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APPLICATION OF THE SELBERG TRACE FORMULA TO THE RIEMANN-ROCH THEOREM

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1. Introduction

The Riemann-Roch theorem is one of the foundational results in the theory of Riemann surfaces. Many proofs of this theorem are known, some traditional, others less so. The objective of this note is to give a proof in the latter vein, the primary tool being the Selberg trace formula.

Thus let X be a compact Riemann surface of genus $g > 1$. Then we may write $X = \Gamma \setminus \mathbf{D}$, $\mathbf{D}$ the open unit disk, Γ a discrete, strictly hyperbolic, cocompact subgroup of $H = G/\{\pm I\}$, $G = \mathrm{SU}(1,1)$. When applied to automorphic forms on Γ, the Riemann-Roch theorem and the Selberg trace formula say about the same thing. Consequently, it should not come as too much of a surprise that the one can be derived from the other.

For us, it will be convenient to regard the Riemann-Roch theorem as a statement about holomorphic line bundles on X. In turn, to get this into a group-theoretic context, it is necessary to use the language of automorphy factors. Once this transcription has been accomplished, it is technically simplest to pass to the $(g-1)$-fold covering group of G. Since the irreducible unitary representations of the universal covering group $\tilde{G}$ of G have been classified, no difficulty is encountered in doing so. Applying now the Selberg trace formula to suitable coefficients or quasi-coefficients then leads easily to the Riemann-Roch theorem.

It will be clear that what is said here can be said more generally. Nevertheless, we shall stay the course and not take up these side issues, interesting as they may be. Let us say only that Shimura [25] has proved a Riemann-Roch theorem for the traces of the Hecke operators. Agreeing to place ourselves in

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the setting of [28] (hence, in particular, automorphic forms become intertwining operators), the techniques utilized in this note allow one to recover those results with very little additional effort. In fact, associated with a pair (D, D^*) of rational divisors on X, satisfying certain standard assumptions, is a pair $(A(D), A(D^*))$ of spaces of automorphic forms on X. The Riemann-Roch theorem is then an expression for the difference

$$\dim(A(D)) - \dim(A(D^*)).$$

If now ζ is in the commensurator of Γ and if $H(\Gamma\zeta\Gamma)$ is the corresponding Hecke operator, then, proceeding geometrically, Shimura [25] explicitly evaluated

$$\operatorname{tr}(H(\Gamma\zeta\Gamma) \mid A(D)) - \operatorname{tr}(H(\Gamma\zeta\Gamma)^* \mid A(D^*))$$

in terms of the data at hand. On the other hand, using the Selberg trace formula, this difference can also be interpreted as a finite sum of orbital integrals, these being computable entities, as their Fourier transforms are known. And, of course, when all is said and done, the two approaches yield the same final results.

2. Irreducible unitary representations of the universal covering group of SU(1,1)

Denote by $\tilde{G}$ the universal covering group of $G = \mathrm{SU}(1,1)$. Then the irreducible unitary representations of $\tilde{G}$ were classified infinitesimally by Pukanszky [21] and realized globally by Sally [23]. They fall into several series, as follows.

Principal series. The representations $\pi(\sigma, \nu)$ in this series are indexed by two parameters σ and ν, where

$$-1/2 < \sigma \leqslant 1/2, \qquad \nu \in \sqrt{-1}\,\mathbf{R},$$

excluding the pair $(1/2, 0)$. Call $\Theta_{\sigma,\nu}$ the corresponding character.

Complementary series. The representations $\pi(\sigma, \nu)$ in this series are indexed by two parameters σ and ν, where

$$-1/2 < \sigma < 1/2, \qquad 0 < \nu < 1/2 - |\sigma|.$$

Call $\Theta_{\sigma,\nu}$ the corresponding character.

Discrete series. The representations $\pi(\tau)$ in this series are indexed by a parameter τ, where

$$\tau > 1/2, \qquad \tau < -1/2.$$

Call Θ_τ the corresponding character.

Pseudo discrete series. The representations $\pi(\tau)$ in this series are indexed by a parameter τ, where

$$0 < \tau < 1/2, \qquad 0 > \tau > -1/2.$$

Call Θ_τ the corresponding character.

