J. DIFFERENTIAL GEOMETRY
25 (1987) 23–53

THE POLYSYMPLECTIC HAMILTONIAN FORMALISM IN FIELD THEORY AND CALCULUS OF VARIATIONS I: THE LOCAL CASE

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Abstract

An invariant geometric Hamiltonian formalism for multiple integral variational problems and field theories is presented. The formalism is based on the notion of a polysymplectic form, which is a vector valued generalization of symplectic forms. Hamiltonian equations, canonical transformations, Lagrange systems, symmetries, field theoretic momentum mappings and a classification of G-homogeneous field theoretic systems on a generalization of coadjoint orbits are investigated.

Introduction

The subject of this article is the Hamiltonian formalism for multiple integral variational problems and field theory.

In classical mechanics (see e.g. [1], [3]), the Hamiltonian formalism for single integral variational problems is the central structure and the base for the theory of symmetries, for statistical mechanics and for quantum mechanics. The geometric setting of the Hamiltonian formalism in terms of symplectic geometry lead to substantial progress, particularly in systems with symmetry groups, interaction models with gauge fields (minimal coupling), and the relation between classical and quantum systems. Thus Hamiltonian systems on symplectic manifolds are now the generally accepted fundamental frame for the dynamics of particle theories including quantum mechanics.

For field theory such a frame has been missing. Many of the quandries of quantum field theory may be due in part to the lack of a satisfactory

Received April 14, 1986.
Key words and phrases. Hamiltonian formalism, calculus of variations, field theory.

Hamiltonian formalism. While locally the Hamiltonian equations for multiple integral variational problems have been known for a long time, their variant geometric meaning has never been fully clarified and a concise formalism based on these equations seems to be still missing.

In this article the Hamiltonian formalism for field theories and multiple integral variational problems is presented in a global geometric setting. The power of the Hamiltonian formalism in mechanics is a consequence of some natural properties of the theory, which an appropriate field theoretic formalism should share. In order to provide a fundamental frame for field theories, a multidimensional Hamiltonian formalism should satisfy the following conditions.

H.0. For each field system an evolution space can be constructed, which describes the states of the system completely.

H.1. The evolution space carries a geometric structure, which assigns to each function (Hamiltonian density) its Hamiltonian equations.

H.2. The geometry of the evolution space gives 'canonical transformations,' i.e. the general symmetry group of a system independently of the choice of a Hamiltonian density.

H.3. The formalism is covariant, i.e. no special coordinates or coordinate systems on the parameter space are used to construct the Hamiltonian equations.

H.4. There is an equivalence between regular Lagrange systems and certain (regular) Hamiltonian systems.

H.5. For one dimensional parameter space the theory reduces to the ordinary Hamiltonian formalism on symplectic manifolds in classical mechanics.

The formalism presented in this article meets all six criteria and the construction of the phase- or evolution-space and the Hamiltonian equations is canonical in the same sense as in mechanics.

The key idea for this generalized Hamiltonian formalism is to replace the symplectic form in classical mechanics by a vector valued, so called polysymplectic form. The evolution space of a classical field will appear as the dual of a jet bundle, which carries naturally a polysymplectic structure. The polysymplectic form will assign to each function on the evolution space the Hamiltonian equations via the 'musical morphisms.' Canonical transformations are bundle isomorphisms leaving this polysymplectic form invariant.

This paper treats the local case, i.e. the Hamiltonian formalism for fields as functions or sections in a trivial fiber bundle. The global case, the Hamiltonian formalism for fields as sections in an arbitrary fiber bundle, will be studied in the forthcoming second part. In §1, field theoretic phase spaces are introduced as 'homogeneous cojet spaces,' which are generalizations of the cotangent bundles. In §2, polysymplectic manifolds are introduced and the canonical polysymplectic structure on the cojet bundle is constructed. The Hamiltonian equations on polysymplectic manifolds are studied in §3. In §4, the equivalence of Hamiltonian and Lagrangian systems (with the usual restrictions) is proven. §5 treats infinitesimal canonical transformations that, in the multiple integral case, are not identical with Hamiltonian systems. §§6 and 7 study systems with symmetries. In §6 polysymplectic actions are studied, a generalized Noether theorem is proven, and momentum mappings for fields are introduced. §7 presents a reduction theorem for polysymplectic manifolds and culminates in a classification of homogeneous polysymplectic spaces by generalized coadjoint orbits. Finally, in §8, a few remarks on the global case are included.

The polysymplectic approach not only recovers all classical results and concepts of local field theory in a global and much more transparent theory, but leads globally and locally to many new results generalizing the Noether theorem based on canonical transformations, the existence of momentum mappings, the Lie algebra structure of the space of currents, the reduction procedure, and the classification of G-homogeneous systems.

There have been several other approaches to Hamiltonian field theories. None of them satisfies all conditions H.0–H.5. The "3 + 1 formalism" singles out a special time coordinate and uses methods of infinite dimensional symplectic geometry [1]. While this is very successful for Galilean systems, the approach is not covariant, i.e. does not satisfy H.3, and, therefore, causes problems in relativistic theories.

A natural geometric approach is the Hamilton-Cartan formalism by Goldschmidt and Sternberg [24] and Ouzilou [39]. Their approach does not satisfy H.1 and H.2. The theories of Garcia and Perez-Rendon and the 'multisymplectic' approach by Kijowski and Tulczyjew, based on a more general theory by Dedecker, also do not satisfy H.1 and H.2. The bibliography of the present article tries to give a survey of these geometric approaches.

This work was inspired by the symplectic formulation of classical mechanics [1] and [51] and by the work of Edelen [9] and Rund [48] on a local Hamiltonian formulation of field theory. The article of Edelen can be seen now as a coordinate version of the local polysymplectic approach, which is presented in the present paper.

The mathematical framework developed in this paper is used in a separate publication to provide a rigorous foundation for field theory. This will also help to clarify some of the present conceptual problems in quantum field theory.

I thank G. Emch for his continuous encouragement, his constructive critical remarks and patient interest during all stages of this work.

Notation. The notation follows Abraham and Marsden [1] with some modifications. In particular, we denote by $\mathscr{F}M$ the smooth functions on M and by $\mathscr{X}M$ the vector fields on M. In general, bundles are denoted by Latin types and the sections in a bundle by script types. The space of bounded linear maps between two Banach spaces $\mathbf{Q}$ and $\mathbf{S}$ are denoted by $Lin(\mathbf{Q}, \mathbf{S})$ while $\mathrm{Hom}(E, F)$ is the bundle of fiberwise linear maps between two vector bundles E and F over the same base.

In the following, let $U \subset \mathbf{R}^n$ be an open neighborhood of the origin, and let *vol* be a volume form on U. U will be the *parameter space* of the theory, in the case of classical field theory U is a neighborhood in Minkowski-space. Classical mechanics is included as the special case $n = 1$. Let Q be an arbitrary smooth Banach manifold with typical tangent space $\mathbf{Q}$, a reflexive Banach space. Q is the space in which the field takes its values. As an example, for scalar fields $Q = \mathbf{R}$. In the special case $n = 1$, which represents classical Newtonian mechanics, Q is the configuration space, and U a time interval. In classical theories, Q is finite dimensional; the infinite dimensional case is included here for possible applications to quantum field theory.

Fields will be considered as sections in the trivial fiber bundle $E = U \times Q$ over U, i.e. as maps $U \overset{\varphi}{\to} Q$. Denote by $p: E \to U$ the canonical projection.

1. Homogeneous jets and cojets

Since the field equations contain the values of the fields and their first derivatives, the space Q of all field values together with all derivatives of fields $\varphi: U \to Q$ will be made into a manifold, which is a generalization of the velocity space in classical mechanics. This is done by the following standard procedure:

1.1. On the set of smooth maps $\psi: U \to Q$ an equivalence relation $\sim$ is defined by:

$$\psi_1 \sim \psi_2 \quad \text{if and only if} \quad T_0\psi_1 = T_0\psi_2.$$

$T_0\psi$ denotes the tangent map of ψ at the origin $O \in U \subset \mathbf{R}^n$.

The equivalence classes $[\psi]$ are the *one-jets* of smooth maps $U \to Q$.

Define I^nQ as the collection of all these equivalence classes. By standard arguments I^nQ is a smooth vector bundle over Q, with typical fiber $Lin(\mathbf{R}^n, \mathbf{Q})$ (the bounded linear maps $\mathbf{R}^n \to \mathbf{Q}$). I^nQ is called the *homogeneous* 1*-jet bundle*. By construction

$$I^nQ \cong \mathrm{Hom}((\mathbf{R}^n, TQ) \simeq TQ \otimes \mathbf{R}^{n*},$$

where $\mathrm{Hom}(\mathbf{R}^n, TQ)$ denotes the bundle of linear maps from $\mathbf{R}^n$ into the tangent spaces of Q. Let τ_Q^n be the *natural projection* $I^nQ \to Q$.

Note. If a base $e_1, \cdots, e_n$ of $\mathbf{R}^n$ is chosen, there is the isomorphy: $I^nQ \simeq \oplus_1^n TQ$. Therefore, elements of I^nQ can also be interpreted as *n-tangentvectors* of Q.

1.2. For any smooth map $\psi\colon U \to Q$, $T\psi\colon TU \to TQ$ is a smooth section $U \to I^nQ$ along ψ and $T_u\psi \in \mathrm{Hom}(\mathbf{R}^n, TQ)$ for $u \in U$.

$$\begin{tikzcd} & Q \arrow[d, "\tau_Q^n"] \\ U \arrow[ur, "T\psi"] \arrow[r, "\psi"'] & Q \end{tikzcd}$$

Sections $Q \overset{X}{\to} I^nQ$ are considered to be partial differential operators $X\colon\mathscr{F}(Q, \mathbf{R}^{n*})$ by $X(f)(q) = df \circ X_q \in \mathbf{R}^{n*}$ or partial differential equations for sections $\psi\colon U \to Q$.

