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CONSTRUCTION OF SINGULAR HOLOMORPHIC VECTOR FIELDS AND FOLIATIONS IN DIMENSION TWO

ALCIDES LINS NETO

0. Introduction

In this paper we construct holomorphic differential equations or foliations in two different situations:

Case 1. *Singularities of vector fields* (local case).

Case 2. *Ricatti foliations in* $\overline{\mathbf{C}}\times\overline{\mathbf{C}}$ (global case).

In Case 1 we consider singular vector fields defined in a neighborhood U of $0 \in \mathbf{C}^2$. Suppose that 0 is an isolated singularity of X. In this case, as is well known, the singularity can be solved by a finite number of blowing-ups (cf. [4], [5], and [12]). Let us consider for simplicity the case where X is solved by one blowing-up. After blowing-up $0 \in \mathbf{C}^2$, we obtain a complex line bundle $\tilde{\mathbf{C}}^2 \to \overline{\mathbf{C}}$, $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, a proper projection $\pi\colon \tilde{\mathbf{C}}^2 \to \mathbf{C}^2$, and a singular holomorphic foliation $\mathscr{F}$ on $\tilde{U} = \pi^{-1}(U)$, where:

(i) $\pi^{-1}(0) = \overline{\mathbf{C}}$, the zero section of $\tilde{\mathbf{C}}^2$, and $\pi\colon \tilde{\mathbf{C}}^2 - \overline{\mathbf{C}} \to \mathbf{C}^2 - \{0\}$ is a diffeomorphism.

(ii) π sends nonsingular leaves of $\mathscr{F}$ in $\tilde{U} - \overline{\mathbf{C}}$ onto integral curves of the complex differential equation $\dot{x} = X(x)$. The singularities of $\mathscr{F}$ are in $\overline{\mathbf{C}}$ and are all simple (cf. §1.1 for the definition). Set S = set of singularities of $\mathscr{F}$.

In some cases (nondicritical cases) $\overline{\mathbf{C}}$ is invariant by $\mathscr{F}$, that is, $\overline{\mathbf{C}}$ is the union of S and a leaf of $\mathscr{F}$, $\overline{\mathbf{C}}-S$. Therefore it is possible to consider the holonomy group of the leaf $\overline{\mathbf{C}}-S$ (in some transversal section). This group is called the projective holonomy of the singularity and we denote it by $\mathscr{H}(\mathscr{F})$. In §2 of this paper we prove a slightly more general version of the following result.

Theorem 1. *Let $G = \{g_1,\cdots,g_k\}$ be a set of germs at $0 \in \mathbf{C}$ of holomorphic diffeomorphisms which leave 0 fixed and such that $g_1,\cdots,g_k$ and $g_1 \circ \cdots \circ g_k$ are linearizable, not necessarily in the same coordinate system. Then*

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there is a germ of vector field X, with a singularity at $0 \in \mathbf{C}^2$, such that its projective holonomy is conjugated (holomorphically) to the group generated by G.

The proof of this theorem is based in a theorem of Grauert (cf. [1]) and a construction done in §2.3. In §2.4 we prove a generalization of Theorem 1 for several blowing-ups. I wish to thank D. Cerveau, who motivated me in the problem solved by Theorem 1, and R. Moussu, who told me about Grauert's theorem, which simplified a lot the original version of the proof.

In Case 2 we consider Ricatti equations in the form

$$(1) \qquad \frac{dx}{dT} = p(x), \qquad \frac{dy}{dT} = a(x) + b(x)y + c(x)y^2,$$

where p, a, b, and c are polynomials, $(x, y) \in \mathbf{C}^2$, and T is a complex time. Let $\tilde{\mathscr{F}}$ be the singular foliation on $\mathbf{C}^2$ whose leaves are the solutions of (1). It is clear that the vertical $\{x\} \times \mathbf{C}$ is invariant for $\tilde{\mathscr{F}}$ if and only if $p(x) = 0$. If $p(x) \neq 0$, then the vertical $\{x\} \times \mathbf{C}$ is transverse to $\tilde{\mathscr{F}}$. On the other hand the change of variables $v = 1/y$ transforms (1) into

$$(1') \qquad \frac{dx}{dT} = p(x), \qquad \frac{dv}{dT} = -a(x)v^2 - b(x)v - c(x)$$

which implies that $\tilde{\mathscr{F}}$ extends to a foliation $\hat{\mathscr{F}}$ on $\mathbf{C} \times \overline{\mathbf{C}}$. Clearly $\hat{\mathscr{F}}$ is transverse to all fibers $\{x\} \in \overline{\mathbf{C}}$ such that $p(x) \neq 0$. Since p, a, b, and c are polynomials, $\hat{\mathscr{F}}$ can be extended to a foliation $\mathscr{F}$ in $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$. This goes as follows: the change of variables $u = 1/x$ transforms equation (1) into

$$(2) \qquad \frac{du}{dT} = -u^2 p\left(\frac{1}{u}\right), \qquad \frac{dy}{dT} = a\left(\frac{1}{u}\right) + b\left(\frac{1}{u}\right)y + c\left(\frac{1}{u}\right)y^2.$$

Let $d = \max\{\mathrm{dg}(a), \mathrm{dg}(b), \mathrm{dg}(c), \mathrm{dg}(p) - 2\}$ (dg = degree). If we multiply the vector field associated to (2) by u^d, we obtain a new Ricatti equation without poles, which extends $\hat{\mathscr{F}}$ to a neighborhood of $\{x = \infty\} \subset \overline{\mathbf{C}} \times \overline{\mathbf{C}}$. Observe that the line $\{x = \infty\}$ is invariant by $\mathscr{F}$ if and only if $\mathrm{dg}(p) < d + 2$. We call $\mathscr{F}$ a *Ricatti foliation* on $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$. The fibers $\{x\} \times \overline{\mathbf{C}}$, where $p(x) = 0$ (or $\{\infty\} \times \overline{\mathbf{C}}$ if $\mathrm{dg}(p) < d + 2$) are called the *invariant fibers*. We say that an invariant fiber $\{x\} \times \overline{\mathbf{C}}$ is *simple* if x is a simple root of $p(x)$.

Let $S = p^{-1}(0)$ or $S = p^{-1}(0) \cup \{\infty\}$ if $\mathrm{dg}(p) < p + 2$. Since $\mathscr{F}$ is transverse to the fibers of $(\overline{\mathbf{C}} - S) \times \overline{\mathbf{C}} \xrightarrow{P_1} \overline{\mathbf{C}} - S$, it follows that we can define a global holonomy in some transverse section $\{q\} \times \overline{\mathbf{C}}$, $q \notin S$. This holonomy is a representation of $\pi_1(\overline{\mathbf{C}} - S, q) \mapsto \mathrm{Diff}(\{q\} \times \overline{\mathbf{C}})$ and it is defined as follows: Take a curve $\gamma \in \pi_1(\overline{\mathbf{C}} - S, q)$ and a point $(q, y) \in \{q\} \times \overline{\mathbf{C}}$. Lift γ to a curve γ_y, contained in the leaf L_y of $\mathscr{F}$ through (q, y), and such that $\gamma_y(0) = (q, y)$ and $P_1(\gamma_y(t)) = \gamma(t)$. Define $f_\gamma(q, y) = \gamma_y(1)$. It can be verified that f_γ is a diffeomorphism of $\{q\} \times \overline{\mathbf{C}}$ which depends only on the homotopy class of γ in $\pi_1(\overline{\mathbf{C}} - S, q)$. Moreover $\gamma \mapsto f_\gamma$ is a homomorphism of groups. In our case, since f_γ is holomorphic, it follows that f_γ is a Moebius transformation of the fiber $\{q\} \times \overline{\mathbf{C}}$.

In §3 we prove the following result.

Theorem 3. *Let $f_1, \cdots, f_k$ be Moebius transformations, where $k \geqslant 1$. Let $x_0, \cdots, x_k$ be $k+1$ points in $\overline{\mathbf{C}}$, where $k \geqslant 1$. There exists a Ricatti foliation $\mathscr{F}$ on $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$ with the following properties*:

(i) *The invariant fibers of $\mathscr{F}$ are $\{x_0\} \times \overline{\mathbf{C}}, \cdots, \{x_k\} \times \overline{\mathbf{C}}$. If one of the f_j's is not parabolic, then these invariant fibers are simple.*

(ii) *The holonomy of $\mathscr{F}$ is conjugated to the subgroup of* $\mathrm{PSL}(2, \mathbf{C})$ *generated by $f_1, \cdots, f_k$.*

(iii) *If $f_1, \cdots, f_k$ and $f_0 = (f_1 \circ \cdots \circ f_k)^{-1}$ are not parabolic or elliptic, then all the singularities of $\mathscr{F}$ are of Poincaré type.*

We say that a singularity p of $\mathscr{F}$ is of Poincaré type if $\mathscr{F}$ can be defined in a neighborhood of p by a vector field X such that the eigenvalues λ_1, λ_2 of $DX(p)$ satisfy $\lambda_1/\lambda_2 \notin \mathbf{R}$.

The proof of this theorem is based on the classification of fiber bundles over $\overline{\mathbf{C}}$ with fiber $\overline{\mathbf{C}}$ and in a construction sketched in §3.1, which is in fact a slight modification of the construction in §2.3.

In §4 we apply Theorem 3 to study some aspects of the structural stability problem for singular foliations on $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$.

I should say that, after writing this paper, J. P. Ramis pointed out that Theorem 3 can be proved from the results of Birkhoff about linear differential equations in [2] and [3]. Nevertheless, I decided to include it here since the method for the construction is almost the same as the one we use for constructing the singularities in §2.

I wish to thank C. Camacho and X. Gomez-Mont for helpful conversations and ideas about Case 2.

1. The blowing-up method and preliminary results for Case 1

1.1. The blowing-up method. Let $Z(x, y) = A(x, y)\partial/\partial x + B(x, y)\partial/\partial y$ be a holomorphic vector field defined in an open set $U \subset \mathbf{C}^2$, such that $0 \in U$ and 0 is a singularity of Z, i.e., $Z(0) = 0$. We say that 0 is a *simple* singularity of Z if the eigenvalues λ_1, λ_2 of its linear part at 0 satisfy one of the following conditions:

$$\lambda_1 \cdot \lambda_2 \neq 0 \quad \text{and} \quad \lambda_1/\lambda_2 \notin \mathbf{Q}_+, \tag{3}$$

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 \neq 0. \tag{4}$$

By definition, the multiplicity of Z at 0 is the order of the first nonzero jet of Z at 0.

The blow-up of $0 \in \mathbf{C}^2$ consists in replacing 0 by a one-dimensional projective line P, the set of complex directions at 0. The total space $\mathbf{C}^2$ is then replaced by a line bundle $\tilde{\mathbf{C}}^2$, whose zero section is P and such that $\tilde{\mathbf{C}}^2 - P$ is diffeomorphic to $\mathbf{C}^2 - \{0\}$. Formally this goes as follows: $\tilde{\mathbf{C}}^2$ is covered by two coordinate charts $((t, x), E_0)$ and $((s, y), E_\infty)$, where $E_0 = \tilde{\mathbf{C}}^2 - l_\infty$, $E_\infty = \mathbf{C}\tilde{\mathrm{H}}^2 - l_0$, l_0 is the y-axis, and l_∞ the x-axis. In the first chart the fibers of $\tilde{\mathbf{C}}^2$ are represented by the lines $t =$ constant and in the second by the lines $s =$ constant. The change of coordinates from the first chart to the second is given by $s = 1/t$, $y = tx$, so that the Chern class of this bundle is -1, as can be easily verified. The projection $\pi\colon \tilde{\mathbf{C}}^2 \to \mathbf{C}^2$ is defined by $\pi(t, x) = (x, t \cdot x)$ in the first chart and by $\pi(s, y) = (s \cdot y, y)$ in the second. This projection sends fibers of $\tilde{\mathbf{C}}^2$ into lines passing through the origin of $\mathbf{C}^2$ and its restriction to $\tilde{\mathbf{C}}^2 - P$ is a diffeomorphism onto $\mathbf{C}^2 - \{0\}$.

Now, let $\mathscr{F}$ be the singular foliation in $U - \{0\}$ whose leaves are the integral curves of the vector field Z. Let $\mathscr{F}^* = \pi^*(\mathscr{F})$ be the coinduced foliation on $\pi^{-1}(U) - P$. It is not difficult to prove that $\mathscr{F}^*$ extends to a singular foliation on $\tilde{U} = \pi^{-1}(U)$ with a finite number of singularities, all of them in P (cf. [4], [5], and [12]). We denote this extended foliation by $\mathscr{F}^{(1)}(Z)$. Two situations can happen:

(i) *Non dicritical case—P is invariant for $\mathscr{F}^{(1)}(Z)$.* In this case, if we denote by S the set of singularities of $\mathscr{F}^{(1)}(Z)$, then $P - S$ is a leaf of $\mathscr{F}^{(1)}(Z)$.

(ii) *Dicritical case—P is not invariant for $\mathscr{F}^{(1)}(Z)$.* In this case $\mathscr{F}^{(1)}(Z)$ is transverse to P, except in a finite number of points. Some of these tangency points are singularities.

The foliation $\mathscr{F}^{(1)}(Z)$ can be expressed near each singularity by a holomorphic vector field (cf. [12]). Therefore the process can be repeated in a neighborhood of each singularity. If we do this, a new foliation $\mathscr{F}^{(2)}(Z)$ is found in a neighborhood of a union of projective lines having normal crossings. The foliation $\mathscr{F}^{(2)}(Z)$ has again a finite number of singularities. The process can be repeated as long as we want, so that after k blowing-ups we have a foliation $\mathscr{F}^{(k)}(Z)$ defined in a neighborhood $U^{(k)}$ of a union $\mathscr{P}^{(k)}$ of projective lines having normal crossings. Moreover the process gives us a proper analytic projection $\pi^{(k)}\colon U^{(k)} \to U$ such that $\pi^{(k)}(\mathscr{P}^{(k)}) = \{0\}$ and $\pi^{(k)}\colon U^{(k)} - \mathscr{P}^{(k)} \to U - \{0\}$ is a holomorphic diffeomorphism which sends leaves of $\mathscr{F}^{(k)}(Z)$ onto integral surfaces of Z. We will write $(U^{(k)}, \pi^{(k)}, \mathscr{P}^{(k)}, \mathscr{F}^{(k)}(Z))$ to denote a sequence of k blowing-ups, beginning at $0 \in \mathbf{C}^2$. The map $\pi^{(k)}$ will be called the blowing-up projection and $\mathscr{P}^{(k)}$ its divisor. The divisor $\mathscr{P}^{(k)}$ is a union of projective lines such that two of them intersect transversally in at most one point, called a *corner*.

We observe that when 0 is a simple singularity of Z, then all singularities of $\mathscr{F}^{(k)}(Z)$ are also simple, so that we shall consider a simple singularity as a final object in the blowing-up method. A remarkable fact about this method is the following:

Desingularization theorem [14]. *Let $0 \in \mathbf{C}^2$ be a singularity of a vector field Z. Then there exists a blowing-up $(U^{(k)}, \pi^{(k)}, \mathscr{P}^{(k)}, \mathscr{F}^{(k)}(Z))$ of Z at 0, such that all singularities of $\mathscr{F}^{(k)}(Z)$ are simple.*

Here we are more interested in constructing the vector field Z from the foliation $\mathscr{F}^{(k)}$. In this direction we have the following

Proposition 1. *Let U be an open polidisk with $0 \in U \subset \mathbf{C}^2$ and $\mathscr{F}$ be a holomorphic foliation defined in $U - \{0\}$. Then there exists a vector field Z in U with at most one singularity at 0 and such that the integral surfaces of Z in $U - \{0\}$ are the leaves of $\mathscr{F}$.*

The following corollary follows easily from Proposition 1.

