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IRRATIONALITY AND THE h-COBORDISM CONJECTURE

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I. Introduction

Underlying each smooth complex projective variety there is a compact differentiable manifold, and Hodge Theory links the topological cohomology of the space with its algebraic geometry. Despite this link there remains a striking difference between the successes of the two theories—the differential topology of manifolds and algebraic geometry of varieties—in problems of classification. For varieties there is a clearer picture in low dimensions, algebraic curves and surfaces, while for manifold topology high dimensions are more tractable. Smale's h-cobordism theorem gives a practical method for comparing the diffeomorphism types of simply-connected manifolds of dimension 5 or more, but for 4-manifolds the proof of the h-cobordism theorem breaks down.

In four dimenisons the first order Yang-Mills equations for gauge fields, like the Laplace equations of Hodge theory, give a path through partial differential equations between geometry and topology. On the one hand the solutions are, roughly speaking, generalizations of the holomorphic bundles over a complex surface. On the other hand their moduli spaces carry topological information. In this paper we exploit this path to define and calculate a new invariant for certain smooth 4-manifolds: we will see that the new invariant goes beyond the classical ones and our results indicate that there is a detailed structure in the differential topology of 4-manifolds quite analogous to that in the geometry of complex surfaces.

The manifold we used to test our invariant was discovered by I. Dolgachev in 1965 [4]. Dolgachev was motivated by the Castelnuovo-Enriques criterion

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for the rationality of an algebraic surface S:

$$\left.\begin{array}{l} q(S)=0 \\ P_2(S)=0 \end{array}\right\} \Rightarrow S \text{ rational}$$

[14, p. 536]. The "irregularity" $q(S)$ is half the first Betti number and hence a homotopy invariant. But, while the "geometric genus"

$$p_g(S)=P_1(S)=\dim \Gamma(K_S)$$

is similarly identified by Hodge Theory as an (oriented) homotopy invariant, the plurigenus $P_2(S)=\dim \Gamma(K_S^2)$ is not, since the holomorphic sections of K_S^2 are not tied to the de Rham cohomology. Severi asked whether the modified conditions

$$\pi_1(S)=1, \quad p_g(S)=0,$$

depending only on the homotopy type of S, imply that the surface is rational. Dolgachev's surface Z—irrational but homotopy equivalent to $\mathbf{P}^2 \# 9\overline{\mathbf{P}}^2$—shows that this is not the case. We will use our new invariant to prove, in a similar spirit to this algebraic geometry discussion, that Z is not *diffeomorphic* to $\mathbf{P}^2 \# 9\overline{\mathbf{P}}^2$.

Wall showed in [30] that there is an h-cobordism between two simply connected, homotopy equivalent 4-manifolds. So Z and $\mathbf{P}^2 \# 9\overline{\mathbf{P}}^2$ are h-cobordant nondiffeomorphic manifolds and they give a counterexample to the "h-cobordism conjecture" for smooth 4-manifolds. That is, the failure of the usual proof of the h-cobordism theorem is irreparable. Of course, by Freedman's classification [11] the two manifolds are homeomorphic and the results we obtain here distinguishing different smooth structures on the same topological 4-manifold are complimentary to those on the nonexistence of smoothings found using gauge fields. (Examples of nonsimply-connected 4-manifolds which are homeomorphic but not diffeomorphic have been found by Cappell and Shaneson [3].)

The main result of this paper was published in [7] and most of the technical background has been developed in [8], [9]. The first half of the paper gives the definition of the invariant, culminating in Theorem (2.15). The invariant is fairly complicated so it is worth pointing out here that one can define a whole range of invariants for various 4-manifolds [10] and these are in some ways simpler than the rather special case discussed here. The advantage of this special case lies in our ability to do calculations which take up the second half of the paper. In the Appendix we extend the discussion to allow for fundamental groups.

The approach we take here is slightly different from that in [7] (rearranging the use of Poincaré Duality in §II(b)) and takes advantage of a suggestion of T. Cochrane and C. Taubes. The author is also grateful to M. F. Atiyah and M. Reid for useful discussions and to M. Ville for translating [7].

II. The invariant Γ

(a) Let X be a smooth, simply-connected, closed oriented 4-manifold whose intersection form has type $(1, n)$. So the positive cone Ω in $H^2(X; R)$,

(2.1) $$\Omega = \{\Theta \in H^2(X; R) \mid \theta \cdot \theta > 0\},$$

has two connected components. For each element e in $H^2(X; Z)$ with $e \cdot e = -1$ let W_e denote the intersection of the hyperplane $e^{\perp}$ in $H^2(X; R)$ with Ω. The union of these "walls" forms a locally finite system of hypersurfaces partitioning Ω into a set $\mathscr{C}_X$ of "chambers": the connected components of $\Omega \setminus (\bigcup_e W_e)$.

The invariant Γ_X of X which we will define in this section is a map

$$\Gamma_X : \mathscr{C}_X \to H^2(X; Z)$$

with the following formal properties:

(2.2) (i) $\Gamma_X(-C) = -\Gamma_X(C)$.

(ii) If C_{-1}, C_1 are chambers in the same component of Ω and if a path between C_{-1}, C_1 meets walls $W_{e_1}, W_{e_2}, \cdots, W_{e_k}$, where $e_i \cdot C_{-1} < 0 < e_i \cdot C_1$, then

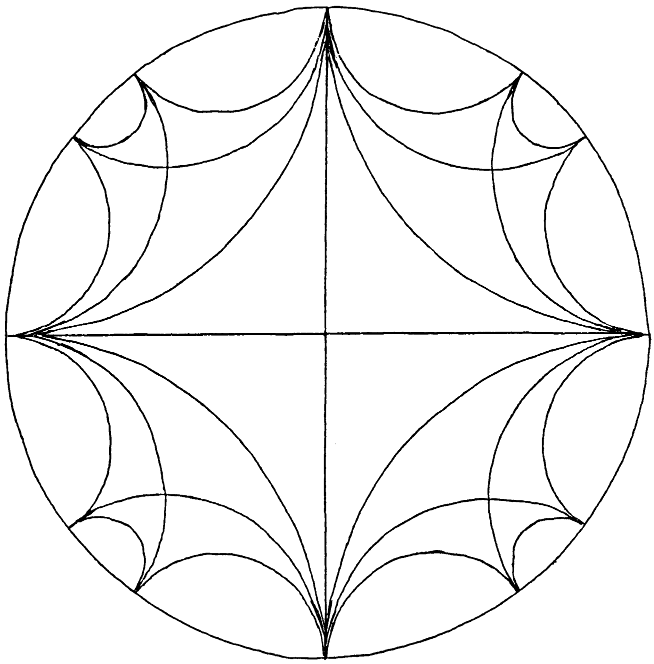
$$\Gamma_X(C_1) = \Gamma_X(C_{-1}) + 2\sum_{i=1}^{k} e_i.$$

(iii) If $f: X_1 \to X_2$ is an orientation preserving diffeomorphism between two such manifolds, then $\Gamma_{X_1}(f^*(C)) = f^*(\Gamma_{X_2}(C))$.

(2.3) Remarks. (i) If $n > 1$, then Theorem B of [8] asserts that the intersection matrix, in a suitable integral basis, is diag$(1, -1, \cdots, -1)$. If $n = 1$, there are two possbilities:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(ii) The quotient $\Omega/\mathbf{R}^*$ is a model for n-dimensional hyperbolic space $\mathfrak{h}_n$ and the quotients of the walls W_e divide $\mathfrak{h}_n$ into chambers in a similar way. To each chamber in $\mathfrak{h}_n$ there corresponds two elements $\pm C$ of $\mathscr{C}_X$. Property (2.2)(i) means that, up to a sign, we can regard our invariant as a function of the chambers in hyperbolic space. For example when $n = 2$ we obtain the configuration of walls in the hyperbolic plane shown in Diagram 1.



(Pattern repeats to infinity)

DIAGRAM 1

(b) Roughly speaking the cohomology classes $\Gamma_X(C)$ will be defined by the homology classes of suitable Yang-Mills moduli spaces. We will assume some familiarity with the references [13], [8], [9]. Let $E \to X$ be an $SU(2)$ bundle with $c_2(E) = 1$. Denote by $\mathcal{B}$ the infinite dimensional space of gauge equivalence classes of connections on E and by $\mathcal{B}^* \subseteq \mathcal{B}$ the dense open subset representing irreducible connections. For each Riemannian metric g on X the moduli space $M = M(g)$ is the subset of $\mathcal{B}$ defined by the anti-self-duality (ASD) equation

$$*_g F_A = -F_A$$

using the metric g. The metric also determines a line in $\Omega \cup \{0\} \subset H^2(X; \mathbf{R})$ made up of the cohomology classes represented by g-self-dual harmonic forms. Choosing a component of the cone Ω is equivalent to choosing a definite basis element ω for the self-dual harmonic forms—normalized so that $\int \omega \wedge \omega = 1$, say. Write $[\omega]$ for the corresponding point in Ω. It is elementary from Hodge theory that the moduli space contains a reducible connection precisely when

$[\omega]$ does *not* lie in one of the chambers. For each wall $W_e = W_{-e}$ containing $[\omega]$ there is a unique point in $M(g) \cap [\mathscr{B} \setminus \mathscr{B}^*]$ corresponding to a reduction $E = L_e \oplus L_e^{-1}$, $c_1(L_e) = e$. We will use the following three properties of the Yang-Mills moduli spaces $M(g)$.

(α) For generic metrics g on X the moduli space is a smooth manifold cut out transversely by the ASD equations [13, Proposition 3.20]. This means that the cohomology of the Atiyah-Hitchin-Singer deformation complex vanishes in dimension 0 and 2 for all solutions A, and so $\dim H_A^1 = \dim M = 2$ agrees with the virtual dimension predicted by the index theorem. In particular, for such generic metrics the line of self-dual harmonic forms does not lie in any wall and there are no reducible ASD connections on E.

(β) A choice of component of Ω, i.e. harmonic form ω, defines an orientation of M. We fix our conventions in line with [9] so that X is given the "homology orientation"

$$-1 \wedge \omega \in \Lambda^2(H^0 \oplus H_+^2)$$

and we use this to define a "standard orientation" of M as in [9, §§III, IV]. Reversing the choice of component of Ω reverses the standard orientation of M.