$\psi\colon U \to Q$ is a *solution* of $X \in \mathscr{S}ec\, I^nQ$ iff $T_u\psi = X(\psi(u))$ for all $u \in U$.

1.3. By Frobenius' theorem $X \in \mathscr{S}ec\, I^nQ$ has a solution, i.e. is *integrable* iff for the canonical base $\partial/\partial X_1 \cdots \partial/\partial X_n$ of $\mathbf{R}^n$ with injections $i_1 \cdots i_n\colon \mathbf{R} \hookrightarrow \mathbf{R}^n$ the vector fields on Q, $X \circ i_j$, $X \circ i_k$, commute for all $j, k = 1, \cdots, n$. In other words, X is a Lie algebra morphism $\mathbf{R}^n \to \mathscr{X}Q$. By the theorem of Palais [40], in this case X has a local solution flow: $U \times V \overset{\psi}{\to} Q$, $V = V^0 \subset Q$, $T_u\psi(q) = X(\psi(q))$, $\psi_u\colon V \to \psi_u(V)$ is a diffeomorphism, and $\psi_{u+u'}(q) = \psi_u \circ \psi_{u'}(q)$.

1.4. Functoriality of I^n. If $f\colon Q_1 \to Q_2$ is a smooth map, then $I^nf\colon I^nQ_1 \to I^nQ_2$, $I^nf(X_q) = [f \circ \psi]_{f(q)}$ is a vector bundle morphism. ($\psi\colon U \to Q$ is the representative of X_q as in 1.1.) Obviously I^n is a covariant functor this way. For $n = 1$, $I^1 = T$, i.e. $I^1Q = TQ$.

1.5. Definition. Denote by $\mathscr{H}om(\mathbf{R}^n, TQ) = \mathscr{S}ec\, I^nQ$ the smooth sections of I^nQ. The 'pointwise' commutator

$$[X, Y](q)(x) := [X(x), Y(x)](q),$$
$$X, Y \in \mathscr{H}om(\mathbf{R}^n, TQ),\ x \in \mathbf{R}^n,\ q \in Q,$$

defines a Lie algebra structure on $\mathscr{H}om(\mathbf{R}^n, TQ)$.

1.6. Definition. Let $\alpha \in \mathscr{A}^pQ$ be a p-form on Q, and let $X \in \mathrm{Hom}(\mathbf{R}^n, TQ)$.

$$X \lrcorner \alpha \in \mathscr{A}^{p-1}(Q, \mathbf{R}^{n*}) = \mathscr{H}om\left(\mathbf{R}^n, \bigwedge^{p-1} T^*Q\right)$$

is defined by $(X \lrcorner \alpha)(x) = X(x) \lrcorner \alpha$ ($\lrcorner$ is the inner product or contraction.) $\mathbf{L}_X\alpha \in \mathscr{A}^p(Q, \mathbf{R}^{n*}) = \mathscr{H}om(\mathbf{R}^n, \wedge^p T^*Q)$, the Lie derivative, is also 'pointwise' defined: $\mathbf{L}_X\alpha(x) := \mathbf{L}_{X(x)}\alpha$.

1.7. Lemma. $[X,Y]\lrcorner\alpha = \mathbf{L}_X Y\lrcorner\alpha - Y\lrcorner d\alpha$, $\mathbf{L}_X\alpha = X\lrcorner d\alpha + dX\lrcorner\alpha$, $\mathbf{L}_X f = df\circ X = 0$ *iff f is constant along any solution of X.*

Sections in I^nQ play the role of directional fields. In classical mechanics $\mathscr{H}om(\mathbf{R}^n, TQ)$ reduces to the vector fields on the configuration space Q. For the Hamiltonian formulation covectors have to be introduced, the field theoretic generalization are cojets:

1.8. Definition. The bundle $\mathrm{Hom}(TQ,\mathbf{R}^n)$ over Q is called *the homogeneous cojet bundle*.

Note. We have $\mathrm{Hom}(TQ,\mathbf{R}^n) \simeq T^*Q\otimes\mathbf{R}^n$. With respect to a base of $\mathbf{R}^n$, $\mathrm{Hom}(TQ,\mathbf{R}^n) \simeq \oplus_1^n T^*Q$.

Denote the sections of $\mathrm{Hom}(TQ,\mathbf{R}^n)$ by $\mathscr{H}om(TQ,\mathbf{R}^n)$. $\mathrm{Hom}(TQ,\mathbf{R}^n)$ will in the Hamiltonian formalism of field theory play the same role as the phase space in mechanics. In particular, $\mathrm{Hom}(TQ,\mathbf{R}^n)$ is the dual of $\mathrm{Hom}(\mathbf{R}^n,TQ)$; the duality is given by the trace of linear maps $\mathbf{R}^n\to\mathbf{R}^n$:

1.9. Proposition. *There are natural isomorphisms*

$$\mathrm{Hom}(\mathbf{R}^n,TQ)\to\mathrm{Hom}(TQ,\mathbf{R}^n)^*, \qquad X_q\mapsto \mathrm{tr}(\cdot\circ X_q),$$

$$\mathrm{Hom}(TQ,\mathbf{R}^n)\to\mathrm{Hom}(\mathbf{R}^n,TQ)^*, \qquad \varphi_q\mapsto \mathrm{tr}(\varphi\circ\cdot),$$

where $\cdot$ denotes the variable in $\mathrm{Hom}(TQ,\mathbf{R}^n)$, *resp.* $\mathrm{Hom}(\mathbf{R}^n,TQ)$, *and the trace is defined as* $\mathrm{tr}\,1 = 1$ (*normalized*). *Therefore, the cojet bundle can be interpreted as the dual of I^nQ.*

2. Polysymplectic structures

The keystone of the Hamiltonian formalisms of classical mechanics is the symplectic structure on the phase space. For field theory we will show that this statement remains true, provided the symplectic form is replaced by a vector valued form, which will be called polysymplectic.

In analogy to the canonical forms on the cotangent bundle, the cojet space $\mathrm{Hom}(TQ,\mathbf{R}^n)$ carries a natural $\mathbf{R}^n$-valued one-form Θ_0 and the associated two-form $\Omega_0 = -d\Theta_0$: Let Q be a reflexive Banach manifold. Consider the canonical projections $\tau_Q^{*n}\colon\mathrm{Hom}(TQ,\mathbf{R}^n)\to Q$ and $\tau_Q\colon TQ\to Q$, and the tangent map of $\tau_Q^{*n}\colon T\,\mathrm{Hom}(TQ,\mathbf{R}^n)\to TQ$ and $\tau_{\mathrm{Hom}}\colon T\,\mathrm{Hom}(TQ,\mathbf{R}^n)\to \mathrm{Hom}(TQ,\mathbf{R}^n)$:

$$\begin{tikzcd}
& T(\mathrm{Hom}(TQ,\mathbf{R}^n)) \arrow[dl, "\tau_{\mathrm{Hom}}"'] \arrow[dr, "T\tau_Q^{*n}"] & \\
\mathrm{Hom}(TQ,\mathbf{R}^n) \arrow[dr, "\tau_Q^{*n}"'] & & TQ \arrow[dl, "\tau_Q"] \\
& Q &
\end{tikzcd}$$

A $\mathbf{R}^n$-valued one-form Θ_0 on $\mathrm{Hom}(TQ, \mathbf{R}^n)$ is defined by

$$\Theta_0\left(W_{m_q}\right) = m_q \circ T\tau_Q^{*n}\left(W_{m_q}\right),$$

where $W_{m_q} \in T\,\mathrm{Hom}(TQ, \mathbf{R}^n)$ and $m_q = \tau_{\mathrm{Hom}}(W_{m_q}) \in \mathrm{Hom}(TQ, \mathbf{R}^n)$.

2.1. Definition. Θ_0 is the *canonical one-form* and $\Omega_0 := -d\Theta_0$ is the *canonical polysymplectic form* on $\mathrm{Hom}(TQ, \mathbf{R}^n)$.

2.2. Proposition. *Ω_0 is closed and nondegenerate in the sense that $\Omega_0^b : TM \to \mathrm{Hom}(TQ, \mathbf{R}^n)$, $\Omega_0^b(v)(W) = \Omega_0(v_1 w)$ ($M := \mathrm{Hom}(TQ, \mathbf{R}^n)$), is injective.*

Proof. We introduce *natural coordinates* on $\mathrm{Hom}(TQ, \mathbf{R}^n) : (q, p)$ with $q : Q \supset V \to q(V) \subset \mathbf{Q}$ a chart of Q and

$$(q, p) : V \times Lin(\mathbf{Q}, \mathbf{R}^n) \to q(V) \times Lin(\mathbf{Q}, \mathbf{R}^n),$$

$$V \times Lin(\mathbf{Q}, \mathbf{R}^n) \simeq \left(\tau_Q^{*n}\right)^{-1}(V).$$

Similarly, we introduce the natural coordinates for $TM : (q, p, \dot{q}, \dot{p})$.

In these coordinates we have

$$\theta_0(q, p; \dot{q}, \dot{p}) = \tau_M(q, p; \dot{q}, \dot{p}) \circ T\tau_Q^{n*}(q, p; \dot{q}, \dot{p}) = (q, p) \circ (q, \dot{q}) = p(\dot{q}).$$

Therefore, $\Omega_0(q, p)((\dot{q}_1, \dot{p}_1), (\dot{q}_2, \dot{p}_2)) = \dot{p}_2(\dot{q}_1) - \dot{p}_1(\dot{q}_2)$. Hahn Banach's theorem gives the desired result.