Corollary. *Let $(U^{(k)}, \pi^{(k)}, \mathscr{P}^{(k)})$ be a sequence of k blowing-ups beginning at $0 \in \mathbf{C}^2$, where $\pi^{(k)}(U^{(k)}) = U$ is a neighborhood of 0 and $\pi^{(k)}(\mathscr{P}^{(k)}) = \{0\}$. Suppose that $\hat{\mathscr{F}}$ is a singular holomorphic foliation in $U^{(k)}$, whose singularities are in $\mathscr{P}^{(k)}$. Then there exists a vector field Z in U such that $\mathscr{F}^{(k)}(Z) = \hat{\mathscr{F}}$, where $\mathscr{F}^{(k)}(Z)$ is as before.*

Proof. Since $\pi^{(k)}\colon U^{(k)} - \mathscr{P}^{(k)} \to U - \{0\}$ is a diffeomorphism, then $\mathscr{F} = \pi_*(\hat{\mathscr{F}})$ is a foliation of $U - \{0\}$. Now apply Proposition 1 to $\mathscr{F}$.

1.2. Proof of Proposition 1. Given a point $p \in U - \{0\}$, there exist a neighborhood $V \subset U - \{0\}$ of p and a vector field $Z^p = A\,\partial/\partial x + B\,\partial/\partial y$ in V, whose integral surfaces are the leaves of $\mathscr{F}$ in V. Let $f\colon V \to \overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ be the *slope* of Z^p, $f(q) = B(q)/A(q)$. Since $V \subset U - \{0\}$, for any $q \in V$ we have $A(q) \neq 0$ or $B(q) \neq 0$. Hence $f\colon V \to \overline{\mathbf{C}}$ is a well-defined holomorphic function. Now if $f\colon V \to \overline{\mathbf{C}}$ and $f'\colon V' \to \overline{\mathbf{C}}$ are the slopes of $\mathscr{F}$ in V and V', where $V \cap V' \neq \varnothing$, then clearly $f \equiv f'$ in $V \cap V'$. Therefore the slope function $f\colon U - \{0\} \to \overline{\mathbf{C}}$ of $\mathscr{F}$ is well defined and holomorphic. It follows from Levi's extension theorem (cf. [8]) that there exist holomorphic functions $P, Q\colon U \to \mathbf{C}$ such that $f(q) = Q(q)/P(q)$ for any $q \in U - \{0\}$. Now, it is not difficult to see that the leaves of $\mathscr{F}$ will be the integral curves of the holomorphic vector field $Z = P\,\partial/\partial x + Q\,\partial/\partial y$.

2. Construction of nondicritical singularities

Let Z be a holomorphic vector field defined in a neighborhood U of $0 \in \mathbf{C}^2$ and such that 0 is an isolated singularity of Z. Suppose that the first blowing-up of Z at 0, say $(\tilde{U}, \pi, P, \mathscr{F}^{(1)})$, is nondicritical. In this case as we saw before, if $S \subset P$ is the set of singularities of $\mathscr{F}^{(1)}$, then $P - S$ is a leaf of

$\mathscr{F}^{(1)}$ and so it makes sense to consider the holonomy of $P - S$, with respect to a transversal section Σ, where $\Sigma \cap (P - S) = \{p_0\}$. This holonomy is a representation of $\pi_1(P - S, p_0)$ in the group of germs of transformations of Σ which leaves p_0 fixed, defined as follows: Let $[\gamma] \in \pi_1(P - S, p_0)$ and γ be a loop whose class in $\pi_1(P - S, p_0)$ is $[\gamma]$. Let $\rho\colon \tilde{\mathbf{C}}^2 \to P$ be the projection of the bundle defined by the first blowing-up. If $p \in \mathbf{C}$ is near p_0, then we can lift γ to a curve γ_p contained in the leaf of $\mathscr{F}^{(1)}$ which passes through p and such that $\rho \circ \gamma_p = \gamma$. The endpoint of γ_p will depend only of $[\gamma]$ and will be denoted by $[\gamma](p)$. The correspondence $p \mapsto [\gamma](p)$ is a holomorphic diffeomorphism between two neighborhoods of p_0 in Σ. Moreover if $[\alpha], [\beta] \in \pi_1(P - S, p_0)$, then $([\alpha] * [\beta])(p) = [\alpha]([\beta](p))$, if both members are defined, where $*$ is the product in $\pi_1(P - S, p_0)$.

Now suppose that $S = \{p_1, \cdots, p_{k+1}\}$ (observe that $k \geqslant 0$). In this case $\pi_1(P - S, p_0)$ is a free group with k generators. Hence the holonomy of $P - S$ at Σ is generated by k germs $f_1, \cdots, f_k\colon (\Sigma, p_0) \to (\Sigma, p_0)$, corresponding to the k generators of $\pi_1(P - S, p_0)$.

Here we prove the following result.

Theorem 1. *Let $g_1, \cdots, g_k$ be germs at $0 \in \mathbf{C}$ of holomorphic diffeomorphisms which leave 0 fixed. Suppose that for any $j \in \{1, \cdots, k\}$, g_j is conjugated with its linear part at 0, $z \mapsto g_j'(0) \cdot z$. Suppose that the composition $g_0 = g_k^{-1} \circ \cdots \circ g_1^{-1}$ is also linearizable. Let $l_1, \cdots, l_{k+1}$ be distinct complex lines through $0 \in \mathbf{C}^2$. Then there exists a germ at $0 \in \mathbf{C}^2$ of holomorphic vector field Z which satisfies the following properties*:

(i) *Z has exactly $k + 1$ analytic invariant manifolds, which are contained in the l_j's.*

(ii) *Z is solved after one blowing-up, which is nondicritical, and the projective holonomy of $\mathscr{F}^{(1)}(Z)$ is conjugated to the group of germs generated by $g_1, \cdots, g_k$.*

(iii) *The multiplicity of Z at 0 is k.*

(iv) *$\mathscr{F}^{(1)}(Z)$ has $k + 1$ singularities in the divisor and all such singularities are linearizable.*

Remark 1. The case where some of the g_i's are periodic is not excluded in the construction.

Remark 2. The same theorem (without (iv)) can be proved when $g_0, g_1, \cdots, g_k$ can be realized as local holonomies of nondegenerated singularities, in the following sense: We say that the germ $g\colon (\mathbf{C}, 0) \to (\mathbf{C}, 0)$ can be realized as local holonomy of a nondegenerated singularity if there exists a differential equation in a ball B, $0 \in B \subset \mathbf{C}^2$,

$$\begin{aligned} \dot{x} &= \lambda_1 x(1 + R_1(x, y)), \\ \dot{y} &= \lambda_2 y(1 + R_2(x, y)) \end{aligned} \tag{5}$$

such that:

(i) $\lambda_1, \lambda_2 \neq 0$ and $R_1(0,0) = R_2(0,0) = 0$.

(ii) $(0,0)$ is the unique singularity of (5) in B.

(iii) The holonomy of the invariant manifold $\{y = 0\} \cap B - \{(0,0)\}$ in some transversal section $\Sigma = \{(x_0, y);\ |y| < \delta\}$ is analytically conjugated to g.

The hypothesis in Theorem 1 that $g_0, \cdots, g_k$ are linearizable implies that each g_j is realizable as a local holonomy of a nondegenerated singularity.

Remark 3. The construction that will be done in §2.3 for the proof of Theorem 1 can be applied also to prove the following result:

Let M be a compact Riemann surface and $S = \{p_0, p_1, \cdots, p_k\} \subset M$, $k \geqslant 1$. Let $\{g_1, \cdots, g_k\}$ be as in the hypothesis of Theorem 1. Let $l \in \mathbf{Z}$. Then there exist a complex 2-dimensional manifold $V \supset M$ and a singular foliation $\mathscr{F}$ on V such that:

(i) The singular set of $\mathscr{F}$ is S and these singularities are linearizable.

(ii) $M - S$ is a leaf of $\mathscr{F}$.

(iii) The holonomy of $M - S$ with respect to $\mathscr{F}$ is conjugated to the group generated by $\{g_1, \cdots, g_k\}$.

(iv) The Chern class of the normal bundle of M in V is l.

At the end of §2.3 we will indicate how to prove this result from the construction.

We observe that, although the C^∞ structure of V is determined completely by l, we have no control on its holomorphic structure (unless in the special case $M = \overline{\mathbf{C}}$ and $l < 0$).

2.1. Preliminaries for the proof of Theorem 1. The proof of Theorem 1 will be based in Proposition 1 and in the following theorem due to Grauert [1]:

Theorem. *Let M^2 be a complex manifold of dimension 2 and $S \subset M^2$ be a compact Riemann surface. Suppose that the Chern class of the normal bundle of S is negative. Let $(TS)^\perp$ be the normal bundle of S in M and S_0 be the null section of $(TS)^\perp$. Then there are neighborhoods V of S in M and W of S_0 in $(TS)^\perp$ which are diffeomorphic by a holomorphic diffeomorphism $\varphi\colon V \to W$ such that $\varphi(S) = S_0$.*

Now let $S \subset M$ be a Riemann surface of genus 0 and suppose that its Chern class is -1. Since the normal bundle $(TS)^\perp$ is linear and has Chern class -1, it follows that $(TS)^\perp$ is equivalent to the bundle $\tilde{\mathbf{C}}^2 \to P$, obtained by blowing-up at $0 \in \mathbf{C}^2$ (cf. [8]). The equivalence is a holomorphic diffeomorphism $\varphi\colon (TS)^\perp \to \tilde{\mathbf{C}}^2$ which sends fibers to fibers linearly. As a consequence of Grauert's theorem we have the following:

Corollary 1. *Let $S \subset M^2$ be a projective plane embedded in M with Chern class -1. Let $\tilde{\mathbf{C}}^2 \to P$ be the line bundle obtained by blowing-up at $0 \in \mathbf{C}^2$.*

Then there are neighborhoods V of S in M and W of P in $\tilde{\mathbf{C}}^2$ which are diffeomorphic by a holomorphic diffeomorphism $\varphi\colon V \to W$ such that $\varphi(S) = P$.

In order to prove Theorem 1 completely we shall need a small refinement of Corollary 1. Let $S \subset M^2$ be as in Corollary 1 and suppose that $\mathscr{G}$ is a nonsingular holomorphic foliation of complex dimension 1, which is defined in a neighborhood V_1 of S and is transverse to S.

Corollary 2. *Let $S \subset M$, and let $\mathscr{G}$, $\tilde{\mathbf{C}}^2$, and P be as above. Then there exists a diffeomorphism $\varphi\colon V \to W$, as in Corollary* 1, *such that the image of any leaf of $\mathscr{G}/V$ by φ is contained in a fiber of $\tilde{\mathbf{C}}^2 \to P$.*

Proof. Let $\tilde{\varphi}\colon \tilde{V} \to \tilde{W}$, $\tilde{W} \supset P$, be as in Corollary 1. Let $\tilde{\mathscr{G}} = \tilde{\varphi}_*(\mathscr{G})$ be the foliation induced by $\mathscr{G}$ in $\tilde{W}$. Let $\pi\colon \tilde{\mathbf{C}}^2 \to \mathbf{C}^2$ be the projection associated to the blowing-up of $0 \in \mathbf{C}^2$. Let $\mathscr{G}_* = \pi_*(\tilde{\mathscr{G}})$. By Proposition 1, $\mathscr{G}_*$ is defined by a vector field Z in $W_* = \pi(\tilde{W})$. Since the leaves of $\tilde{\mathscr{G}}$ are transverse to P, it follows that the linear part of Z at 0 can be taken as $DZ(0) = L = x\,\partial/\partial x + y\,\partial/\partial y$. Now by Poincaré's linearization theorem [1], it follows that there is a diffeomorphism $\psi\colon U_1 \to U_2$, such that $0 \in U_1 \cap U_2$ and $\psi_*(Z) = L$. Now the integral curves of L are lines passing through $0 \in \mathbf{C}^2$. Let $\tilde{\psi}\colon \tilde{U}_1 \to \tilde{U}_2$ be the blowing-up of ψ, $\tilde{U}_i = \pi^{-1}(U_i)$, $i = 1,2$. It follows that $\varphi = \tilde{\psi} \circ \tilde{\varphi}$ satisfies the properties needed.

2.2. Idea of the proof of Theorem 1. The idea of the proof is to construct a manifold M of complex dimension 2, by glueing several local models of linear foliations in such a way that at the end a singular foliation $\mathscr{F}$ will be defined in M which will have an invariant set $P \subset M$, diffeomorphic to a projective line and with the Chern class of the normal bundle equal to -1. The holonomy of $P - \{\text{singular set of } \mathscr{F}\}$ with respect to $\mathscr{F}$ will be conjugated to the given group, generated by $g_1, \cdots, g_k$. Hence by Corollary 1 of Grauert's theorem this foliation $\mathscr{F}$ will be equivalent to a foliation $\tilde{\mathscr{F}}$ in a neighborhood $\tilde{V}$ of P in $\tilde{\mathbf{C}}^2$ and therefore there will be a vector field Z, defined in a neighborhood of 0, whose blowing-up is $\tilde{\mathscr{F}}$. The projective holonomy of the singularity 0 of Z will be conjugated to the given group, generated by $g_1, \cdots, g_k$. Moreover the construction of $\mathscr{F}$ will be done in such a way that its separatrices, not contained in P, will be leaves of a foliation transverse to P, and so by Corollary 2 the equivalence between $\mathscr{F}$ and $\tilde{\mathscr{F}}$ will be chosen in such a way that the separatrices of $\tilde{\mathscr{F}}$ will be contained in the fibers of $\tilde{\mathbf{C}}^2 \to P$. It will follow that the separatrices of the vector field Z will be contained in $k+1$ complex lines through the origin of $\mathbf{C}^2$. The fact that the multiplicity of Z at 0 is k will follow from Theorem 1 of [4].

2.3. Construction of the manifold M and the foliation $\mathscr{F}$. Let $z_0^0 = 0$ and $z_1^0, \cdots, z_k^0$ be arbitrary k points in $\mathbf{C}$, and for each $j \in \{0, \cdots, k\}$ let D_j be a open disk of radius r and center z_j^0, where r is chosen so that $|z_i^0 - z_j^0| > 2r$

for any $i \neq j$, $0 \leqslant i, j \leqslant k$. For each $j \in \{1, \cdots, k\}$, let us choose a point $z_j' \in D_j - \{z_j^0\}$ and a point $z_j'' \in D_0 - \{0\}$, where

$$z_j'' = \frac{r}{2} \exp\left(\frac{2\pi i(j-1)}{k}\right), \quad z_j' = z_j^0 + \frac{r}{2}. \tag{6}$$

Let $\alpha_1, \cdots, \alpha_k \colon I \to \mathbf{C}$, $I = [0,1]$, be simple curves in $\mathbf{C}$ satisfying the following properties:

(a) $\alpha_j(0) = z_j''$, $\alpha_j(1) = z_j'$.

(b) $\alpha_j(I) \cap D_i = \varnothing$ if $0 \neq i \neq j$.

(c) $\alpha_i(I) \cap \alpha_j(I) = \varnothing$ if $i \neq j$.

(d) For any $j \in \{1, \cdots, k\}$, $\alpha_j(I) \cap D_0$ and $\alpha_j(I) \cap D_j$ are segments of straight lines contained in diameters of D_0 and D_j respectively.

Let $A_1, \cdots, A_k$ be small strips around $\alpha_1, \cdots, \alpha_k$ respectively which satisfy the following properties:

(b′) $A_j \cap D_i = \varnothing$ if $0 \neq i \neq j$.

(c′) $A_i \cap A_j = \varnothing$ if $i \neq j$.

(d′) $A_j \cap D_0$ and $A_j \cap D_j$ are contained in sectors of D_0 and D_j, $1 \leqslant j \leqslant k$ (see Figure 1).