(γ) There is a finite-dimensional model for the end of M using Taubes' construction of a map:

$$\tau : X \times (0, \varepsilon) \to \mathscr{B}^*$$

parametrizing "concentrated" solutions of an "infinite dimensional part" of the ASD equations. Choose a harmonic form ω as above and let $\Lambda \to X \times (0, \varepsilon)$ be the 3-plane bundle $\pi_1^*(\Lambda^2_{+,X})$. Then there is a section Φ of Λ such that τ maps the zero set $Z = Z(\Phi) \subset X \times (0, \varepsilon)$ to M, and the complement of $\tau(Z)$ in M is compact. Furthermore regular zeros of Φ correspond to regular solutions of the ASD equations and the standard orientation of M corresponds to a standard orientation in the finite-dimensional model (see [26], [8], [9]).

These three properties are all that is needed to define the invariant Γ_X, although for calculations we will also make use of the following explicit approximation to Φ. If ω is the section Λ given by our choice of harmonic form, then

$$\Phi(x, \lambda) = 8\pi^2\lambda^2\omega(x) + h(x, \lambda), \qquad (x, \lambda) \in X \times (0, \varepsilon), \tag{2.4}$$

where h, $\partial h/\partial X$, and $\lambda \partial h/\partial \lambda$ are $O(\lambda^3)$ [8, Lemma 5.4]. Use the coordinate $t = \sqrt{\lambda}$ in the $\mathbf{R}$-factor of $X \times (0, \varepsilon)$ to adjoin a boundary, $t = 0$. Then (2.4) implies that $\lambda^{-2}\Phi$ has a C^1 extension to $X \times [0, \varepsilon)$, equal to ω on $X \times \{0\}$. So if ω vanishes transversely on a submanifold γ in X, then the end of M is a collar over γ. For generic metrics on X the harmonic form ω will have this property, by the discussion of [8, §VI].

(c) Fix a generic metric g on X and a g-self-dual harmonic form ω. We will define an invariant depending, at the outset, on g and ω using three simple topological constructions.

First, we can construct a universal bundle $\mathbf{E}$ over $\mathscr{B}^* \times X$ with structure group $U(2)$ [8, Proposition (2.20)]. Define a map

$$\hat{\mu} : H_2(\mathscr{B}^*; \mathbf{Z}) \to H^2(X; \mathbf{Z}) \tag{2.5}$$

by $\hat{\mu}(l) = l \setminus c_2(\mathbf{E})$. This is the adjoint of the map

$$\mu : H^2(X; \mathbf{Z}) \to H^2(\mathscr{B}^*; \mathbf{Z})$$

used in [8] and so, for the simply-connected manifold X, μ determines $\hat{\mu}$. In any case $\hat{\mu}$ is independent of the choice of universal bundle $\mathbf{E}$.

Now suppose the moduli space $M(g)$ happens to be compact. Then it carries a fundamental homology class $[M(g)]$ in $H_2(\mathscr{B}^*)$ (using the standard orientation of (β)) and we can use the universal bundle to define a cohomology class $\hat{\mu}([M(g)]$ in $H^2(X)$. The second construction we need identifies a correction term associated to the description of the end of the moduli space which will give a class in $H^2(X)$ when M is not compact. (In fact the contribution from the moduli space will be *twice* the fundamental class).

For each "level" α in $(0, \varepsilon)$ write $X_\alpha = X \times \{\alpha\} \subset X \times (0, \varepsilon)$, $Y_\alpha = X_\alpha \cap Z$, $Z_\alpha = Z \cap X \times [\alpha, \varepsilon)$. By Sard's Theorem the intersection $X_\alpha \cap Z$ is transverse for almost all α and in that case Z_α is a manifold with boundary Y_α. Let $N_\alpha = \tau(Y_\alpha) \subset M$ and M_α be the union of $\tau(Z_\alpha)$ and the compact piece $M \setminus \tau(Z)$. So M_α is a manifold with boundary N_α (see Diagram 2).

Fix α in general position and suppose a relative class

$$\hat{e}_\alpha \in H_2(X_\alpha, Y_\alpha)$$

is given, which maps to *twice* the fundamental class in $H_1(Y_\alpha)$ under the boundary map of the pair (X_α, Y_α). Then, with the appropriate orientation conventions, the difference

$$2[M_\alpha] - \tau_*(\hat{e}_\alpha) \tag{2.6}$$

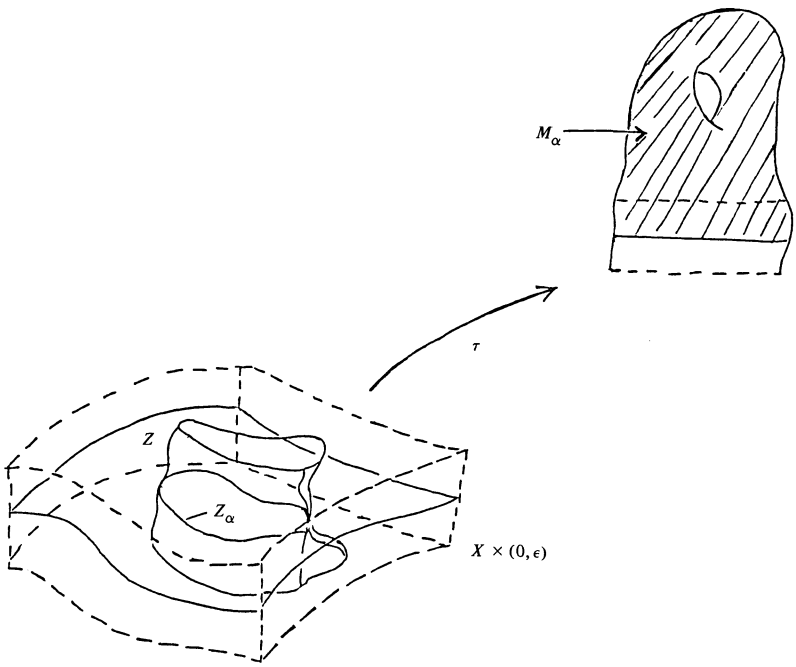


DIAGRAM 2

in $H_2(\mathscr{B}^*, N_\alpha)$ will have a unique lift $l(\alpha, M, \hat{e}_\alpha)$ to $H_2(\mathscr{B}^*)$ (since $H_2(N_\alpha) = 0$). Similarly, suppose $\alpha < \beta$ are two levels in general position and $\hat{e}_\alpha$, $\hat{e}_\beta$ are given as above. Write $X_{\alpha\beta} = X \times [\alpha, \beta]$ and $Z_{\alpha\beta} = X_{\alpha\beta} \cap Z$. The combination $2[Z_{\alpha\beta}] - \hat{e}_\alpha + \hat{e}_\beta$ can be regarded as an element in $H_2(Z_{\alpha\beta} \cup X_\alpha \cup X_\beta)$. If there is a class $\hat{e}_{\alpha\beta}$ in $H_3(X_{\alpha\beta}, Z_{\alpha\beta} \cup X_\alpha \cup X_\beta)$, mapping to $2[Z_{\alpha\beta}] - \hat{e}_\alpha + \hat{e}_\beta$ under the boundary map, then it follows that $l(\alpha, M, \hat{e}_\alpha) = l(\beta, M, \hat{e}_\beta)$ in $H_2(\mathscr{B}^*)$.

The classes $\hat{e}_\alpha$, $\hat{e}_{\alpha\beta}$ posited above can be found in the topology of the model for the end of the moduli space. Let $\Phi^\perp \subset \Lambda|_{X_\alpha \setminus Y_\alpha}$ be the orthogonal complement of Φ. (It is a 2-plane bundle with a standard orientation, although we postpone discussing these signs for a moment.)

The Euler class $e(\Phi^\perp)$ lies in $H^2(X_\alpha \setminus Y_\alpha)$; we take $\hat{e}_\alpha$ to be its dual homology class in $H_2(X_\alpha, Y_\alpha)$. Since Y_α is a submanifold of X_α, the duality isomorphism $H_2(X_\alpha, Y_\alpha) \cong H^2(X_\alpha \setminus Y_\alpha)$ used here can be defined by the composite of the excision

$$H_2(X_\alpha, Y_\alpha) \cong H_2(X_\alpha, \partial\nu_\alpha), \qquad H^2(X_\alpha \setminus Y_\alpha) \cong H^2(X_\alpha \setminus \nu_\alpha)$$

of a tubular neighborhood ν_α of Y_α, with Lefschetz Duality $H_2(X_\alpha \setminus \nu_\alpha) \cong H^2(X_\alpha, \partial\nu_\alpha)$. To check the boundary properties of $\hat{e}_\alpha$ we use the Thom isomorphism for the 2-sphere bundle $\partial\nu_\alpha \to Y_\alpha$. The section Φ vanishes transversely on Z so the bundle $\Phi^\perp$ can be identified with the tangent bundle along the fibers over $\partial\nu_\alpha$ and this means that $e(\Phi^\perp)$ restricts to $e(S^2) = 2$ times the generator on H^2(fiber). Going over to homology this calculation verifies that $\hat{e}_\alpha$ has the desired properties. In just the same way the Euler class of the bundle $\Phi^\perp$ over $X_{\alpha\beta} \setminus Z_{\alpha\beta}$ dualizes to give the required relative homology class $\hat{e}_{\alpha\beta}$. (Of course this argument is essentially the same as that showing that twice the Euler class of a 3-plane bundle vanishes [22, Property 9.4, p. 98].)

So, with the fixed metric and harmonic form, we define in this way an element l of $H_2(\mathscr{B}^*; \mathbf{Z})$. If ω does *not* vanish anywhere on X, then by (2.4) M is compact and l has two distinct contributions, one from $e(\Phi^\perp)$ and one from $2[M]$, in (2.6). But only this combination makes good sense for general metrics. The third step in our definition is to check the orientations and to give an explicit description of the contribution from Φ in the case when ω does not vanish.

For the orientations we suppose, as in [9, (4.10), Example (iii)], that (x_0, x_1, x_2, x_3) is a system of oriented local coordinates around a point in the zero set γ of ω (relative to some metric) and that there is a local frame $\varepsilon_1, \varepsilon_2, \varepsilon_3$ for $\Lambda^2_{+,X}$ such that

$$\omega = \sum_{i=1}^{3} x_i \varepsilon_i.$$

The coordinate x_0 is chosen so that $\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3$ is the orientation of Λ^2_+ defined in [9, §III(c)]. Then the standard orientation of M was calculated (loc.cit.) to be given by

$$\frac{\partial}{\partial x_0} \wedge n,$$

where n is the vector pointing "into" the moduli space, and $\partial/\partial x_0$ corresponds to translations along γ.

