J. DIFFERENTIAL GEOMETRY
27 (1988) 67–80

# EUCLIDEAN DECOMPOSITIONS OF NONCOMPACT HYPERBOLIC MANIFOLDS

D. B. A. EPSTEIN & R. C. PENNER

In this paper, we introduce a method for dividing up a noncompact hyperbolic manifold of finite volume into canonical Euclidean pieces. The construction first arose in the setting of surfaces (see [7]), and in this case one gets a canonical cell decomposition of the surface and a canonical Euclidean structure. (The Euclidean structure, of course, is not complete.) The conformal structure underlying this Euclidean structure does not agree with the underlying hyperbolic structure, but the two conformal structures are probably not too distant (cf. Sullivan's theorem [5] for an analogous result).

This investigation arose from an attempt to understand the coordinates and cell decomposition of Teichmüller space due to Harer and Mumford [6] and independently to Thurston. Such coordinates and cell decompositions are also provided in [3] and [7]; in the latter, the action of the mapping class group on the coordinates is considered. We would like to thank J. Harer for the inspiration of his work and for several helpful remarks.

Our method is to work in Minkowski space and to represent a cusp by a point on the light-cone. The orbit of this point turns out to be discrete (even though the action of the group on the light-cone is ergodic), and we take the convex hull of the orbit. The boundary of this convex hull is decomposed into affine pieces, and one should think of the convex hull boundary as a kind of piecewise linear approximation to the upper sheet of the hyperboloid in Minkowski space. Each piece has a natural Euclidean structure. The suggestion that this might be possible first arose in a conversation between the authors and Lee Mosher. We thank Mosher for his contribution to this crucial idea. Comments by Brian Bowditch have also been helpful on a number of occasions. As a final credit, we wish to thank Bill Thurston. Much of this work as been discussed at various points with him, and the exposition has gained substantially from his comments.

Received July 31, 1986 and, in revised form, October 28, 1986.

There are a large number of interesting problems and applications which arise from our work. Some of these are explored in [7].

## 1. Minkowski space and the hyperboloid model

Let $V$ be a real vector space of dimension $(n+1)$ with a nondegenerate quadratic form $\langle\,\cdot\,,\cdot\rangle$ of type $(n,1)$, i.e., there is a positive definite subspace of dimension $n$ and a negative definite subspace of dimension 1. All such quadratic forms are equivalent, and we can choose a basis $(e_0, e_1, \cdots, e_n)$ with $\langle e_i, e_j\rangle = 0$ if $i \neq j$, $\langle e_0, e_0\rangle = -1$, and $\langle e_i, e_i\rangle = 1$ if $i \geqslant 1$. The corresponding metric on $V$ admits an expression

$$ds^2 = -dx_0^2 + dx_1^2 + \cdots + dx_n^2,$$

and we define *Minkowski* space $M^{n+1}$ to be $V$ equipped with this metric.

If $S$ is an affine subspace of $M^{n+1}$ of codimension 1, then the restriction of the metric on $M^{n+1}$ to $S$ may be positive definite, singular, or of type $(n-1,1)$. If we write

$$S = \{x \in M^{n+1}m \colon \langle x, s\rangle = \lambda\},$$

where $0 \neq s \in M^{n+1}$ and $\lambda \in \mathbb{R}$, then these cases correspond to $\langle s, s\rangle < 0$, $\langle s, s\rangle = 0$, and $\langle s, s\rangle > 0$, respectively. In the positive definite case, $S$ has an induced Euclidean structure, and we can speak of Euclidean distances between points of $S$, and so on.

The hyperboloid

$$\{v \in V \colon \langle v, v\rangle = -1\} = \{x \in M^{n+1} \colon -x_0^2 + x_1^2 + \cdots + x_n^2 = -1\}$$

has two components. The upper sheet, where $x_0 \geqslant 1$, is a model for hyperbolic $n$-space $\mathbb{H}^n$. The form of the metric on $M^{n+1}$, restricted to a tangent space of the hyperboloid, is positive definite, and the hyperboloid inherits the structure of a Riemannian manifold. An isometry of the upper sheet with the Poincaré disk model of hyperbolic $n$-space is given by radial projection from $(-1, 0, \cdots, 0)$ to the unit disk in the plane $x_0 = 0$. An isometry with the projective or Klein model is given by radial projection from the origin of $M^{n+1}$ to the unit disk in the plane $x_0 = 1$ with center $(1, 0, 0)$.

We define the *light-cone* $L$ of $V$ by

$$L = \{v \in V \colon \langle v, v\rangle = 0\} = \{x \in M^{n+1} \colon x_0^2 = x_1^2 + \cdots + x_n^2\}$$

and the *positive light-cone* $L^+$ to be the component of $L - \{0\}$ where $x_0 > 0$. A ray from the origin in $L^+$ corresponds to a point of $S_\infty^{n-1}$, the sphere at infinity of $\mathbb{H}^n$: the points at infinity can be thought of as the boundary of the

Poincaré disk model (or the boundary of the projective disk model), and radial projection from $(-1, 0, \cdots, 0)$ (or the origin) establishes the correspondence. A point $v \in L^+$ corresponds to a horosphere

$$\{ w \in \mathbb{H}^n : \langle w, v \rangle = -1 \},$$

and the corresponding horoball is

$$\{ w \in \mathbb{H}^n : 0 > \langle w, v \rangle \geqslant -1 \}.$$

The center of the horosphere is the ray through $v$, and as $v$ moves away from the origin along the ray, the horoball contracts towards the center of the horoball. This bijection between the points of $L^+$ and the set of horoballs becomes a homeomorphism if we use the geometric topology on the space of horoballs in $\mathbb{H}^n$. In order to avoid the unpleasant features of the geometric topology on noncompact spaces, we work with closures of horoballs in the closed unit disk employing the Hausdorff metric on closed subspaces of the unit disk.