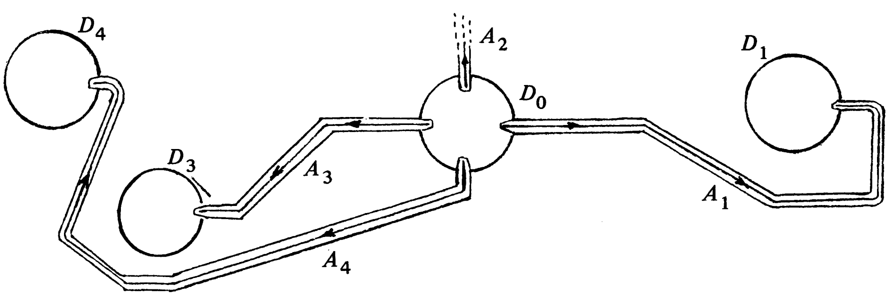

FIGURE 1

We also set $U = (\bigcup_{i=1}^k A_i) \cup (\bigcup_{i=0}^k D_i)$ and $\gamma = \partial U$. From the construction, γ is a simple curve in $\mathbf{C}$. Let T be a tubular neighborhood of γ and set $V = (\overline{\mathbf{C}} - U) \cup T$, where $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. It follows that $\{A_1, \cdots, A_k, D_0, \cdots, D_k, V\}$ is a covering of $\overline{\mathbf{C}}$ by open sets. For each $j = 1, \cdots, k$ let us consider in $A_j \times \mathbf{C}$ coordinates (z, v_j), $z \in A_j$, $v_j \in \mathbf{C}$, and for each $i = 0, \cdots, k$ coordinates (z, u_i) in $D_i \times \mathbf{C}$, $z \in D_i$, $u_i \in \mathbf{C}$. In $V \times \mathbf{C}$ we take coordinates (w, y) where $w = 1/z \in V$ and $y \in \mathbf{C}$.

Now the idea is to take in each set of the form $V \times \mathbf{C}$, $D_i \times \mathbf{C}$, $A_j \times \mathbf{C}$ a local model of foliation and glue them together in order to obtain a manifold M and a singular foliation $\mathscr{F}$ in M as in §2.2. In $A_j \times \mathbf{C}$ we take the horizontal foliation $\hat{\mathscr{F}}_j$, whose leaves are of the form $v_j = \text{constant}$, $j = 1, \cdots, k$. If $V \times \mathbf{C}$ we take also the horizontal foliation $\hat{\mathscr{F}}$, whose leaves are of the form $y = \text{constant}$. The local models $\hat{\mathscr{F}}_j$ in $D_j \times \mathbf{C}$, $j = 0, \cdots, k$, will be singular foliations induced by linear vector fields in $D_j \times \mathbf{C}$ of the form

$$\frac{dz}{dT} = z - z_j^0, \qquad \frac{du_j}{dT} = \alpha_j u_j. \tag{7}$$

The numbers α_j will be chosen according to the generators $g_1, \cdots, g_k$ of the holonomy group. Let $g_j(x) = \lambda_j x + \cdots$, where $\lambda_j = g_j'(0)$. We take α_j so that $e^{2\pi i \alpha_j} = \lambda_j$, $j = 1, \cdots, k$, and $\alpha_0 = -1 - \sum_{i=1}^k \alpha_i$. Let $\gamma_j(\theta) = r_j e^{i\theta} + z_j^0$, $0 \leqslant \theta \leqslant 2\pi$, where $r_j < r$. Let $\Sigma_j = \{p_j\} \times C$, $p_j \in \gamma_j[0, 2\pi]$. It is easy to verify that the holonomy of the curve γ_j in Σ_j, with respect to the foliation $\tilde{\mathscr{F}}_j$, is of the form $u_j \mapsto \lambda_j u_j$, where $\lambda_0 = \lambda_1^{-1} \cdots \lambda_k^{-1}$. We have also, from the hypothesis, that the transformation $\dot{g}_0 = g_k^{-1} \circ \cdots \circ g_1^{-1}$ is linearizable, and so we can choose the coordinates (z, u_0) in $D_0 \times \mathbf{C}$ so that $g_0(u_0) = \lambda_0 u_0$.

Now let us define the diffeomorphisms of identification, in order to glue together the sets $A_j \times \mathbf{C}$ and $D_j \times \mathbf{C}$, $j = 1, \cdots, k$. Since $A_j \cap D_j$ is simply connected and $z_j^0 \notin A_j \cap D_j$, let us consider the coordinate system $(z, \tilde{u}_j)$ in $(A_j \cap D_j) \times \mathbf{C}$, where

$$\tilde{u}_j = u_j \exp\left(-\alpha_j \lg\left(\frac{z - z_0^j}{r/2}\right)\right). \tag{8}$$

Here lg is the branch of the logarithm in $\mathbf{C} - \{x + iy;\ x \leqslant 0\}$ such that $\lg(1) = 0$. Since $z_j' = z_0^j + r/2$, we have that $\tilde{u}_j(z_j', u_j) = u_j$ and $\tilde{u}_j(z, 0) = 0$. Moreover the leaves of the foliation $\tilde{\mathscr{F}}_j$ restricted to $(A_j \cap D_j) \times \mathbf{C}$ are the level surfaces $\tilde{u}_j = \text{constant}$, as can be easily seen from (8). Let us identify the point $(z, v_j) \in (A_j \cap D_j) \times \mathbf{C} \subset A_j \times \mathbf{C}$ with the point $(z, u_j) \in (A_j \cap D_j) \times \mathbf{C} \subset D_j \times \mathbf{C}$, where

$$u_j = v_j \exp\left(\alpha_j \lg\left(\frac{z - z_0^j}{r/2}\right)\right). \tag{9}$$

Clearly (9) is equivalent to identifying (z, v_j) with $(z, \tilde{u}_j)$ and so, with (9), we are glueing together plaques of the foliation $\hat{\mathscr{F}}_j$ in $(A_j \cap D_j) \times \mathbf{C}$ with plaques of $\tilde{\mathscr{F}}_j$ in $(A_j \cap D_j) \times \mathbf{C}$. Observe that this identification sends the fiber $\{z = c\} \subset A_j \times \mathbf{C}$, $c \in A_j \cap D_j$, in the fiber $\{z = c\} \subset D_j \times \mathbf{C}$. Moreover the holonomy of the curve $\beta_j = \alpha_j * \gamma_j * \alpha_j^{-1}$ in the section $\Sigma_j'' = \{z_j''\} \times C \subset A_j \times \mathbf{C}$, with respect to the foliation obtained by glueing together $\hat{\mathscr{F}}_j$ with $\tilde{\mathscr{F}}_j$, is linear of the form $v_j \mapsto \lambda_j v_j$. Let us call this foliation $\tilde{\mathscr{F}}_j$ also.

Now let $h_j\colon B_j \to C_j$ be a holomorphic diffeomorphism, where $h_j(0) = 0 \in B_j \cap C_j$, and let us glue together the new foliation $\tilde{\mathscr{F}}_j$ with $\tilde{\mathscr{F}}_0$ in $(A_j \cap D_0) \times \mathbf{C}$, but now using h_j instead of the identity. More specifically, let us identify the points $(z, v_j) \in (A_j \cap D_0) \times B_j$ with $(z, u_0) \in (A_j \cap D_0) \times \mathbf{C}$ by

$$u_0 = h_j(v_j) \exp\left(\alpha_0 \lg\left(z/z_j''\right)\right). \tag{9'}$$

As above, identification (9′) glues together plaques of $\tilde{\mathscr{F}}_j$ with plaques of $\tilde{\mathscr{F}}_0$ and this defines a new foliation in a complex manifold of complex dimension two, which contains $D_0 \cup A_j \cup D_j$ as a leaf of this new foliation. The holonomy of the curve β_j, in the section $\{z_j''\} \times \mathbf{C} \subset D_0 \times \mathbf{C}$ is given by

$$u_0 \mapsto h_j\big(\lambda_j h_j^{-1}(u_0)\big). \tag{10}$$

Now let $\gamma_0(\theta) = r/2e^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$, and for each $j = 1, \cdots, k$, let μ_j be the segment of γ_0 between $r/2$ and z_j'' (in the positive sense). Let $\delta_j = \mu_j * \beta_j * \mu_j^{-1}$ and $\Sigma_0 = \{r/2\} \times \mathbf{C}$.

It is easy to verify that the holonomy of the curve δ_j in Σ_0 is of the form,

$$u \mapsto \tilde{h}_j\big(\lambda_j \tilde{h}_j^{-1}(u)\big), \tag{10′}$$

where $\tilde{h}_j = a_j^{-1} h_j$, $a_j = \exp(2\pi i \alpha_0(j-1)/k)$.

Since g_j is linearizable we can choose h_j so that $\tilde{h}_j^{-1} \circ g_j \circ \tilde{h}_j(u_j) = \lambda_j u_j$. In the section Σ_0 the holonomy of δ_j is therefore $g_j(u_0) = \lambda_j u_0 + a_2^j u_0^2 + \cdots$.

Now let $\tilde{M}$ be the manifold obtained by glueing together all the foliations $\tilde{\mathscr{F}}_1, \cdots, \tilde{\mathscr{F}}_k$ as indicated above. Let $\tilde{\mathscr{F}}$ be the foliation in $\tilde{M}$ obtained in this way. From the construction, $\tilde{\mathscr{F}}$ satisfies the following properties:

(a) $U = (\bigcup_{i=1}^k A_i) \cap (\bigcup_{j=0}^k D_j)$ is a leaf of $\tilde{\mathscr{F}}$.

(b) The holonomy of U in Σ_0 is generated by $g_1, \cdots, g_k$. This follows from the fact that $g_0 = g_k^{-1} \circ \cdots \circ g_1^{-1}$.

(c) The holonomy of the curve $\delta_1 * \cdots * \delta_k * \gamma_0$ is the identity. This follows also from $g_0 = g_k^{-1} \circ \cdots \circ g_1^{-1}$.

(d) $\tilde{M}$ admits another foliation $\tilde{\mathscr{G}}$, transversal to U, without singularities. This foliation is obtained by glueing together, in each step of the construction, the vertical foliations $z =$ constant of $A_j \times \mathbf{C}$ and $D_j \times \mathbf{C}$ and $D_0 \times \mathbf{C}$. Any leaf of $\tilde{\mathscr{G}}$ cuts U in exactly one point and so we can define a projection $\tilde{p}$: $\tilde{M} \to U$ so that $\tilde{p}^{-1}(z)$ is the leaf of $\tilde{\mathscr{G}}$ through $(z, 0)$.

(e) Let $\tilde{l}_0, \cdots, \tilde{l}_k$ be the separatrices of the singularities of $\tilde{\mathscr{F}}$ which are transversal to U (the equation of $\tilde{l}_j$ in $D_j \times \mathbf{C}$ is $z = z_0^j$). Then $\tilde{l}_0, \cdots, \tilde{l}_k$ are leaves of $\tilde{\mathscr{G}}$. Moreover $\tilde{\mathscr{G}}$ is transverse to $\tilde{\mathscr{F}}$ in $\tilde{M} - \bigcup_{j=0}^k \tilde{l}_j$.

Now let $A = T \cap U$, where T is the tubular neighborhood of $\gamma = \partial U$ considered before. Then A is clearly an annulus. Moreover, if δ is a closed curve in A which generates the homotopy of A, then the holonomy of δ with respect to $\tilde{\mathscr{F}}$ (in some transversal section) is trivial. This follows from (c) and the fact that δ is homotopic to the curve $\delta_1 * \cdots * \delta_k * \gamma_0$ in $U - \bigcup_{j=0}^k z_0^j$. It follows from Reeb's stability theorem (cf. [13]) that the restricted foliation $\tilde{\mathscr{F}}/\tilde{A}$, $\tilde{A} = \tilde{p}^{-1}(A)$, is diffeomorphic to a product foliation, that is, there exists a diffeomorphism φ: $W \to A \times D$, of some neighborhood W of A in $\tilde{A}$ onto $A \times D$, where $D \subset \mathbf{C}$ is a disk, such that φ sends leaves of $\tilde{\mathscr{F}} \mid W$ onto leaves of the trivial foliation $A \times \{c\}$, $c \in D$. This map φ can be chosen so that $\varphi(\tilde{p}^{-1}(z) \cap W) = \{z\} \times D$.

In order to complete the construction of M and $\mathscr{F}$ it is sufficient to glue together the foliations $\tilde{\mathscr{F}}$ in $\tilde{M}$ and $\hat{\mathscr{F}}$ in $V \times D$ by using φ, that is, if we identify a point $q \in W$ with $\varphi(q) \in V \times D$, we obtain a manifold M_1 which contains a projective space $U \cup V = \overline{\mathbf{C}} = P$. Since φ sends leaves of $\tilde{\mathscr{F}}/W$ onto leaves of the horizontal foliation in $A \times D$, it follows that the foliation $\tilde{\mathscr{F}}$ extends to a foliation $\mathscr{F}_1$ in M_1, where P is invariant by $\mathscr{F}_1$. Observe that the foliation $\tilde{\mathscr{G}}$ can be extended also to M, since $\varphi(\tilde{p}^{-1}(z) \cap W) = \{z\} \times D$. Let us call this extension $\mathscr{G}_1$. The leaves of $\mathscr{G}_1$ are transverse to P and each leaf intersects P in exactly one point, hence $\tilde{p}$ can be extended to a projection p: $M_1 \to P$, such that $p^{-1}(z)$ is a leaf of $\mathscr{G}_1$ for any $z \in S$. Observe that some of the leaves of $\mathscr{G}_1$ are diffeomorphic to $\mathbf{C}$, whereas others are diffeomorphic to disks. Nevertheless, it is easy to see that we can take a small neighborhood M of P in M_1 so that p/M: $M \to P$ is a fibration with fibers diffeomorphic to disks. To conclude the construction it is sufficient to take $\mathscr{F} = \mathscr{F}_1/M$ and $\mathscr{G} = \mathscr{G}_1/M$.

Let us prove that the Chern class of the normal bundle of P in M is -1. This follows from the formula:

$$\text{Chern class of } TP^{\perp} = \sum_{i=0}^{k} i\left(z_j^0, P\right),$$

where $i(z_j^0, P)$ is the index of the singularity of $\mathscr{F}$ with respect to the invariant manifold P (cf. [5]). In [5] it is shown that $i(z_j^0, P) = \alpha_j$ and so

$$\text{Chern class of } TP^{\perp} = \sum_{j=0}^{k} \alpha_j = -1.$$

This concludes the proof of Theorem 1.

In order to prove the assertion in Remark 3, we observe that if M is a Riemann surface and $p_0, \cdots, p_k \in M$, then there is a disk $U \subset M$ such that $\{p_0, \cdots, p_k\} \subset U$. From the construction it is possible to construct a singular foliation $\mathscr{F}_1$ on $U \times D$ such that:

(a) The singularities of $\mathscr{F}_1$ are $p_0, \cdots, p_k$ and $\mathscr{F}_1$ is linearizable in a neighborhood of each singularity.

(b) $(U - \{p_0, \cdots, p_k\}) \times \{0\}$ is a leaf of $\mathscr{F}_1$ and the holonomy of this leaf is conjugated to the group of germs generated by $g_1, \cdots, g_k$.

(c) $\sum_{i=0}^{k} i(p_j, U \times \{0\}) = l$.

(d) The holonomy of a simple closed curve near the boundary of ∂U is trivial, that is the foliation restricted to $A \times D$, where A is a tubular neighborhood of ∂U, is trivial.

Now, as before, glue $\mathscr{F}_1$ with the foliation of $(M - (U - A)) \times D$ whose leaves are the horizontals $(M - (U - A)) \times \{z\}$, $z \in D$. The foliation obtained by this process will be holomorphic and will satisfy properties (i), (ii), (iii), and (iv) of Remark 3.

2.4. Generalization of Theorem 1 for several blowing-ups. Observe that in the construction of §2.3 we could take $\alpha_0, \cdots, \alpha_k$ so that $\sum_{i=0}^{k} \alpha_i = n$, $n \in \mathbf{Z}$. The difference is that the Chern class of the normal bundle to P would be n in this case.