Limit of discrete series. The representations $\pi(\pm 1/2)$ in this series are the two irreducible constituents of $\pi(1/2, 0)$. Call $\Theta_{\pm 1/2}$ the corresponding character.

The list is then completed by adjoining the trivial one-dimensional representation $\pi(0)$ with the character Θ_0.

Of the above, the elements of the complementary series and the pseudo discrete series, together with $\pi(0)$ are nontempered.

In passing, we mention that one can then define the $\pi(\sigma, \nu)$ for all real σ and all complex ν. Of course, in terms of the characters, there is an equality

$$\Theta_{\sigma+1,\nu} = \Theta_{\sigma,\nu}.$$

Moreover, the various embeddings at points of reducibility can be determined in the usual way (cf. Molchanov [14] and Sally [24]).

In the sequel, we shall need to deal primarily with the $\pi(\tau)$, the principal and complementary series playing virtually no role at all. Because of this, it will be best to consider the $\pi(\tau)$ in a little more detail.

Since $\overline{\Theta}_\tau = \Theta_{-\tau}$ for $\tau \geqslant 0$, the representation conjugate to $\pi(\tau)$ is $\pi(-\tau)$. Accordingly, there is no loss of generality in taking τ nonnegative. Let $\mathbf{D}$ be the open unit disk in $\mathbf{C}$. Then, $\forall \tau \geqslant 0$, Sally [23] introduced a certain Hilbert space $H_\tau(\mathbf{D})$ of holomorphic functions in $\mathbf{D}$ and realized $\pi(\tau)$ on $H_\tau(\mathbf{D})$ as a multiplier representation. To be specific, it is necessary to describe $\tilde{G}$.

Thus, starting with

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in G,$$

set

$$\Omega = \beta/\alpha, \qquad \omega = \arg(\alpha),$$

where ω is defined mod 2π. Plainly, $|\Omega| < 1$ and

$$\alpha = \frac{e^{\sqrt{-1}\,\omega}}{\sqrt{1 - |\Omega|^2}}, \qquad \beta = \frac{e^{\sqrt{-1}\,\omega}\Omega}{\sqrt{1 - |\Omega|^2}}.$$

Consequently, G can be parametrized by the (Ω, ω), namely

$$G = \{(\Omega, \omega): |\Omega| < 1, -\pi < \omega \leqslant \pi\}.$$

The same parameters serve to describe $\tilde{G}$, the only difference being that distinct ω give rise to distinct elements of $\tilde{G}$, i.e.,

$$\tilde{G} = \{(\Omega, \omega): |\Omega| < 1, -\infty < \omega < +\infty\}.$$

The canonical projection $p: \tilde{G} \to G$ is the map

$$p(\Omega, \omega) = (\Omega, \omega(\operatorname{mod} 2\pi))$$

and has for its kernel the discrete, central subgroup $\{(0, 2n\pi): n \in \mathbf{Z}\}$.

The group G operates on $\mathbf{D}$ by linear fractional transformations, hence, by projection, so does $\tilde{G}$:

$$(\Omega, \omega) \cdot \zeta = e^{2\sqrt{-1}\omega}\left(\frac{\zeta + \Omega}{\zeta\bar{\Omega} + 1}\right) \qquad (\zeta \in \mathbf{D}).$$

That being, consider now the function

$$J(\tau : ?): \tilde{G} \times \mathbf{D} \to \mathbf{C}^{\times}$$

defined by the rule

$$J(\tau: (\Omega, \omega), \zeta) = e^{2\sqrt{-1}\omega\tau}(1 - |\Omega|^2)^{\tau}(1 + \zeta\bar{\Omega})^{-2\tau}.$$

Then $J(\tau)$ is a multiplier in that

$$J(\tau: (\Omega', \omega')(\Omega'', \omega''), \zeta) = J(\tau: (\Omega', \omega'), (\Omega'', \omega'') \cdot \zeta)J(\tau: (\Omega'', \omega''), \zeta).$$