The group of linear isomorphisms of $M^{n+1}$ preserving the quadratic form is the Lie group $O(1, n)$. The subgroup $O^+(1, n)$ of elements preserving the upper sheet of the hyperboloid has index two in $O(1, n)$ and is equal to the group of isometries of hyperbolic $n$-space. $O^+(1, n)$ has two components; the identity component consists of orientation-preserving isometries of hyperbolic $n$-space. Such elements are also characterized by the property that they preserve the orientation of $M^{n+1}$. $O(1, n)$ has four components, $SO(1, n)$ two components, and $SO(1, n) \cap O^+(1, n)$ is the identity component, preserving the sheets and the orientation. This group is sometimes called the group of Möbius transformations of hyperbolic $n$-space; we denote it by $SO^+(1, n)$.

**Lemma 1.1.** *$O^+(1, n)$ acts transitively on $L^+$, and the stabilizer of a point is noncompact if $n \geqslant 1$. The stabilizer of a point of $L^+$ under $SO^+(1, n)$ is noncompact if $n \geqslant 1$, and the action is transitive if $n > 1$.*

*Proof.* Let $v_0 \in L^+$ be an arbitrary point. Let $v_1 \in L^+$ be chosen so that $v_1$ is not a scalar multiple of $v_0$. It follows that $v_0 + v_1$ lies inside the light-cone, so

$$2\langle v_0, v_1 \rangle = \langle v_0 + v_1, v_0 + v_1 \rangle < 0.$$

Changing $v_1$ by a positive scalar, we may assume that $\langle v_0, v_1 \rangle = -1$. The subspace spanned by $v_0$ and $v_1$ has type $(1, 1)$ (consider the basis $v_0 + v_1, v_0 - v_1$), and its orthogonal complement has type $(n - 1, 0)$, with basis $(v_2, \cdots, v_n)$. Our basis has the following properties:

$$\langle v_0, v_0 \rangle = \langle v_1, v_1 \rangle = 0, \quad \langle v_0, v_1 \rangle = -1, \quad \langle v_i, v_j \rangle = \delta_{ij} \text{ if } i, j \geqslant 2.$$

Any two such bases are equivalent under $O(V) = O(1, n)$, and, if we insist that $v_0 \in L^+$, then any two such basis are equivalent under $O^+(1, n)$. Therefore $O^+(1, n)$ acts transitively on $L^+$.

Let $N$ be the stabilizer of $v_0$. $N$ is homeomorphic to the space of special bases $(v_0, v_1, \cdots, v_n)$ with $v_0$ fixed. The space $C$ of possible choices for $v_1$ is the positive light-cone surface $L^+$, from which the ray from the origin through $v_0$ has been removed. $N$ fibers over $C$ with fiber $O(n-1)$, the space of possible choices for $(v_2, \cdots, v_n)$, given $v_0$ and $v_1$. Since $C$ is contractible, $N$ is homeomorphic to $C \times O(n-1)$. This proves the lemma for $O^+(1, n)$.

The case of $SO^+(1, n)$ is identical except that the orientation class of $(v_0, v_1, \cdots, v_n)$ is specified. If $n > 1$, we can ensure that the orientation class is correct, replacing $v_n$ by $-v_n$ if necessary. q.e.d.

Hyperbolic (sometimes called loxodromic) transformations of $\mathbb{H}^n$ are those elements of $O^+(1, n)$ with an eigenvalue $\lambda$ not lying on the unit circle. It follows that $\lambda$ must be real and positive, and the corresponding eigenvector is unique and lies on $L^+$. There is exactly one other linearly independent eigenvector on $L^+$, and the corresponding eigenvalue is $\lambda^{-1}$. All other eigenvectors lie outside the light-cone. Parabolic transformations have a unique eigenvector (ray) in $L^+$ and no eigenvector in the interior of the light-cone. The ray in $L^+$ is fixed pointwise—the eigenvalue is 1. Elliptic transformations have all eigenvalues on the unit circle and at least one fixed point in $\mathbb{H}^n$. Any element of $O^+(1, n)$ is the identity or is elliptic, parabolic, or hyperbolic.

## 2. The action of discrete subgroups on the light-cone

If $\Gamma$ is a discrete subgroup of $O^+(1, n)$, then $\Gamma$ acts properly discontinuously on $\mathbb{H}^n$. Conversely, a group acting properly discontinuously by isometries on $\mathbb{H}^n$ is a discrete subgroup of $O^+(1, n)$. This nice correspondence results from the fact that the stabilizer in $O^+(1, n)$ of a point of $\mathbb{H}^n$ is compact (isomorphic to the orthogonal group $O(n)$).

In contrast, the action of $\Gamma$ on $L^+$ is usually not proper.

**Theorem 2.1.** *If* $\Gamma \backslash O^+(1, n)$ (*or equivalently* $\Gamma \backslash \mathbb{H}^n$) *has finite volume, then the action of* $\Gamma$ *on* $L^+$ *is ergodic, with respect to Lebesgue measure on* $L^+$.