Let us consider some manifold $U^{(k)}$ obtained after k blowing-ups as indicated in §1. Then a projection is defined, $\pi^{(k)}: U^{(k)} \to U$, where U is a neighborhood of $0 \in \mathbf{C}^2$, $(\pi^{(k)})^{-1}(0) = \mathscr{P}^{(k)}$ is a union of projective spaces, and $\pi^{(k)} | U^{(k)} - \mathscr{P}^{(k)} \to U - \{0\}$ is a diffeomorphism. In this process, $\mathscr{P}^{(k)}$ is in fact a tree of projective spaces so that if $\mathscr{P}^{(k)} = \bigcup_{i=1}^{k} P_i$, where $P_1, \cdots, P_k$ are projective spaces, then $P_i \cap P_j$ is empty or consists of exactly one point (a corner of $\mathscr{P}^{(k)}$). Moreover, we have no cycles, in the sense that if $P_{i_1}, \cdots, P_{i_l}$ is a chain of projective spaces such that $P_{i_r} \cap P_{i_{r+1}} \neq \varnothing$, $r = 1, \cdots, l-1$, then $P_{i_1} \cap P_{i_l} = \varnothing$.

Let us take in each P_j a set $\{p_0^j, \cdots, p_{r_j}^j\} = S_j$, where this set contains all the intersections of P_j with the other P_i's. Let us also take for each j a group of germs of diffeomorphisms H_j, generated by $g_1^j, \cdots, g_{r_j}^j$. Suppose that $g_1^j, \cdots, g_{r_j}^j$ and $g_0^j = (g_1^j \circ \cdots \circ g_{r_j}^j)^{-1}$ are all linearizable (not necessarily in the same coordinate system). Then by the construction of §2.3 it is possible to obtain a manifold M_j and a foliation $\mathscr{F}_j$ in M_j with the following properties:

(a) $P_j \subset M_j$ and the Chern class of $TP_j^{\perp}$ in M_j is equal to the Chern class of $TP_j^{\perp}$ in $U^{(k)}$.

(b) The set of singularities of $\mathscr{F}_j$ is S_j and all such singularities have a neighborhood where $\mathscr{F}_j$ can be written as in (7).

Now let us suppose that $P_i \cap P_j = \{p\} \neq \varnothing$ (this intersection is in $\mathscr{P}^{(k)}$) and suppose that $\mathscr{F}_i$ is written in a neighborhood $W_i \subset M_i$ of p as

$$\frac{dx}{dT} = x, \qquad \frac{dy}{dT} = \alpha y, \tag{11}$$

where (x, y) is a coordinate system such that $p = (0,0)$ and $P_i \cap W_i = \{y = 0\}$. Similarly, suppose that $\mathscr{F}_j$ can be written in a neighborhood $W_j \subset M_j$ of p as

$$\frac{du}{dT} = u, \qquad \frac{dv}{dT} = \beta v, \tag{11'}$$

where $W_j \cap P_j = \{v = 0\}$. If $g_s^i \in H_i$ and $g_{s'}^j \in H_j$ are the holonomy elements of $\mathscr{F}_i$ and $\mathscr{F}_j$ relative to $p \in P_i$ and $p \in P_j$ respectively, then we have

$(g_s^i)'(0) = e^{2\pi i\alpha}$ and $(g_{s'}^j)'(0) = e^{2\pi i\beta}$. Let us suppose that the following equation of compatibility is satisfied:

$$\alpha \cdot \beta = 1. \tag{12}$$

In this case the foliations defined by (11) and (11′) can be glued together by the diffeomorphism $\varphi(x, y) = (y, x)$. Therefore we can glue together the manifolds M_i and M_j in order to obtain a new manifold $M_i \cup^{\varphi} M_j = M_{ij} \supset P_i \cup P_j$ and a foliation $\mathscr{F}_{ij}$ in M_{ij} such that $P_i \cup P_j$ is invariant by $\mathscr{F}_{ij}$ and the holonomies of $P_i - S_i$ and $P_j - S_j$ are exactly H_i and H_j.

If the compatibility equation (12) is satisfied in all the corners of $\mathscr{P}^{(k)}$ it is clear that we can glue together all the manifolds M_j's and foliations $\mathscr{F}_j$'s in order to obtain a manifold $M^{(k)} \supset \mathscr{P}^{(k)}$ and a foliation $\mathscr{F}^{(k)}$ in $M^{(k)}$ such that the holonomy of $P_j - S_j$ is exactly H_j and the Chern classes of $TP_j^{\perp}$ in $M^{(k)}$ and in $U^{(k)}$ are the same.

Now let us observe that the Chern class of the last projective space obtained by the blowing-up process considered is -1. Hence, by Grauert's theorem the manifold $M^{(k)}$ can be blown down to a manifold $M^{(k-1)} \supset P_1 \cup \cdots \cup P_{k-1}$. The Chern class of each $TP_j^{\perp}$ in $M^{(k-1)}$ clearly coincides with the Chern class of $TP_j^{\perp}$ in $U^{(k-1)}$, the corresponding manifold obtained by blowing down $U^{(k)}$. Moreover by the corollary of Proposition 1 in §1, the foliation $\mathscr{F}^{(k)}$ can be blown down to a foliation $\mathscr{F}^{(k-1)}$ in $M^{(k-1)}$. If we continue this process inductively we obtain finally a foliation $\mathscr{F}^{(0)}$ in a neighborhood of $0 \in \mathbf{C}^2$, which by Proposition 1 can be represented by a vector field Z defined in $U - \{0\}$. We have proved the following result.

Theorem 2. *Let $(U^{(k)}, \pi^{(k)}, \mathscr{P}^{(k)})$ be a sequence of k blowing-ups beginning at $0 \in \mathbf{C}^2$, where $\pi^{(k)}(U^{(k)}) = U$. Let $P_1, \cdots, P_k$ be the projective spaces contained in $\mathscr{P}^{(k)}$ and S a finite subset of $\mathscr{P}^{(k)}$ which contains properly all the corners of $\mathscr{P}^{(k)}$. For each $j = 1, \cdots, k$, let H_j be a group of germs at $0 \in \mathbf{C}$ of holomorphic diffeomorphisms which leave 0 fixed and satisfy the following properties*:

(i) *For each $p \in S \cap P_j$ there exists a germ $g_p \in H_j$ which is linearizable and such that the set $A_j = \{g_p \mid p \in S \cap P_j\}$ generates H_j.*

(ii) *If $S \cap P_l = \{p_1, \cdots, p_r\}$, then we have $g_{p_1} \circ \cdots \circ g_{p_r} = $ identity. Moreover for each p_j there exists $\alpha_j \in \mathbf{C}$ such that $g'_{p_j}(0) = e^{2\pi i\alpha_j}$ and $\sum_{j=1}^r \alpha_j = c(P_l)$.*

(iii) *If $P_l \cap P_j = p$ is a corner, and $f_p \in H_l$, $g_p \in H_j$, where $f'_p(0) = e^{2\pi i\alpha}$, $g'_p(0) = e^{2\pi i\beta}$ (α and β as in (ii)), then $\alpha \cdot \beta = 1$.*

Then there exists a vector field Z in U, such that if $\mathscr{F}^{(k)}$ is the singular foliation of $U^{(k)}$ associated to Z then,

(a) *$\mathscr{P}^{(k)}$ is invariant by $\mathscr{F}^{(k)}$.*

(b) *The set of singularities of* $\mathscr{F}^{(k)}$ *is* S.

(c) *The holonomy of* $P_j - S$ *with respect to* $\mathscr{F}^{(k)}$ *is* H_j.

(d) *The multiplicity of* Z *at* $0 \in \mathbf{C}^2$ *is* $\dot{v} = \#S - \#(corners) - 1$.

We observe that (d) follows from Theorem 1 of [4].

3. Construction of Ricatti foliations in $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$

In this section we prove Theorem 3 (stated in the Introduction). The idea of the proof is to construct a singular foliation $\mathscr{F}$ in a fiber bundle E over $\overline{\mathbf{C}}$ with fiber $\overline{\mathbf{C}}$ satisfying conditions (i), (ii), and (iii) of Theorem 3, by glueing together local pieces as in §2.3. This process is sketched in §3.1. In §3.2 we prove that the glueing process can be done in such a way that at the end $E = \overline{\mathbf{C}} \times \overline{\mathbf{C}}$.

3.1. Construction of E **and** $\mathscr{F}$**.** Here we use the same notations of §2.3. Let $D_0, \cdots, D_k$ be disks around $x_0 = 0, \cdots, x_k$, and $A_1, \cdots, A_k$ be strips which satisfy (b′), (c′), and (d′) of §2.3 (see Figure 1). Let V be as in §2.3, so that $\{A_1, \cdots, A_k, D_0, \cdots, D_k, V\}$ is a covering of $\overline{\mathbf{C}}$. We take coordinate systems (x, v_j) for $A_j \times \mathbf{C}$, $j = 1, \cdots, k$, (x, u_i) for $D_i \times \mathbf{C}$, $i = 0, \cdots, k$, and (w, y) for $V \times \mathbf{C}$, $w = 1/x$. In a neighborhood of $A_j \times \infty \subset A_j \times \overline{\mathbf{C}}$ we take coordinates $(x, \hat{v}_j)$, $\hat{v}_j = 1/v_j$. Analogously we put $\hat{u}_i = 1/u_i$, $i = 0, \cdots, k$, and $\hat{y} = 1/y$.

Let us define the local models for $\mathscr{F}$:

(i) In $A_j \times \overline{\mathbf{C}}$ we consider the trivial foliation, whose leaves are of the form $A_j \times p$, $p \in \overline{\mathbf{C}}$, $j = 1, \cdots, k$. The same in $V \times \overline{\mathbf{C}}$.

(ii) Let us fix $l \in \{0, \cdots, k\}$. As is well known, there is a coordinate system ξ in $\overline{\mathbf{C}} - \{\text{point}\}$ such that f_l can be written in one of the following forms:

(a) $f_l(\xi) = \lambda_l \xi$ if f_l is not parabolic.

(b) $f_l(\xi) = \xi - 1$ if f_l is parabolic.

In case (a) we consider a local model of the form:

$$\frac{dx}{dT} = x - x_l, \quad \frac{du_l}{dT} = \alpha_l u_l, \quad \left(\frac{d\hat{u}_l}{dT} = -\alpha_l \hat{u}_l\right), \tag{13}$$

where $e^{2\pi i \alpha_l} = \lambda_l$.

In case (b) we consider the local model:

$$\frac{dx}{dT} = x - x_l, \quad \frac{du_l}{dT} = \frac{-1}{2\pi i}, \quad \left(\frac{d\hat{u}_l}{dT} = \frac{1}{2\pi i}(\hat{u}_l)^2\right). \tag{14}$$

Clearly the holonomies of (13) and (14) around a circle in D_l containing x_l are as in (a) and (b) respectively.

Now let us glue together the foliation on $A_j \times \overline{\mathbf{C}}$ and the foliations on $D_0 \times \overline{\mathbf{C}}$ and $D_j \times \overline{\mathbf{C}}$. Suppose first that f_0 and f_j are not parabolic. In this case we use the same identifications as in (9) and (9′) of §2.3, where in (9′)

we take $h_j \in \mathrm{PSL}(2, \mathbf{C})$ such that $f_j(z) = a_j^{-1} h_j(\lambda_j h_j^{-1}(a_j z))$, where $a_j = \exp(2\pi i \alpha_0 (j-1)/k)$. With this choice the holonomy of the curve δ_j in the section $\Sigma_0 = \{r/2\} \times \overline{\mathbf{C}}$ will be of course $u \mapsto f_j(u)$ (see Figure 2).

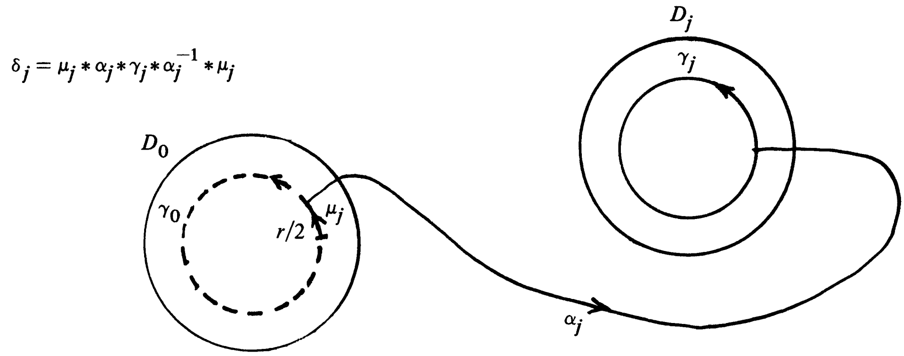

FIGURE 2

In the case where f_j or f_0 are parabolic the identifications in (9) and (9′) are, respectively,

$$u_j = v_j - \frac{1}{2\pi i} \lg\left(\frac{x - x_j}{r/2}\right), \tag{15}$$

$$u_0 = h_j(v_j) - \frac{1}{2\pi i} \lg\left(\frac{x}{z_j''}\right) \tag{15'}$$

(see §2.3 for the definition of z_j'').

It can be verified easily that h_j can be taken in such a way that the holonomy of δ_j in Σ_0 is f_j.

Now the extension of $\mathcal{F}$ to $V \times \overline{\mathbf{C}}$ is done in the same way as in §2.3. We leave the details to the reader.

At the end of the process we obtain a fiber bundle $E \xrightarrow{\pi} \overline{\mathbf{C}}$, with fiber $\overline{\mathbf{C}}$, and a foliation $\mathcal{F}$ on E whose leaves are transversal to the fibers in $\pi^{-1}(\overline{\mathbf{C}} - \{x_0, \cdots, x_k\})$ and such that the fibers $\pi^{-1}(x_0), \cdots, \pi^{-1}(x_k)$ are invariant by $\mathcal{F}$. Observe that in the case where $f_0, \cdots, f_k$ are not elliptic or parabolic then all the singularities of $\mathcal{F}$ are of Poincaré type (see (13)).

3.2. How to obtain $E = \overline{\mathbf{C}} \times \overline{\mathbf{C}}$. We use here the classification of ruled surfaces over $\overline{\mathbf{C}}$ (cf. [8]) which is a consequence of Grothendieck's theorem on the classification of holomorphic vector bundles over $\overline{\mathbf{C}}$ (cf. [9]). The classification of ruled surfaces over $\overline{\mathbf{C}}$ can be summarized as follows:

For each integer $k \geqslant 0$ there exists a unique fiber bundle E_k over $\overline{\mathbf{C}}$ with fiber $\overline{\mathbf{C}}$ which is characterized by the property that E_k is the projectivization of $F_k \oplus F_0$, where F_j is the line bundle over $\overline{\mathbf{C}}$ with Chern class $-j$. Every ruled surface over $\overline{\mathbf{C}}$ is holomorphically equivalent to E_k for some k. In terms of sections of E we have the following characterization.

Proposition 2. *Let E be a ruled surface over $\overline{\mathbf{C}}$. Then $E \approx E_k$ $(k \geqslant 0)$ if and only if E has a holomorphic section σ: $\overline{\mathbf{C}} \to E$ such that the Chern class $c(\sigma)$ of the normal bundle of $\sigma(\overline{\mathbf{C}})$ in E is $-k$. If $k \geqslant 1$ then this section is the unique one with the property $c(\sigma) < k$.*

Proof. Observe that $F_k \oplus F_0$ can be covered by two coordinate charts (x, y_1, y_2) and (u, v_1, v_2), where $u = 1/x$, $v_1 = x^k y_1$, and $v_2 = y_2$. When we projectivize these charts we get $(x, (y_1 : y_2))$, $(u, (v_1 : v_2))$, where $u = 1/x$ and $(v_1 : v_2) = (x^k y_1 : y_2)$. This implies that E_k can be covered by four coordinate charts (x, y_1), (x, y_2), (u, v_1), and (u, v_2) such that the transitions are given by the equations: $u = 1/x$, $y_2 = 1/y_1$, $v_2 = 1/v_1$, $v_1 = x^k y_1$, and $v_2 = x^{-k} y_2$. It follows that the section σ which is expressed in the first chart as $\sigma(x) = 0$ and in the third as $\sigma(u) = 0$ has $c(\sigma) = -k$.