And, using $J(\tau)$, one can realize $\pi(\tau)$ on $H_\tau(\mathbf{D})$ by putting

$$\pi(\tau)(\Omega, \omega)f(\tau) = J(\tau: (\Omega, \omega)^{-1}, \zeta)f((\Omega, \omega)^{-1} \cdot \zeta).$$

Let G_n be the covering group of G of degree n. Then

$$G_n = \{(\Omega, \omega): |\Omega| < 1, -n\pi < \omega \leqslant n\pi\}$$

and there is a commutative triangle:

$$\begin{tikzcd} \tilde{G} \arrow[r] \arrow[d] & G \\ G_n \arrow[ur] & \end{tikzcd}$$

The $\pi(\tau)$ which drop to G_n correspond to the $\tau \in (2n)^{-1} \cdot \mathbf{N}$. Suppose that $\tau = m/2n$. We then have:

(1) $m > 2n \Rightarrow \pi(\tau)$ is in the discrete series and is integrable.

(2) $n < m \leqslant 2n \Rightarrow \pi(\tau)$ is in the discrete series but is not integrable.

(3) $m = n \Rightarrow \pi(\tau)$ is in the limit of the discrete series.

(4) $m = 1, 2, \ldots, n - 1 \Rightarrow \pi(\tau)$ is in the pseudo discrete series.

(5) $m = 0 \Rightarrow \pi(\tau)$ is the trivial one-dimensional representation.

3. Parametrization of the line bundles on X

Let X be a compact Riemann surface of genus $g > 1$. Then, by the uniformization theorem, $X = \Gamma \setminus \mathbf{D}$, where Γ is a discrete, strictly hyperbolic, cocompact subgroup of $H = G/\{\pm I\}$. Owing to a classical theorem of Petersson [19], Γ can be lifted to G in the sense that there is an injective morphism $i: \Gamma \to G$ such that it, followed by the canonical projection $G \to H$, is the identity on Γ. This has been reproved by Patterson [16] who then goes on to show that Γ can actually be lifted to $H_{2(g-1)}$, the covering group of H of degree $2(g-1)$. Since $H_{2(g-1)}$ can be identified with G_{g-1}, we can and will assume that Γ sits inside G_{g-1}.

Recall now that by definition, a factor of automorphy for Γ is a function

$$I: \Gamma \times \mathbf{D} \to \mathbf{C}^{\times}$$

such that $\forall \gamma \in \Gamma$, $I(\gamma, ?): \mathbf{D} \to \mathbf{C}^{\times}$ is holomorphic and satisfies the condition

$$I(\gamma'\gamma'', \zeta) = I(\gamma', \gamma'' \cdot \zeta) I(\gamma'', \zeta).$$

The automorphy factors for Γ form a group under multiplication. Furthermore,

$$\begin{aligned} \operatorname{Pic}(X) &= \text{group of holomorphic line bundles on } X \\ &= \frac{\text{group of automorphy factors for } \Gamma}{\text{group of trivial automorphy factors for } \Gamma}, \end{aligned}$$

an automorphy factor being termed trivial if

$$I(\gamma, \zeta) = \frac{h(\gamma \cdot \zeta)}{h(\zeta)},$$

h a holomorphic function on $\mathbf{D}$ without zeros. In fact, attached to every I is a holomorphic line bundle on X:

$$\mathscr{L}_I = \Gamma \setminus \mathbf{D} \times \mathbf{C},$$

where

$$\gamma \cdot (\zeta, z) = (\gamma \cdot \zeta, I(\gamma, \zeta) z).$$

Equivalent automorphy factors give rise to isomorphic line bundles and, as is well known (cf. Gunning [9]), every $\mathscr{L} \in \operatorname{Pic}(X)$ is of the form $\mathscr{L}_I$ for some I. It is clear that the space of holomorphic cross sections for $\mathscr{L}_I$ can be identified with the holomorphic functions $f: \mathbf{D} \to \mathbf{C}$ such that

$$f(\gamma \cdot \zeta) = I(\gamma, \zeta) f(\zeta).$$

Agreeing to write $I(\tau) = 1/J(\tau)$, let $\chi \in \hat{\Gamma}$, the unitary character group of Γ. Then