Orient $\omega^\perp$ by the rule that $\varepsilon_1^\perp$ has orientation $\varepsilon_2 \wedge \varepsilon_3$. Then the dual of $e(\omega^\perp)$ is represented locally by the half-plane $P = \{x_2 = x_3 = 0,\ x_1 > 0\}$, taken with multiplicity 2 and orientation $\partial/\partial x_0 \wedge \partial/\partial x_1$. So the oriented boundaries of $2M$ and P are equal and this rule is the correct one for orienting $\Phi^\perp$ to achieve cancellation.

On the other hand, at points where ω does *not* vanish it defines an *almost complex structure* on the 4-manifold (such that $2\omega/|\omega|$ is the standard metric 2-form) and this rule corresponds to the identification $\omega^\perp \cong \Lambda^2_{\mathbf{C}} TX$ (i.e., the

(0, 2) forms; see [9, §III]). So, if (for some other metric) ω does not vanish, the approximation (2.4) to Φ shows that $e(\Phi^{\perp})$ is $-c_1(K_X)$, where K_X is the canonical bundle of the almost complex structure.

Finally, the duality formula equating the composite

$$H_2(X;\mathbf{Z}) \underset{\mu}{\to} H^2(\mathscr{B}^*;\mathbf{Z}) \underset{\tau^*}{\to} H^2(X;\mathbf{Z}) \tag{2.7}$$

with Poincaré Duality [8, §III] leads to the adjoint form:

$$\begin{gathered} H_2(X) \underset{\tau_*}{\to} H_2(\mathscr{B}^*;\mathbf{Z}) \underset{\hat{\mu}}{\to} H^2(X;\mathbf{Z}), \\ \hat{\mu}\tau_* = \text{P.D.} \end{gathered} \tag{2.8}$$

So applying $\hat{\mu}$ to the contribution from $\Phi^{\perp}$ gives a contribution of $c_1(K_X)$ to Γ.

In sum we have:

Proposition (2.9). *If g is a generic metric on X and ω is a g self-dual-closed form, then the construction above defines a cohomology class*

$$\Gamma(g,\omega) = \hat{\mu}\Big(l\big(\alpha, M, e(\Phi^{\perp})\big)^{\wedge}\Big)$$

in $H^2(X;\mathbf{Z})$. If ω is nowhere zero, so M is compact, then $\Gamma(G,\omega) = 2\hat{\mu}([M]) + c_1(K_X)$ where M has the standard orientation defined by ω and K_X in the canonical bundle of the almost complex structure for which $2\omega/|\omega|$ is the metric form.

Note that

$$\Gamma(g,-\omega) = -\Gamma(g,\omega), \tag{2.10}$$

since changing the sign of ω changes the orientations throughout.

(d) We now compare the elements $\Gamma(g,\omega)$ in $H^2(X)$ for different Riemannian metrics and show that their totality gives an invariant Γ_X with properties (2.2)(i)–(iii). The comparison involves generic paths in the space of Riemannian metrics and two cases arise, according to the presence of reducible solutions to the ASD equations.

Proposition (2.11). *Let g_{-1}, g_1 be generic metrics on X which can be joined by a path g_t ($t \in [-1,1]$) of metrics for which the cohomology classes $[\omega_t]$ of the self-dual harmonic forms lie in a single chamber C in Γ_X. Then $\Gamma(g_{-1},\omega_{-1}) = \Gamma(g_1,\omega_1)$.*

Proof. This is a simple transversality argument, similar to [16, Chapter 3]. The hypothesis on the metrics g_t means that all the moduli spaces $M(g_t)$ lie in $\mathscr{B}^*$. Work with the space $\mathscr{R}$ of C^k Riemannian metrics ($k \gg 0$) and let $\mathscr{A}$ be extended to include L_1^p connections (some $p > 2$). These are Banach manifolds so we can use the ordinary implicit function theorem.

Consider first the ends of the moduli spaces. We may regard the self-dual harmonic forms, for all different metrics, as defining a section of a 3-plane bundle Λ over $X \times \mathscr{R}$. According to [8, §VI] this section vanishes transversely on a codimension 3 submanifold $G \subset X \times \mathscr{R}$, say. The zero sets $\gamma \subset X$ for the different metrics are the fibers of the projection of G to $\mathscr{R}$. The generic metrics $g_{-1}, g_1 \in \mathscr{R}$ were chosen so that $X \times \{g_{-1}\}$, $X \times \{g_1\}$ are transverse to G. It follows that, since X is compact, there is a map

$$H : [-1,1] \times B^N \to \mathscr{R}$$

(some $N > 0$ and $B^N \subset \mathbf{R}^N$ the unit ball), with $H(t,\eta) = g_t$ if $\eta = 0$ or $|t| = 1$, such that $H \times 1_X$ is transverse to G. So $(H \times 1_X)^{-1}(G)$ is a submanifold of $X \times [-1,1] \times B^N$ and we can apply Sard's Theorem to find arbitrarily small regular values ξ of the projection

$$(H \times 1_X)^{-1}(G) \to B^N.$$

This means that we can suppose the path g_t chosen so that their harmonic forms ω_t—considered as a section of a bundle over $X \times [-1,1]$—vanish transversely. For if the original path did not have this property a nearby path $H(t,\xi)$ will do.

Let

$$\mathscr{M} = \bigcup_{t \in [-1,1]} (M(g_t), t) \subset \mathscr{B}^* \times [-1,1].$$

The boundary description of §III(b)(γ) extends to describe the end of this family of moduli spaces. The end is modelled on the zero set $\mathscr{Z}$ of a section Φ of the bundle $\Lambda \to X \times [-1,1] \times (0,\varepsilon)$ and Ψ is approximated by ω_t. So if ω_t vanishes transversely the end is a collar on the 2-dimensional zero set.

In [13, §3] Freed and Uhlenbeck show that the "universal" moduli space—a subset of $\mathscr{B}^* \times \mathscr{R}$—is a submanifold. So, repeating the transversality argument above, we can perburb the path g_t slightly to smooth the compact part of $\mathscr{M}$ not already covered by the description of its end. Thus we can suppose $\mathscr{M}$ is a 3-manifold with two boundaries $M(g_{-1})$, $M(g_1)$. Now choose a level α in $(0,\varepsilon)$ with $X_\alpha \times [-1,1]$ transverse to $\mathscr{Z}$ and $X_\alpha \times \{-1\}$, $X_\alpha \times \{1\}$ transverse to the sets $Z_\alpha^{(-1)}$, $Z_\alpha^{(1)}$ considered in (c) above. Then $\mathscr{Y}_\alpha = (X_\alpha \times [-1,1]) \cap \mathscr{Z}$ is a cobordism between $Y_\alpha^{(-1)}$ and $Y_\alpha^{(1)}$. The dual of

$$e(\Psi^\perp) \in H^2(X_\alpha \times [-1,1] \setminus \mathscr{Y}_\alpha)$$

is a class in $H_3(X_\alpha \times [-1,1], X_\alpha \times \{-1,1\} \cup \mathscr{Y}_\alpha)$ with boundary $\hat{e}^{(1)} - \hat{e}^{(-1)} + 2\mathscr{Y}_\alpha$. So $2\mathscr{M}_\alpha - \tau_*(e(\Psi^\perp)^\wedge)$ gives a homology in $\mathscr{B}^* \times [-1,1]$ between $l^{(-1)} \times \{-1\}$ and $l^{(1)} \times \{1\}$; thus, projecting to $\mathscr{B}^*$, $l^{(-1)} = l^{(1)}$ in $H_2(\mathscr{B}^*; \mathbf{Z})$.

In general we have:

Proposition (2.12). *Let g_{-1}, g_1 be generic metrics on X and choose corresponding harmonic forms ω_{-1}, ω_1 so that $[\omega_{-1}]$, $[\omega_1]$ lie in the same component of Ω. Then*

$$\Gamma(g_1, \omega_1) = \Gamma(g_{-1}, \omega_{-1}) + 2\sum_{i=1}^{k} e_i,$$

where $\{e_1, \cdots, e_k\} = \{e \in H^2(X; \mathbf{Z}) \mid e \cdot e = -1,\ \omega_{-1} \cdot e < 0 < \omega_1 \cdot e\}$.

Proof. The discussion of [8, §VI] shows that there is a path of metrics g_t from g_{-1} to g_1 such that the corresponding path $[\omega_t]$ in Ω passes transversely through any wall it meets. Suppose, for simplicity, that $[\omega_t]$ meets just one wall W_e, $\omega_{-1} \cdot e < 0 < \omega_1 \cdot e$, and crosses it when $t = 0$. Then the moduli space $M(g_0)$ contains a point associated to a reducible connection A, compatible with a splitting $E = L_e^{-1} \oplus L_e$. The deformation theory models a neighborhood of this point in $M(g_0)$ on a quotient $\phi^{-1}(0)/S^1$, where $\phi : \mathbf{C}^{q+2} \to \mathbf{R} \oplus \mathbf{C}^q$ is an equivariant map (see [8, §IV]). In just the same way the behavior of the family of moduli spaces $M(g_t)$ near this reduction, for $|t| < \tau$, is modelled by a map $\chi : \mathbf{C}^{q+2} \times (-\tau, \tau) \to \mathbf{R} \oplus \mathbf{C}^q$ extending ϕ. For the discussion of homology it is not loss to suppose that $q = 0$ (adding a small perturbation by hand if necessary, as in [5], [13, §4]. Then χ takes its values in the copy of the 1-dimensional space $H^2_+(X) = \mathbf{R} \cdot \omega_0$ corresponding to the trivial factor in the Lie algebra bundle $\mathfrak{g}_E = \mathbf{R} \oplus L_e^2$. As in [9] we fix signs so that the $\mathbf{R}$ factor acts with positive weight on L_e. Now $\chi(Z, t)$ represents the ω_0 component of the curvature of a connection $A + \tilde{p}(Z, t)$ which is reducible when $Z = 0$. It has the property that

$$F(A + \tilde{p}(0, t)) - \chi(0, t)\omega_0$$

is g_t ASD so

$$\omega_t \cdot (F(A + \tilde{p}(0, t))) = \chi(0, t)\omega_t \cdot \omega_0.$$

The curvature $F(A + \tilde{p}(0, t))$ represents $-2\pi c_1(L_e) = -2\pi e$ and $\omega_t \cdot \omega > 0$. So the condition that ω_t passes transversely through W_e, and the sense in which it crosses, implies that $\partial\chi/\partial t < 0$ when $t = Z = 0$. This means that projection to the Z-factor maps a small neighborhood of $(0,0)$ in $\chi^{-1}(0)$ equivariantly and diffeomorphically to a small ball in $\mathbf{C}^2$. Cut out the corresponding neighborhood from the moduli space $\mathscr{M} = \bigcup_t (M(g_t), t) \subset \mathscr{B} \times [-1, 1]$—the remainder is contained in $\mathscr{B}^* \times [-1, 1]$. Using the argument in Proposition (2.11), away from the reduction, we get a homology in $\mathscr{B}^*$ between $l^{(-1)}$ and $l^{(1)} \pm 2\mathbf{P}_e^1$, where $\mathbf{P}_e^1 \subset \mathscr{B}^*$ is the 2-sphere corresponding to $\{|Z| = \delta\}/S^1$ in the local model. Given $\mathbf{P}_e^1$ its usual complex orientation: by