**Remark.** $\Gamma$ does not preserve Lebesgue measure on $L^+$, but it does preserve the measure class, i.e. the transformed measure has the same sets of measure zero.

*Proof.* We need only prove that some subgroup of $\Gamma$ acts ergodically, so we may assume that $\Gamma \subset SO^+(1, n)$. Let $N$ be the stabilizer in $SO^+(1, n)$ of a point in $L^+$, so that $L^+ = SO^+(1, n)/N$. We need only show that if $\Gamma$ acts on

$SO^+(1,n)$ on the left and $N$ acts on the right, then the action of $\Gamma \times N$ on $SO^+(1,n)$ is ergodic. Equivalently, we show that the action of $N$ on $\Gamma \setminus SO^+(1,n)$ is ergodic; as before, it suffices to show that some subgroup of $N$ acts ergodically on $\Gamma \setminus SO^+(1,n)$. Since $\Gamma \setminus SO^+(1,n)$ has finite volume by hypothesis, the result follows upon appealing to [9, p. 19].

**Corollary 2.2.** *If $\Gamma \setminus \mathbb{H}^n$ has finite volume, then almost every point of $L^+$ has a dense $\Gamma$-orbit.*

*Proof.* Let $U$ be a nonempty open subset of $L^+$. The difference $L^+ - \Gamma U$ then has zero measure. Letting $U_i$ run over a countable basis of $L^+$, it follows that

$$L^+ - \bigcap_i \Gamma U_i = \bigcup_i (L^+ - \Gamma U_i)$$

has zero measure. Finally, this is exactly the set of points whose orbits are not dense.

**Theorem 2.3.** *If $\Gamma$ is a discrete subgroup of $O^+(1,n)$ and $\Gamma \setminus O^+(1,n)$ (or equivalently $\Gamma \setminus \mathbb{H}^n$) is compact, then the orbit of every point of $L^+$ under $\Gamma$ is dense.*

*Proof.* We may assume that $\Gamma$ preserves orientation. If $N$ is the stabilizer of a point of $L^+$, then $N$ acts strictly ergodically on $\Gamma \setminus SO^+(1,n)$ by [4]. q.e.d.

Our basic starting point is the following result, which is somewhat unexpected in view of the ergodicity results above.

**Theorem 2.4.** *Let $\Gamma$ be a discrete subgroup of $O^+(1,n)$ such that for some point $v \in L^+$, the stabilizer of $v$ in $\Gamma$ is nontrivial. If $n \geqslant 4$, we also suppose that $\Gamma \setminus \mathbb{H}^n$ has finite volume and that $\Gamma$ is finitely generated. Then the orbit of $v$ under $\Gamma$ is discrete and is closed in $M^{n+1}$.*

**Remarks.** (1) To say that the orbit is closed in $M^{n+1}$ means that it is closed in $L^+$ and does not accumulate at 0.

(2) If $p \in S_\infty^{n-1}$ is a parabolic fixed point, then a horoball centered at $p$ is said to be *uniform* (with respect to $\Gamma$) if $\gamma H \cap H \neq \varphi$ implies that $\gamma H = H$ for $\gamma \in \Gamma$. Our proof is based on the existence of uniform horoballs. Apanasov has examples for $n > 4$ where uniform horoballs do not exist. These examples all require an infinite number of generators for $\Gamma$. The situation for finitely generated groups is unknown. With a little care, these examples can be modified so that the orbit of a parabolic fixed point is not discrete. This example is explained in the Appendix.

(3) If $\Gamma \setminus \mathbb{H}^n$ has finite volume, then $\Gamma$ is finitely generated. This result has been proved by Brian Bowditch.

*Proof.* If $\Gamma \backslash \mathbb{H}^n$ has finite volume and $\Gamma$ is finitely generated, then, by going to a finite cover, we may assume $\Gamma$ is torsion free. It follows that $\Gamma \backslash \mathbb{H}^n$ is a finite union of cusps and a compact part (see [8] or [9]). Lifting to the universal cover, each of the cusps gives a uniform horoball. If $n = 2$ or $n = 3$, an orientation-preserving parabolic subgroup consists of translations (in the upper half-space model for example). The existence of uniform horoballs centered on any parabolic fixed point then follows from the Margulis Lemma or Jørgensen Inequality.

Clearly the orbit of a fixed uniform horoball is discrete and closed in the space of all horoballs. Moreover, the Euclidean radius of the horoballs in the Poincaré disk model is bounded away from one, and this implies that the origin in the Minkowski space $M^{n+1}$ is not a limit point of the orbit of the parabolic fixed point $v \in L^+$.

## 3. The convex hull construction

Let $\Gamma$ be a discrete subgroup of $O^+(1, n)$ such that the quotient of $\mathbb{H}^n$ by $\Gamma$ has finite volume and such that there is at least one parabolic fixed point in $L^+$ (i.e., at least one cusp in $\Gamma \backslash \mathbb{H}^n$). We choose one orbit $B_i \subset L^+$ of parabolic fixed points corresponding to each cusp of $\Gamma \backslash \mathbb{H}^n$, $i = 1, \cdots, p$. Theorem 2.4 guarantees that each $B_i$ is discrete in $M^{n+1}$.

Let $C$ be the closed convex hull in $M^{n+1}$ of $B_1 \cup \cdots \cup B_p$. We now have a series of lemmas to elucidate the nature of $C$.