$$\chi(\gamma) I\left(\frac{k}{2(g-1)} : \gamma, \zeta\right) \qquad (k \in \mathbf{Z})$$

is an automorphy factor for Γ, thus determines a holomorphic line bundle on X, call it $\mathscr{L}(\chi, k)$, the Chern class of which is exactly k itself. In particular, $I(1\colon \gamma, \zeta)$ is the canonical automorphy factor, thereby determining the canonical bundle

$$\mathscr{K} = \mathscr{L}(1, 2(g-1)).$$

Lemma 3.1. *Given a holomorphic line bundle $\mathscr{L}$ on X, there exists a unique pair (χ, k) such that $\mathscr{L}$ is isomorphic to $\mathscr{L}(\chi, k)$.*

(Up to language and notation, this is due to Gunning [6].)

So, the $\mathscr{L}(\chi, k)$ exhaust the elements of $\mathrm{Pic}(X)$. Naturally, on abstract grounds,

$$\mathrm{Pic}(X) \sim \hat{\Gamma} \times \mathbf{Z},$$

and this is reflected in the relations

$$\mathscr{L}(\chi', k')\mathscr{L}(\chi'', k'') = \mathscr{L}(\chi'\chi'', k' + k''), \qquad \mathscr{L}(\chi, k)^{-1} = \mathscr{L}(\bar{\chi}, -k).$$

Bear in mind too that $\hat{\Gamma} \sim \mathrm{Jac}(X)$.

With $\kappa = k/2(g-1)$, put

$$\mathfrak{o}(\chi, \kappa) = \dim\big(H^0(X, \mathscr{L}(\chi, k))\big).$$

Then the Riemann-Roch theorem asserts that

$$\mathfrak{o}(\chi, \kappa) - \mathfrak{o}(\bar{\chi}, 1 - \kappa) = (2\kappa - 1)(g - 1).$$

In a word, our objective is to prove this using representation-theoretic techniques (cf. infra.)

4. Deduction of the Riemann-Roch theorem

For the sake of simplicity, we shall henceforth write G in place of G_{g-1}. With this understanding, let $\chi \in \hat{\Gamma}$ and denote by $L^2(\Gamma \setminus G;\ \chi)$ the representation space for $\mathrm{Ind}_\Gamma^G(\chi)$. Then the Selberg trace formula says that for any $\alpha \in \mathscr{C}^1(G)$, the L^1 Schwartz space of Harish-Chandra, the operator $L_{\Gamma\setminus G;\chi}(\alpha)$ is trace class, its trace being on the one hand

$$\sum_{\pi \in \hat{G}} m(\chi, \pi)\Theta_\pi(\alpha),$$

and on the other hand

$$\sum_{\{\gamma\}_\Gamma} \chi(\gamma)\mathrm{vol}(\Gamma_\gamma \setminus G_\gamma) \cdot \int_{G_\gamma \setminus G} \alpha(x^{-1}\gamma x)\, d_{G_\gamma\setminus G}(x).$$

Here, we use the customary notation of the subject (cf. [15]), which can therefore remain unexplained. Note that the function

$$m\colon \hat{\Gamma} \times \hat{G} \to \mathbf{N}$$
$$(\chi, \pi) \to m(\chi, \pi)$$

possesses the symmetries

$$m(\chi, \pi) = m(\bar{\chi}, \bar{\pi}), \qquad m(\bar{\chi}, \pi) = m(\chi, \bar{\pi}).$$

Suppose now that **D** is equipped with the G-invariant Riemannian structure in which it has curvature -1. Then the volume of $X = \Gamma \backslash \mathbf{D}$ is $4\pi(g-1)$. Assigning to Γ counting measure then determines a Haar measure on G by the requirement of compatibility

$$\int_G = \int_{\Gamma\backslash G} \int_\Gamma.$$

In terms of the Iwasawa decomposition $G = K \cdot A \cdot N$, this determination is the same as the specification

$$\int_G f(x)\, d_G(x) = \int_K \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\theta, t, \xi) e^t \, d\theta \, dt \, d\xi,$$

where $\mathrm{vol}(K) = 1$.