2.3. Corollary. *In the natural bundle coordinates the canonical forms on $\mathrm{Hom}(TQ, \mathbf{R}^n)$ have the local representation*:

$$\Theta_0 = \sum_{i=1}^{n} p_i dq \otimes \frac{\partial}{\partial x_i}, \qquad \Omega_0 = \sum_{i=1}^{n} dq \wedge dp_i \otimes \frac{\partial}{\partial x_i}.$$

The proof for the following result is similar to the case $n = 1$ of symplectic geometry:

2.4. Proposition. 1. *Θ_0 is the unique $\mathbf{R}^n$-valued one-form on $\mathrm{Hom}(TQ, \mathbf{R}^n)$ such that $\beta^*\Theta_0 = \beta$ for any vector valued form $\beta \in \mathscr{A}^1(\mathbf{Q}, \mathbf{R}^n)$.*

2. *If $f : Q \to Q$ is a diffeomorphism, then*

$$I^{n*}f : \mathrm{Hom}(TQ, \mathbf{R}^n) \to \mathrm{Hom}(TQ, \mathbf{R}^n).$$

$I^{n*}f(m_q) = m_q \circ I^n f$, $m_q \in Lin(T_qQ, \mathbf{R}^n) = Lin(\mathbf{R}^n, T_qQ)^*$ is a diffeomorphism and leaves Θ_0 and Ω_0 invariant, i.e.

$$\left(I^{n*}f\right)^*\Theta_0 = \Theta_0, \qquad \left(I^{n*}f\right)^*\Omega_0 = \Omega_0.$$

As in classical mechanics, the Hamiltonian formalism can be extended to situations where the phase space is not of the form $\mathrm{Hom}(TQ, \mathbf{R}^n)$ for some manifold Q. This requires the definition of a polysymplectic manifold:

2.5. Definition. A closed nondegenerate $\mathbf{R}^n$-valued two-form Ω on a manifold M is called a polysymplectic form. The pair (M, Ω) is a *polysymplectic manifold*.

Unlike the case of symplectic forms, the classification of linear polysymplectic forms is not trivial. Therefore, two polysymplectic forms are not necessarily locally equivariant. Since those forms, which can be locally represented as in 2.3, are of particular interest (those give locally "canonical" equations), so called standard forms are introduced:

2.6. Definition. A polysymplectic form Ω on a manifold M is called a *standard* form iff M has an atlas of canonical charts for Ω, i.e. charts in which locally Ω is written as the canonical evaluation form on $Q \times Lin(\mathbf{Q}, \mathbf{R}^n)$. (M, Ω) is called a *standard polysymplectic manifold.*

Note. Standard forms take the representation of 2.3 in canonical charts.

Morphisms of polysymplectic manifolds are introduced in the natural way:

2.7. Definition. Let (M_i, Ω_i), $i = 1, 2$, be polysymplectic manifolds. A smooth map $f\colon M_1 \to M_2$ is *polysymplectic* iff

$$f^*\Omega_2 = \Omega_1.$$

The group of all *polysymplectic diffeomorphisms* of (M, Ω) is denoted by $\mathrm{Pspl}(M, \Omega)$ (the polysymplectomorphism group). Sometimes the elements of $\mathrm{Pspl}(M, \Omega)$ are called *canonical transformations.*

Remark. By 2.4.2, I^{n*} defines an embedding

$$\mathit{Diff}(Q) \hookrightarrow \mathrm{Pspl}(\mathrm{Hom}(TQ, \mathbf{R}^n), \Omega_0).$$

2.8. Lemma. *Polysymplectic maps are immersions.*

Proof. If f is polysymplectic and $Tf(v) = 0$, then

$$(f^*\Omega_2)(v, w) = \Omega_2(Tf(v), Tf(w)) = 0 \quad \text{for all } w \in TM_1.$$

This implies $v = 0$ since $\Omega_1 = f^*\Omega_2$ is nondegenerate. q.e.d.

The polysymplectic structure provides the procedure which assigns to a function on M, the Hamiltonian, its associated Hamiltonian equations. The main tools are the *musical morphisms*:

2.9. Definition. Let (M, Ω) be a polysymplectic manifold.

$\Omega^b\colon TM \to \mathrm{Hom}(TM, \mathbf{R}^n)$, $\Omega^b(v_m)(w_m) = \Omega(v_m, w_m)$.

$\Omega^\#\colon \mathrm{Hom}(\mathbf{R}^n, TM) \to T^*M$, $\Omega^\#(X_m) = \mathrm{tr}(\Omega^b \circ X_m)$, where $(\mathrm{tr}(\Omega^b \circ X_m)) \cdot v_m = -\mathrm{tr}(\Omega^b(v_m) \circ X_m)$.

${}^b\colon \mathscr{X}M \to \mathscr{H}om(TM, \mathbf{R}^n) = \mathscr{S}ec\,\mathrm{Hom}(TM, \mathbf{R}^n)$, $v^b(m) = \Omega^b(v_m)$.

${}^\#\colon \mathscr{H}om(\mathbf{R}^n, TM) \to \mathscr{X}^*M = \mathscr{A}^1M$, $X^\#(m) = \Omega^\#(X(m))$.

Note that $\Omega^\#$ is the adjoint map of Ω^b by the identification of $\mathrm{Hom}(\mathbf{R}^n, TM)$ with $\mathrm{Hom}(TM, \mathbf{R}^n)^*$ by 1.9.

2.10. Remark. Define the components Ω_i of Ω by $\Omega_i = \mathrm{pr}_i \circ \Omega$, where $\mathrm{pr}_i\colon \mathbf{R}^n \to \mathbf{R}$ is the ith projection. Then each Ω_i is a presymplectic form on M. Moreover, Ω is standard iff locally:

1. $(\mathrm{Ker}\,\Omega_i^b)^\perp \simeq V \times \mathbf{Q}$, $V \subset \mathbf{Q}$ ($^\perp$ the polysymplectic annihilator).
2. $\Sigma\,\mathrm{Ker}\,\Omega_i \simeq \mathbf{Q}^*$ ($\mathbf{Q}^*$ is the Banach dual of $\mathbf{Q}$).

3. Canonical equations

In classical mechanics, a symplectic form ω on phase space assigns via the musical morphism to each function H on phase space a Hamiltonian vector field ${}^{b-1}(dH) =: X_H$, i.e. $X_H \lrcorner \omega = dH$. The flow of X_H is, in canonical coordinates, determined by the Hamiltonian equations.

For polysymplectic manifolds, a corresponding construction assigns to each function on the polysymplectic manifold (M, Ω) a system of first order partial differential equations, the Hamiltonian equations:

For $H \in \mathscr{F}M$, $dH \in \mathscr{A}^1 M$ is a one-form on M. An affine subbundle of $\mathrm{Hom}(\mathbf{R}^n, TM)$ is defined by

$$\Omega^{\#-1}(dH) := \left\{ X_m \in \mathrm{Hom}(\mathbf{R}^n, TM) \,|\, \Omega^{\#}(X_m) = dH(m) \right\}.$$

(Note that $\mathrm{Hom}(\mathbf{R}^n, TM)$ is a bundle over M.)

3.1. Definition. $\Omega^{\#-1}(dH)$ is called the *system of Hamiltonian partial differential equations associated with the Hamiltonian function H.* A smooth map $\psi: U \to M$, $U = U^0 \subset \mathbf{R}^n$, is a *solution* of $\Omega^{\#-1}(dH)$ iff $T_u\psi \in \Omega^{\#-1}(dH(\psi(u)))$ for all $u \in U$.

In order to obtain the classical local formulation of the Hamiltonian equations we have to restrict ourselves to standard forms. For standard polysymplectic forms Ω on M canonical neighborhoods of M have the form $M \supset V \simeq W \times Lin(\mathbf{Q}, \mathbf{R}^n)$. For $\psi: U \to V$, locally ψ is given by $\psi(u) = (q(u), p(u))$ with

$$q: U \to W \subset Q \quad \text{and} \quad p: U \to Lin(\mathbf{Q}, \mathbf{R}^n).$$

Then $Dq: U \to \mathrm{Hom}(\mathbf{R}^n, TW)$ and $Dp: U \to Lin(\mathbf{R}^n, Lin(\mathbf{Q}, \mathbf{R}^n))$. Therefore, $Dp: U \times \mathbf{Q} \to \mathrm{End}(\mathbf{R}^n)$. Thus taking the trace gives $\mathrm{tr}\, Dp: U \times \mathbf{Q} \to \mathbf{R}$ or $\mathrm{tr}\, Dp: U \to \mathbf{Q}^*$ (Banach-dual).

3.2. Theorem. *Let (M, Ω) be a standard polysymplectic manifold, (p, q) canonical coordinates for Ω on M, and $H \in \mathscr{F}M$ a Hamiltonian function.*

A smooth map $\psi: U \to M$ is a solution of $\Omega^{\#-1}(dH)$ iff in canonical coordinates $\psi(u) = (q(u), p(u))$ and

$$\boxed{\mathrm{tr}\, dp(u) = -\frac{\partial H}{\partial q}(\psi(u))}, \qquad \boxed{Dq(u) = \frac{\partial H}{\partial p}(\psi(u))}.$$

If a base $e_1 \cdots e_u$ of $\mathbf{R}^n$ is chosen and $p(u) = (p_1(u), \cdots, p_n(u))$ with respect to this base, then the equations take the form:

$$\boxed{\sum_{i=1}^{n} \frac{\partial p_i}{\partial x_i}(u) = -\frac{\partial H}{\partial q}(\psi(u))}, \qquad \boxed{\frac{\partial q}{\partial x_i}(u) = \frac{\partial H}{\partial p_i}(\psi(u))}.$$

These are the classical local Hamiltonian equations associated with a regular multiple integral variational problem [48]. The equivalence of these equations with the Euler-Lagrange equations in the regular case will be shown in §4.