[8, Lemma (2.28)] we have

(2.13) $$\langle \mu(\alpha), \mathbf{P}_e^1 \rangle = -\langle \alpha, e \rangle$$

for α in $H_2(X; \mathbf{Z})$. Since $\pi_1(X) = 1$ and $H_2(X)$ is free this implies that

(2.14) $$\hat{\mu}(\mathbf{P}_e^1) = -e.$$

To find the right sign for the change in $\Gamma(g_t, \omega_t)$ consider a case when $\chi(Z_1, Z_2, t) = |Z_1|^2 + |Z_2|^2 - t$. Then as t increases through 0, the moduli space *gains* a copy of $\mathbf{P}_e^1$ as a new component. The results of [9] (especially Example (4.5)) show that the standard orientation of M on this component is opposite to the usual complex orientation of $\mathbf{P}_e^1$. So

$$l^{(1)} = l^{(-1)} - 2[\mathbf{P}_e]$$

in $H_2(\mathscr{B}^*)$ and $\Gamma(g_1, \omega_1) = \Gamma(g_{-1}, \omega_{-1}) + 2e$. Now Proposition (2.12), in general, follows from this special case by counting, with the correct signs, all the walls crossed. (It does not matter if the path $[\omega_t]$ crosses many walls simultaneously since the different reductions are separated in $\mathscr{B}$.)

So we have obtained the main theorem of this paper.

Theorem (2.15). *There is a unique way to define a map $\Gamma_X \colon \mathscr{C}_X \to H^2(X; \mathbf{Z})$ with the three properties in* (2.2) *such that for any generic metric g on X and g-self-dual form ω, $\Gamma_X(C) = \Gamma(g, \omega)$, where C is the chamber containing* $[\omega]$.

The proof follows immediately from (2.9)–(2.12).

III. Stable bundles over algebraic surfaces

(a) Review of complex geometry. In this section we suppose that the 4-manifold X admits a complex structure and Kähler metric. For such a metric the holonomy group of the cotangent bundle is reduced from $SO(4)$ to $U(2)$ and the bundle of self-dual 2-forms decomposes as

(3.1) $$\Lambda_+^2 = \mathbf{R} \cdot \omega \oplus K_X.$$

Here ω is the usual metric form and K_X is the canonical bundle of complex $(2,0)$ forms.

One consequence of this differential geometry for the manifold in the large is the Hodge Index Theorem

(3.2) $$b_2^+(X) = 1 + 2p_g.$$

That is, the self-dual harmonic forms split into a 1-dimensional piece spanned by ω and a piece identified with the holomorphic 2-forms. So the manifold has an intersection form of type $(1, n)$—as in §II—if and only if its geometric genus p_g is zero. Also, the complex structure picks out a preferred component of the positive cone ω; the cone containing the cohomology class of the Kähler metric.

Suppose also that X is an algebraic surface (for topology this is no real loss of generality since it is known that any Kähler surface deforms to an algebraic one [18, Theorem 16.1]). Then it is natural to consider "Hodge metrics"—Kähler metrics whose form ω defines the cohomology class $c_1(H) \in H^2(X; \mathbf{Z})$ of an ample line bundle $H \to X$. So the sections of some positive power H^d given an embedding $X \hookrightarrow \mathbf{P}^N$ whose hyperplane sections are curves in X Poincaré dual to $d[\omega]$. Conversely we can identify the cohomology classes of ample line bundles. An element of $H^2(X; \mathbf{Z})$ is the first Chern class of a holomorphic bundle if it has type $(1,1)$ and on any compact complex surface S we have the "Nakai criterion."

Proposition (3.3) [2, *p.* 127]. *A line bundle $\mathscr{L} \to S$ is ample if and only if $c_1(\mathscr{L})^2 > 0$ and $c_1(\mathscr{L}) \cdot c > 0$ for every complex curve c in S.*

(b) The ASD moduli spaces. If g is a Hodge metric on an algebraic surface X there is an algebro-geometric description of the moduli spaces of g—ASD connections similar to the description of the harmonic forms in the Hodge Index Theorem. Most of the account following applies to general surfaces; for those of interest in this paper, with $p_g = 0$, the moduli space $M(X, g)$ will be compact since the harmonic ω is the Kähler form and does not vanish anywhere.

If $d_A : \Omega^0(E) \to \Omega^1(E)$ is the covariant derivative of a unitary connection A on a complex vector bundle $E \to X$, we define derivatives ∂_A, $\bar\partial_A$ in the holomorphic and anti-holomorphic directions:

$$d_A = \partial_A \oplus \bar\partial_A : \Omega^0(E) \to \Omega^{1,0}(E) \oplus \Omega^{0,1}(E).$$

As usual, there is a similar operator

$$\bar\partial_A : \Omega^{0,1}(E) \to \Omega^{0,2}(E).$$

If A is ASD then the bundle decomposition (3.1) implies that $\bar\partial_A^2 = 0$ and a version of the Newlander-Nirenberg Theorem ([23], [1, §5]) asserts that this is the integrability condition for $\bar\partial_A$. If it is satisfied, there are local trivializations of E by sections s satisfying $\bar\partial_A s = 0$, and these define a *holomorphic structure* on E. Since X is projective algebraic this holomorphic structure will also be algebraic—admitting a system of rational transition functions [14, pp. 171, 207].

Thus the integrability theorem gives a map from the moduli space of ASD connections on an $SU(2)$ bundle $E \to X$ to the set of equivalence classes of algebraic 2-plane bundles $\mathscr{E}$ with trivial determinant.

The main theorem of [6] describes this map more precisely. Such an algebraic bundle $\mathscr{E}$ is said to be *stable* if all line bundles $\mathscr{L}$ admitting nonzero holomorphic maps $\mathscr{L} \to \mathscr{E}$ have

$$\deg \mathscr{L} \equiv \mathscr{L} \cdot H < 0. \tag{3.4}$$

Here H is an ample line bundle compatible with the metric and we adopt the usual notation confusing line bundles and their cohomology classes. Then, assuming for simplicity that there are no reducible ASD connections on E, we have

Proposition (3.5) [6]. *The forgetful map assigning a $\bar{\partial}$-operator to a connection induces a* (1-1) *correspondence between a moduli space of ASD connections on E and the equivalence classes of stable bundles $\mathscr{E}$ with $\Lambda^2 \mathscr{E} \cong \mathcal{O}_X$ and $c_2(\mathscr{E}) = c_2(E)$.*

This gives an algebro-geometric description of the ASD moduli spaces as sets. To get our hands on their fundamental homology classes we enhance the description by comparing the deformation theories of ASD connections and algebraic bundles.

The points in the ASD moduli space in the neighbohrood of an orbit $[A]$ are described by the Kuranishi method, splitting the equations into finite and infinite dimensional parts. This can be done in many different ways and we will use a minor variation of the set-up in [8, §IV]. Working in the transversal slice

$$\{A + a \mid d_A^* a = 0\}$$

we consider, for small a, any smooth family

$$u_a : H_A^2 \to \Omega_+^2(\mathfrak{g}_E)$$

of maps transverse to $\operatorname{Im} d_A^+$. Then the nonlinear equation

$$F_{A+a}^+ \in \operatorname{Im}(u_a)$$

can be solved by the implicit function theorem: the solutions, for small a, are parametrized by a map:

$$H_A^1 \to \bar{\mathscr{A}}, \qquad p \mapsto \mathscr{A} + i(p).$$

Relative to this parametrization the nearby points in the moduli space are the zeros of the map

$$\phi : H_A^1 \to H_A^2$$

given by the remaining component of F^+ (so $F_{A+i(p)}^+ = u_{i(p)}(\phi(p))$). One sees readily that changing the choice of transversals U_a and parametrization $i(p)$ changes ϕ to a different model ϕ^1 with $\phi^1(p) = \gamma_p \circ \phi \circ \delta(p)$, where δ is a diffeomorphism defined on a small neighborhood of 0 in H_A^1 and $p \mapsto \gamma_p$ is a smooth map from a similar neighborhood to $\mathrm{GL}(H_A^2)$.

On the algebraic, or complex analytic, side we use the analogue for bundles of Kuranishi's original construction [20]. Let E be the C^∞ bundle underlying a given stable holomorphic bundle $\mathscr{E}$ with $\Lambda^2\mathscr{E} \cong \mathscr{O}_X$. Denote by $\bar{\mathscr{A}}$ the complex affine space of $\bar\partial$-operators:

$$\bar\partial : \Omega^0(E) \to \Omega^{0,1}(E),$$
$$\bar\partial(f \cdot s) = \bar\partial f \cdot s + f \cdot \bar\partial s \cdot ,$$

compatible with the given trivialization of Λ^2, and by $\mathscr{G}^{\mathbf{C}}$ the "complex gauge group" of special linear automorphisms of E. Then $\mathscr{E}$ is represented by an operator $\bar\partial_0$ in $\bar{\mathscr{A}}$ and the set of holomorphic structures on E is the quotient by $\mathscr{G}^{\mathbf{C}}$ of the subset $\{F^{0,2} = 0\}$ of $\bar{\mathscr{A}}$. Here

$$F^{0,2} : \bar{\mathscr{A}} \to \Omega^{0,2}(\mathrm{End}_0 E)$$

is the map assigning the tensor $\bar\partial^2$ to a $\bar\partial$-operator ($\mathrm{End}_0 E \subset \mathrm{End}\, E$ denotes trace-free endomorphisms). Stability is an open condition so, for the local discussion of a neighborhood of $[\mathscr{E}]$ in the moduli space M_S of stable bundles, we need not distinguish between the moduli of arbitrary holomorphic bundles and the stable ones.