**Lemma 3.1.** *The dimension of $C$ is $(n + 1)$.*

*Proof.* We will assume that $C$ has dimension less than $(n + 1)$ and derive a contradiction. Let $W$ be the vector subspace in $M^{n+1}$ which is parallel to the affine hull of $C$. Since $C$ is invariant under $\Gamma$ and $\Gamma$ acts linearly, $W$ is also invariant under $\Gamma$. The quadratic form on $W$ must be nondegenerate, for otherwise the kernel will be an invariant one-dimensional space, giving rise to a fixed point for $\Gamma$ in $S_\infty^{n-1}$. However, there is no such fixed point.

Thus, $W$ is nondegenerate and so is $W^\perp$; furthermore, both are invariant under the action of $\Gamma$. One of these (call it $W$) has a quadratic form of type $(s, 1)$ with $s < n$, so $W$ gives a $\Gamma$-invariant subspace of $\mathbb{H}^n$. This means that the limit set of $\Gamma$ is contained in a sphere of dimension $(s - 1)$. Since $\Gamma \backslash \mathbb{H}^n$ has finite volume, the limit set is the whole of $S_\infty^{n-1}$. This contradiction proves the result.

**Proposition 3.2.** *Suppose that $\Gamma \backslash \mathbb{H}^n$ has finite volume and let $x \in L^+$. The origin is not an accumulation point for $\Gamma x$ if and only if $x$ is a parabolic fixed point.*

*Proof.* We have already proved one implication in Theorem 2.4. To prove the other implication, we suppose that the origin is not an accumulation point.

The complement in $\Gamma \setminus \mathbb{H}^n$ of the set of cusps is compact, and we let $R$ be greater than the diameter of this compact set. Let $z_0$ be a point of $\mathbb{H}^n$ mapping into this compactum and let $K$ be the ball centered at $z_0$ of radius $R$. Now, choose a height (i.e., first coordinate in $M^{n+1}$) so that every point of $L^+$ above this height corresponds to a horoball which is disjoint from $K$.

By hypothesis, we can multiply $x$ by a large enough scalar so that the $\Gamma$-orbit of $x$ lies entirely above this height. It follows that the horoball corresponding to $x$ projects to one of the cusps of $\Gamma \setminus \mathbb{H}^n$. Therefore, this horoball lies entirely inside some standard horoball about some cusp. This can only happen if $x$ is itself a parabolic fixed point.

**Lemma 3.3.** *$L^+ \cap C$ is the set of points of the form $\alpha z$, where $\alpha \geqslant 1$ and $z \in B_i$ for some $i = 1, \cdots, p$.*

*Proof.* Suppose that $x \in L^+$ is not of the stated form and choose $\alpha > 1$ so that $\alpha x$ is also not of the stated form. Let $A$ be the horizontal subspace (i.e., contained in a level set of $x_0$) of dimension $(n - 1)$ through $\alpha x$ tangent to $L^+$. The tangent plane to $L^+$ at $\alpha x$ can be rotated slightly around $A$ to give a plane $P$ which intersects $L^+$ in a long thin ellipsoid nearly equal to the ray $[0, \alpha x]$ and containing $\alpha x$. Since there are only finitely many points of $B_1 \cup \cdots \cup B_p$ below the height of $\alpha x$, we can ensure that the rotation about $A$ is so small that $x$ lies on the opposite side of $P$ from $B_1 \cup \cdots \cup B_p$. It follows that $x \notin C$.

It remains to show that if $x \in B_i$ for some $i = 1, \cdots, p$ and $\alpha > 1$, then $\alpha x \in C$. The rays of $L^+$ through points of $B_i$ are dense in the space of all rays in $L^+$ (since the action of $\Gamma$ on $S_\infty^{n-1}$ is minimal), so we can choose a sequence of such rays $r_j$ converging to the ray through $x$. Let $\gamma_j x \in r_j$. Since $B_i$ is discrete, $\gamma_j x$ tends to infinity, the segment $[x, \gamma_j x]$ is contained in $C$ by definition, and clearly contains points arbitrarily near $\alpha x$ for $j$ large. Since $C$ is closed, $\alpha x \in C$, as desired.

**Lemma 3.4.** *Each ray from the origin lying on $L^+$ meets $\partial C$ exactly once.*

*Proof.* Let $r$ be a ray from the origin inside $L^+$. By considering the projective disk model and again using minimality of the action of $\Gamma$ on $S_\infty^{n-1}$, we see that there are points $z_1, \cdots, z_k \in B_i$ with the point of $\mathbb{H}^n$ represented by $r$ in the interior of the hyperbolic convex hull of the points in $S_\infty^{n-1}$ corresponding to $z_1, \cdots, z_k$. It follows that $r$ meets the convex hull of $\{z_1, \cdots, z_k\}$ and hence meets $C$. Let $z$ be the first such intersection as we proceed from the origin along $r$.

We wish to prove that as we proceed along $r$, we eventually reach and remain in the interior of $C$. To see this, take $k$ rays in $L^+$ through points of $B_i$

very near to the rays through $z_1, \cdots, z_k$, such that their convex hull contains the ray of $z$ in its interior. Since $B_i$ is discrete, the points of $B_i$ in these nearby rays can be assumed to lie above any pre-assigned height. It follows that every point of $r$ above $z$ is in the interior of $C$.