Proof. Let $X \in \mathscr{S}ec\,\mathrm{Hom}(\mathbf{R}^n, TM)$ with $X(\psi(u)) = D\psi(u) \in Lin(\mathbf{R}^n, T_{\psi(u)}M)$. Since Ω is a standard form the fibers of TM are isomorphic with $\mathbf{Q} \times Lin(\mathbf{Q}, \mathbf{R}^n)$ and in canonical coordinates X can be written as $X(m) = X_q(m) + X_p(m)$, where $X_q(m) \in Lin(\mathbf{R}^n, \mathbf{Q})$ and $X_p(m) \in Lin(\mathbf{R}^n, Lin(\mathbf{Q}, \mathbf{R}^n))$.

$v \in \mathscr{X}M$ can be written in canonical coordinates as $v(m) = \dot{q}(m) + \dot{p}(m)$, $\dot{q}(m) \in Q$, $\dot{p}(m) \in Lin(\mathbf{Q}, \mathbf{R}^n)$. Since $\Omega^{\#}(X) \cdot v = \mathrm{tr}\,\Omega^b \circ X(v) = -\mathrm{tr}\,\Omega^b(v) \circ X$ and $\Omega^b(\dot{q}, \dot{p}) \cdot (\underline{\dot{q}}, \underline{\dot{p}}) = \underline{\dot{p}}(\dot{q}) - \dot{p}(\underline{\dot{q}})$ for $(\dot{q}, \dot{p}) \in TM$ one obtains

$$\Omega^{\#}(X)(\dot{q}, \dot{p}) = -\mathrm{tr}\big(X_p(\dot{q}) - \dot{p} \circ X_q\big).$$

Now let

$$dH = \frac{\partial H}{\partial q} dq + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} dp_i.$$

Then $dH(\dot{q}, \dot{p}) = -\mathrm{tr}(X_p(\dot{q}) - \dot{p} \circ X_q)$ is equivalent to $-\mathrm{tr}\,X_p = \partial H/\partial q$ and $\partial H/\partial p = X_q$. q.e.d.

3.3. Example (*Scalar field*). Let $n = 4$, $Q = \mathbf{R}$, and $M = \mathbf{R} \times \mathbf{R}^4$ with coordinates $(q, p_1, \cdots, p_4)$. Let $H(q_1, p_1, \cdots, p_4) = \frac{1}{2}\sum_1^4 p_i^2 + mq^2$ be a Hamiltonian on M. The canonical polysymplectic form Ω is given by

$$\Omega = \sum_1^4 dq \wedge dp_i \otimes \frac{\partial}{\partial x_i}.$$

The Hamiltonian equations for a scalar field

$$\psi(x_1, \cdots, x_4) = \big(q(x_1, \cdots, x_4),\ p_1(x_1, \cdots, x_4), \cdots, p_4(x_1, \cdots, x_4)\big)$$

are

$$\sum_{i=1}^{4} \frac{\partial p_i}{\partial x_i} = mq \quad \text{and} \quad \frac{\partial q}{\partial x_i} = p_i. \quad \text{q.e.d.}$$

The fundamental object for the field theoretic Hamilton formalism is the polysymplectic form Ω. Once a polysymplectic manifold (M, Ω) is given, a field theoretic system is (abstractly) determined and its Hamiltonian systems, infinitesimal transformations, and symmetries can be studied.

3.4. Definition. Let (M, Ω) be a polysymplectic manifold. Then

$\mathscr{H}_{\mathrm{loc}}M := \{X \in \mathscr{S}ec\,\mathrm{Hom}(\mathbf{R}^n, TM) \mid \Omega^{\#}(X) \text{ is closed}\}$,

$\mathscr{H}M := \{X \in \mathscr{S}ec\,\mathrm{Hom}(\mathbf{R}^n, TM) \mid \Omega^{\#}(X) = dH \text{ for some } H \in \mathscr{F}M \text{ (exact)}\}$,

$\mathscr{C}M := \{X \in \mathscr{S}ec\,\mathrm{Hom}(\mathbf{R}^n, TM) \mid \Omega^{\#}(X) = 0\} = \mathscr{S}ec\,\mathrm{Ker}\,\Omega^{\#}$.

For $H \in \mathscr{F}M : \mathscr{X}_H := \{X \in \mathscr{H}M \mid \Omega^{\#}(X) = dH\}$.

Elements of $\mathscr{H}M$ are the "total Hamiltonian equations" on M. An element of $\mathscr{X}_H$ is a section of the bundle $\Omega^{\#-1}(dH)$ over M. If $X \in \mathscr{X}_H$ is integrable, a unique solution $\psi: U \to M$ with $\psi(0) = m$ and $D(\psi(u)) = X(\psi(u))$ is determined. For $X_0 \in \mathscr{X}_H$, one has $\mathscr{X}_H = X_0 + \mathscr{C}M$, an affine subspace of $\mathscr{H}M$.

In the conventional case $n = 1$ of classical mechanics, $\mathscr{H}_{\text{loc}}M$ and $\mathscr{H}M$ are the local and global Hamiltonian vector fields and $\mathscr{C}M$ is reduced to zero.

3.5. Proposition. *The map $\tilde{\mathscr{X}}_\bullet: \mathscr{F}M \to \mathscr{H}M/\mathscr{C}M$ given by $\tilde{\mathscr{X}}_H = \{X + \mathscr{C}M \mid X \in \mathscr{X}_H\}$ for $H \in \mathscr{F}M$ is surjective and the sequence*

$$0 \to \mathbf{R} \hookrightarrow \mathscr{F}M \to \mathscr{H}M/\mathscr{C}M \to 0$$

is an exact sequence of $\mathbf{R}$*-vector spaces.*

We now introduce the concept of energy-momentum tensors.

3.6. Definition. Let (M, Ω) be a polysymplectic manifold, $X \in \mathscr{H}M$, and $\Omega^{\#}(X) = dH$ for $H \in \mathscr{F}M$. $\mathbf{H} \in \mathscr{F}(M, \operatorname{End}(\mathbf{R}^n))$ is called an *energy-momentum tensor* of $\mathscr{X}_H$ iff $\operatorname{tr} d\mathbf{H} = dH$.

Remark. H is an energy-momentum tensor iff $\operatorname{tr} d\mathbf{H} = \operatorname{tr}(X \lrcorner \Omega) + \text{const}$ by definition of $\Omega^{\#}$. For $X \in \mathscr{H}M$, $X \lrcorner \Omega$ is in general not closed unless X is an infinitesimal polysymplectic flow (cf. §6).

3.7. Proposition. *Let $M = \operatorname{Hom}(TQ, \mathbf{R}^n) = T^*Q \otimes \mathbf{R}^n$ and let Ω_0, Θ_0 be the canonical forms (cf. 2.1) on M. Then:*

1. *$X \in \mathscr{C}M$ implies $X \lrcorner \Theta_0 = 0$.*
2. *$X \in \mathscr{H}M$ iff $d(\operatorname{tr} \mathbf{L}_X \Theta_0) = 0$.*
3. *$X \in \mathscr{H}M$ implies $\operatorname{tr}(\mathbf{L}_X \Theta_0) = -d(H - \operatorname{tr}(X \lrcorner \Theta_0))$.*

Proof. 1. (local in canonical coordinates): If $\Theta_0 = \sum p_i dq \otimes \partial/\partial x_i$ and $X = X_q \partial/\partial q + \sum X_{p_i} \partial/\partial p_i$, then

$$X \lrcorner \Theta_0 = \sum p_i X_q \otimes \frac{\partial}{\partial x_i}.$$

But $X \in \mathscr{C}M$ iff $\operatorname{tr}(X \lrcorner \Omega) = 0$ iff

$$\left(X_q \frac{\partial}{\partial q} + \sum X_{p_i} \frac{\partial}{\partial p_i} \right) \lrcorner \Omega = 0,$$

$$\left(X_q \frac{\partial}{\partial q} + \sum X_{p_i} \lrcorner \frac{\partial}{\partial p_i} \right) \lrcorner \Omega = \sum \left(\left(X_q \circ u_i \right) dp_i - \left(X_{p_i} \circ u_i \right) dq, \right.$$

where u_i are the coordinate embeddings $\mathbf{R} \hookrightarrow \mathbf{R}^n$. This implies $X_q = 0$ and $\sum p_i \circ u_i = 0$. Thus $X \lrcorner \Theta_0 = 0$.

2. $\operatorname{tr} \mathbf{L}_X \Theta_0 = \operatorname{tr}(dX \lrcorner \Theta_0 + X \lrcorner d\Theta_0)$ is exact iff $X \lrcorner d\Theta_0$ is exact.

3. $\operatorname{tr}((dX \lrcorner \Theta_0) + X \lrcorner d\Theta_0) = -dH + \operatorname{tr} dX \lrcorner \Theta_0$. q.e.d.

Remark. These results remain true for standard exact polysymplectic forms.