We construct a sheaf of rings $\mathscr{O}_{M_S}$ on M_S from a presheaf Q. If U is open in M_S then an element of $Q(U)$ is represented by a $\mathscr{G}^{\mathbf{C}}$-invariant holomorphic function f on a subset $V \subset \bar{\mathscr{A}}$ lying over U. Two such representations (f_1, V_1), (f_2, V_2) are equivalent if there is a g in $\mathscr{G}^{\mathbf{C}}$ such that $V_2 = g(V_1)$ and $f_1 - f_2 \circ g$ is induced by an invariant function of $F^{0,2}|_{V_1}$. (The precise definition of holomorphic functions on these infinite-dimensional spaces is not very critical. We could, for example, take a version of $\bar{\mathscr{A}}$ based on Banach spaces.)

If we write

$$\bar{\mathscr{A}} = \left\{ \bar\partial_0 + \alpha \,|\, \alpha \in \Omega^{0,1}(\mathrm{End}_0 E) \right\},$$

then

$$F^{0,2}\left(\bar\partial_0 + \alpha\right) = \bar\partial_0 \alpha + \tfrac{1}{2}[\alpha, \alpha] \tag{3.6}$$

(extending $\bar\partial_0$ to act on $\mathrm{End}\, E$) and the complex gauge group $\mathscr{G}^{\mathbf{C}}$ acts by

$$g\left(\bar\partial_0 + \alpha\right) = \bar\partial_0 + \left\{ g\alpha g^{-1} + g\bar\partial_0\left(g^{-1}\right)\right\}. \tag{3.7}$$

Kuranishi's method applies the implicit function theorem to (3.6), (3.7) to give a finite-dimensional model for the sheaf $\mathscr{O}_{M_S}$ in terms of the Dolbeault cohomology groups:

$$H^i\left(\mathrm{End}_0 E; \bar\partial_0\right) = H^i(\mathrm{End}_0 \mathscr{E}).$$

Using the same notation as before we get holomorphic maps:

$$H^1(\mathrm{End}_0\,\mathscr{E}) \to \overline{\mathscr{A}},$$
$$(3.8)\qquad \pi \mapsto \bar\partial_0 + j(\pi),$$
$$\psi : H^1(\mathrm{End}_0\,\mathscr{E}) \to H^2(\mathrm{End}_0\,\mathscr{E})$$

such that $F^{0,2}(\bar\partial_0 + j(\pi)) = V_\pi(\psi(\pi))$ for small π in $H^1(\mathrm{End}_0\,\mathscr{E})$. (Once again the model ψ depends upon a choice of transversal maps V_π from $H^2(\mathrm{End}_0\,E)$ to $\Omega^{0,2}(\mathrm{End}_0\,E)$.) The stable bundle $\mathscr{E}$ has $H^0(\mathrm{End}_0\,\mathscr{E}) = 0$ [24, p. 172] so the stabilizer of $\bar\partial_0$ in $\mathscr{G}^{\mathbf{C}}$ is $\{\pm 1\}$. It follows then that the stalk of $\mathscr{O}_{M_S}$ at $[\mathscr{E}]$ is naturally isomorphic to the germs of holomorphic functions at 0 in $H^1(\mathrm{End}_0\,\mathscr{E})$ divided by the ideal generated by the components of ψ. That is, to the structure sheaf of the "universal local deformation" $T = \{\psi = 0\} \subset H^1(\mathrm{End}_0\,\mathscr{E})$. Moreover, as Kuranishi shows in [20], a local universal family—a locally free sheaf over a neighborhood of $\{0\} \times X$ in the complex analytic space $T \times X$—exists, and the universal properties of this characterize T intrinsically. Restricting to the stable bundles means that these local deformations fit together to define a separated analytic moduli space M_S (in fact it is a quasi-projective variety). When $c_2(\mathscr{E})$ is odd this is a "fine" moduli space, admitting a universal sheaf $\mathbb{E}$ over $M_S \times X$, by the same argument used in [8, Proposition (2.20)], [24].

To compare M_S with the ASD moduli space M fix a Hermitian metric on E, compatible with the trivialization of Λ^2. Then $\overline{\mathscr{A}}$ is naturally identified with the space $\mathscr{A}$ of $SU(2)$ connections via the $\bar\partial$-operator $\bar\partial_A$ associated to a connection. For any such connection A we use (3.1) to write

$$F_A^+ = i\hat F_A \cdot \omega \oplus F_A^{0,2}$$

(of course $\mathrm{End}_0\,E = \mathfrak{g}_E \otimes \mathbf{C}$ and the bundles of $(2,0)$ and $(0,2)$ forms are isomorphic as real bundles). Proposition (3.3) asserts that the $\mathscr{G}^{\mathbf{C}}$ orbits in $\{F^{0,2} = 0\}$ containing solutions of $\hat F_A = 0$ are exactly those of the stable bundles, and the solution in each orbit is unique up to $\mathscr{G}$, the special unitary gauge group. Locally, in the neighborhood of an irreducible ASD connection A, this follows easily from the implicit function theorem. If we write $a = \alpha - \alpha^*$, where α is in $\Omega^{0,1}(\mathrm{End}_0\,E)$, the gauge fixing equation $d_A^* a = 0$ combined with the condition $F_{A+a} = 0$ gives an equation of the form

$$(3.9)\qquad \bar\partial_A^*\alpha + \{\alpha, \alpha\} = 0,$$

where $\{\ ,\ \}$ is an algebraic bilinear term. The derivative of the $\mathscr{G}^{\mathbf{C}}$ action (3.7) at $g = 1$, $\alpha = 0$ (with $\bar\partial_0 = \bar\partial_A$) is

$$(u, \alpha) \mapsto \alpha - \bar\partial_A u.$$

So the solutions of (3.9) give a smooth transversal to the $\mathscr{G}^{\mathbf{C}}$ action, since the Laplacian $\bar\partial_A^*\bar\partial_A = \frac{1}{2}d_A^*d_A$ on $\Omega^0(\mathrm{End}_0 E)$ is invertible. That is, for any connection $A + (\beta - \beta^*)$ close to A ($\beta \in \Omega^{0,1}(\mathrm{End}_0 E)$), there exists a unique element g_β of $\mathscr{G}^{\mathbf{C}}$, close to 1, such that the $(0,1)$ part of $g_\beta(A + \beta - \beta^*) - A$ satisfies (3.9).

The Kähler identities give natural isomorphisms

$$(3.10) \qquad H^i(\mathrm{End}_0 \mathscr{E}) \cong H_A^i, \qquad i = 1, 2,$$

between the cohomology groups of the Atiyah-Hitchin-Singer deformation complex and sheaf cohomology groups associated to the holomorphic bundle $\mathscr{E}$ defined by A (see [9]). Suppose we define a local model ψ, as above, for M_s using the constant harmonic lift $H_A^{0,2} \subset \Omega^{0,2}(\mathrm{End}_0 E)$, so

$$F^{0,2}\big(\bar\partial_0 + j(\pi)\big) = \psi(\pi) \in H_A^{0,2}.$$

Then $F^{0,2}(g_{i(\pi)}(\bar\partial_0 + j(\pi)))$ lies in $g_{i(\pi)} H_A^{0,2} g_{i(\pi)}^{-1}$. Since $\hat F$ vanishes on the unitary connection $A + \dot{l}(\pi)$ corresponding to $g_{i(\pi)}(\bar\partial_0 + j(\pi))$ our model ϕ for M agrees with ψ (under the isomorphisms (3.10)) if we use the family of transversals

$$U_{\beta-\beta^*} = \mathrm{Ad}\, g_\beta : H_A^{0,2} \to g_\beta H_A^{0,2} g_\beta^{-1}.$$

In the simple case when $H^2(\mathrm{End}_0 \mathscr{E})$ vanishes for all points $[\mathscr{E}]$ in M_S (and also there are no reducible solutions) the moduli space M is smooth and cut out transversely by the ASD equations. The Kähler metric is then one of the generic metrics considered in §2. The "standard orientation" of M corresponding to the homology orientation $-1 \wedge \omega$ of X was chosen in [9, §III(c)] to agree with the complex orientation of M_S and we can calculate the contribution to Γ using the holomorphic universal family

$$(3.11) \qquad \hat\mu(M) = [M_S] \backslash c_2(\mathbb{E}).$$

So for the homology calculation we do not need to know the ASD connections explicitly. (Another way of saying this is that for any family T of indecomposable holomorphic bundles there is an associated homotopy class of maps $T \to \mathscr{B}^*$ defined by choosing any smooth family of Hermitian metrics on the bundles.) So we have:

Proposition (3.12). *If X is a simply-connected algebraic surface with $p_g(X) = 0$ and $H \to X$ is an ample line bundle such that all H-stable bundles $\mathscr{E}$ with $c_2(\mathscr{E}) = 1$, $\Lambda^2\mathscr{E} \cong \mathscr{O}_X$ have $H^2(\mathrm{End}_0 \mathscr{E}) = 0$ and there are no line bundles $\mathscr{L}$ with $\mathscr{L} \cdot H = 0$, $\mathscr{L} \cdot \mathscr{L} = -1$, then*

$$\Gamma_X(C) = 2[M_S] \backslash c_2(\mathbb{E}) + c_1(K_X),$$

where C is the chamber containing $c_1(H)$.

If $H^2(\mathrm{End}_0\,\mathscr{E})$ does not always vanish and the contribution to Γ is defined by the moduli space M' for a nearby generic metric, it is reasonably clear that the class $\hat{\mu}(M')$ can still be extracted from the ringed space $(M_S, \mathscr{O}_{M_S})$, together with the universal family over $M_S \times X$. Rather than attempt this in general however we shall stick to a simple example which arises in applications [12], [25]. Suppose there are no reducible solutions; for every point $[\mathscr{E}]$ in M_S we have $\dim H^2(\mathrm{End}_0\,\mathscr{E}) = q$, $\dim H^1(\mathrm{End}_0\,\mathscr{E}) = 1 + q$, and a local model ψ has the shape:

$$\psi(z_0, z_1, \cdots, z_q) = (z_1^{m_1}, \cdots, z_q^{m_q}).$$

Thus the reduced space M_S^{red} is a smooth Riemann surface, and M is identified with this as a point set.