**Proposition 3.5.** *The boundary of $C$ in $M^{n+1}$ is the union of $C \cap L^+$ (described in Lemma 3.3) and a countable set of codimension-one faces $F_1$, $F_2, \cdots$. Each $F_i$ is the convex hull of a finite number of points in $B_1 \cup \cdots \cup B_p$. The affine hull $A_i$ of $F_i$ is Euclidean, and the intersection of $A_i$ with $L^+$ is spherical with respect to the Euclidean structure on $A_i$ (ellipsoidal with respect to the usual Euclidean structure on $M^{n+1}$). The set of faces $F_i$ is locally finite in the interior of the light-cone.*

*Proof.* Let $z_0 \in \partial C - L^+$ and let $W$ be a support plane for $C$ at $z_0$. $W$ cannot contain 0, for otherwise the ray from 0 through $z_0$ could not meet the interior of $C$, which contradicts Lemma 2.4. Therefore

$$W = \{x \in M^{n+1}: \langle x, w\rangle = -1\}$$

for some vector $w \in M^{n+1}$, and

$$C \subset \{x \in M^{n+1}: \langle x, w\rangle \leqslant -1\}.$$

We claim that $w$ lies inside $L^+$ and presently rule out the other cases in turn.

Suppose that $\langle w, w\rangle > 0$ and choose $x'$ so that $\langle x', w\rangle = 0$, $\langle x', x'\rangle = -\langle w, w\rangle$ and so that $x'$ lies inside the positive light-cone. It follows that $x' + w \in L^+$ and $\langle x' + w, w\rangle = \langle w, w\rangle > 0$. Let $N$ be a neighborhood of $x' + w$ such that if $y \in N$, then $\langle y, w\rangle > \langle w, w\rangle/2$. We then choose $z \in B_i$ so that for some large $\alpha$, $z/\alpha \in N$. It follows that $\langle z, w\rangle > \alpha/2\langle w, w\rangle$, and so $z \notin C$, a contradiction.

Suppose that $\langle w, w\rangle = 0$. Insofar as $\langle z_0, w\rangle = -1$ and $z_0$ lies inside $L^+$, $w$ must lie in the positive rather than the negative light-cone surface. If $z \in B_1 \cup \cdots \cup B_p$, then $\langle w, z\rangle \leqslant -1$; hence $w$ is not a multiple of any such $z$. By Proposition 3.2, we can find a sequence $\gamma_j \in \Gamma$ so that $\gamma_j w$ converges to 0. Finally, we have

$$-1 \geqslant \langle \gamma_j^{-1} z, w\rangle = \langle z, \gamma_j w\rangle \to 0,$$

which is a contradiction.

We are left with the only possible case: $\langle w, w\rangle < 0$, so $W$ meets $L^+$ in an ellipsoid and has a Euclidean structure. We see that $W \cap L^+$ is an $(n-1)$ sphere in this Euclidean structure by conjugating in $SO^+(1, n)$ so that $W$ is horizontal.

We next claim that there is a support plane through $z_0$ which contains $n$ affinely independent points of $B_1 \cup \cdots \cup B_p$. Recall that $W$ is an arbitrary support plane through $z_0$ and suppose that some affine subspace $A$ of $W$ containing $W \cap (B_1 \cup \cdots \cup B_p)$ has dimension $(n-1)$. Rotate $W$ about $A$ as far as possible without passing through any point of $B_1 \cup \cdots \cup B_p$, and continue to denote the new support plane by $W$. As before, $W$ meets $L^+$ in a compactum lying below some definite height, so only a finite number of points of $B_1 \cup \cdots \cup B_p$ were at issue when deciding how far to rotate about $A$. It follows that $W$ contains at least one more point of $B_1 \cup \cdots \cup B_p$ than it did before rotation and that the affine hull of $W \cap (B_1 \cup \cdots \cup B_p)$ has increased dimension. Finally, if we choose $W$ to be a support plane through $z_0$ so that the affine hull of $W \cap (B_1 \cup \cdots \cup B_p)$ has maximal dimension, then this dimension is $n$. Thus, $W \cap C$ is a closed convex set of dimension $n$ and is one of the faces $F_i$ in the statement of the proposition.

We must lastly prove that the set of faces $F_i$ is locally finite inside $L^+$. To this end, let $K$ be any compactum inside $L^+$ and suppose that $F_1, F_2, \cdots$ is a sequence of distinct faces meeting $K$. Choose $x_i \in F_i \cap K$ converging to some point $x$, and such that the affine hulls $A_i$ of $F_i$ converge to a limit $A$ containing $x$. Since each $A_i$ is a support plane for $C$, so is $A$, whence $A$ meets $L^+$ in an ellipsoid. By compactness, the set $(A_1 \cup A_2 \cup \cdots) \cap (B_1 \cup \cdots \cup B_p)$ is finite, so there are only a finite number of distinct faces $F_i$. This contradiction establishes the proposition.

Since the construction of $C$ was equivariant under the group $\Gamma$, the decomposition into faces is $\Gamma$-invariant. The cell structure on $\partial C - L^+$ is locally finite and each cell is Euclidean. This cell structure projects homeomorphically to the projective disk model of $\mathbb{H}^n$ and gives a locally finite tesselation. This tesselation may be altered if we change the initial $\Gamma$-orbits $B_1, \cdots, B_p$ by $\alpha_1 B_1, \cdots, \alpha_p B_p$, where each $\alpha_i > 0$; however, if $\alpha_1 = \cdots = \alpha_p$, then it is unchanged. We therefore get a $(p-1)$-parameter family of tesselations of $\mathbb{H}^n$. Each cell has totally geodesic faces and there are no zero-dimensional cells (they all lie in $L^+$). In particular, if $p = 1$, there is a canonical such tesselation.