3.8. Definition. Let $M = \mathrm{Hom}(TQ, \mathbf{R}^n)$, and let Ω_0 and Θ_0 be canonical forms. For $H \in \mathscr{F}M$ denote by d^pH the vertical derivative of H with respect to the projection $\tau_Q^{n*}: M \to Q$ (compare 2.1). Define the *inverse Legendre transformation* $\mathbf{F}H: M \to \mathrm{Hom}(\mathbf{R}^n, TQ)$ by

$$\mathrm{tr}\big(m' \circ \mathbf{F}H(m)\big) = d^pH(m) \cdot m'$$

(i.e. $\mathbf{F}H$ is the vertical derivative d^pH followed by the isomorphism between $\mathrm{Hom}(TQ, \mathbf{R}^n)^*$ and $\mathrm{Hom}(\mathbf{R}^n, TQ)$ in 1.9). H is called *regular* iff $\mathbf{F}H$ is a local diffeomorphism, and H is called *hyperregular* iff $\mathbf{F}H$ is a global diffeomorphism.

3.9. Definition. Let $H \in \mathscr{F}M$ be regular. Define the *action tensor* by $\mathbf{G}: M \to \mathrm{End}\,\mathbf{R}^n$ with $\mathbf{G}(m) = m \circ \mathbf{F}H(m)$, and the *action* of H by $G: M \to \mathbf{R}$ with $G := \mathrm{tr}\,\mathbf{G}$.

3.10. Proposition. *Let $X \in \mathscr{H}M$ be a Hamiltonian system, $\Omega^\#(X) = dH$, $M = \mathrm{Hom}(TQ, \mathbf{R}^n)$ and Θ_0 is the fundamental one-form. Then*

1. $X \lrcorner \Theta_0 = \mathbf{G}$.
2. $-\mathbf{L}_X\Theta_0 + \mathbf{G}$ *is the energy-momentum tensor.*

The proof depends on the following lemma.

3.11. Lemma. *Let $T\tau_Q^{n*}: TM = T\,\mathrm{Hom}(TQ, \mathbf{R}^n) \to TQ$ be the projection in* 2.1 *and $X \in \mathscr{X}_H$, i.e.* $\mathrm{tr}\,X - \Omega = dH$. *Then*

$$\boxed{T\tau_Q^{n*} \circ X = \mathbf{F}H}\,.$$

Proof. From 3.2 locally

$$X = X_q\frac{\partial}{\partial q} + X_p\frac{\partial}{\partial p}$$

and

$$X_q\frac{\partial}{\partial q} = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} = \mathbf{F}H\frac{\partial}{\partial q} : T\tau_Q^{n*}\left(X_q\frac{\partial}{\partial q} + X_p\frac{\partial}{\partial p}\right) = X_q\frac{\partial}{\partial q}.$$

This gives the result.

Proof of 3.10. 1. By the definition of $\Theta_0 : \Theta_0 \circ X(m) = m \circ T\tau_Q^{n*} \circ X$ and by the definition of $\mathbf{G}: \mathbf{G}(m) = m \circ \mathbf{F}H(m)$. 3.11 gives the result.

2. $\mathbf{L}_X\theta_0 = dX \lrcorner \Theta_0 + X \lrcorner d\Theta_0$.

$$\mathrm{tr}(\mathbf{L}_X\Theta_0 - d\mathbf{G}) = \mathrm{tr}\,dX \lrcorner \Theta_0 + \mathrm{tr}\,X \lrcorner d\Theta_0 - \mathrm{tr}\,d(X \lrcorner \Theta_0) = -dH.$$

3.12. Corollary. $\mathbf{H} = \mathbf{G} + H$ *is the energy-momentum tensor.*

Proof. $d\,\mathrm{tr}\,\mathbf{H} = d\,\mathrm{tr}\,G - d\,\mathrm{tr}\,\mathbf{G} + dH = dH.$

3.13. Example (*Scalar field*). $M = \mathbf{R} \times \mathbf{R}^4$ and $H(q, p_1, \cdots, p_4) = \frac{1}{2}\Sigma_1^2 p_i^2 = q^2$ as in 3.3. The energy-momentum tensor can be calculated by 3.12.

$$\mathbf{G}(m) = m \circ \mathbf{F}H(m) \quad \text{for } m \in M, \mathbf{F}H = \sum_1^4 p_i,$$

$$G_{ij} = p_i p_j, \qquad \operatorname{tr} \mathbf{G} = \sum_1^4 p_i^2;$$

therefore

$$\mathbf{H}_{ij} = p_i p_j - \left(\frac{1}{2} \sum_1^4 p_i^2 + mq^2 \right).$$

In the case $n = 1$ of classical mechanics $\mathbf{H}$ is, up to a constant, identical with the Hamiltonian H.

4. Lagrange theory

Originally the local form of the canonical equations (3.2) was derived as an equivalent form of the Euler-Lagrange equations associated with a variational problem with regular Lagrangian. In this section the alternative description of field equations as the Euler-Lagrange equations on I^nQ and the relation with the canonical formalism is studied.

A *Lagrangian* is defined as a smooth function $I^nQ \to \mathbf{R}$. In analogy with 3.8 the *Legendre transformation* $\mathbf{F}L$ of L,

$$\mathbf{F}L : I^nQ \to \operatorname{Hom}(TQ, \mathbf{R}^n),$$

is defined by

$$\operatorname{tr}(\mathbf{F}L(\varphi) \circ \psi) = d^{\mathrm{v}}L(\varphi) \bullet \psi,$$

where $\varphi, \psi \in I^nQ = \operatorname{Hom}(\mathbf{R}^n, TQ)$ and $d^{\mathrm{v}}L$ is the vertical derivative of L with respect to $\tau^n : I^nQ \to Q$. L is called *regular* (resp. *hyperregular*) iff $\mathbf{F}L$ is a local (resp. global) diffeomorphism.

The canonical forms Θ_0 and Ω_0 on $\operatorname{Hom}(TQ, \mathbf{R}^n)$ can be pulled back to forms Θ_L and Ω_L on I^nQ via $\mathbf{F}L$ and for regular $\mathbf{F}L$ canonical equations on I^nQ can be established; these are the *Lagrange equations*:

4.1. Definition. Let $L : I^nQ \to \mathbf{R}$ be a smooth Lagrangian and let $\mathbf{F}L : I^nQ \to \operatorname{Hom}(TQ, \mathbf{R}^n)$ be Legendre transformation. Define $\Theta_L : \mathbf{F}L^*\Theta_0$ to be the *Lagrange one-form* and $\Omega_L := \mathbf{F}L^*\Omega_0$ to be the *Lagrange two-form*.

Clearly Ω_L is a closed form and if L is regular, then Ω_L is a polysymplectic form on I^nQ, moreover, by 2.10, in this case Ω_L is standard.

In generalization of the case of classical mechanics, $n = 1$, an energy function E will now be defined, which acts as a Hamiltonian on (I^nQ, Ω_L). But like in 3.9 in the field theoretic case tensors are obtained.

4.2. Definition. Let $L: I^nQ \to \mathbf{R}$ be a Lagrangian with Legendre transformation $\mathbf{F}L: I^nQ \to I^*Q$. Define:

the action tensor by $\mathbf{A}: I^nQ \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ *with* $\mathbf{A}(\varphi) = \mathbf{F}L(\varphi) \circ \varphi$,

the action density by $A: I^nQ \to \mathbf{R}$ *with* $A = \operatorname{tr} \mathbf{A}$,

the energy-momentum tensor by $\mathbf{E}: \mathbf{A} - L$,

the energy-momentum density by $E: \operatorname{tr} \mathbf{E}$.

4.3. Proposition. *Let $\dot\varphi \in TI^nQ$, and let $\tau_{I^nQ}: TI^nQ \to I^nQ$ and $\tau_Q^n: I^nQ \to Q$ be the natural projections. Then*

1. $\Theta_L(\dot\varphi) = \mathbf{F}L(\tau_{I^nQ}\dot\varphi) \circ T\tau_Q^n(\dot\varphi)$.

2. *In natural bundle coordinates on $I^nQ: (q, y_i)$ with $y_i = \partial q/\partial x_i$, Θ_L and $\mathbf{A}$ are of the form*

$$\Theta_L = \sum_{i=1}^{n} \frac{\partial L}{\partial y_i} dq \otimes \frac{\partial}{\partial x_i}, \qquad \mathbf{A} = \sum_{i=1}^{n} y_i \frac{\partial L}{\partial y_i}.$$

Proof.

$$\begin{aligned} 1.\ \mathbf{F}L^*\Theta_0(\dot\varphi) &= \Theta_0(T\mathbf{F}L(\dot\varphi)) \\ &= \tau_{\mathrm{Hom}} \circ T\mathbf{F}L(\dot\varphi) \circ T\tau_Q^{n*} \circ T\mathbf{F}L(\dot\varphi) \quad (\text{cf. } 2.1) \\ &= \mathbf{F}L \circ \tau_{I^nQ}(\dot\varphi) \circ T\left(\tau_Q^{n*} \circ \mathbf{F}L\right)(\dot\varphi) \\ &= \mathbf{F}L \circ \tau_{I^nQ}(\dot\varphi) \circ T\tau_Q^n(\dot\varphi) \quad (\text{since } \mathbf{F}L \text{ is bundle morphism over } 1_Q). \end{aligned}$$

$$\begin{aligned} 2.\ \mathbf{F}L^*\left(\sum p_i dq \otimes \frac{\partial}{\partial x_i}\right) &= \sum (p_i \circ \mathbf{F}L)(dq \circ T\mathbf{F}L) \otimes \frac{\partial}{\partial x_i} \\ &= \sum \frac{\partial L}{\partial y_i} dq \otimes \frac{\partial}{\partial x_i}. \end{aligned}$$

4.4. Definition. Let $L: I^nQ \to \mathbf{R}$ be a Lagrangian.

$$\Omega^{n-1}(dE) \subset I^n(I^nQ)$$

are the Euler-Lagrange equations associated with L.