Proposition (3.13). *In this case*

$$\Gamma_X(c) = 2\left(\prod_{i=1}^{q} m_i\right)[M_S^{\mathrm{red}}] \setminus c_2(\mathbb{E}) + c_1(K_X).$$

Proof. The vector spaces $H^2(\mathrm{End}_0\,\mathscr{E}) \cong H_A^2$ fit together to define a bundle $H^2 \to M$, which can be regarded by the choice of lifting as a subbundle of that defined by the Ω_+^2 spaces.

Similarly we can choose a family of q-dimensional subspaces of the $H^1(\mathrm{End}_0\,\mathscr{E})$'s, transverse to the zeros of the ψ's, defining a bundle $Q \to M$. Putting together the maps i on the fibers gives an embedding

$$I : (N \subset Q) \to \mathscr{B}^*$$

of a disc bundle N in Q. The local models combine to give a map:

$$\Phi : N \to H^2.$$

To calculate homology we need not restrict ourselves to the manifolds defined by ASD equations. If σ is any section of the bundle $\Omega_+^2 \to \mathscr{B}^*$ which is a "compact perturbation" of F^+ then the solutions of $F_A^+ + \sigma(A) = 0$ serve equally well (cf. [5], [9]). We consider sections σ which extend a fiber-preserving map

$$\varepsilon : N \to H_+^2$$

over $I(N)$. (More precisely, we use the transversals u to embed H_+^2 in Ω_+^2 over N and suppose these are chosen to be compatible with the holomorphic description, as above.) Then if σ is small the solutions of $F_A^+ + \sigma(A) = 0$ in $\mathscr{B}^*$ and of $\Phi + \varepsilon = 0$ in N correspond under I. Choosing ε in general position makes the zero set of $\Phi + \varepsilon$ a 2-manifold M^* and if ε is small this does not meet ∂N. Since N retracts onto M the homology class of $I(M^*)$ is a multiple $d[M]$, where $d = m_1 \cdots m_q$ is the degree of the map Φ on the boundary of a fiber of the disc bundle. Then extend ε over $\mathscr{B}^*$ to define σ.

(3.14) Remark. Although we have used abstract perturbations here it is possible to predict the behavior of the moduli space under a small change in the Riemannian structure—defined by $\mu: \Lambda^2_- \to \Lambda^2_+$, say (cf. [8, §VI]). For $[A]$ in M let $\varepsilon_\mu(A)$ be the projection of $\mu(F_A)$ to H^2_A. Then if ε_μ vanishes transversely on M and μ is small the zero set of $\Phi + \varepsilon_\mu$ in N will be diffeomorphic to the moduli space of the new metric.

(b) Calculations and corollaries.

(i) *Dolgachev surfaces.* Let $(f = 0)$, $(g = 0)$ be two general cubic curves in $\mathbf{CP}^2$ and Y be the algebraic surface obtained by blowing up the nine common zeros of f and g. From its definition Y is a rational surface and the underlying smooth oriented 4-manifold is the connected sum $\mathbf{CP}^2 \# 9\overline{\mathbf{CP}}^2$. At the same time Y is an "elliptic surface"—the rational map

$$f/g : \mathbf{CP}^2 \longrightarrow \mathbf{CP}^1$$

induces a holomorphic map $Y \to \mathbf{CP}^1$ whose fibers are copies of the cubic curves $(f + \lambda g = 0)$. By choosing f, g in general position we can arrange that the only singular cubics in this pencil are irreducible curves with one ordinary double point.

In [4] Dolgachev constructs a new complex surface Z by performing logarithmic transformations on two smooth fibers in Y. We will only consider the simplest case, when the multiplicities of the logarithmic transformations are 2 and 3 (for other cases see [13], [25]). So the new surface also admits a holomorphic map $Z \to \mathbf{CP}^1$ and, except for a pair of multiple fibers F_2, F_3, the fibers in Z can be identified with those in Y. The homology and linear equivalence classes of F_2, F_3 are related to those of a general smooth fiber F in Z by

$$[F] = 2[F_2] = 3[F_3]. \tag{3.15}$$

This Dolgachev surface Z gave a negative answer to the question of Severi mentioned in §I. We next outline a proof of this, together with a number of other standard facts.

Proposition (3.16). (i) *Z is an irrational algebraic surface with $p_g(Z) = 0$. Moreover there is an ample line bundle $H \to Z$ such that for any effective divisor D, not equivalent to a multiple of K_Z, $H \cdot D > H \cdot K_Z$.*

(ii) *The 4-manifold underlying Z is simply connected and homotopy equivalent, smoothly h-cobordant and homeomorphic to that underlying Y.*

Proof. Both parts of the proposition follow from the description of the canonical classes K_Y, K_Z in terms of the elliptic fibrations

$$\begin{aligned} K_Y &= -F, \\ K_Z &= -F + F_2 + F_3 = +\tfrac{1}{6}F = F_2 - F_3 \end{aligned} \tag{3.17}$$

(see [14, pp. 147, 187, 572]). Here we have made an abuse of notation, identifying the general fibers F in the two manifolds, and use the standard representation of line bundles by divisors. This means that $p_g(Z) = h^0(K_Z) = 0$ for the zero divisor of a section of K_Z would be a positive linear combination of F, F_2, F_3 (since all fibers are irreducible) but $\frac{1}{6}F$ cannot be represented in this way. Note in passing that K_Z generates the subgroup $\langle F, F_2, F_3 \rangle$ of $H_2(Z)$ and then an argument of Kodaira [19, Lemmas 2 and 6] shows that the homology class of K_Z is primitive.

To prove (i) we observe that if D is any effective divisor, then $F \cdot D \geqslant 0$ and $F \cdot D = 0$ if and only if D is equivalent to a combination of fibers. Choose a homology class c with $c \cdot K_Z = 1$. Replacing c by $c + nK_Z$ if necessary we can suppose $c^2 > 0$. Since $p_g(Z) = 0$, c is a $(1,1)$ class, represented by a holomorphic line bundle $\mathscr{L}$. The Riemann-Roch formula [2, p. 21] implies that when m is large $h^0(\mathscr{L}^m) + h^0(K_Z \otimes \mathscr{L}^{-m}) \gg 0$, but $F \cdot c_1(K_Z \otimes \mathscr{L}^{-m}) < 0$ so $h^0(K_Z \otimes \mathscr{L}^{-m}) = 0$ and $\mathscr{L}^m$ has a large space of holomorphic sections. Hence $c \cdot D \geqslant 0$ for any effective divisor D and this means that the bundle $H = \mathscr{L} \otimes K_Z^2$ has the desired properties. For $c_1(H)^2 > 0$ and $c_1(H) \cdot D = c \cdot D + 2K_Z \cdot D$ is positive because each term is nonnegative and if $K_Z \cdot D = 0$, $D \sim dK_Z$ and $c \cdot D > 0$. Thus the Nakai criterion (Proposition (3.2)) shows that H is ample, and if D is effective,

$$c_1(H) \cdot D \geqslant 2 > 1 = H \cdot K_Z.$$

Finally, X is certainly irrational since there is a holomorphic section of K_Z^2, vanishing on the fiber F_3.

For part (ii) of the proposition we refer to Kodaira's argument in [19] to show that, like Y, is simply connected. Then, since K_Z is nonzero in $H^2(Z, \mathbf{Z}/2)$ and represents $w_2(Z)$, the intersection form of Z is odd. So the assertions follow (using the classification of forms [28] and the theorems of Milnor [21], Wall [29], and Freedman [11]) if we show that Z has an intersection form of type $(1, 9)$. But the Hodge index formula gives

$$b_2^+(Z) = 1 + 2p_g(Z) = 1,$$

and Riemann-Roch (Noether's formula) yields

$$1 = 1 - 2b_1(Z) + P_g(Z) = \tfrac{1}{12}\left(c_1(Z)^2 + c_2(Z)\right).$$

So the Euler characteristic $c_2(Z)$ is 12 and $b_2(Z) = 10$, as required.

(3.18) Remark. Roughly speaking the logarithmic transformations change the complex geometry of the manifold by "changing the sign" of the canonical bundle. Positive multiples of K_Z are represented by effective divisors, so $K_Z \cdot H > 0$, while *negative* multiples of K_Y are effective. In general rationality of surfaces is linked to some positivity of the anti-canonical bundle, hence positivity of the curvature of Kähler metrics [17].

(ii) There is a general method which analyses the rank 2 holomorphic bundles $\mathscr{E}$ over a surface using the sections of twisted bundles $\mathscr{E} \otimes \mathscr{L}$ (see [14, pp. 726–731], [15], [24]). If a section vanishes at isolated points, the bundle can be recovered from local data associated to the points and global data in the cohomology of a line bundle. For the bundles we study in this paper the power of this method is increased by the following observation, which effectively goes back to Schwarzenberger [26].

Lemma (3.19). *If X is an algebraic surface with $q(X) = p_g(X) = 0$ and $\mathscr{E} \to X$ is a rank 2 holomorphic bundle with $c_2(\mathscr{E}) = 1$, $\Lambda^2(\mathscr{E}) \cong \mathscr{O}_X$, which is stable with respect to some ample line bundle, then $\mathscr{E} \otimes K_X$ has a nonzero holomorphic section.*

Proof. Riemann-Roch gives

$$h^0(\mathscr{E}) + h^2(\mathscr{E}) \geqslant \chi(\mathscr{E}) = 2\chi(\mathscr{O}) - c_2(\mathscr{E}) = 1.$$

But if $\mathscr{E}$ is stable, $h^0(\mathscr{E}) = 0$ (take $\mathscr{L} = \mathscr{O}_X$ in Definition (3.4)) so $h^2(\mathscr{E}) > 0$. Then, by Serre duality,

$$h^2(\mathscr{E}) = h^0\left(\mathscr{E}^* \otimes K_X\right) = h^0(\mathscr{E} \otimes K_X),$$

since the trivialization of Λ^2 defines an isomorphism $\mathscr{E} \cong \mathscr{E}^*$.