In case $\Gamma$ is torsion-free, then $\Gamma \backslash \mathbb{H}^n$ is a hyperbolic manifold, and the interior of a face of $C$ of any dimension injects into $\Gamma \backslash \mathbb{H}^n$ by Lemma 3.4. Thus, the tesselations of $\mathbb{H}^n$ descend to a natural $(p-1)$-parameter family of decompositions of $\Gamma \backslash \mathbb{H}^n$. (The decompositions are not CW decompositions since part of the boundary of a cell may well be at infinity.) Furthermore, each cell has a natural Euclidean structure (up to change of scale), but, for $n > 2$, these do not generally combine to give Euclidean structure on $\Gamma \backslash \mathbb{H}^n$ since the total angle in the link of an $(n-2)$-dimensional decomposition element will not have an angle $2\pi$.

In case $n = 2$ with $\Gamma$ torsion-free, $\Gamma \backslash \mathbb{H}^2$ does inherit a Euclidean structure since there are no zero cells.

We summarize with

**Theorem 3.6.** *Suppose that $\Gamma$ is a (finitely generated) discrete subgroup of $O^+(1, n)$ so that $\Gamma \setminus \mathbb{H}^n$ has finite volume and p cusps, $p \geqslant 1$. The convex hull construction associates a $(p-1)$-parameter family of locally finite $\Gamma$-invariant tesselations of $\mathbb{H}^n$. In particular, if $l'$ is torsion-free, then the tesselations descend to a canonical $(p-1)$-parameter family of decompositions of the hyperbolic manifold $\Gamma \setminus \mathbb{H}^n$. Furthermore, if $n = 2$, then the surface $\Gamma \setminus \mathbb{H}^2$ has a $(p-1)$-parameter family of (incomplete) Euclidean structures, and for each parameter, there is a canonical decomposition of the surface by disjointly embedded geodesics into a finite number of cells.*

**Remark.** In case $n = 2$ and $p = 1$ with $\Gamma$ torsion-free, one can fix the topological type $\Delta$ of such a decomposition on a surface and consider $C(\Delta) = \{\Gamma \in$ Teichmüller space of the genus $g$ once punctured surface: the convex hull construction for $\Gamma$ gives $\Delta\}$. The collection of all $C(\Delta)$ as $\Delta$ varies gives a decomposition of the Teichmüller space. Each $C(\Delta)$ turns out to be contractible, giving a cell decomposition of the Teichmüller space itself (see [6]).

## 4. The Ford domain

Our construction is dual to the classical Ford domain. Usually duality means a $(1,1)$-correspondence between cells of dimension $r$ in one decomposition with cells of dimension $n - r$ in another decomposition, reversing the relation of inclusion. In our case the correspondence is much more precise, and has a metric quality. We will not explore the correspondence in detail, but will content ourselves with showing that each vertex of the Ford domain is *equal* to the center of a top dimensional face of our convex hull $C$.

First we recall the classical definition of the Ford domain. Let $\Gamma$ be a discrete group of isometries of hyperbolic space $\mathbb{H}^n$ such that $\Gamma \setminus \mathbb{H}^n$ has finite volume. We also assume that $\Gamma$ is finitely generated (Brian Bowditch has recently proved that this is a redundant hypothesis). We suppose that $\Gamma \setminus \mathbb{H}^n$ has one cusp. We use the upper half-space model, and assume that $\Gamma_\infty$, the stabilizer of infinity, is nontrivial.

Each element of $\gamma \in \Gamma - \Gamma_\infty$ has an isometric sphere $S_\gamma$ which is orthogonal to the boundary of the upper half-space. Let $B_\gamma$ be the half ball bounded by $S_\gamma$, and let the Euclidean center be $p_\gamma$, lying on the boundary. Let $F = \mathbb{H}^n \setminus \bigcup_\gamma B_\gamma$. Then $F$ is invariant under $\Gamma_\infty$, which acts by Euclidean motions on $\mathbb{H}^n$.

Classically the *Ford domain* is the intersection of $F$ with a fundamental domain for the action of $\Gamma_\infty$ on $F$. Such a choice of fundamental domain is not canonical, and we prefer to work with $\Gamma_\infty \setminus F$, which is canonical. This is a hyperbolic manifold with boundary. The boundary has a finite number of faces, and $\Gamma \backslash \mathbb{H}^n$ can be obtained by pairing faces of $\Gamma_\infty \backslash F$.

A coordinate free way of describing $\Gamma_\infty \backslash F$ is to take a horosphere in $\Gamma \backslash \mathbb{H}^n$ and let it expand until it collides with itself. We can regard the horosphere as a balloon which is gently expanded, coming to rest where it meets itself. Formally, we fix a cusp and a small horosphere in $\mathbb{H}^n$ corresponding to this cusp. Then $F$ is the set of points in $\mathbb{H}^n$ which are nearer to this horosphere than to any translate of the horosphere under $\Gamma$. The collision locus in $\mathbb{H}^n$ is the set of points for which two or more of these translates are equidistant. The collision locus in $\Gamma \backslash \mathbb{H}^n$ is the image of the collision locus in $\mathbb{H}^n$. Then $\Gamma_\infty \backslash F$ is the space obtained by cutting $\Gamma \backslash \mathbb{H}^n$ open along the collision locus. This is proved as follows.