4.5. Remark. $I^n(I^nQ) = \operatorname{Hom}(\mathbf{R}^n, \operatorname{Hom}(\mathbf{R}^n, TQ))$ is a bundle over I^nQ and by the functionality of I^n there are two natural projections $\tau_{I^nQ}^n: I^n(I^nQ) \to I^nQ$ and $I^nI^nQ: I^n(I^nQ) \to I^nQ$:

$$\begin{tikzcd} & I^n(I^nQ) \arrow[dl, "\tau^n_{I^nQ}"'] \arrow[dr, "I^n\tau^n_Q"] & \\ I^nQ \arrow[dr, "\tau_Q"'] & & I^nQ \arrow[dl, "\tau_Q"] \\ & Q & \end{tikzcd}$$

If ψ is a smooth map $U \to Q$, then $T(T\psi)(u) \in I^n(I^nQ)$, $\tau^n_{I^nQ}(T(T\psi))(u) = T\psi(u)$, and $I^n\tau^n_Q(T(T\psi))(u) = T(\tau_Q(\psi(u))) = T\psi(u)$. In analogy with the case $n = 1$ of ordinary differential equations, a section $X: I^nQ \to I^nI^nQ$ is called a *second order section* of I^nI^nQ iff $\tau^n_{I^nQ} \circ X = I^n\tau^n_Q \circ X = 1_{I^nQ}$, i.e., X is a section for $\tau^n_{I^nQ}$ and $I^n\tau^n_Q$.

An integrable section $X: I^nQ \to I^nI^nQ$ is a *second order section* iff for every map $\sigma: U \to I^nQ$ with $T_u\sigma = X(\sigma(u))$ for all $u \in U = U^0 \subset \mathbf{R}^n$: $T(\tau^n_Q \circ \sigma) = \sigma$. $\tau^n_Q \circ \sigma$ is called the (*base*) *solution* of the second order equation X. If $X: I^nQ \to I^nI^nQ$ has the form $X(q, y) = (q, y, X_q, X_y)$ in natural coordinates, then X is a second order equation iff $y = X_q$.

4.6. Lemma. *Let $X: I^nQ \to I^nI^nQ$ be an integrable section of $\tau^n_{I^nQ}$, i.e. through every point $m \in I^nQ$ there exists a solution of $\sigma: U \to I^nQ$ of $X: T_u\sigma = X(\sigma(u))$, $u \in U = U^0 \subset \mathbf{R}^n$ and $\sigma(0) = m$.*

If X is a second order section, then X has a base solution $\psi: U \to Q$ with $T(T\psi)(u) = X(T_u\psi)$. In particular, if in natural coordinates X has the form

$$X(q, y) = (q, y, y, X_y),$$

then $X_y \in Lin(\mathbf{R}^n, Lin(\mathbf{R}^n, \mathbf{Q}))$ is symmetric.

Proof. Locally $\sigma: U \to I^nQ$ has the form $\sigma(u) = (\sigma_q(u), \sigma_y(u))$, where $\sigma_q(u) \in \mathbf{Q}$ and $\sigma_y(u) \in Lin(\mathbf{R}^n, \mathbf{Q})$. Thus locally $T_u\sigma = (\sigma_q(u), \sigma_y(u))$. If X is second order, it follows that $\sigma_y(u) = D\sigma_q(u)$. This implies $D\sigma_y(u) = DD\sigma_q(u)$. q.e.d.

A subbundle (or fibered submanifold) F of $I^nI^nQ \xrightarrow{\tau^n_{I^nQ}} I^nQ$ is called a *second order partial differential equation* iff every section X of F is a second order section of I^nI^nQ.

Now all preparations for a local formulation of the Euler-Lagrange equations are made:

4.7. Theorem. *If $\Omega^{\#-1}_L(dE)$ is a second order equation, then a smooth map $\sigma: U \to I^nQ$ ($U = U^0 \subset \mathbf{R}^n$) is a solution of the Euler-Lagrange equation $\Omega^{\#-1}_L(dE)$ iff, in natural bundle coordinates of I^nQ and I^nI^nQ, $\sigma(u) = (\sigma_q(u), \sigma_y(u))$, $u \in U$, and*

$$\frac{d}{du}\big(\sigma_q(u)\big) = \sigma_y(u), \tag{1}$$

$$\frac{d}{du}D_yL\big(\sigma_q(u), \sigma_y(u)\big) - D_qL\big(\sigma_q(u), \sigma_y(u)\big) = 0, \tag{2}$$

where D_q and D_y are the partial derivatives with respect to the coordinate decomposition $\mathbf{Q} \oplus Lin(\mathbf{R}^n, \mathbf{Q}) \supset V \hookrightarrow I^nQ$.

If $u_1, \cdots, u_n$ is a base of $\mathbf{R}^n$ *and y_i are the components of $y \in$* $\operatorname{Hom}(\mathbf{R}^n, TQ) \simeq \oplus_1^n TQ$, *then equation* (2) *can be written*

$$\sum_{i=1}^{n} \frac{d}{du_i} \frac{\partial L}{\partial y_i} - \frac{\partial L}{\partial q} = 0.$$

Proof. The proof is a generalization of the proof for the case $n = 1$ as given in Abraham and Marsden [1]:

Let $X: I^n Q \to I^n I^n Q$ be a section of $\Omega_L^{\#-1}(dE)$, i.e. $\Omega^{\#}(X) = dE$. In natural coordinates, $X = (X_q, X_y)$ with $X_q \in Lin(\mathbf{R}^n, \mathbf{Q})$, $X_y \in Lin(\mathbf{R}^n(Lin(\mathbf{R}^n, \mathbf{Q})))$. $E = \operatorname{tr} \mathbf{A} - L$. Therefore, $DE = D(\operatorname{tr} \mathbf{A}) - DL$ and by definition of $\mathbf{A}$

$$\begin{aligned} DE(q, y) \cdot (\dot{q}_1, \dot{y}_1) &= D\big(D_y L(q, y) \cdot (\dot{q}_1, \dot{y}_1)\big) \cdot y + D_y L(q, y) \cdot \dot{y}_1 \\ &\quad - D_q L(q, y) \cdot \dot{q}_1 - D_y L(q, y) \cdot \dot{y}_1 \\ &= D_q D_y L(q, y) \cdot \dot{q}_1 \cdot y + D_y D_y(q, y) \cdot \dot{y}_1 \cdot y - D_q L(q, y) \cdot \dot{q}_1. \end{aligned}$$

By definition of Ω_0

$$\begin{aligned} &\Omega_L(q, y)((\dot{q}_1, \dot{y}_1)(\dot{q}_2, \dot{y}_2)) \\ &\quad = \Omega_0(q, y)\big(D\mathbf{F}(q, y) \cdot (\dot{q}_1, \dot{y}_1),\ D\mathbf{F}L(q, y) \cdot (\dot{q}_2, \dot{y}_2)\big) \\ &\quad = D\mathbf{F}_p L(q, y)(\dot{q}_2, \dot{y}_2) \cdot \dot{q}_1 - D\mathbf{F}_p L(q, y)(\dot{q}_1, \dot{y}_1) \cdot \dot{q}_2 \end{aligned}$$

($\mathbf{F}_p L(q, y) \in Lin(\mathbf{Q} \times Lin(\mathbf{R}^n, Q)), Lin(\mathbf{Q}, \mathbf{R}^n))$ denotes the second component of $\mathbf{F}L$; note $\mathbf{F}_q L = 1_{\mathbf{Q}}$ in this notation)

$$\begin{aligned} &= D_q \mathbf{F}_p L(q, y) \cdot \dot{q}_2 \cdot \dot{q}_1 + \big(D_y \mathbf{F}_p L(q, y) \cdot \dot{y}_2\big) \cdot \dot{q}_1 \\ &\quad - D_q \mathbf{F}_p L(q, y) \cdot \dot{q}_1 \cdot \dot{q}_2 - \big(D_y \mathbf{F}_p L(q, y) \cdot \dot{y}_1\big) \cdot \dot{q}_2. \end{aligned}$$

Thus

$$\begin{aligned} \Omega_L^{\#}(q, y)(X_q, X_y) \cdot (\dot{q}, \dot{y}) &= \operatorname{tr} \Omega_L(q, y) \cdot \big((X_q, X_y), (\dot{q}, \dot{y})\big) \\ &= \operatorname{tr}\Big(\big(D_q \mathbf{F}_p L(q, y) \cdot \dot{q}\big) \circ X_q + \big(D_y \mathbf{F}_p L(q, y) \circ \dot{y}\big) \circ X_q \\ &\quad - \big(D_q \mathbf{F}_p L(q, y) \circ X_q\big) \cdot \dot{q} - \big(D_y \mathbf{F}_p L(q, y) \circ X_y\big) \cdot \dot{q}\Big) \\ &= D_q D_y L(q, y) \cdot \dot{q} \cdot X_q + D_y D_y L(q, y) \cdot \dot{y} \cdot X_q \\ &\quad - D_q D_y(q, y) \cdot X_q \cdot \dot{q} - D_y D_y L(q, y) \cdot \tilde{X}_y \cdot \dot{q}, \end{aligned}$$

where $\tilde{X}_y(x, y) = X_y(x, y)$, $x, y \in \mathbf{R}^n$.