Put

$$d(\kappa) = \frac{2|\kappa| - 1}{4\pi}.$$

Then, for $|k| > g - 1$, $d(\kappa)$ is just the formal degree of the discrete series representation $\pi(\kappa)$ (in the agreed to normalization of the Haar measure on G).

Turning to our proof of the Riemann-Roch theorem, recall first that for $\kappa \geqslant 0$,

$$o(\chi, \kappa) = m(\chi, \pi(\kappa)),$$

this by the duality theorem (cf. Mackey [12] or Maurin and Maurin [13]). Next, observe that there is no loss of generality in taking $\kappa \geqslant 0$ throughout. For if the contention has been established for $\kappa \geqslant 0$, then $\kappa < 0 \Rightarrow 1 - \kappa > 0$, so, switching the roles of χ and $\bar{\chi}$,

$$o(\bar{\chi}, 1 - \kappa) - o(\chi, 1 - (1 - \kappa)) = (2(1 - \kappa) - 1)(g - 1)$$

or still

$$o(\chi, \kappa) - o(\bar{\chi}, 1 - \kappa) = (2\kappa - 1)(g - 1),$$

as desired. Of course it is automatic that for negative κ,

$$o(\chi, \kappa) = 0,$$

$k < 0$ being the Chern class of $\mathscr{L}(\chi, k)$.

It will be convenient to distinguish three cases.

(1) $k > 2(g - 1)$. In this situation, $\pi(\kappa)$ is integrable. Therefore one can proceed as in Langlands [11], feeding into the Selberg trace formula a coefficient

$$\alpha_\kappa(x) = d(\kappa) \overline{(\pi(\kappa)(x) v_\kappa, v_\kappa)},$$

v_κ a normalized K-finite function in $H_\kappa(\mathbf{D})$. Thanks to the Schur orthogonality relations,

$$\Theta_\pi(\alpha_\kappa) = \begin{cases} 1 & \text{if } \pi = \pi(\kappa), \\ 0 & \text{if } \pi \neq \pi(\kappa). \end{cases}$$

But, in view of the Selberg principle, all the orbital integrals

$$\int_{G_\gamma \backslash G} \alpha_\kappa(x^{-1}\gamma x)\, d_{G_\gamma \backslash G}(x)$$

vanish except for the one corresponding to $\gamma = 1$. Consequently, we end up with the equality

$$m(\chi, \pi(\kappa)) = d(\kappa)\mathrm{vol}(\Gamma \backslash G),$$

that is,

$$\mathrm{o}(\chi, \kappa) = (2\kappa - 1)(g - 1),$$

which is the Riemann-Roch theorem in this case.

To make further progress, we need a lemma.

Lemma 4.1. *Let $g - 1 < k \leqslant 2(g - 1)$. Then there exists a K-finite $\alpha_\kappa \in C_c^\infty(G)$ such that for every tempered π,*

$$\Theta_\pi(\alpha_\kappa) = \begin{cases} 1 & \text{if } \pi = \pi(\kappa), \\ 0 & \text{if } \pi \neq \pi(\kappa). \end{cases}$$

(For k in the indicated range, $\pi(\kappa)$ is square integrable but not integrable. Thus, α_κ is a quasi-coefficient for $\pi(\kappa)$. As for the lemma, it is a special case of a general result due to Clozel and Delorme [2] (applicable even though, strictly speaking, G is not linear). In our particular setting, it can also be deduced by an easy extension of the classical methods of Ehrenpreis and Mautner [3].)