Since $\gamma$ is equal to inversion in $S_\gamma$, followed by a Euclidean isometry, the image under $\gamma$ of a horosphere centered at $p_\gamma$ is a horizontal horosphere. Therefore these horospheres in $\mathbb{H}^n$, which correspond under an element $\gamma \in \Gamma - \Gamma_\infty$, are obtained by hyperbolic reflection in the hyperbolic hyperplane $S_\gamma$. As these horospheres about infinity and $p_\gamma$ expand, they collide along $S_\gamma$. In this way we see that the boundary of $F$ is exactly the collision locus for the expanding horospheres.

Now let us see how this construction ties up without convex hull construction. Let $p$ be a vertex on the boundary of $F$ in $\mathbb{H}^n$. This means that $n$ expanding horospheres, whose centers do not lie in the same vertical hyperplane, and which correspond under $\Gamma$, meet each other and the expanding horizontal horosphere at $p$. Moreover, at the moment of collision $p$ is not yet contained in the interior of any other expanding horosphere in the same orbit under $\Gamma$. The point $p$ gives a vertex of $\Gamma_\infty \backslash F$ and also gives a vertex in any classical Ford domain.

Each horosphere corresponds to a point $v$ in a positive light-cone surface, with $\langle v, v \rangle = 0$. Let $\{v_0, \cdots, v_n\}$ correspond to the $(n+1)$ horospheres of the preceding paragraphs. The point $p$ of the preceding paragraph lies on the upper hyperboloid and, by the results of §1, we have

$$\langle v_0, p \rangle = \cdots = \langle v_n, p \rangle = -1.$$

The $n$-dimensional affine space $A$ containing $\{v_0, \cdots, v_n\}$ is $\{v \colon \langle v, p \rangle = -1\}$.

We choose coordinates so that $p = (1, 0, \cdots, 0)$ in Minkowski space. Then $A$ is horizontal, and meets the positive light-cone surface in an $(n-1)$-dimensional sphere $S$.

If $v$ is any parabolic fixed point in the same orbit as $\{v_0, \cdots, v_n\}$, then $\langle p, v \rangle \leqslant -1$. For if $\langle p, v \rangle > -1$, then $p$ is contained in the interior of the expanding horoball centered on $[v]$. But we have assumed this is not the case.

Hence $v$ must lie on or above $A$. It follows that $A$ intersects $\partial C$ in an $n$-dimensional face $B$.

All the vertices of $B$ lie on the $(n-1)$-dimensional sphere $S$ and $\{v_0, \cdots, v_n\} \subset B \cap S$. $B \cap S$ is finite, but may have more then $(n+1)$ elements if $p$ happens to lie on additional horospheres in the same orbit under $\Gamma$ as those we have already considered.

Thus we have shown that $p$ is the center of the circumsphere for the Euclidean polyhedron $B$. The converse discussion works in a similar way.

Thus the tesselation of $\mathbb{H}^n$ by the faces of $C$ is dual to the cell complex given by the faces of $F$.

## 5. Tesselations by regular ideal polyhedra

It is interesting to relate our construction with the tesselations of $\mathbb{H}^3$ described in [8]. Suppose we have a tesselation of $\mathbb{H}^3$ by regular ideal hyperbolic polyhedra (i.e. with vertices at infinity). This is possible with tetrahedra, cubes, octahedra, and dodecahedra, but not with icosahedra. The tesselation is invariant under symmetries of the polyhedron and also under reflection in each of its faces.

For example, the tesselation of $\mathbb{H}^3$ related to the hyperbolic structure on the complement of the figure eight knot is by regular ideal hyperbolic tetrahedra.

**Theorem.** *Let $\Gamma$ be the group of symmetries of a tesselation of $\mathbb{H}^3$ by regular ideal polyhedra. Let $C$ be the convex hull in Minkowski space of the orbit under $\Gamma$ of a parabolic fixed point in the positive light-cone surface, as described in* §3. *Then the intrinsic Euclidean structure on each face is isometric* (*up to change of scale*) *with the regular Euclidean polyhedron corresponding to the given regular ideal hyperbolic polyhedron.*

*Proof.* Let $P$ be a regular ideal hyperbolic polyhedron. We will consider the tesselation of $\mathbb{H}^3$ generated by reflection in the faces of $G$. Let $p \in P$ be the unique fixed point for the symmetries of $P$ (i.e. for the finite group of hyperbolic isometries of $P$). We may take coordinates in Minskowski space, with $p = (1, 0, 0, 0)$.

The symmetries of $P$ fix $p$ and therefore fix (setwise) the horizontal plane $x_0 = 1$, since this plane, which we call $A$, is given by

$$A = \{v : \langle v, p \rangle = -1\}.$$

A vertex of $P$ corresponds to a ray on the positive light-cone. Let $v_0$ be the intersection of $A$ with this ray. Then $v_0$ is a parabolic fixed point for $\Gamma$. The orbit of $v_0$ under $\Gamma$ contains all vertices of $P$.