$\Omega_L^{\#}(X) = dE$ therefore implies

$$\begin{aligned} &D_q D_y L(q, y) \cdot \dot{q} \cdot X_q + D_y D_y L(q, y) \cdot \dot{y} \cdot X_q \\ &\quad - D_q D_y L(q, y) \cdot X_q \cdot \dot{q} - D_y D_y L(q, y) \cdot \tilde{X}_y \cdot \dot{q} \\ &= D_q D_y L(q, y) \cdot \dot{q} \cdot y + D_y D_y L(q, y) \cdot \dot{y} \cdot y - D_q L(q, y) \cdot \dot{q}. \end{aligned}$$

If X is a second order section, then $X_q = y$. Then the condition $\Omega_L^\#(X) = dE$ is equivalent to

$$-D_q D_y L(q, y) \cdot y \cdot \dot{q} - D_y D_y L(q, y) \cdot \tilde{X}_y \cdot \dot{q} = -D_q L(q, y) \cdot \dot{q}.$$

If $s: U \to V \subset Q$ is a smooth map, $s(u) = q$, $Ds(u) = y$, then s is a base solution of X iff

$$D_q D_y L(s(u), Ds(u)) \cdot \dot{q} + D_y D_y L(q, y) \cdot D^2 s(u) \cdot \dot{q} = D_q L(s(u), Ds(u)),$$

or

$$\frac{d}{du} D_y L(s(u), Ds(u)) - D_q L(s(u), Ds(u)) = 0,$$

which is the Lagrange equation. q.e.d.

Note. If L is regular (i.e. $D_y D_y L$ is nondegenerate), then setting $\dot{q} = 0$ in $\Omega_L^\#(X) = dE$ implies $X_q = y$. Thus one has:

4.8. Corollary. *If L is regular, then $\Omega_L^{\#-1}(dE)$ is a second order equation.*

If L is hyperregular, the Hamiltonian and Lagrangian formulations for extremals of the variational problem $\int L \cdot \mathrm{vol} \to$ extrem. are equivalent. In particular H is hyperregular iff L is hyperrregular, and then

$$\mathbf{A} = Z_E \lrcorner \Theta_L = \mathbf{F}^* L(Z_H \lrcorner \Theta_0), \qquad LH = \mathbf{F}L^{-1}.$$

Furthermore:

4.9. Proposition. *Let $L: I^n Q \to \mathbf{R}$ be a hyperregular Lagrangian, $H = E \circ \mathbf{F}L$. Then the set of base solutions $\tau_Q^n \circ \sigma: U \to Q$, $T_u \sigma \in \Omega_L^{\#-1}(dE)$ for all $u \in U \subset \mathbf{R}^n$, and $\tau_Q^{n*} \circ \psi: U \to Q$, $T_u \psi \in \Omega_L^{\#-1}(dH)$ for $\psi: U \to I^* Q$, of the Lagrange and Hamiltonian equations are equal. Furthermore, $\mathbf{F}L$ maps solutions of $\Omega_L^{\#-1}(dE)$ into solutions of $\Omega_0^{\#-1}(dH)$. Finally, $\Omega_L^{\#-1}(dE) = \mathbf{F}L^*(\Omega_0^{\#-1}(dH))$.*

Remark. Even when L is regular, $\Omega_L^{\#-1}(dE)$ and $\Omega_0^{\#-1}(dH)$ may have solutions $U \to I^n Q$ or $U \to I^* Q$, which are not derivatives of base solutions $U \to Q$. Therefore as Hamiltonian systems, i.e. as first order equations, $\Omega_L^{\#-1}(dE)$ and $\Omega_0^{\#-1}(dH)$ have more solutions than the Lagrange equations in 4.7. Nevertheless, in the hyperregular case, $\mathbf{F}L$ still gives a 1-1 correspondence between all solutions of $\Omega_L^{\#-1}(dE)$ and $\Omega_0^{\#-1}(dH)$. Then solutions of $\Omega_0^{\#-1}(dH)$ which correspond to base solutions of the Euler Lagrange equations are characterized by

$$\mathbf{F}L \circ T\left(\tau_Q^* \circ \psi\right) = \psi \quad \text{for } \psi: U \to I^* Q.$$

5. Hamiltonian vector fields, Poisson brackets

One of the important properties of the Hamiltonian formalism in mechanics is the fact that the canonical equations describe the flow of infinitesimal morphisms, i.e. Hamiltonian vector fields. In the Hamiltonian formalism for field theories the relation between canonical equations and infinitesimal morphisms is more complicated: every polysymplectic flow is the flow of a poly-Hamiltonian system $X \in \mathscr{H}_{\mathrm{loc}}M$, but not every integrable poly-Hamiltonian system $X \in \mathscr{H}_{\mathrm{loc}}M$ has a polysymplectic flow, since, for $r \in \mathbf{R}^n$, $X(r) \in \mathscr{X}M$ is in general not an infinitesimal polysymplectic transformation.

In this section, the relation between infinitesimal symmetries and poly-Hamiltonian systems is studied.

5.1. Definition. Let (M, Ω) be a polysymplectic manifold

$$\mathscr{X}^H_{\mathrm{loc}}M := \{v \in \mathscr{X}M \mid \mathbf{L}_v\Omega = 0\}$$

be the *local Hamiltonian vector fields* (or infinitesimal morphisms), let

$$\mathscr{X}^HM := \{v \in \mathscr{X}M \mid v \lrcorner \Omega = dF \text{ for } F: M \to \mathbf{R}^n\}$$

be the *Hamiltonian vector fields*, and let

$$\mathscr{F}h(M\mathbf{R}^n) := \left\{F \in \mathscr{F}(M, \mathbf{R}^n) \mid dF(m) \in \operatorname{Image}\Omega^b_m \text{ for all } m \in M\right\}$$

be the *currents*.

5.2. Proposition. 1. $[\mathscr{X}^H_{\mathrm{loc}}M, \mathscr{X}^H_{\mathrm{loc}}M] \subset \mathscr{H}^HM$ *are, in particular,* $\mathscr{X}^H_{\mathrm{loc}}M$, *and* $\mathscr{X}^HM$ *are Lie subalgebras of* $\mathscr{X}M$.

2. *The map* $\mathscr{F}h(M, \mathbf{R}^n) \to \mathscr{X}^HM$, $F \mapsto \Omega^{b-1}(dF) := v_F$ *is well defined and the sequence of* $\mathbf{R}$-*vector spaces*

$$0 \to \mathbf{R}^n \hookrightarrow \mathscr{F}h(M, \mathbf{R}^n) \to \mathscr{X}^HM \to 0$$

is exact. (*Compare with* 3.5.)

If one considers $\mathbf{R}^n$ as a commutative Lie algebra, a natural Lie algebra structure is induced on $\mathscr{F}h(M, \mathbf{R}^n)$, such that the sequence in 5.2 is an exact sequence of Lie algebras. This leads to the introduction of Poisson brackets:

5.3. Definition. Let (M, Ω) be a polysymplectic manifold, $f, g \in \mathscr{F}M$, $F, G \in \mathscr{F}h(M, \mathbf{R}^n)$:

$$\{F, G\} := \Omega(v_G, v_F) = \mathbf{L}_{v_F}G = -\mathbf{L}_{v_G}F \in \mathscr{F}h(M, \mathbf{R}^n),$$
$$\{f, G\} := -\mathbf{L}_{v_G}f = \operatorname{tr}\Omega^b(v_G) \circ Z_f = \operatorname{tr}\mathbf{L}_{Z_f}G \in \mathscr{F}M,$$
$$\{f, g\} := \operatorname{tr}\Omega(Z_g, Z_f) \in \mathscr{F}M.$$

$\{\cdot, \cdot\}$ are called the *Poisson brackets*. $\mathscr{F}h(M, \mathbf{R}^n)$ is called the *algebra of currents*.

5.4. Lemma. *Let* $f, g \in \mathscr{F}M$, $F, G \in \mathscr{F}h(M, \mathbf{R}^n)$. *Then*:
1. $d\{f, G\} = -\mathrm{tr}([Z_f, v_G] \lrcorner \Omega)$;
2. $d\{f, g\} = \mathrm{tr}([Z_f, Z_q] \lrcorner \Omega) - \mathrm{tr}(Z_f \lrcorner \mathbf{L}_{Z_g} \Omega - Z_g \lrcorner \mathbf{L}_{Z_f} \Omega)$;
3. $\{F, g\} = 0$ *iff* g *is constant along the trajectories of* v_F;
4. $\{F, G\} = 0$ *iff* G *is constant along the trajectories of* v_F *and reverse*;
5. $\{F, g\} = 0$ *iff* $\mathrm{tr}(dF \circ Z_G) = \mathrm{tr}\,\Omega(v_F, Z_g) = 0$;
6. *If* $X \in \mathscr{H}M$, $\mathrm{tr}\, X \lrcorner \Omega = dH$, *then*

$$\mathbf{L}_X H = X \lrcorner dH = X \lrcorner \mathrm{tr}(X \lrcorner \Omega) = 0.$$

5.5. Remark. Let $v \in \mathscr{X}^H M$, $v - \Omega = dF$, and for a base $u_1, \cdots, u_n$ of $\mathbf{R}^n$ let $\mathrm{pr}_i : \mathbf{R}^n \to \mathbf{R}$ be the ith projection. There is a natural injection $\bar{u}_i : \mathscr{X}^H M \to \mathscr{H}M$ such that $\Omega^{\#}(\bar{u}_i(v_F)) = d(\mathrm{pr}_i \circ F)$, $\bar{u}_i$ is given by the embedding $TM \hookrightarrow I^n M \simeq \oplus_1^n TM$.