In order to complete the proof it is sufficient to prove that if $k \geqslant 1$ and θ is another section of E_k, then $c(\theta) \geqslant k$. It is easy to verify that θ can be represented in the above charts as

$$(16)\qquad \begin{aligned} y_1 &= \frac{p(x)}{q(x)}, & y_2 &= \frac{q(x)}{p(x)}, \\ v_1 &= u^{s-r-k}\frac{\tilde{p}(u)}{\tilde{q}(u)}, & v_2 &= u^{r+k-s}\frac{\tilde{q}(u)}{\tilde{p}(u)}, \end{aligned}$$

where p and q are polynomials without common factors, $\mathrm{dg}(p) = r$, $\mathrm{dg}(q) = s$, $\tilde{p}(u) = u^r p(1/u)$, and $\tilde{q}(u) = u^s q(1/u)$. It is sufficient to prove that the self-intersection number of the section given by (16) is at least k. This can be done by considering a small perturbation $\tilde{\theta}$ of θ, expressed in the chart (x, y_1) as $y_1 = (1 + \varepsilon)p(x)/q(x)$, where $|\varepsilon| < 1$. The intersection number of $\tilde{\theta}$ with θ is $r + s + t$, where $t = 0$ if $s = r + k$, $t = s - r - k$ if $s > r + k$, or $t = r + k + s$ if $r + k > s$. In any case it is clear that this number is at least k, which proves the proposition.

Now let us consider a point $p \in E_k - \sigma(\overline{\mathbf{C}})$, where σ is the section given by Proposition 2 $(k \geqslant 1)$. Let F be the fiber of E_k through p. Since F is a fiber we have $c(F) = 0$. When we blow up at p we obtain a new manifold $\tilde{E}_k$, a proper map $\tilde{\pi}$: $\tilde{E}_k \to E_k$, and a projective space $P \subset \tilde{E}_k$ such that $c(P) = -1$, $\tilde{\pi}(P) = p$, and $\tilde{\pi}/\tilde{E}_k - P$: $\tilde{E}_k - P \to E_k - \{p\}$ is a diffeomorphism.

Assertion. *There exists a projective space $\tilde{F} \subset \tilde{E}_k$ such that $\tilde{\pi}(\tilde{F}) = F$, $\tilde{F}$ crosses P transversally, and $c(\tilde{F}) = -1$.*

This assertion follows from the following more general lemma.

Lemma 1. *Let M be a 2-dimensional complex manifold and $S \subset M$ be a Riemann surface such that the Chern class of the normal bundle of S in M is*

$c(S)$. Let $\tilde{M} \xrightarrow{\tilde{\pi}} M$ be the manifold obtained by blowing-up once at p. Then $\tilde{\pi}^{-1}(S) = \tilde{S} \cup P$, where $\tilde{\pi}(P) = p$, $\tilde{\pi}(\tilde{S}) = S$, $\tilde{S}$ is diffeomorphic to S, $\tilde{S}$ crosses P transversally at one point, and $c(\tilde{S}) = c(S) - 1$.

For the proof see [5].

From the assertion, we know that $\tilde{F} \subset \tilde{E}_k$ satisfies $c(\tilde{F}) = -1$. It follows from Grauert's theorem (see §2.1) that we can blow down a neighborhood of $\tilde{F}$ to a neighborhood of $0 \in \mathbf{C}^2$. In this way we obtain a new manifold $\hat{E}_k$ and a proper map $\hat{\pi}$: $\tilde{E}_k \to \hat{E}_k$ such that $\hat{\pi}(\tilde{F})$ is a point $\hat{p} \in \hat{E}_k$ and $\hat{\pi} \mid \tilde{E}_k - \tilde{F}$ is a diffeomorphism. Let $\hat{P} = \hat{\pi}(P)$. Then it is easy to see that $\hat{P}$ is a projective space embedded in $\hat{E}_k$ and from the lemma we have $c(\hat{P}) = 0$.

Proposition 3. *The manifold $\hat{E}_k$ is a fiber bundle over $\overline{\mathbf{C}}$ with fiber $\overline{\mathbf{C}}$ and $\hat{E}_k \approx E_{k-1}$. Moreover if we put $\psi = \hat{\pi} \circ (\tilde{\pi} \mid \tilde{E}_k - P)^{-1}$, then ψ: $E_k - F \to \hat{E}_k - \hat{P}$ is a diffeomorphism which sends fibers to fibers.*

Proof. Since $\tilde{\pi} \mid \tilde{E}_k - P$ and $\hat{\pi} \mid \tilde{E}_k - \tilde{F}$ are diffeomorphisms, it is clear that ψ is a diffeomorphism. Let π_k: $E_k \to \overline{\mathbf{C}}$ be the projection of the bundle E_k. Define $\hat{\pi}_k$: $\hat{E}_k - \hat{P} \to \overline{\mathbf{C}} - \{x_0\}$, where $x_0 = \pi_k(F)$, by $\hat{\pi}_k = \pi_k \circ \psi^{-1}$. Clearly $\hat{\pi}_k$: $\hat{E}_k - \hat{P} \to \overline{\mathbf{C}} - \{x_0\}$ defines a fiber bundle structure in $\hat{E}_k - \hat{P}$. We can suppose $x_0 \neq \infty$. If D is a small neighborhood of x_0, then it is not difficult to see that $\hat{\pi}_k^{-1}(D - \{x_0\}) \cup \hat{P} = U_D$ is a neighborhood of $\hat{P}$. Moreover as the diameter of D tends to zero, U_D tends to $\hat{P}$. In particular $\hat{\pi}_k$ is bounded in $U_D - \hat{P}$ and so it can be extended holomorphically to $\hat{P}$ as $\hat{\pi}_k(\hat{P}) = x_0$. It follows that $\hat{\pi}_k$: $\hat{E}_k \to \overline{\mathbf{C}}$ is a fiber bundle. It remains to prove that $\hat{E}_k \approx E_{k-1}$.

Let us consider the section σ: $\overline{\mathbf{C}} \to E_k$, given by Proposition 2, with $c(\sigma) = -k$. Since the point p where we did the blowing-up at the beginning is not in $\sigma(\overline{\mathbf{C}})$, we obtain an embedded projective space $\tilde{\Sigma} = \tilde{\pi}^{-1}(\sigma(\overline{\mathbf{C}})) \subset \tilde{E}_k$. The Chern class of the normal bundle of $\tilde{\Sigma}$ is of course $c(\tilde{\Sigma}) = c(\sigma) = -k$. Let $\hat{\Sigma} = \hat{\pi}(\tilde{\Sigma}) \subset \hat{E}_k$. From the lemma it follows that $c(\hat{\Sigma}) = -k + 1$. Let us prove that $\hat{\Sigma}$ is the image of some section $\hat{\sigma}$: $\overline{\mathbf{C}} \to \hat{E}_k$. Define $\hat{\sigma}$: $\overline{\mathbf{C}} - \{x_0\} \to \hat{E}_k - \hat{P}$ by $\hat{\sigma} = \psi \circ \sigma$. It is not difficult to see that $\hat{\sigma}$ is bounded in a punctured neighborhood of x_0 and so $\hat{\sigma}$ can be extended holomorphically to x_0, where $\hat{\sigma}(x_0) \in \hat{P}$. Moreover $\hat{\sigma}(\overline{\mathbf{C}}) = \hat{\Sigma}$, which implies that $c(\hat{\sigma}) = -k + 1$. It follows from Proposition 2 that $\hat{E}_k \approx E_{k-1}$. (Figure 3 illustrates the process.)

Now let us consider the singular foliation $\mathscr{F}$ on E constructed in §3.1 and let us apply to E the process described above in the case where $E \not\approx E_0$, say $E \approx E_l$.

Suppose first that some of the generators, say f_1, of the holonomy group is not parabolic. In §3.1 we have chosen a local model for the foliation of the form

$$(13_1) \qquad \frac{dx}{dT} = x + x_1, \quad \frac{du_1}{dT} = \alpha_1 u_1, \quad \left(\frac{d\hat{u}_1}{dT} = -\alpha_1 \hat{u}_1, \; \hat{u}_1 = \frac{1}{u_1} \right),$$

where $e^{2\pi i\alpha_1} = \lambda_1$, $f_1(\xi) = \lambda_1\xi$. Observe that the glueing process was done in such a way that the projective spaces defined by $\{x = \text{constant}\}$ are fibers of the bundle E. The fiber $F = \{x = x_1\}$ contains two singularities of $\mathscr{F}$, namely $\{x = x_1, u_1 = 0\}$ and $\{x = x_1, \hat{u}_1 = 0\}$. On the other hand the section σ, given by Proposition 2, has an expression of the form $u_1 = \sigma(x)$ in the chart (x, u_1) $(x \in D_1)$, where σ is meromorphic. Let us suppose that $\sigma(x_1) \neq 0$. In this case the point $p = (x_1, 0) \notin \sigma(\overline{\mathbf{C}})$ and we can apply the argument illustrated in Figure 3 to it.

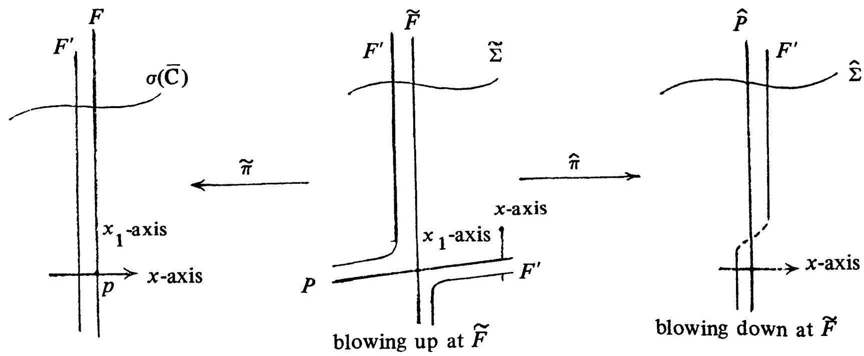

FIGURE 3

Let us consider the blowing-up at $(x_1, 0)$ given by $u_1 = t(x - x_1)$ and $x - x_1 = su_1$, $s = 1/t$. The open set $\tilde{\pi}^{-1}(D_1 \times \overline{\mathbf{C}}) \subset \tilde{E}_k$ can be covered by three charts (s, u_1), $(s, \hat{u}_1)$, and (t, x), where $t = 1/s$ and $x = s \cdot u_1 + x_1 = s\hat{u}_1^{-1} + x_1$. Here $\tilde{F} = \{(s, u_1) \mid s = 0\} \cup \{(s, \hat{u}_1) \mid s = 0\}$ and

$$P = \{(s, u_1) \mid u_1 = 0\} \cup \{(t, x) \mid x = x_1\}.$$

Now, when we blow down $\tilde{F}$, the open set $\hat{\pi}(\tilde{\pi}^{-1}(D_1 \times \overline{\mathbf{C}})) \subset \hat{E}_k$ can be covered by two charts (x, t), (x, s) where $s = 1/t$ and the inverse blowing-up is given by $x - x_1 = u_1 s$, $s = \hat{u}_1(x - x_1)$ (see Figure 3). We have $\hat{\pi}(\tilde{F}) = q = \{s = 0, x = x_1\}$ and $\hat{\pi}(P) = \hat{P} = \{(x, t) \mid x = x_1\} \cup \{(x, s) \mid x = x_1\}$. The map ψ can be expressed by $\psi(x, u_1) = (x, t)$, $t = u_1/(x - x_1)$, or $\psi(x, \hat{u}_1) = (x, s)$, $s = \hat{u}_1(x - x_1)$. The differential equation in (13) is thus transformed by ψ into:

$$(13_1') \qquad \frac{dx}{dT} = x - x_1, \quad \frac{dt}{dT} = (\alpha_1 - 1)t, \quad \left(\frac{ds}{dT} = (1 - \alpha_1)s\right).$$

Observe also that the section $\hat{\sigma}$ of Proposition 3 is expressed in the chart (x, t) as $t = \sigma(x)/(x - x_1)$, and since $\sigma(x_1) \neq 0$, $\hat{\sigma}$ has a pole at $x = x_1$. Therefore we can apply the same process again if $c(\hat{\sigma}) < 0$, blowing-up at $\hat{p} = \{x = x_1, t = 0\}$.

In the case where $\sigma(x_1) = 0$ we begin the process at $p = \{x = x_1, \hat{u}_1 = 0\}$ and we obtain at the end the local model:

$$(13_1'') \qquad \frac{dx}{dT} = x - x_1, \quad \frac{dt}{dT} = (-\alpha_1 - 1)t, \quad \left(\frac{ds}{dT} = (1 + \alpha_1)s\right).$$

Observe that the holonomy of $(13_1')$ or $(13_1'')$ is the same as the holonomy of (13_1) and so the blow-up, blow-down process does not affect the glueing maps. Therefore the above argument implies that if we had chosen the local model as $(13_1')$ or $(13_1'')$ in the construction of §3.1, instead of (13_1), then the bundle obtained at the end of the construction would be E_{l-1} instead of E_l. This proves the following lemma:

Lemma 2. *Let $f_0, \cdots, f_k$ be Moebius transformations, where $k \geqslant 1$ and $f_0 = (f_1 \circ \cdots \circ f_k)^{-1}$. Suppose that f_1 is not parabolic. Choose local models as in* (13) *or* (14) *which realize f_l as local holonomy in the normal form for $l \neq 1$. Choose also Moebius transformations $h_1, \cdots, h_k$ such that $f_j(z) = a_j^{-1}h_j(\lambda_j h_j^{-1}(a_j z))$ if f_j is not parabolic, or $f_j(z) = a_j^{-1}h_j(h_j^{-1}(a_j z) - 1)$ if f_j is parabolic, where $a_j = \exp(2\pi i \alpha_0(j-1)/k)$, $1 \leqslant j \leqslant k$. Then there exists α_1 with $e^{2\pi i \alpha_1} = \lambda_1$ such that the bundle obtained at the end of the construction of* §3.1 *is $E_0 = \overline{\mathbf{C}} \times \overline{\mathbf{C}}$. Moreover, if no f_j is elliptic or parabolic, then all singularities of $\mathscr{F}$ are of Poincaré type.*

In the case where all f_j's are parabolic the argument is analogous. At the end we obtain a foliation $\mathscr{F}$ on $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$ with the desired holonomy. However there is a difference in the local models near the invariant fibers. These local models can be obtained from (14) by applying the change of variables ψ several times and by multiplying the final equation by some power of $x - x_l$ in order to cancel the pole, if necessary. Since these computations are straightforward, we leave them to the reader. In order to complete the proof of Theorem 3, we prove the following result.

Proposition 4. *The foliation $\mathscr{F}$ obtained above is of Ricatti type. In other words, there is a Ricatti equation*:

$$(17) \qquad \frac{dx}{dT} = p(x), \qquad \frac{dy}{dT} = a(x) + b(x)y + c(x)y^2,$$

where $\mathrm{dg}(p) = k + 1$, $\max\{\mathrm{dg}(a), \mathrm{dg}(b), \mathrm{dg}(c)\} \leqslant k - 1$, *and such that its compactification in $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$ is exactly $\mathscr{F}$.*

Proof. Let us consider in $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$ coordinate systems (x, y), (x, v), (u, y), (u, v), where $u = 1/x$, $v = 1/y$. We choose these coordinates in such a way that the invariant fibers are the verticals $\{x = x_j\}$, where $x_j \neq \infty$, $0 \leqslant j \leqslant k$. The image of the chart (x, y) is $\mathbf{C} \times \mathbf{C}$, therefore it induces a singular foliation $\tilde{\mathscr{F}}$ on $\mathbf{C} \times \mathbf{C}$ which is transverse to all verticals $x = c$, where $c \neq x_j$, $0 \leqslant j \leqslant k$. The verticals $\{x = x_j\}$, $0 \leqslant j \leqslant k$, are $\tilde{\mathscr{F}}$ invariant.

