intersection form and $\mathbf{F}_1$ is of type $(1, 1)$, deformation type and diffeomorphism type coincide. The case $n \geqslant 2$ follows from the case $n = 2$, by the remark above. Finally, to handle the case $n = 2$, using the remark above, it suffices to note that the blowup at $\mathbf{F}_0$ at one point is the same as the blowup of $\mathbf{F}_1$ at any point not on the exceptional curve. Once again deformation type and diffeomorphism type coincide.

There are no simply connected surfaces of type $(1, n)$ and Kodaira dimension 0.

If Y has Kodaira dimension 1, its minimal model $\overline{Y}$ is elliptic. If in addition $\overline{Y}$ is simply connected and of type $(1, n)$, then it is a Dolgachev surface [6]. Hence, Y itself is a blown up Dolgachev surface. By Theorem 3 and I(3.8), the theorem is true for these surfaces.

If Y has Kodaira dimension 2, then its minimal model $\overline{Y}$ is of general type. Suppose Y is simply connected and of type $(1, n)$. By the Noether formula

$$c_1^2(\overline{Y}) + c_2(\overline{Y}) = 12 \qquad \chi(\mathcal{O}_{\overline{Y}}) = 12.$$

As $c_2(\overline{Y})$ is the Euler characteristic of $\overline{Y}$,

$$c_2(\overline{Y}) = 2 + b_2(\overline{Y}) \geqslant 3.$$

Thus $c_1^2(\overline{Y}) \leqslant 9$. By a theorem of Bombieri [3], the line bundle $\mathcal{O}_{\overline{Y}}(5K_{\overline{Y}})$ defines a birational morphism of $\overline{Y}$ to its image in $\mathbf{P}^N$ of degree $5(K_{\overline{Y}})^2 = 5(c_1(\overline{Y}))^2$, where $N = 10(K_{\overline{Y}})^2$.

From the general theory of Hilbert schemes, the collection of such subsets of $\mathbf{P}^N$ is parametrized by a finite union of quasi-projective varieties. The same will be true for the blown up varieties Y. □

Remark 3.5. The discussion of surfaces of general type shows that the elliptic surfaces and their blow ups are the only class of surfaces that can give us an infinity of diffeomorphism types in any given homotopy type.

In light of (3.4), it is natural to make the following conjecture.

Conjecture 3.6. The forgetful map from smooth algebraic surfaces, modulo deformation type, to smooth oriented 4-manifolds, modulo diffeomorphism, is finite-to-one.

It follows from (3.5) that (3.6) reduces to the analogous conjecture for blown up elliptic surfaces. Moreover, it is sufficient to prove (3.6) for blown up simply connected elliptic surfaces, via an explicit calculation of the fundamental group and an analysis similar to the proof of I(3.8).

Remark 3.7. The analogous conjecture for higher dimensional algebraic varieties is false, even if one only considers simply connected algebraic varieties. On the other hand, Kollår has proven this conjecture for algebraic (or equivalently, Kähler) varieties with $b_2 = 1$.

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