Corollary (3.20). *There are no stable bundles $\mathscr{E}$ with $c_2(\mathscr{E}) = 1$, $\Lambda^2(\mathrm{E}) \cong \mathscr{O}_Y$ over the rational surface Y.*

K_Y^{-1} has holomorphic sections (with zero divisors the fibers of $Y \to \mathbf{CP}^1$) so $K_Y \cdot H < 0$ for any polarization $H \to Y$ and Lemma (3.19) contradicts the stability of $\mathscr{E}$.

Theorem (3.21). *Fix an ample line bundle $H \to Z$ as in Proposition* (3.16). *For every H-stable bundle $\mathscr{E} \to Z$ with $c_2(\mathscr{E}) = 1$, $\Lambda^2\mathscr{E} \cong \mathscr{O}_Z$ we have $H^2(\mathrm{End}_0\,\mathscr{E}) = 0$. There is a holomorphic equivalence between the moduli space M_S of such bundles and the multiple fiber $F_2 \subset Z$. The second Chern class of the universal family $\mathbb{E} \to M_S \times Z$ is the Poincaré dual of the "diagonal" $\Delta_{F_2} \subset F_2 \times F_2 \subset F_2 \times Z$.*

Proof. Let s be a nonzero section of $\mathscr{E} \otimes K_Z$, as in Lemma (3.19). We see first that s has isolated zeros; for if it vanishes on an effective divisor D there would be a nontrivial map

$$K_Z^{-1} \otimes (D) \to \mathscr{E}$$

and, since $(K_Z - D) \cdot H < 0$, we would get a contradiction to the stability of $\mathscr{E}$. Then $c_2(\mathscr{E} \otimes K_Z) = c_2(\mathscr{E}) + c_1(K_Z)^2 = 1$ and the formula expressing c_2 as the sum over the zeros of multiplicities shows that there is just one zero and S vanishes transversely there.

In sum, the section s expresses the sheaf $\mathcal{O}(\mathcal{E})$ as an extension

$$0 \to \mathcal{O}\left(K_Z^{-1}\right) \xrightarrow{s\cdot} \mathcal{O}(\mathcal{E}) \to \mathcal{O}(K_Z) \otimes \mathcal{I}_z \to 0, \tag{3.22}$$

where $\mathcal{I}_z$ is the ideal sheaf of a point z in Z. Such extensions of sheaves are classified by a group $\mathrm{Ext}^1(\mathcal{I}_z \otimes K_Z, K_Z^{-1})$ which fits into an exact sequence:

$$\begin{array}{c} H^1\left(K_Z^{-2}\right) \to \mathrm{Ext}^1\left(\mathcal{I}_z \otimes K_Z, K_Z^{-1}\right) \to \left(K_Z^{-3}\right)_z \to H^0\left(K_Z^3\right)^*. \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \cong \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \mathbf{C} \end{array} \tag{3.23}$$

The last map of the sequence is the evaluation at z of a section of K_Z^3, and an element of $\mathrm{Ext}(\mathcal{I}_z \otimes K_Z, K_Z^{-1})$ gives a locally free sheaf as the middle term if and only if it does not map to zero in $(K_Z^{-3})_z$. $\mathcal{O}(\mathcal{E})$ is locally free so the 1-dimensional space of holomorphic sections of K_Z^3 constrains z to lie on their zero divisor $F_2 \subset Z$. Take the tensor product of (3.22) with K_Z to get

$$0 \to \mathcal{O} \to \mathcal{O}(\mathcal{E} \otimes K_Z) \to \mathcal{O}\left(K_Z^2\right) \otimes \mathcal{I}_z \to 0.$$

The space of global holomorphic sections of $\mathcal{O}(K_Z^2) \otimes \mathcal{I}_z$ vanishes (since F_2 and F_3 are disjoint) so the section S is unique up to scalars, and changing s to λs multiplies the extension class by λ.

Conversely, Riemann-Roch gives

$$h^1\left(K_Z^{-2}\right) = h^0\left(K_Z^3\right) - h^0\left(K_Z^{-2}\right) - 1 = 0,$$

so for each point z in F_2 we can construct a bundle $\mathcal{E}_z$, unique up to isomorphism, such that $\mathcal{E}_z \otimes K_Z$ has a section vanishing at z.

Next we see that all the bundles $\mathcal{E}_z$ are H-stable. Suppose $\mathcal{L}$ is a line bundle with $\mathcal{L} \cdot H \geqslant 0$ and $\alpha : \mathcal{L} \to \mathcal{E}_z$ is a holomorphic map. Then the composite $\mathcal{L} \to \mathcal{E}_z \to K_Z \otimes \mathcal{I}_z$ is zero, since otherwise $K_Z \otimes \mathcal{L}^{-1}$ is represented by an effective divisor D and so $(K_Z - (\mathcal{L})) \cdot H = D \cdot H > K_Z \cdot H$ (by the choice of H in Proposition (3.16)(i)), contradicting $\mathcal{L} \cdot H \geqslant 0$. Thus α factors through $K_Z^{-1} \xrightarrow{s} \mathcal{E}_z$. But $H \cdot (\mathcal{L} + K_Z) < 0$ so $H^0(\mathrm{Hom}(\mathcal{L}, K_Z^{-1})) = 0$ and α is identically zero. Hence $\mathcal{E}_z$ satisfies the condition for H-stability.

Put together this means that the moduli space M_S of an H-stable bundle can be identified with the Riemann surface F_2. It is also a reduced space, with $H^2(\mathrm{End}_0 \mathcal{E}_z) = 0$ for all z. This can be seen from the exact sequences

$$0 \to \mathcal{O}(K_Z) \to \mathcal{O}\left(\mathcal{E} \otimes K_Z^2\right) \to \mathcal{O}\left(K_Z^3\right) \otimes \mathcal{I}_z \to 0,$$

$$0 \to \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E} \otimes \mathcal{E} \otimes K_Z) \to \mathcal{O}\left(\mathcal{E} \otimes K_Z^2\right) \otimes \mathcal{I}_z \to 0.$$

The section ϕ of K_2^3 vanishes at z so the first sequence shows that $H^0(\mathcal{E} \otimes K_Z^2)$ is 1-dimensional but, since ϕ has a simple zero, $H^0(\mathcal{E} \otimes K_Z^2 \otimes \mathcal{I}_z) = 0$ and the second sequence shows that $H^0(\mathcal{E} \otimes \mathcal{E} \otimes K_Z) \cong H^2(\mathrm{End}\,\mathcal{E})^*$ is 0.

Finally, we can construct the universal family $\mathbb{E} \to F_2 \times Z$ by the 3-dimensional version of the same method. Let $\mathscr{I}_{\Delta_{F_2}} \subset \mathscr{O}_{F_2 \times Z}$ be the ideal sheaf of $\Delta_{F_2} = \{(z, z) \in F_2 \times Z \mid z \in F_2\}$. The normal bundle ν of Δ_{F_2} in $F_x \times Z$ is $\mathscr{O} \oplus \pi_2^*(K_Z^3)$ and $\Lambda^2 \nu \cong \pi_1^*(K_Z^3 \mid_{F_2})$. There is a class in

$$\mathrm{Ext}^1\left(\pi_2^*\left(K_Z^{-1}\right) \otimes \mathscr{I}_{\Delta_{F_2}}, \pi_1^*\left(K_Z^3 \mid_{F_2}\right) \otimes \pi_2^*\left(K_Z^{-1}\right)\right),$$

which gives a locally free extension

$$0 \to \pi_1^*\left(K_Z^3 \mid_{F_3}\right) \otimes \pi_2^*\left(K_Z^{-1}\right) \to \mathbb{E} \to \Pi_2^*\left(K_Z\right) \otimes \mathscr{I}_{\Delta_{F_2}} \to 0$$

of sheaves over $F_2 \times Z$ (see [24, pp. 93–107, and remark at top of page 111]). On each slice $\{z\} \times Z$, $\pi_1^*(K_Z^3)$ is trivial and this extension restricts to (3.22). So from this construction we see that $c_2(\mathbb{E}) = c_2(\pi_1^*(K_Z^{-3} \mid_{F_2}) \otimes \pi_2^*(K_Z) \otimes \mathscr{E})$ is Poincaré dual to the zero set Δ_{F_2}.

Combining these stable bundle calculations with our main theorems gives, in contrast to Proposition (3.16)(ii):

Theorem (3.24). *The Dolgachev surface Z is not diffeomorphic to the rational surface Y.*

To prove this we compare the invariants Γ_Y, Γ_Z. If C_Y is a chamber in $H^2(Y)$ containing the class of an ample line bundle, then

$$\Gamma_Y(C_Y) = c_1(K_Y)$$

since the moduli space is empty. On the other hand, if C_Z is a corresponding chamber containing the class of the ample bundle of Proposition (3.16)(i)

$$\Gamma_Z(C_Z) = c_1(K_Z) + 2\,\mathrm{P.D.}(F_2) = 7c_1(K_Z).$$

Here P.D. denotes the Poincaré dual and we have used the formula:

$$[F_2]/\mathrm{P.D.}\left(\Delta_{F_2} \subset F_2 \times Z\right) = \mathrm{P.D.}(F_2).$$

At this point there are a number of ways of seeing that there can be no diffeomorphism $f\colon Y \to Z$ compatible with the Γ-invariants. Perhaps the most natural is to use a *sign* attached to the invariant. (For much more general information see [13] and [25].) A theorem of Wall [30, Theorem 2] says that all the isometries of $H^2(Y; \mathbf{Z})$ are realized by diffeomorphisms of Y so these diffeomorphisms act transitively on the chambers in $H^2(Y)$ (consider the reflections in the walls W_e). Hence we may suppose $f^*(C_Z) = C_Y$. Then $\Gamma_Y(C_Y) \cdot \omega_Y < 0$ for any ω_Y in C_Y (because $\Gamma_Y(C_Y)$ is a null vector so the sign is constant on C_Y and if ω_Y is a Kähler class we can use Remark (3.18)); while $\Gamma_Z(C_Z) \cdot \omega_Z > 0$ for any ω_Z in C_Z (same reasoning). This contradicts the existence of f.

Appendix

We have worked with simply-connected 4-manifolds in this paper for convenience but the results extend easily to take account of fundamental groups. This follows the lines of [9] very closely so we will leave some details for the reader to fill in.