Now, since $\tilde{\mathscr{F}}$ is transverse to the verticals in the set $U \times \mathbf{C}$, $U = \mathbf{C} - \{x_0, \cdots, x_k\}$, it follows that $\tilde{\mathscr{F}}$ can be defined in $U \times \mathbf{C}$ by a differential equation of the form $dy/dx = f(x, y)$, where f: $U \times \mathbf{C} \to \mathbf{C}$ is holomorphic (f is the slope of $\tilde{\mathscr{F}}$ at (x, y)). Since $\tilde{\mathscr{F}}$ can be compactified to $U \times \overline{\mathbf{C}}$, it follows that f is a polynomial in the variable y. The fact that $\mathscr{F}$ is transverse to the fibers $x = c$, $c \in U$, at the points of the form $\{x = c, y = \infty\} = \{x = c, v = 0\}$ implies that the degree of f with respect to y is at most 2. Therefore we can write $f(x, y) = A(x) + B(x)y + C(x)y^2$, where A, B, C: $U \to \mathbf{C}$ are holomorphic. Since $\tilde{\mathscr{F}}$ extends to the vertical $x = x_j$, $0 \leqslant j \leqslant k$, as a singular foliation, it follows from Proposition 1 of §1 (or from the construction) that the points $x_0, \cdots, x_k$ are poles of A, B, C. Therefore we can write $A = a/p$, $B = b/p$, $C = c/p$, where p is a polynomial whose roots are $x_0, \cdots, x_k$ and a, b, c: $\mathbf{C} \to \mathbf{C}$ are holomorphic. Hence $\tilde{\mathscr{F}}$ can be defined by equations (17). In order to prove that a, b, and c are polynomials with $\max\{\mathrm{dg}(a), \mathrm{dg}(b), \mathrm{dg}(c)\} \leqslant k - 1$, it is sufficient to use the fact that $\mathscr{F}$ extends to the line $x = \infty$, and that this vertical is not invariant.

4. Applications

In this section we study perturbations of Ricatti foliation on $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$.

Let M be an n-dimensional complex manifold. A singular foliation $\mathscr{F}$ on M is given by a covering $\{U_\alpha\}_{\alpha \in I}$ of M by open sets and a collection $X = \{X_\alpha\}_{\alpha \in I}$ such that:

(i) For each $\alpha \in I$, X_α is a holomorphic vector field on U_α, whose singular set S_α has codimension at least 2.

(ii) If $U_\alpha \cap U_\beta \neq \varnothing$, then there exists a function $\lambda_{\alpha\beta}$: $U_\alpha \cap U_\beta \to \mathbf{C}^*$ such that $X_\alpha = \lambda_{\alpha\beta} \cdot X_\beta$.

Let $F(M)$ be the set of singular foliations on M. Given $\mathscr{F} \in F(M)$ as above, we define the singular set of $\mathscr{F}$ as $S = \bigcup_{\alpha \in I} S_\alpha$. Clearly $\mathscr{F}$ is a foliation, in the usual sense, on $M - S$. The nonsingular leaves of $\mathscr{F}/U_\alpha$ are the nonsingular integral curves of X_α.

Let us suppose that M is compact. In this case we can suppose that $I = \{1, \cdots, m\}$ is finite and that each U_α is the domain of a coordinate system φ_α: $U_\alpha \to B_2$, $B_r = \{(x_1, \cdots, x_n);\ |x_j| < r,\ j = 1, \cdots, m\}$, where the set $\{V_\alpha\}_{\alpha \in I}$, $V_\alpha = \varphi_\alpha^{-1}(\overline{B}_1)$, is also a covering of M. Let us fix $\mathscr{F} \in F(M)$ and these coverings. Given $\tilde{\mathscr{F}} \in F(M)$, for each $\alpha \in I$ there exists a vector field $\tilde{X}_\alpha$ on U_α such that the leaves of $\tilde{\mathscr{F}}\,|U_\alpha$ are the integral curves of $\tilde{X}_\alpha$. This follows from an argument analogous to that of Proposition 1 (cf. [7]). Let us define the ε neighborhood of $\mathscr{F}$, $\mathscr{U}(F, X, \varepsilon)$, as the set of all $\tilde{\mathscr{F}} \in F(M)$ such that for each $\alpha \in I$ there exists a function $\mu_\alpha \colon U_\alpha \to \mathbf{C}^*$ satisfying

$$\sup\left\{ |X_\alpha(x) - \mu_\alpha(x) \cdot \tilde{X}_\alpha(x)|;\ x \in V_\alpha \right\} < \varepsilon.$$

It can be verified easily that the set $\{\mathscr{U}(\mathscr{F}, X, \varepsilon);\ \mathscr{F} \in F(M),\ \varepsilon > 0$ and $X = \{X_\alpha\}_{\alpha \in I}$, where $\mathscr{F} \mid U_\alpha$ is represented by $X_\alpha\}$ is a base for a topology in $F(M)$.

Let us consider the case where $M = \overline{\mathbf{C}} \times \overline{\mathbf{C}}$ and $\mathscr{F}$ is a Ricatti foliation. Let $\{x_j\} \times \overline{\mathbf{C}}$, $j = 0, \cdots, k$, be the invariant fibers of $\mathscr{F}$. If $\overline{D} \subset \overline{\mathbf{C}} - \{x_0, \cdots, x_k\}$ is a closed disk, then $\mathscr{F}$ is transverse to all fibers $\{x\} \times \overline{\mathbf{C}}$, $x \in \overline{D}$. Since $\overline{D}$ is compact, it follows that there exists a neighborhood $\mathscr{U}$ of $\mathscr{F}$ in $F(\overline{\mathbf{C}} \times \overline{\mathbf{C}})$ such that if $\tilde{\mathscr{F}} \in \mathscr{U}$, then $\tilde{\mathscr{F}}$ is also transverse to all fibers $\{x\} \times \overline{\mathbf{C}}$, $x \in \overline{D}$. From this fact it is not difficult to prove that $\tilde{\mathscr{F}}$ is also a Ricatti foliation. For the proof just use the same computations made in the proof of Proposition 4. So we have the following result.

Proposition 5. *The set of Ricatti foliations is an open set of* $F(\overline{\mathbf{C}} \times \overline{\mathbf{C}})$.

In this section we prove the following results.

Theorem 4. *Let* $k \geqslant 3$. *There exists an open set* $\mathscr{U} \subset F(\overline{\mathbf{C}} \times \overline{\mathbf{C}})$ *with the following properties*:

(i) *Any* $\mathscr{F} \in \mathscr{U}$ *is a Ricatti foliation with* k *invariant fibers. All singularities of* $\mathscr{F}$ *are of Poincaré type.*

(ii) *If* $\mathscr{F}$ *and* $\mathscr{G} \in \mathscr{U}$ *are topologically equivalent, then their holonomies are conformally conjugated.*

We say that $\mathscr{F}$ and $\mathscr{G}$ are topologically equivalent if there exists a homeomorphism $h\colon \overline{\mathbf{C}} \times \overline{\mathbf{C}} \to \overline{\mathbf{C}} \times \overline{\mathbf{C}}$ which sends leaves of $\mathscr{F}$ onto leaves of $\mathscr{G}$ and the singular set of $\mathscr{F}$ onto the singular set of $\mathscr{G}$.

Theorem 5. *Let* $f_1, \cdots, f_k$ *be Moebius transformations such that the group* G *generated by them is free and structurally stable in the sense of* [15]. *Let* $f_0 = (f_1 \circ \cdots \circ f_k)^{-1}$ *and* $\mathscr{F}$ *be a Ricatti foliation constructed as in Theorem* 3 *from* $f_0, \cdots, f_k$. *Then* $\mathscr{F}$ *is structurally stable.*

We say that $\mathscr{F} \in F(\overline{\mathbf{C}} \times \overline{\mathbf{C}})$ is structurally stable if there exists a neighborhood $\mathscr{U}$ of $\mathscr{F}$ such that any $\mathscr{G} \in \mathscr{U}$ is topologically equivalent to $\mathscr{F}$.

Remark. Let $D_1, \cdots, D_{2k}$ be disjoint closed disks and $f_1, \cdots, f_k$ be Moebius transformations such that:

(i) $f_j(\partial D_j) = \partial D_{k+j}$, $j = 1, \cdots, k$,

(ii) $f_j(\overline{\mathbf{C}} - (D_j \cup D_{j+k})) \subset D_j$ and $f_j^{-1}(\overline{\mathbf{C}} - (D_j \cup D_{j+k})) \subset D_{j+k}$, $j = 1, \cdots, k$.

Then $f_1, \cdots, f_k$ are loxodromic or hyperbolic and the group generated by them is free. This type of group is known as a Schottky group (cf. [11]) and is structurally stable, since all nearby representations are free (cf. [15]).

4.1. Proof of Theorem 4. We begin by proving that if $\mathscr{F}$ and $\mathscr{G}$ are topologically equivalent, then their holonomies are topologically conjugated.

Let h be an equivalence between $\mathscr{F}$ and $\mathscr{G}$. Then h sends invariant fibers of $\mathscr{F}$ onto invariant fibers of $\mathscr{G}$. So, if the invariant fibers of $\mathscr{F}$ are $\{x_j\} \times \overline{\mathbf{C}}$, $j = 0, \cdots, k$, then $h(\{x_j\} \times \overline{\mathbf{C}})$ is an invariant fiber of $\mathscr{G}$, which we can suppose is $\{x_j'\} \times \overline{\mathbf{C}}$. Put $S = \{x_0, \cdots, x_k\}$, $S' = \{x_0', \cdots, x_k'\}$. For any fiber $\Sigma_q = \{q\} \times \overline{\mathbf{C}}$, $q \notin S$, its image $h(\Sigma_q)$ is a topological sphere which is topologically transverse to $\mathscr{G}$ (that is, $h(\Sigma_q)$ has a product neighborhood whose fibers are disks on the leaves of $\mathscr{G}$). Let us fix two fibers $\Sigma = \Sigma_q$, $\Sigma' = \Sigma_{q'}$, $q \notin S$, $q' \in S'$. We are going to prove that there exists a homotopy ψ: $I \times \overline{\mathbf{C}} \to (\overline{\mathbf{C}} - S') \times \overline{\mathbf{C}}$ with the following properties:

(i) $\psi_0(\overline{\mathbf{C}}) = h(\Sigma)$, $\psi_1(\overline{\mathbf{C}}) = \Sigma'$ and ψ_0: $\overline{\mathbf{C}} \to h(\Sigma)$, ψ_1: $\overline{\mathbf{C}} \to \Sigma'$ are homeomorphisms ($\psi_t(z) = \psi(t, z)$).

(ii) For $z \in \overline{\mathbf{C}}$, $\psi(I \times z)$ is contained in the leaf of $\mathscr{G}$ through $\psi_0(z)$.

This homotopy can be constructed easily by considering the universal covering $W \times \overline{\mathbf{C}} \overset{\pi}{\hookrightarrow} (\overline{\mathbf{C}} - S') \times \overline{\mathbf{C}}$, where $W = \mathbf{C}$ or $W = \{x \in \mathbf{C};\ |x| < 1\}$. Let $\pi^*(\mathscr{G}) = \mathscr{G}_*$ be the foliation coinducted by $\mathscr{G}$. Then $\mathscr{G}_*$ is transverse to the fibers $\{x\} \times \overline{\mathbf{C}}$, $x \in W$. It follows from a theorem of Ehresman that $\mathscr{G}_*$ is equivalent to the trivial foliation on $W \times \overline{\mathbf{C}}$, whose leaves are of the form $W \times \{z\}$, $z \in \overline{\mathbf{C}}$ (cf. [6]). We can suppose therefore that $\mathscr{G}_*$ is this foliation. Let $\hat{\Sigma}'$ and $\hat{\Sigma}$ be connected submanifolds such that $\pi(\hat{\Sigma}') = \Sigma'$ and $\pi(\hat{\Sigma}) = h(\Sigma)$. Since $\hat{\Sigma}'$ is transverse and $\hat{\Sigma}$ is topologically transverse to $\mathscr{G}_*$, there exist functions α, β: $\mathbf{C} \to W$, α analytic and β continuous, such that $\hat{\Sigma}' = \{(\alpha(z), z);\ z \in \overline{\mathbf{C}}\}$ and $\hat{\Sigma} = \{(\beta(z), z);\ z \in \overline{\mathbf{C}}\}$.

This assertion is clear for $\hat{\Sigma}'$. Let us prove it for $\hat{\Sigma}$. It is sufficient to prove that each leaf $L = W \times \{z\}$ of $\mathscr{G}_*$ cuts $\hat{\Sigma}$ in exactly one point. Clearly each leaf L cuts $\hat{\Sigma}'$ in exactly one point. So we can consider a map p: $\hat{\Sigma} \to \hat{\Sigma}'$ defined by $P(q) = L \cap \hat{\Sigma}'$, where $q \in \hat{\Sigma}$ and L is the leaf of $\mathscr{G}_*$ through q. Since $\hat{\Sigma}$ is topologically transverse to $\mathscr{G}_*$ ($\hat{\Sigma}$ has a product neighborhood whose fibers are disks on the leaves of $\mathscr{G}_*$), it follows that P is a covering map. This proves the assertion, because $\hat{\Sigma}' \approx \overline{\mathbf{C}}$ and $\hat{\Sigma}$ is connected.

Now it is sufficient to put $\psi(t, z) = \pi(t\alpha(z) + (1 - t)\beta(z), z)$. It is easy to verify that ψ satisfies (i) and (ii).

We are going to prove that the homeomorphism $\theta = \psi_1 \circ \psi_0^{-1} \circ h$: $\Sigma \to \Sigma'$ is a conjugation between the holonomies of $\mathscr{F}$ in Σ and $\mathscr{G}$ in Σ'.

For each point $p' \in h(\Sigma)$, let $\alpha_{p'}$ be the curve on the leaf of $\mathscr{G}$ through p', defined by $\alpha_{p'}(t) = \psi_t \circ \psi_0^{-1}(p')$. Let γ be a loop in $\pi_1(\overline{\mathbf{C}} - S, q)$ and for $p = (q, y) \in \Sigma$, let γ_p be the lifting of γ on the leaf L_p of $\mathscr{F}$ through p such that $\gamma_p(0) = p$. By definition we have $\gamma_p(1) = f_{[\gamma]}(p)$, where $f_{[\gamma]}$ is the holonomy transformation associated to γ. Let $p' = h(p)$, $p'' = h(f_{[\gamma]}(p))$, and $\gamma'_{\theta(p)} = \alpha_{p'}^{-1} * (h \circ \gamma_p) * \alpha_{p''}$ (see Figure 4). We have

$$\gamma'_{\theta(p)}(0) = \alpha_{p'}(1) = \theta(p), \qquad \gamma'_{\theta(p)}(1) = \alpha_{p''}(1) = \theta\big(f_{[\gamma]}(p)\big).$$

Moreover $\gamma'_{\theta(p)}$ is a curve contained in the leaf $L'_{h(p)}$ and so it is the lifting on this leaf of the loop $P_1(\gamma'_{\theta(p)}) = \gamma'_p$, where P_1: $(\overline{\mathbf{C}}-S') \times \overline{\mathbf{C}} \to \overline{\mathbf{C}}-S'$ is the first projection. Hence $\gamma'_{\theta(p)}(1) = g_{[\gamma'_p]}(\theta(p))$, where $g_{[\gamma'_p]}$ is the holonomy transformations of $\mathscr{G}$ associated to $[\gamma'_p] \in \pi_1(\overline{\mathbf{C}}-S', q')$. Now observe that the homotopy class of $\gamma'_p \in \pi_1(\overline{\mathbf{C}}-S', q')$ does not depend on p. Moreover, since h: $\overline{\mathbf{C}}-S \times \overline{\mathbf{C}} \to \overline{\mathbf{C}}-S' \times \overline{\mathbf{C}}$ is a homeomorphism, the map $[\gamma] \in \pi_1(\overline{\mathbf{C}}-S, q) \mapsto [\gamma'_p] \in \pi_1(\overline{\mathbf{C}}-S', q')$ is an isomorphism and from the above construction we have that $\theta \circ f_{[\gamma]}(p) = g_{[\gamma'_p]} \circ \theta(p)$. This proves the assertion.