(2) $0 \leqslant k < g - 1$ or $g - 1 < k \leqslant 2(g - 1)$. By symmetry, it will evidently be enough to look at the k in the range $g - 1 < k \leqslant 2(g - 1)$ (since $0 \leqslant \kappa < 1/2 \Rightarrow 1/2 < 1 - \kappa \leqslant 1$). Take now α_κ per the lemma and insert it into the Selberg trace formula. Then, of necessity,

$$\Theta_\pi(\alpha_\kappa) = 0$$

for all $\pi \in \hat{G}$ with the exception of

$$\pi = \pi(\kappa), \qquad \pi = \pi(\kappa - 1).$$

Tacitly, an obvious continuation argument vis-à-vis the links has been invoked (cf. [22]). The connection between $\pi(\kappa)$ and $\pi(\kappa - 1)$ is exhibited by the reducibility of the principal series representation $\pi(\kappa, \kappa - 1/2)$, giving the character identities

$$\Theta_{\kappa-1,\kappa-1/2} = \Theta_\kappa + \Theta_{\kappa-1} \qquad (1/2 < \kappa < 1),$$
$$\Theta_{0,1/2} = \Theta_1 + \Theta_{-1} + \Theta_0 \qquad (\kappa = 1).$$

Evaluating at α_κ, we conclude that

$$0 = \Theta_\kappa(\alpha_\kappa) + \Theta_{\kappa-1}(\alpha_\kappa) \qquad (1/2 < \kappa \leqslant 1).$$

Because Γ is strictly hyperbolic, the Fourier transform of the orbital integral

$$\int_{G_\gamma\backslash G} \alpha_\kappa(x^{-1}\gamma x)\, d_{G_\gamma\backslash G}(x) \qquad (\gamma \neq 1)$$

admits an expansion in terms of the characters of the principal series, hence must vanish. Accordingly, all that remains is the equality

$$m(\chi, \pi(\kappa)) - m(\chi, \pi(\kappa - 1)) = \alpha_\kappa(1)\mathrm{vol}(\Gamma \backslash G).$$

By the Plancherel theorem for G, $\alpha_\kappa(1) = d(\kappa)$ and, as before,

$$d(\kappa)\mathrm{vol}(\Gamma \backslash G) = (2\kappa - 1)(g - 1).$$

But $m(\chi, \pi(\kappa)) = \mathfrak{o}(\chi, \kappa)$, so, to finish up, we have only to note that

$$m(\chi, \pi(\kappa - 1)) = m(\bar{\chi}, \overline{\pi(\kappa - 1)}) = m(\bar{\chi}, \pi(1 - \kappa)),$$

the term on the right being precisely $\mathfrak{o}(\bar{\chi}, 1 - \kappa)$.

A similar, albeit slightly simpler argument will serve to complete the discussion.

(3) $k = g - 1$. The contention is that

$$\mathfrak{o}(\chi, 1/2) = \mathfrak{o}(\bar{\chi}, 1/2)$$

or, what amounts to the same, that

$$m(\chi, \pi(1/2)) = m(\bar{\chi}, \pi(1/2))$$

or still

$$m(\chi, \pi(1/2)) = m(\chi, \pi(-1/2)).$$

This is familiar. Indeed, it is well known (and follows from what is said in Clozel and Delorme [2] or Ehrenpreis and Mautner [3]) that one can find a K-finite function $\alpha_0 \in C_c^\infty(G)$ such that for every tempered π,

$$\Theta_\pi(\alpha_0) = \begin{cases} \pm 1 & \text{if } \pi = \pi(\pm 1/2), \\ 0 & \text{if } \pi \neq \pi(\pm 1/2), \end{cases}$$

so, by linking, $\Theta_\pi(\alpha_0) = 0$ for all nontempered π too. Our assertion is thus immediate.

5. Some remarks on the multiplicites

The calculation of the multiplicity $m(\chi, \pi(\kappa))$ is an important problem that is not yet completely solved. A few words on this subject thus seem to be in order.

We can and will assume that κ is nonnegative. Then

$$\kappa > 1 \Rightarrow m(\chi, \pi(\kappa)) = (2\kappa - 1)(g - 1),$$
$$1 \geqslant \kappa \geqslant 1/2 \Rightarrow m(\chi, \pi(\kappa)) \geqslant (2\kappa - 1)(g - 1).$$

The two extreme possibilities in the range $0 \leqslant \kappa \leqslant 1$ are actually direct since

$$\kappa = 1 \Rightarrow m(\chi, \pi(\kappa)) = \begin{cases} g & \text{if } \chi = 1, \\ g - 1 & \text{if } \chi \neq 1, \end{cases}$$

$$\kappa = 0 \Rightarrow m(\chi, \pi(\kappa)) = \begin{cases} 1 & \text{if } \chi = 1, \\ 0 & \text{if } \chi \neq 1. \end{cases}$$

The crucial case is therefore when $0 < \kappa < 1$, i.e., in terms of the Chern class, when $0 < k < 2(g - 1)$.