Suppose X^4 has a form of type $(1, n)$ and that $H_1(X; \mathbf{R}) = 0$. Let $\hat{A}$ be the finite group $\mathrm{Hom}(H_1(X; \mathbf{Z}), S^1)$ and write $|\hat{A}| = a + 2b$, where a is the number of elements z in $\hat{A}$ (or equivalently $H_1(X; \mathbf{Z})$) with $2z = 0$. We still have 2-dimensional moduli spaces $M(g)$ defined by ASD connections over X but the ends of these may be complicated by the presence of representations $\pi_1(X) \to SU(2)$. However, just as in [9, Corollary (2.11), Proposition (2.12), (2.13)] we can find a small perturbation σ' of the ASD equations defining a moduli space $M^{\sigma'}$ with boundary components corresponding only to the representation $\pi_1(X) \to \{\pm 1\}$. Regarded as $SO(3)$ connections these are all isomorphic so the boundary description is the same as in §II above. We can define classes $\Gamma(g, \omega, \sigma')$ in $H^2(X; \mathbf{Z})$, as before, so that

$$\Gamma(g, \omega, \sigma') = ac_1(K_Z) + 2\hat{\mu}\left(\left[M^{\sigma'}\right]\right)$$

in the case when ω is nowhere vanishing and $M^{\sigma'}$ is compact. (Here we have to extend formula (2.8) to the case when $H_1(X; \mathbf{Z}) \neq 0$.)

If the small perturbation σ', supported near the end of $M(g)$, is varied continuously in a 1-parameter family the cohomology class $\Gamma(g, \omega, \sigma')$ will be unaffected so long as $M^{\sigma'}(g)$ does not gain any more ends. The irreducible representations cause no problem here: their moduli space has virtual dimension -6 so, after perturbation of the equations, irreducible solutions can be avoided in a 1-parameter family. This gives the corresponding control of the ends of the spaces $M^{\sigma^*}(g)$, as in [9, §II(d)]. On the other hand, at the b remaining abelian reductions there are two distinct "directions" in which the ASD equations can be deformed, corresponding to a choice of bundle "L" in the splitting $L \oplus L^{-1}$. (This is the sign of $E(\varepsilon)$ in the discussion of [9, Proposition (2.12)].)

To see this clearly suppose that $\pi_1(X)$ is abelian and that for each flat connection $L \oplus L^{-1}$, $H^2_+(X; L^2)$ is two dimensional. (After perturbation we reach this picture anyway.) Then Theorem (5.5) of [8] describes the corresponding end of $M(g)$ as a subset of $\Sigma \times (0, \varepsilon)$, where $\Sigma \to X$ is the 2-sphere bundle

$$\Sigma = \mathrm{Hom}\left(\mathbf{R} \oplus L^2, \Lambda^2_{+,X}\right)/S^1.$$

Under the map $\tau_L : \Sigma \times (0, \varepsilon) \to \mathscr{B}^*$ defined by Taubes construction, the end of $M(g)$ corresponds to the zero set

$$Z = \{f = s_1 = s_2 = 0\} \subset \Sigma \times (0, \varepsilon).$$

Here f is a real valued function and s_1, s_2 are sections of the complex line bundle over $\Sigma \times (0, \varepsilon)$ associated to the circle action on $\mathrm{Hom}(\mathbf{R} \oplus L^2, \Lambda^2_+)$. This is $T \otimes L^{-2}$ where T is the standard line bundle over Σ, of degree 2 on the fibers. Now for $\delta \neq 0$ consider the perturbed equations

$$\{f = \delta,\ s_1 = s_2 = 0\}.$$

These close off the end of the moduli space and the difference in homology between the two cases $\delta > 0$, $\delta < 0$ is clearly given by the homology class of $\{s_1 = s_2 = 0\}$ in $\Sigma \times \{\varepsilon/2\}$, i.e.,

$$2\hat{\mu}\tau_{L*}(\phi) \in H^2(X; \mathbf{Z}),$$

where ϕ in $H_2(\Sigma)$ is the Poincaré dual of $c_1(T \otimes L^{-2})$. One can compute that $\hat{\mu}\tau_{L*}(\phi)$ is a multiple, $m \cdot c_1(L)$ say, in $H^2(X; \mathbf{Z})$.

So if we choose a line bundle L in the splitting and the perturbation σ', corresponding to $\delta > 0$ say, the class

$$\Gamma(g, \omega) = \Gamma(g, \omega, \sigma') + mc_1(L)$$

is independent of the choice $L^{\pm 1}$. That is, we take the "average" of the two possibilities.

In this way we associate a cohomology class to the perturbed moduli space. As the metric varies there are now $|\hat{A}|$ reductions at which singularities appear each time the period points cross a wall. Formula (2.14) extends to the case when $H_1(X) \neq 0$. In sum we obtain:

Theorem (4.1). *If X is a smooth, compact, oriented, 4-manifold with $H_1(X; \mathbf{Z}) = 0$, $b_2^+(X) = 1$, then the construction above defines a map*

$$\Gamma_X : \mathscr{C}_X \to H^2(X; \mathbf{Z})$$

with properties (i), (ii), (iii) *of* (2.2), *where in* (ii) *we take the sum over the $|\hat{A}|$ elements of $H^2(X; \mathbf{Z})$ associated to a given wall between the chambers.*

Example. Let S be a "Godeaux surface" [2, p. 170]: the quotient of $\{\sum_{i=1}^4 z_i^5 = 0\} \subset \mathbf{CP}^3$ under the $\mathbf{Z}/5$ action generated by $g(z_1, z_2, z_3, z_4) = (\alpha z_1, \alpha^2 z_2, \alpha^3 z_3, \alpha^4 z_4)$ $(\alpha^5 = 1)$. We take the Hodge metric on S defined by the Fubini-Study metric on $\mathbf{P}^3$, with ample line bundle K_S. $\pi_1(S)$ and the torsion part of $H^2(S)$ are isomorphic to $\mathbf{Z}/5$; let L be the flat complex line bundle corresponding to the representation sending g to α. The four line bundles $K_S \otimes L^i$, $i = 1, 2, 3, 4$, each have one-dimensional spaces of sections—corresponding to the coordinate functions z_i. These cut out the curves

$$c_i = \left\{ z_i = 0,\ \sum_{j=1}^4 z_j^5 = 0 \right\} / \mathbf{Z}/5,$$

and each pair c_i, c_j meets transversely in a single point.

If $\mathscr{E} \to S$ is a stable bundle, with $\Lambda^2\mathscr{E} = \mathscr{O}_S$ and $c_2(\mathscr{E}) = 1$, arguments like those in §III above show that $H^0(\mathscr{E} \otimes K_S)$ is one-dimensional and a generator vanishes transversely at the points

$$p_1 = c_1 \cap c_4, \qquad p_2 = c_2 \cap c_3.$$

The relevant extension group is isomorphic to $(K_S^{-3})_{p_1} \oplus (K_S^{-3})_{p_2} \cong \mathbf{C}^2$ and a pair (λ_1, λ_2) gives a bundle if each component λ_i is nonzero. Homothetic vectors in $\mathbf{C}^2$ give isomorphic bundles. Conversely one finds that all these bundles are stable and the moduli space M_S is a copy of $\mathbf{C}^*$ (and is reduced).

This fits in with what we know about the ends of the ASD moduli space. The complex structure $\Lambda^2_+ \cong K_S \oplus \mathbf{R}$ defines two preferred sections $S_+, S_- \subset \Sigma$ (poles of the 2-spheres) and an equatorial S^1 bundle $E \subset \Sigma \xrightarrow{\pi} S$. The homogeneous approximation to the equations $s_1 = s_2 = 0$ modelling the end [8, Lemma 5.4] associated to the reduction $L \oplus L^{-1}$ has solutions:

$$2\left[S_+ \cap \pi^{-1}(c_2)\right] \cup 2\left[S_- \cap \pi^{-1}(c_3)\right] \cup \pi^{-1}(p_2).$$

Here the factor 2 indicates that the surfaces are defined with multiplicity 2. In fact the components lying over c_2, c_3 correspond to "semistable" bundles defined as extensions

$$L^{-1} \to \mathscr{E} \to L \otimes \mathscr{I}_z, \qquad L^{-2} \to \mathscr{E} \to L^2 \otimes \mathscr{I}_z$$

respectively.

The extra equation $(f = 0)$ cuts out the equatorial bundle E and the intersection $E \cap \pi^{-1}(p_2)$ corresponds to one of the ends of $M_S \cong \mathbf{C}^*$. Similarly for the other end, associated to the reduction $L^2 \oplus L^{-2}$.

Let c be the singular curve in the product $\mathbf{P}_1 \times S$:

$$c = \mathbf{P}_1 \times \{p_1\} \cup \mathbf{P}_1 \times \{p_2\} \cup \{0\} \times c_2 \cup \{\infty\} \times c_1.$$

There is an isomorphism

$$K_c \cong K_{\mathbf{P}_1 \times S} \otimes \mathscr{O}_c \otimes \left(\mathscr{O}(1) \boxtimes K_S^2\right),$$

and this defines a bundle $\mathbb{F} = \mathbb{E} \otimes K_S$ over $\mathbb{P}_1 \times S$ with a section vanishing transversely on c. $\mathbb{E}$ restricts to $\mathbf{C}^* \times S \subset \mathbb{P}_1 \times S$ as the universal bundle over the moduli space and restricts to $\{0\} \times S$, $\{\infty\} \times S$ as the semistable bundles

$$L \to \mathscr{E} \to L^{-1} \otimes \mathscr{L}_{p_2}, \qquad L^2 \to \mathscr{E} \to L^{-2} \otimes \mathscr{L}_{p_1}$$

respectively. This corresponds to compactifying the moduli space by deforming the ends into the hemispheres $(f > 0) \subset \pi^{-1}(p_1)$, $\pi^{-1}(p_2)$. There are three different contributions to Γ; the one from the moduli space is

$$2([\mathbf{P}_1] \backslash c_2(\mathbb{E}) = 2c_1(K_S) + 6c_1(L);$$

averaged over the choice $L^{\pm 1}$ gives $2c_1(K_S)$. We get $2 \times 4c_1(K_S)$ from the zero sets of S_i in Σ and $c_1(K_S)$ for the boundary correction term.

So if C is the Chamber containing $c_1(K_S)$ we have

$$\Gamma(C) = 11c_1(K_S).$$

(4.2) Corollary. *There is no diffeomorphism $f: S \to S$ such that $f^*(K_S)$ is isomorphic to $K_S \otimes L^i$ for $i = 1, 2, 3, 4$.*

I. Hambleton and M. Kreck have recently shown that these maps do exist as homeomorphisms.

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