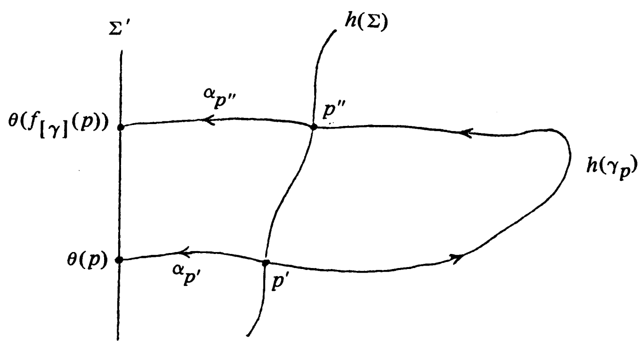

FIGURE 4

Theorem 4 will follow from Theorem 3 and the lemma below.

Lemma 3. *There exist open sets* $\mathscr{U}_1, \mathscr{U}_2 \subset \mathrm{PSL}(2, \mathbf{C})$ *satisfying the following properties*:

(i) *Any element* $f \in \mathscr{U}_1 \cup \mathscr{U}_2$ *is hyperbolic or loxodromic.*

(ii) *If* $f_1 \in \mathscr{U}_1$ *and* $f_2 \in \mathscr{U}_2$, *then* f_1 *and* f_2 *have no common fixed points.*

(iii) *Given* $(f_1, f_2) \in \mathscr{U}_1 \times \mathscr{U}_2$ *and* $g_1, g_2 \in \mathrm{PSL}(2, \mathbf{C})$ *such that there exists a homeomorphism* θ *of* $\overline{\mathbf{C}}$ *satisfying* $\theta \circ f_j = g_j \circ \theta$, $j = 1, 2$, *then* θ *is a conformal map.*

Proof. The idea is to construct open sets $\mathscr{U}_1$, $\mathscr{U}_2$ which satisfy (i) and (ii) and: (iv) For any $(f_1, f_2) \in \mathscr{U}_1 \times \mathscr{U}_2$, the group generated by f_1 and f_2 is not discrete.

Suppose for a moment that we have constructed such $\mathscr{U}_1$ and $\mathscr{U}_2$. Fix $(f_1, f_2) \in \mathscr{U}_1 \times \mathscr{U}_2$ and let Γ be the group generated by f_1 and f_2. Since Γ is not discrete, its closure $\overline{\Gamma}$ contains a one-parameter subgroup $\{f_t\}_{t\in\mathbf{R}}$. Now let θ be a homeomorphism of $\overline{\mathbf{C}}$ such that $g_j = \theta\circ f_j\circ\theta^{-1}$, $j=1,2$, are Moebius transformations. Let Γ' be the group generated by g_1 and g_2, and $\overline{\Gamma}'$ be its closure. Then $g_t=\theta\circ f_t\circ\theta^{-1}$ is a one-parameter subgroup of $\overline{\Gamma}'$. It follows from a theorem of E. Cartan that $t\mapsto g_t$ is real analytic. Hence for any $z\in\overline{\mathbf{C}}$ which is not a fixed point of the family $\{f_t\}_{t\in\mathbf{R}}$ we have that θ is real analytic along the curve $t\mapsto f_t(z)$. This follows from $\theta(f_t(z))=g_t(\theta(z))$. We observe that, since a one-parameter subgroup is abelian, all nontrivial elements of $\{f_t\}_{t\in\mathbf{R}}$ have the same fixed points. The same is true for the family $\{g_t\}_{t\in\mathbf{R}}$.

Now, let $f\in\Gamma$ be such that f and the family $\{f_t\}_{t\in\mathbf{R}}$ have no fixed points in common. It follows that the family $\{\tilde{f}_s=f^{-1}f_sf\}_{s\in\mathbf{R}}$ is contained in $\overline{\Gamma}$ and has no fixed points in common with $\{f_t\}_{t\in\mathbf{R}}$. Hence there exists $z_0\in\overline{\mathbf{C}}$ such that the curves $t\mapsto f_t(z_0)$ and $s\mapsto\tilde{f}_s(z_0)$ are transverses at $t=s=0$. Since transversality is an open property, the same is true for the curves $t\mapsto f_t(z)$ and $s\mapsto\tilde{f}_s(z)$, where $z\in D$, D a neighborhood of z_0. Using the transversality of these curves and the fact that θ is real analytic along them, it is not difficult to prove that θ is C^∞ in D. Since Γ is not discrete, it follows from Montel's theorem that $\bigcup_{h\in\Gamma}h(D)$ covers all of $\overline{\mathbf{C}}$, with possible exception of two points. It follows that there exists $h\in\Gamma$ such that $h(D)$ contains a fixed point z_1 of f_1, for example. Since f_1 is loxodromic or hyperbolic we can suppose that $f_1'(z_1)=\lambda$, $|\lambda|<1$. Let $f_3=h^{-1}f_1h\in\Gamma$. Then $f_3(h^{-1}(z_1))=h^{-1}(z_1)\in D$ and $f_3'(h^{-1}(z_1))=\lambda$. Moreover we can suppose that the fixed points of f_3 are $h^{-1}(z_1)=0$ and ∞, so that $f_3(z)=\lambda z$. Similarly we can suppose that the fixed points of $g_3=\theta\circ f_3\circ\theta^{-1}$ are 0 and ∞, so that $g_3(\omega)=\mu\omega$, $|\mu|<1$. This implies that $\theta(\lambda^n z)=\mu^n\theta(z)$ and this equation together with the fact that θ is C^1 in D, $0\in D$, implies that $\theta(z)=\xi z$ or $\theta(z)=\xi\bar{z}$. Therefore θ is a conformal map. It remains to prove the existence of $\mathcal{U}_1$ and $\mathcal{U}_2$ satisfying (i), (ii), and (iv). This will follow from the lemma below.

Lemma 4. *Let $f_0(z)=\lambda z$, where $|\lambda|>1$ and $|\lambda-1|<1$. Then there exist neighborhoods $\mathcal{V}_1$ of f_0 and $\mathcal{V}_2$ of the identity in* PSL(2, *C*), *such that for any pair $(f_1,f_2)\in\mathcal{V}_1\times\mathcal{V}_2$, the group generated by f_1 and f_2 is discrete if and only if f_1 and f_2 commute.*

Proof. Let us consider the map φ: $\mathrm{PSL}(2,\mathbf{C})\times\mathrm{PSL}(2,\mathbf{C})\to\mathrm{PSL}(2,\mathbf{C})$ given by $\varphi(f,g)=f\circ g\circ f^{-1}\circ g^{-1}$. We take $\varphi_g(f)=\varphi(f,g)$ and $\psi=\varphi_{f_0}$.

Assertion. *ψ is a contraction in a neighborhood $\mathcal{V}_2$ of the identity I, and $\lim_{n\to\infty}\psi^n(f)=I$ for any $f\in\mathcal{V}_2$.*

In fact, $\psi(I)=I$, and for $A\in T_I(\mathrm{PSL}(2,\mathbf{C}))$ we have

$$D\psi(I)\cdot A=A-f_0\cdot A\cdot f_0^{-1}. \tag{18}$$

In formula (18) we are considering f_0 as a matrix of the form $\left(\begin{smallmatrix}\mu & 0\\ 0 & \mu^{-1}\end{smallmatrix}\right)$, where $\mu^2 = \lambda$ and $A = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & -\alpha\end{smallmatrix}\right)$. An easy computation implies that the eigenvalues of $D\psi(I)$ are 0, $1-\lambda$, and $1-\lambda^{-1}$. Since $|1-\lambda| < 1$ and $|1-\lambda^{-1}| = |\lambda^{-1}||1-\lambda| < 1$, it follows that ψ is a contraction in a neighborhood $\mathscr{V}_2$ of I and I is the unique fixed point of ψ in $\mathscr{V}_2$. This proves the assertion.

Let us take $\mathscr{V}_2$ in such a way that for any $f \in \mathscr{V}_2 - \{I\}$ we have $f^2 \neq I$. Let $f \in \mathscr{V}_2$. Then $\psi^n(f) \in \Gamma$, the group generated by f_0 and f. If Γ is discrete the sequence $\{\psi^n(f)\}_{n \geqslant 0}$ stabilizes for $n \geqslant n_0$ and since $\lim_{n\to\infty}\psi^n(f) = I$ we have $\psi^n(f) = I$ for $n \geqslant n_0$. Put $f_j = \psi^j(f)$, $j \geqslant 1$, and let $m = \min\{n;\ f_n = I\}$. We assert that $m = 1$ and hence f and f_0 commute.

In fact, suppose by contradiction that $m > 1$. This implies that $f_{m-1} \neq I$ and f_{m-1} commutes with f_0. Since $f_0(z) = \lambda z$, $|\lambda| > 1$, we have that $f_{m-1}(z) = \rho z$, where $\rho \neq 1$. On the other hand $f_{m-1} = f_{m-2} \circ f_0 \circ f_{m-2}^{-1} \circ f_0^{-1}$ and so $f_0 \circ f_{m-2} \circ f_0^{-1} = f_{m-1}^{-1} \circ f_{m-2}$. It is not difficult to see that this equation together with $f_{m-2} \neq I$ and $f_{m-1}(z) = \rho z$, $\rho \neq 1$, implies that $f_{m-2}^2 = I$, which is a contradiction since $f_{m-2} \in \mathscr{V}_2$.

Now, let $\mathscr{V}_1$ be a neighborhood of f_0 with the following properties:

(a) For any $g \in \mathscr{V}_1$, $\varphi_g | \mathscr{V}_2$ is a contradiction and I is the unique fixed point of φ_g in $\mathscr{V}_2$.

(b) If $g \in \mathscr{V}_1$, then g has a fixed point p such that $|g'(p)| > 1$ and $|g'(p) - 1| < 1$.

It is not difficult to see that $\mathscr{V}_1$ and $\mathscr{V}_2$ satisfy the properties we need.

Now let $X = \{f \in \mathrm{PSL}(2, \mathbf{C});\ f \circ f_0 = f_0 \circ f\}$. Then X is a codimension 2 submanifold of $\mathrm{PSL}(2,\mathbf{C})$, which implies that $\mathscr{V}_2 - X \neq \varnothing$. Let $f_1 \in \mathscr{V}_2 - X$ be loxodromic or hyperbolic, with no common fixed points with f_0. It is not difficult to see that f_0 and f_1 have neighborhoods $\mathscr{U}_1 \subset \mathscr{V}_1$ and $\mathscr{U}_2 \subset \mathscr{V}_2$ which satisfy (i), (ii), and (iii) of Lemma 3. This ends the proof of Theorem 4.

4.2. Proof of Theorem 5. Observe first that since $G = [f_1, \cdots, f_k]$ is free and structurally stable, then there exist neighborhoods $\mathscr{U}_1, \cdots, \mathscr{U}_k$ of $f_1, \cdots, f_k$ respectively, such that for any $(g_1, \cdots, g_k) \in \mathscr{U}_1 \times \cdots \times \mathscr{U}_k = \mathscr{U}$, the group $\tilde{G} = [g_1, \cdots, g_k]$ is also free. Moreover, it follows from the results of [15] that if $(g_1, \cdots, g_k) \in \mathscr{U}$, then $\tilde{G} = [\tilde{g}_1, \cdots, \tilde{g}_k]$ is quasi-conformally conjugated to G. In fact in [15] this is proved for one-parameter families $G_\lambda = [g_{1\lambda}, \cdots, g_{k\lambda}]$, $g_{j0} = f_j$, $j = 1, \cdots, k$, and the result is that the conjugation $\lambda \mapsto h_\lambda$ between G_0 and G_λ can be chosen in such a way that it depends holomorphically on λ and $h_0 = \mathrm{id}$. We observe also that all elements of G are hyperbolic or loxodromic. In fact, since G is free and has nontrivial domains of discontinuity (cf. [15]), it follows that G does not contain elliptic elements. Let us prove that G does not contain parabolic elements.

Given sequences $I = (i_1, \cdots, i_r) \subset \{1, \cdots, k\}$ and $K = \{k_1, \cdots, k_r\} \subset \{-1, 1\}$, let us consider the map $\varphi_{I,K}$: $\mathscr{U} \to \mathrm{PSL}(2, \mathbf{C})$, $\varphi_{I,K}(g_1, \cdots, g_k) = g_{i_1}^{k_1} \circ \cdots \circ g_{i_r}^{k_r}$. Of course we do not consider sequences I and K such that $k_j k_{j+1} = -1$ if $i_j = i_{j+1}$, so that $\varphi_{I,K}(g_1, \cdots, g_k) \neq I$ if $(g_1, \cdots, g_k) \in \mathscr{U}$. In this case it is easy to verify that $\varphi_{I,K}(\mathscr{U})$ is an open set of $\mathrm{PSL}(2, \mathbf{C})$ and that the set

$$\mathscr{A}_{I,K} = \{(g_1, \cdots, g_k) \in \mathscr{U};\ \varphi_{I,K}(g_1, \cdots, g_k) \text{ is not parabolic}\}$$

is open and dense in $\mathscr{U}$. It follows that $\mathscr{A} = \bigcap_{I,K} \mathscr{A}_{I,K}$ is a generic subset of $\mathscr{U}$. If $(g_1, \cdots, g_k) \in \mathscr{A}$, then the group $[g_1, \cdots, g_k]$ does not contain parabolic elements. Since G is rigid, it follows that G does not contain parabolic elements.

Now let us consider a Ricatti foliation $\mathscr{F}$ on $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$, constructed from $f_1, \cdots, f_k$ as in Theorem 3. Since $f_1, \cdots, f_k$ and $f_0 = (f_1 \circ \cdots \circ f_k)^{-1}$ are hyperbolic or loxodromic, it follows that $\mathscr{F}$ has $k + 1$ invariant vertical fibers and $2k + 2$ singularities, all of Poincaré type, where each invariant fiber contains exactly two singularities. Let us suppose that the invariant fibers for $\mathscr{F}$ are $\{x_0\} \times \overline{\mathbf{C}}, \cdots, \{x_k\} \times \overline{\mathbf{C}}$, where for each $j \in \{0, \cdots, k\}$ we have a local model

$$\frac{dx}{dT} = x - x_j, \quad \frac{du}{dT} = \alpha_j u, \quad \left(\frac{d\hat{u}}{dT} = -\alpha_j \hat{u},\ \hat{u} = \frac{1}{u}\right), \tag{19}$$

where $e^{2\pi i \alpha_j}$ is the eigenvalue of Df_j in one of the fixed points of f_j.

Observe that in the proof of Theorem 3, the generators $f_1, \cdots, f_k, f_0$ of the holonomy are associated to fixed generators $\gamma_1, \cdots, \gamma_k, \gamma_0$ of $\pi_1(\overline{\mathbf{C}} - S, q)$, where $S = \{x_0, \cdots, x_k\}$, $\gamma_0 \approx (\gamma_1 * \cdots * \gamma_k)^{-1}$, and $q \notin S$. Let us consider neighborhoods $\mathscr{U}_j$ of f_j, $j = 1, \cdots, k$, as before, and $\mathscr{U}_0 = \{(g_1 \circ \cdots \circ g_k)^{-1};\ g_j \in \mathscr{U}_j,\ j = 1, \cdots, k\}$. Let us also fix curves $\gamma_0, \cdots, \gamma_k$, and $k + 1$ disks $D_0, \cdots, D_k \subset \overline{\mathbf{C}}$, such that $D_i \cap D_j = \varnothing$ if $i \neq j$, $x_j \in D_j$, and $D_i \cap \gamma_j = \varnothing$ for all i, j. There exists a neighborhood $\mathscr{V}$ of $\mathscr{F}$ in $F(\overline{\mathbf{C}} \times \overline{\mathbf{C}})$ with the following properties:

(a) If $\mathscr{G} \in \mathscr{V}$, then $\mathscr{G}$ is of Ricatti type.