This puts us square into the realm of special line bundles (or divisors). For an introduction to this circle of ideas, the reader is referred to Griffiths [5]; a systematic treatment can be found in Arbarello et al. [1] (the diagram on p. 205 of this reference is especially illuminating). The theory, as has been developed by the geometers, can be carried over to $\hat{\Gamma}$ via its identification with the Jacobian of X. Naturally, the fact that $\hat{\Gamma}$ is a complex analytic manifold is decisive here and essentially controls the situation (as one would expect, in view of Torelli's theorem).

Thus, let $0 < k \leqslant g - 1$ and denote by $\hat{\Gamma}_k$ the set of all χ such that $\mathscr{L}(\chi, k)$ admits a nontrivial holomorphic cross section or, equivalently, such that $m(\chi, \pi(\kappa))$ is positive. Then it turns out that $\hat{\Gamma}_k$ is a k-dimensional irreducible complex analytic subvariety of $\hat{\Gamma}$, possibly with singularities (except when $k = 1$). At the nonsingular points of $\hat{\Gamma}_k$, $m(\chi, \pi(\kappa)) = 1$, but on the singular locus, $m(\chi, \pi(\kappa)) > 1$. The complement $\hat{\Gamma} - \hat{\Gamma}_k$ is an open dense set of full measure so, generically, $m(\chi, \pi(\kappa)) = 0$. Still, $m(\chi, \pi(\kappa))$ will be positive whenever $\chi \in \hat{\Gamma}_k$, hence nontempered representations will assuredly occur in $L^2(\Gamma \setminus G; \chi)$. Passing to conjugates, we see that for $g - 1 \leqslant k < 2(g - 1)$,

$$m(\chi, \pi(\kappa)) = (2\kappa - 1)(g - 1),$$

again, generically. Furthermore, in both cases, a priori upper bounds are provided by Clifford's theorem and its variants in that

$$m(\chi, \pi(\kappa)) \leqslant [k/2] + 1 \qquad (0 < k < 2(g - 1))$$

with the usual improvement on the right of X is not hyperelliptic.

To close, let us consider one simple example.

Take χ of order $2(g - 1)$, $k = 1$. Then

$$\mathscr{L}(\chi, 1)^{\otimes 2(g-1)} = \mathscr{K}.$$

Suppose now that

$$m(\chi, \pi(1/2(g-1))) > 0.$$

Then $\mathscr{L}(\chi, 1)$ is isomorphic to a point bundle and, in addition, such a point must be a Weierstrass point for which $2g-1$ is a gap. Because the number of Weierstrass points is finite, this forces an ex ante restriction on the possible occurrence of $\pi(1/2(g-1))$ in $L^2(\Gamma \setminus G;\ \chi)$. In this connection, recall that on a hyperelliptic surface, the gap sequence at a Weierstrass point has the form $1, 2, \cdots, 2g-1$ and that there are, altogether, $2(g+1)$ Weierstrass points. The other side of the coin is a normal surface. Such a surface has $g^3 - g$ Weierstrass points, the map sequence at a Weierstrass point having the form $1, \cdots, g-1, g+1$. Accordingly, if $g > 2$ and X is normal, then $\pi(\frac{1}{2}(g-1))$ cannot occur in $L^2(\Gamma \setminus G;\ \chi)$. Letting $\chi = 1$, this means that $\pi(1 - \frac{1}{2}(g-1))$ must occur in $L^2(\Gamma \setminus G)$ with multiplicity $g-2$. This representation is square integrable but not integrable. Nevertheless, its multiplicity has the form $(2\kappa - 1)(g-1)$, the same as its integrable counterparts, yet the parameter value is to the left of $\kappa = 1$ and the multiplicity of $\pi(1)$ in $L^2(\Gamma \setminus G)$ is always g and not $g-1$.