By symmetry it also follows that $p$ lies in the collision locus of the expanding horosphere. As in §4, we see that $A \cap C$ is a top dimensional face of $C$. $A \cap C$ can be identified with $P$ in the projective model of hyperbolic space. In the natural Euclidean structure on $A$, $A \cap C$ is the convex hull of the vertices of $P$, and is identified with $P$ in the projective model of $\mathbb{H}^3$. Since the group of symmetries of $P$ acts, the polyhedron must be a regular Euclidean polyhedron.

## Appendix

We reproduce a construction due to Apanosov to show that for $n \geqslant 4$, there is a discrete (nonfinitely generated) subgroup $\Gamma$ of the Möbius transformations on $\mathbb{H}^n$ with a parabolic fixed point but with no uniform horoball. This example also shows that the orbit of a parabolic fixed point is not in general discrete (cf. Theorem 2.4). The situation for finitely generated groups remains open.

The construction is given in the upper half-space model with $n = 4$. Parabolic elements fixing infinity can be identified with Euclidean isometries of the horizontal horosphere $H$ at height one. Let $\alpha$ be a parabolic element which corresponds to a rotation through an irrational angle about an axis $A$, followed by a translation along $A$. The vertical plane $P$ through $A$ in the upper half-space is a hyperbolic plane on which $\alpha$ acts as a parabolic in the familiar way. $M = \mathbb{H}^4/\alpha$ is a 4-dimensional hyperbolic manifold. Our counterexample will be constructed by excising a countable set of disjoint half-$\mathbb{H}^4$'s from $M$, and then gluing together the boundary $\mathbb{H}^3$'s in pairs.

Let $x_1, x_2, \cdots$ be a countable sequence of points in the boundary $\mathbb{R}^3$ to $\mathbb{H}^4$, whose Euclidean distances from $P$ tend to infinity. (We will continually pass to subsequences without change of notation.) Fix a point $z_0 \in A$, and let $H_k$ be the horosphere centered at $x_k$ passing through $z_0$. $H_k$ converges to $H$, and we will construct $\Gamma$ with elements $\gamma_k \in \Gamma$ such that $\gamma_i H = H_{k(i)}$, where $k(i)$ converges to infinity with $i$. Since horospheres correspond to points on the light-cone (see §1), we will obtain a nondiscrete orbit of a parabolic fixed point.

We will inductively choose hemispheres $S_1, S_2, \cdots$ centered at $x_{i(1)}, x_{i(2)}, \cdots$, where $i(k) \geqslant k$, and hemispheres $T_1, T_2, \cdots$ such that all the four-dimensional half-balls bounded by any translate under $\{\alpha^n\}$ of any $S_k$ or any $T_k$ are pairwise disjoint.

Having defined $x_{i(1)}, \cdots, x_{i(k-1)}, S_1, \cdots, S_{k-1}$, and $T_1, \cdots, T_{k-1}$, we choose $S_k$ centered at $x_{i(k)}$ of radius 2 so as to satisfy the disjointness conditions. Let $\alpha_k$ be inversion in $S_k$ followed by reflection in a vertical plane through $x_{i(k)}$

(the only point of the reflection is to make $\alpha_k$ preserve orientation). Then $\alpha_k$ sends the horosphere $H_{i(k)}$ to the horizontal horosphere. Let $\beta_k$ be a similarity of the Euclidean structure on the upper half-space taking $\alpha_k H_{i(k)}$ to $H$. By composing with a parabolic transformation keeping infinity fixed, we may assume that $\beta_k S_k = T_k$ satisfies the disjointness condition together with $S_1, \cdots, S_k, T_1, \cdots, T_{k-1}$. Then $\beta_k \alpha_k H_{i(k)} = H$ and $\beta_k \alpha_k S_k = T_k$.

We remove from $M = \mathbb{H}^4/\alpha$ the half-spaces corresponding to $S_k$ and $T_k$ and glue them together with the isometry $\beta_k \alpha_k$. After doing this a countable number of times, we obtain a complete hyperbolic 4-manifold with fundamental group $\Gamma$, where $\Gamma$ is the free group on $\{\alpha, \beta_1 \alpha_1, \beta_2, \alpha_2, \cdots\}$. The orbit of $H$ includes all of the $H_{i(k)}$.

This completes the discussion of the example.

## References

[1] B. N. Apanasov, *Cusp ends of hyperbolic manifolds*, Ann. Global Anal. Geom. **3** (1985) 1–11.
[2] B. H. Bowditch, *On geometrically finite groups*, to appear.
[3] B. H. Bowditch & D. B. A. Epstein, *On natural triangulations associated to a punctured surface*, to appear.
[4] R. J. Ellis & W. Perrizo, *Unique ergodicity of flows on homogeneous spaces*, Israel J. Math. **29** (1978).
[5] D. B. A. Epstein & A. Marden, *Convex hulls in hyperbolic space, a theorem of Sullivan and measured pleated surfaces*, Analytical and Geometric Aspects of Hyperbolic Space (D. B. A. Epstein, ed.), Cambridge University Press, 1987.
[6] J. Harer, *The virtual cohomological dimension of the mapping class groups of orientable surfaces*, preprint.
[7] R. C. Penner, *The Teichmüller space of punctured surfaces*, preprint.
[8] W. P. Thurston, Mimeographed notes from Princeton University, 1979.
[9] ———, Book to be published by Princeton University Press.
[10] R. J. Zimmer, *Ergodic theory and semisimple groups*, Birkhauser, Cambridge, MA, 1984.

UNIVERSITY OF WARWICK, ENGLAND
UNIVERSITY OF SOUTHERN CALIFORNIA