(b) In the chart (x, y) considered in Proposition 4, $\mathscr{G}$ has an expression of the form

$$\frac{dx}{dT} = p(x, \mathscr{G}), \qquad \frac{dy}{dT} = a(x, \mathscr{G}) + b(x, \mathscr{G}) y + c(x, \mathscr{G}) y^2, \tag{17'}$$

where for each $\mathscr{G} \in \mathscr{V}$, $p(x, \mathscr{G})$, $a(x, \mathscr{G})$, $b(x, \mathscr{G})$, and $c(x, \mathscr{G})$ are polynomials such that $\mathrm{dg}(p) = k + 1$, $\max\{\mathrm{dg}(a), \mathrm{dg}(b), \mathrm{dg}(c)\} \leqslant k - 1$, and the correspondences $\mathscr{G} \mapsto a, b, c, p$ are continuous.

These properties follow easily from Propositions 4 and 5. They imply that we can choose $\mathscr{V}$ satisfying the following additional properties:

(c) If $\mathscr{G} \in \mathscr{V}$, then it has $k+1$ invariant fibers $\{x_j(\mathscr{G})\} \times \overline{\mathbf{C}}$, $0 \leqslant j \leqslant k$, where $x_j(\mathscr{G}) \in D_j$ and the map $\mathscr{G} \mapsto x_j(\mathscr{G})$ is continuous for all j. This follows from the fact that the roots of $p(x, \mathscr{F}) = 0$ are simple.

(d) The holonomy of $\mathscr{G} \in \mathscr{V}$ in the section $\{q\} \times \overline{\mathbf{C}}$ is generated by transformations $f^{\mathscr{G}_1}, \cdots, f_k^{\mathscr{G}}$, where $f_j^{\mathscr{G}} \in \mathscr{U}_j$ is the holonomy element of $\mathscr{G}$ relative to $\gamma_j \in \pi_1(\overline{\mathbf{C}} - \bigcup_{i=0}^k D_i, q)$ and the correspondence $\mathscr{G} \mapsto f_j^{\mathscr{G}}$ is continuous. This follows from (b) and the fact that the curves $\gamma_0, \cdots, \gamma_k$ are fixed.

(e) For each $\mathscr{G} \in \mathscr{V}$ the group $[f_1^{\mathscr{G}}, \cdots, f_k^{\mathscr{G}}]$ is conjugated to $[f_1, \cdots, f_k]$ by a homeomorphism $h_{\mathscr{G}}$ of $\overline{C}$ such that $\lim_{\mathscr{G} \to \mathscr{F}} h_{\mathscr{G}} = I$. This follows from Sullivan's results [15].

We remark that the fact that all groups $[f_1^{\mathscr{G}}, \cdots, f_k^{\mathscr{G}}]$ are discrete implies that $h_{\mathscr{G}}$ conjugates $f_j^{\mathscr{G}}$ with f_j for each j. This follows also from Sullivan's techniques.

Another fact that we shall use here is that there exist coordinate systems (x, v) and $(x, \hat{v})$ in $D_j \times \overline{\mathbf{C}}$, $\hat{v} = 1/v$, such that $\mathscr{G}/D_j \times \overline{\mathbf{C}}$ can be expressed as

$$(20) \qquad \frac{dx}{dT} = x - x_j(\mathscr{G}), \quad \frac{dv}{dT} = \alpha_j(\mathscr{G})v, \quad \left(\frac{d\hat{v}}{dT} = -\alpha_j(\mathscr{G})\hat{v}\right),$$

where $\mathscr{G} \mapsto \alpha_j(\mathscr{G})$ is continuous and $\alpha_j(\mathscr{F}) = \alpha_j$.

To prove this, observe first that $p(x, \mathscr{G}) = (x - x_j(\mathscr{G}))Q(x, \mathscr{G})$, where $Q(x, \mathscr{G}) \neq 0$ if $x \in D_j$. Therefore we can divide the right members of (17′) by $Q(x, \mathscr{G})$, thus obtaining a local expression for $\mathscr{G} \mid D_j \times \overline{\mathbf{C}}$ of the form

$$(17'') \quad \frac{dx}{dT} = x - x_j(\mathscr{G}), \qquad \frac{dy}{dT} = A(x, \mathscr{G}) + B(x, \mathscr{G})y + C(x, \mathscr{G})y^2.$$

Now, observe that $\mathscr{G} \mid D_j \times \overline{\mathbf{C}}$ has two invariant manifolds of the form $y = \alpha_1(x)$ and $y = \alpha_2(x)$, which pass through the singularities of $\mathscr{G}$ in $\{x_j(\mathscr{G})\} \times \overline{\mathbf{C}}$. The change of variables $u = (y - \alpha_1(x))/(y - \alpha_2(x))$ changes (17″) to the form

$$(19') \qquad \frac{dx}{dT} = x - x_j(\mathscr{G}), \qquad \frac{du}{dT} = \alpha(x, \mathscr{G})u.$$

Let $\alpha(x_j(\mathscr{G}), \mathscr{G}) = \alpha_j(\mathscr{G})$ and consider the change of variables $u = ve^{\varphi(x, \mathscr{G})}$, where

$$\frac{\partial \varphi}{\partial x}(x, \mathscr{G}) = \big(\alpha(x, \mathscr{G}) - \alpha_j(\mathscr{G})\big)/\big(x - x_j(\mathscr{G})\big).$$

An easy computation shows that if we make this change of variables in (19′), then we get (20).

Now let us construct a topological equivalence between $\mathcal{F}$ and $\mathcal{G} \in \mathcal{V}$. We start from a conjugation $h_{\mathcal{G}}$ between the holonomies of $\mathcal{F}$ and $\mathcal{G}$ in the section $\Sigma_q = \{q\} \times \overline{\mathbf{C}}$. These holonomies are generated by $f_1, \cdots, f_k$ and $g_1, \cdots, g_k$ respectively, where $g_j = f_j^{\mathcal{G}}$, and we have $h \circ f_j = g_j \circ h$, $j = 1, \cdots, k$, $h = h_{\mathcal{G}}$. Let us extend h to a homeomorphism H: $W \times \overline{\mathbf{C}} \to W \times \overline{\mathbf{C}}$ which preserves fibers, where $W = \overline{\mathbf{C}} - \bigcup_{j=0}^{k} D_j$.

Let $(x, y) \in W \times \overline{\mathbf{C}}$ and join the points x and q by a curve β in W. Since $\mathcal{F} \mid W \times \overline{\mathbf{C}}$ is transverse to the vertical fibers, lift β to a curve β_y in the leaf of $\mathcal{F} \mid W \times \overline{\mathbf{C}}$ which covers β. Then $\beta_y(0) = (x, y)$, $\beta_y(1) = (q, y') \in \sigma_q$. Take $(q, y'') = h(q, y')$ and consider the lifting $\beta_{y''}^{-1}$ of the curve β^{-1} in the leaf of $\mathcal{G} \mid W \times \overline{\mathbf{C}}$ through (q, y''). Then $\beta_{y''}^{-1}(0) = (q, y'')$ and $\beta_{y''}^{-1}(1) = (x, y''')$. Using the fact that h is a conjugation between the holonomy groups, it can be proved that y''' does not depend on the curve β chosen and that the correspondence $(x, y) \mapsto (x, y''') = H(x, y)$ is a homeomorphism (cf. [6]).

Now let us consider the restriction $H_j = H \mid \partial D_j \times \overline{\mathbf{C}}$: $\partial D_j \times \overline{\mathbf{C}} \hookleftarrow$. Our problem is to extend H_j to the interior of $D_j \times \overline{\mathbf{C}}$. Observe that if $\mathcal{G}$ is near $\mathcal{F}$, then H_j is near the identity. This follows from the construction. For the sake of simplicity let us suppose that $x_j = 0 = x_j(\mathcal{G})$ and that $D_j = \{x;\ |x| \leqslant 1\}$. Take coordinate systems (x, u) and (x, v) such that $\mathcal{F}/D_j \times \overline{\mathbf{C}}$ are expressed by vector fields $X(x, u) = (x, \alpha_j u)$ and $Y(x, v) = (x, \tilde{\alpha}_j v)$, where $\tilde{\alpha}_j = \alpha_j(\mathcal{G})$ is near α_j.

Let us divide $D_j \times \overline{\mathbf{C}}$ in two polydisks $B_0 = \{(x, u);\ x \in D_j,\ |u| \leqslant 1\}$ and $B_\infty = \{(x, u);\ x \in D_j,\ |u| \geqslant 1\}$. Let $T = \{(x, u);\ x \in D_j,\ |u| = 1\} = \partial B_0 \cap \partial B_\infty$. Then T is a solid torus and the foliation $\hat{\mathcal{F}}$ of T, obtained by intersecting the leaves of X with T, consists of one closed leaf $\gamma = \{(x, u);\ x = 0,\ |u| = 1\}$ and all other leaves are transverse to the boundary and have γ as limit set (see Figure 5). The same is true for the vector field Y and $\hat{T} = \{(x, v);\ x \in D_j,\ |v| = 1\}$.

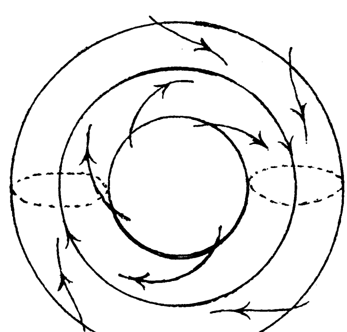

FIGURE 5

Let $V = H_j(\partial T)$. Then V is a topological 2-torus which is near ∂T, since H_j is near the identity. Since V is topologically transverse to $\mathscr{G}$, we can obtain a tubular neighborhood U of V, whose fibers are leaves of $\mathscr{G}\,|\,U$. This neighborhood can be constructed by covering V with a finite number of trivialization charts of $\mathscr{G}$. Since we are supposing H_j near the identity, we can suppose that $\partial T \subset U$, so that if we take $\tilde{V} = \{(x, v);\ |x| = r,\ |v| = 1\}$, where $r < 1$ is near 1, then $\tilde{V} \subset U$ and $\tilde{V}$ intersects each leaf of $\mathscr{G}\,|\,U$ in exactly one point. If L_p is the leaf of $\mathscr{G}\,|\,U$ through $p \in V$, then $L_p \cap \tilde{V}$ is a point $(x(p), v(p))$, where $p \mapsto (x(p), v(p))$ is continuous with p and $|x(p)| = r$, $|v(p)| = 1$. Since L_p is diffeomorphic to a disk for all p, we can join p to $(x(p), v(p))$ by a path inside L_p, say $\rho(t, p) = (x(t, p), v(t, p))$, where $\rho(0, p) = p$, $\rho(1, p) = (x(p), v(p))$, $(t, p) \mapsto \rho(t, p)$ is continuous, $t \mapsto \rho(t, p)$ is C^∞, and $t \mapsto |x(t, p)|$ is decreasing with t. Let $\tilde{T} = \rho([0,1] \times V) \cup \{(x, v);\ |x| \leqslant r,\ |v| = 1\}$. It follows that $\tilde{T}$ is a topological solid torus such that $\partial\tilde{T} = V$ and the real foliation $\hat{\mathscr{G}}$, obtained by intersecting the leaves of Y with $\tilde{T}$, has one closed leaf $\tilde{\gamma} = \{(x, v);\ x = 0,\ |v| = 1\}$ and all other leaves are transverse to $\partial\tilde{T}$ and have $\tilde{\gamma}$ as limit set. By using the foliations $\hat{\mathscr{F}}$ and $\hat{\mathscr{G}}$ constructed above, it is not difficult to extend H_j to T, in such a way that H_j sends leaves of $\hat{\mathscr{F}}$ onto leaves of $\hat{\mathscr{G}}$ and $H_j(T) = \tilde{T}$.

Now $\tilde{T}$ divides $D_j \times \overline{\mathbf{C}}$ in two regions, say $\tilde{B}_0$ and $\tilde{B}_\infty$, where $\{v = 0\} \subset \tilde{B}_0$ and $\{v = \infty\} \subset \tilde{B}_\infty$. The idea for extending H_j to B_0, for example, is to prove the existence of real vector fields X^0 and Y^0 with the following properties:

(a) X^0 and Y^0 are tangent to $\mathscr{F}$ and $\mathscr{G}$ respectively.

(b) The ω-limit set of any orbit of X^0 in B_0 is the singularity $\{x = 0, u = 0\}$ and the ω-limit set of any orbit of Y^0 in $\tilde{B}_0$ is $\{x = 0, v = 0\}$.

Let us suppose for a moment the existence of such X^0 and Y^0. Let X_t^0 and Y_t^0 be the flows of X^0 and Y^0 respectively. Given $p \in B_0 - \{(0,0)\}$, there exists a unique $t(p) \leqslant 0$ such that $p' = X^0_{t(p)}(p) \in \partial B_0$. Define $H_j(p) = Y^0_{-t(p)}(p')$. It is not difficult to see that $H_j\colon B_0 - \{(0,0)\} \to \tilde{B}_0 - \{(0,0)\}$ is a homeomorphism which sends leaves of $\mathscr{F}$ onto leaves of $\mathscr{G}$. Moreover, since $\lim_{p \to (0,0)} t(p) = -\infty$, it follows that

$$\lim_{p \to (0,0)} H_j(p) = \lim_{p \to (0,0)} Y^0_{-t(p)}(p') = (0,0)$$

and hence H_j extends to B_0.

The construction of X^0 is immediate: take $X^0(x, u) = \lambda X(x, u) = (\lambda x, \lambda\alpha_j u)$, where $\mathrm{Re}(\lambda) < 0$ and $\mathrm{Re}(\lambda\alpha_j) < 0$. This is possible because $\alpha_j \notin \mathbf{R}$ (X is of Poincaré type). The difficulty for constructing Y^0 is that $\partial\tilde{B}_0$ has a part which is only continuous, namely $\partial\tilde{B}_0 \cap U$. This difficulty can be bypassed by constructing a real vector field Y^1 on $\tilde{B}_0 \cap U$ satisfying the following properties:

(i) Y^1 is tangent to the leaves of $\mathscr{G}$.

(ii) Y^1 is transverse to $\partial \tilde{B}_0 \cap U$ and points to the interior of $\tilde{B}_0$.

(iii) The orbit of Y^1 through a point $p \in \partial \tilde{B}_0 \cap U$ leaves U in a finite time.

(iv) $Y^1 = \tilde{\lambda} Y$ in a neighborhood of ∂U, where $\mathrm{Re}(\tilde{\lambda}) < 0$ and $\mathrm{Re}(\tilde{\lambda}\tilde{\alpha}_j) < 0$.

Clearly Y^1 can be extended to a vector field Y^0 which satisfies (a) and (b). We leave to the reader the work of constructing Y^1. As a suggestion we observe that:

(A) Each leaf L_p of $\mathscr{G}/U$ intersects $\partial \tilde{B}_0$ in a piecewise C^∞ curve, with two vertices and three C^∞ segments, namely: $L_p \cap (\partial D_j \times \mathbf{C}) \cap \tilde{B}_0$, the curve $t \mapsto \rho(t, p)$, and $L_p \cap \{(x, v);\ |v| = 1,\ |x| \leqslant r\}$.

(B) $\partial \tilde{B}_0 = \partial B_0$ in a neighborhood of ∂U and so the real vector field $\tilde{\lambda} Y$ is transverse to $\partial \tilde{B}_0$ in a neighborhood of ∂U.

(C) U is a tubular neighborhood of V, with fibers L_p, $p \in V$.

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Instituto de Matemática Pura e Aplicada
Rio de Janeiro, Brasil