References

[1] E. Arbarello et al., *Geometry of algebraic curves*, Springer, New York, 1985.

[2] L. Clozel & P. Delorme, *Sur le théorème de Paley-Wiener invariant pour les groupes de Lie réductifs réels*, C. R. Acad. Sci. Paris **300** (1985) 331–333.

[3] L. Ehrenpreis & F. Mautner, *Some properties of the Fourier transform on semisimple Lie groups*, Trans. Amer. Math. Soc. **84** (1957) 1–55.

[4] E. Freitag, *Automorphy factors of Hilbert's modular group*, Tata Inst. Studies in Math. **7** (1975) 9–19.

[5] P. Griffiths, *An introduction to the theory of special divisors on algebraic curves*, CBMS Regional Conf. Ser. in Math., No. 44, Amer. Math. Soc., Providence, RI, 1980, 1–25.

[6] R. Gunning, *The structure of factors of automorphy*, Amer. J. Math. **78** (1956) 357–382.

[7] ______, *Lectures on Riemann surfaces*, Princeton University Press, Princeton, NJ, 1966.

[8] ______, *Lectures on Jacobi varieties*, Princeton University Press, Princeton, NJ, 1972.

[9] ______, *Riemann surfaces and generalized theta functions*, Springer, New York, 1976.

[10] R. Hotta & N. Wallach, *On Matsushima's formula for the Betti numbers of a locally symmetric space*, Osaka J. Math. **12** (1975) 419–431.

[11] R. Langlands, *Dimension of spaces of automorphic forms*, Proc. Sympos. Pure Math., Vol. 9, Amer. Math. Soc., Providence, RI, 1965, 253–257.

[12] G. Mackey, *Induced representations of locally compact groups and applications*, Functional Analysis and Related Fields, Springer, New York, 1970, 132–166.

[13] K. Maurin & L. Maurin, *Duality, imprimitivity, reciprocity*, Ann. Polon. Math. **29** (1975) 309–322.

[14] V. Molchanov, *Plancherel formula for pseudo Riemannian symmetric spaces of the universal covering group of* SL(2, **R**), Siberian Math. J. **25** (1984) 903–917.

[15] S. Osborne & G. Warner, *The theory of Eisenstein systems*, Academic Press, New York, 1981.

[16] S. Patterson, *On the cohomology of Fuchsian groups*, Glasgow Math. J. **16** (1975) 123–140.

[17] H. Petersson, *Zur analytischen Theorie der Grenzkreisgrüppen*. I, Math. Ann. **115** (1938) 23–67.

[18] ______, *Zur analytischen Theorie der Grenzkreisgruppen*. II, Math. Ann. **115** (1938) 175–204.

[19] ______, *Zur analytischen Theorie der Grenzkreisgruppen*. III, Math. Ann. **115** (1938) 518–572.

[20] ______, *Zur analytischen Theorie der Grenzkreisgruppen*. IV, Math. Ann. **115** (1938) 670–709.

[21] L. Pukanszky, *The Plancherel formula for the universal covering group of* SL(2, **R**), Math. Ann. **156** (1964) 96–143.

[22] D. Ragozin & G. Warner, *On a method for computing multiplicities in* $L^2(\Gamma \setminus G)$, Sympos. Math. **22** (1977) 291–314.

[23] P. Sally, *Analytic continuation of the irreducible unitary representations of the universal covering group of* SL(2, **R**), Mem. Amer. Math. Soc. No. 69, 1967, 1–94.

[24] ______, *Intertwining operators and the representations of* SL(2, **R**), J. Functional Analysis **6** (1970) 441–453.

[25] G. Shimura, *On the trace formula for Hecke operators*, Acta Math. **132** (1974) 245–281.

[26] N. Wallach, *On the Selberg trace formula in the case of compact quotient*, Bull. Amer. Math. Soc. **82** (1976) 171–195.

[27] G. Warner, *Selberg's trace formula for nonuniform lattices*: *The* **R**-*rank* 1 *case*, Advances in Math. Studies **6** (1979) 1–142.

[28] ______, *Elementary aspects of the theory of Hecke operators*, preprint.

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