5.6. Proposition (*local characterization of* $\mathscr{X}^H M$ *and* $\mathscr{F}h(M, \mathbf{R}^n)$). *Let* (M, Ω) *be standard.*

(a) *Let* $\varphi \in \mathscr{A}^1(M, \mathbf{R}^n)$ *be a* $\mathbf{R}^n$*-valued one-form on* M. *Then* $\varphi_m \in \Omega^b(T_m M)$ *for* $m \in M$ *iff, locally in canonical coordinates*, $\varphi(\dot{q}, \dot{p}) = l(\dot{q}) - \dot{p}(r)$ *for some* $l \in \mathbf{Q}^* \otimes \mathbf{R}^n$, $r \in Q$. *Therefore* φ *has the coordinate representation*

$$\begin{pmatrix} -\alpha_1, & \tau, & 0, & \cdots & 0 \\ \vdots & 0 & & & \vdots \\ \vdots & & & & 0 \\ \vdots & & & & \\ -\alpha_n, & 0, & 0, & & \tau \end{pmatrix} : TM \to \mathbf{R} \quad \textit{with} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = l.$$

(b) $v \in \mathscr{X}^H M$. $c : (-\varepsilon, \varepsilon) \to M$ *is a trajectory of* $v = v_F$ *iff in canonical coordinates* $c(t) = (q(t), p(t))$ *and*

$$\frac{dq}{dt} = \frac{\partial F_i}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial F_i}{\partial q} \quad (\textit{for any } i = 1, \cdots, n)$$

(*because of* (a)) $\partial F_i / \partial p_i = r$ *for all* $i = 1, \cdots, n$).

6. Polysymplectic actions and symmetries

One of the most important advantages of the Hamiltonian formalism is the fact that the general symmetry group $\mathrm{Pspl}(M, \Omega)$ of a system has a simple geometric characterization and is defined without reference to the choice of a Hamiltonian. In other words, only in the Hamiltonian formalism can canonical transformations be defined.

Indeed, the most powerful results in classical mechanics are obtained for Hamiltonian systems with symmetry, and the progress in symplectic geometry in the last two decades covers mainly systems with symmetry groups: geometric quantization the orbit classication, and completely integrable systems are some examples.

From the results in the last sections, similar achievements can be expected from the Hamiltonian formalism for field theories. In the following the most important results in that direction will be outlined: the polysymplectic version of the Noether theorem, the concept of momentum mappings, the reduction procedure, and finally the classification of polysymplectic homogeneous spaces as coadjoint orbits.

These results are not only interesting by themselves and important for classical fields but they indicate clearly a new way to quantum field theory: via the poly-Hamiltonian theory.

Let (M,Ω) be a polysymplectic manifold, and $\mathrm{Pspl}(M,\Omega) \subset \mathrm{Diff}(M,\Omega)$ the group of polysymplectomorphisms (cf. 2.6). For a smooth action $\Lambda: G \times M \to M$ of a Lie group G with Lie algebra LG, denote by $\lambda: LG \to \mathscr{X}M$ the induced infinitesimal action, i.e. for $v \in LG$, $\lambda(v)$ is a *killing vector field*. Frequently in the following $\lambda(v)$ will also be denoted by $\underline{v}$.

6.1. Definition. A *polysymplectic action* of a Lie group G on (M,Ω) is a smooth action $\Lambda: G \times M \to M$ with $\Lambda_g^*\Omega = \Omega$ for all $g \in G$. Therefore Λ is a homomorphism $G \to \mathrm{Pspl}(M,\Omega)$, and the infinitesimal action is a Lie algebra map

$$\lambda: LG \to \mathscr{X}^H_{\mathrm{loc}}M.$$

Λ is called a *strongly polysymplectic action iff* $\lambda(v) \in \mathscr{X}^H M$ for all $v \in LG$.

$F \in \mathscr{F}h(M,\mathbf{R}^n)$ is an *infinitesimal generator* of $\underline{v} = \lambda(v)$, $v \in LG$ iff $\underline{v} \lrcorner \Omega = dF$.

The Noether theorem in classical mechanics states that infinitesimal generators are conserved quantities. In field theory the relation between infinitesimal generators and conservations laws is based on the following theorem:

6.2. Theorem. *Let $\Lambda: G \times M \to M$ be a strongly polysymplectic action on (M,Ω), $H \in \mathscr{F}M$ a G-invariant function, and $\psi: U \to M$ a solution of the Hamiltonian system $\Omega^{\#-1}(dH)$. Then for every infinitesimal generator F of Λ one has*

$$\mathrm{tr}\, d(F \circ \psi) = 0.$$

Proof.

$$\begin{aligned} \mathrm{tr}\, d(F \circ \psi) &= \mathrm{tr}\, \mathbf{L}_{Z_H} F = \mathrm{tr}(Z_H \lrcorner dF) \\ &= \mathrm{tr}(Z_H \lrcorner_{\underline{v}} \lrcorner \Omega) \qquad (v \in LG, \underline{v} \lrcorner \Omega = dF) \\ &= -\mathrm{tr}(\underline{v} \lrcorner Z_H \lrcorner \Omega) = -\underline{v} \lrcorner (\mathrm{tr}(Z_H \lrcorner \Omega)) = -\mathbf{L}_{\underline{v}} H = 0. \end{aligned}$$

6.3. Remark. If $\psi: U \to Q$ is considered to be a section in the trivial bundle $U \times M$, i.e. $\psi: U \to U \times M$, and F is considered to be a bundle morphism over 1_U, $F: U \times M \to TU = U \times \mathbf{R}^n$, then $F \circ \psi$ is a vector field on U and the equation in 6.2 becomes

$$\boxed{\operatorname{div}(F \circ \psi) = 0}\,.$$

Therefore, 6.2 is one version of the classical Noether theorem, which says: The infinitesimal generators of canonical transformations induce conserved currents. The equation $\operatorname{div}(F \circ \psi)$ is in this context a continuity equation.

The conventional form of the Noether theorem on I^*Q will be given below using the concept of momentum maps.

A group action $\Lambda: G \times M \to M$ induces a Lie algebra morphism $\lambda^n: Lin(\mathbf{R}^n, LG) \to \operatorname{Hom}(\mathbf{R}^n, TM)$ by $\lambda^n(f) = \lambda \circ f$. (Since the Lie algebra structures are pointwise for $u \in \mathbf{R}^n$ defined, λ^n is a Lie algebra morphism.)

Let G act strongly polysymplectic on (M, Ω). For a base $(e_1, \cdots, e_n)$ of $\mathbf{R}^n$, $\lambda^n(f)(e_i) = \lambda(f(e_i)) \in \mathscr{X}^H M$, $f \in Lin(\mathbf{R}^n, LG)$. Let $F_i \in \mathscr{F}h(M, \mathbf{R}^n)$ be the infinitesimal generators of $\lambda(f(e_i))$, i.e. $\lambda^n f(e_i) \lrcorner \Omega = dF_i$. Then for any $x \in \mathbf{R}^n$ one has $\lambda^n f(x) \lrcorner \Omega = d(\sum x_i F_i)$. Therefore, if one defines $\mathbf{E}_f$ by

$$d\mathbf{E}_f = \lambda^n f \lrcorner \Omega = \Omega^b \circ \lambda^n f,$$

$\mathbf{E}_f: M \to \operatorname{End}(\mathbf{R}^n)$ is an energy-momentum tensor. ($\mathbf{E}_f$ is uniquely determined up to a constant.)

6.4. Remark. The definition of λ^n shows that, for any polysymplectic action, $d(\lambda^n(f) \lrcorner \Omega) = 0$ for all $f \in Lin(\mathbf{R}^n, LG)$. If the action is strongly polysymplectic, there exists $\mathbf{E}$ with

$$\lambda^n(f) \lrcorner \Omega = d\mathbf{E}_f.$$

In particular $\operatorname{tr}((\lambda^n f) \lrcorner \Omega) = \Omega^\# \circ (\lambda^n f) = dH$ for $H = \operatorname{rt} \mathbf{E}_f$. Therefore λ^n is a morphism $\lambda^n: Lin(\mathbf{R}^n, LG) \to \mathscr{H}M$.

If $X \in \mathscr{H}M$ is integrable, then there exists a local flow $\phi: U \times M \to M$, $U \subset \mathbf{R}^n$, with $D\phi(m) = X(\phi(m))$. But this flow is in general not a polysymplectic flow as is easily seen from the poly-Hamiltonian equations (3.2): The components $(\partial p_i / \partial u_j)(u)$ are not determined by the Hamiltonian H, therefore there exist $X \in Z_H$ and $b \in \mathbf{R}^n$ with $X(b) \notin \mathscr{X}^H M$. Polysymplectic actions are therefore special cases of poly-Hamiltonian flows.

The map $\mathbf{E}: Lin(\mathbf{R}^n, LG) \to \mathscr{F}(M, \operatorname{End}(\mathbf{R}^n))$ is called a *momentum-tensor map* associated with the strongly polysymplectic action. The trace of $\mathbf{E}$ is a generalization of the momentum map in symplectic geometry. This will be specified in the following definitions.

6.5. Definition. Let $\Lambda: G \times M \to M$ be a (strongly) polysymplectic action. A map

$$\mu: LG \to \mathscr{F}h(M, \mathbf{R}^n)$$

is a *momentum map* of Λ, iff $d(\mu(v)) = \lambda(v) \lrcorner \Omega$ for all $v \in LG$, i.e. the diagram

$$\begin{tikzcd}
\mathcal{F}h(M,\mathbf{R}^n) \arrow[rr, "\Omega\#^{-1}\circ d"] & & \mathfrak{X}^H M \\
& & LG \arrow[u, "\lambda"'] \arrow[ull, "\mu"]
\end{tikzcd}$$

commutes.

6.6. Remark. There is a dual characterization of the momentum map via λ^n:

$\mu^n : Lin(\mathbf{R}^n, LG) \to \mathcal{F}M$ is a momentum map iff

$$d(\mu^n(f)) = \mathrm{tr}(\lambda^n(f) \lrcorner \Omega) = \Omega^{\#}(\lambda^n(f)),$$

i.e. the diagram

$$\begin{tikzcd}
\mathcal{F}M \arrow[rr, "\tilde{\Omega}^{\#-1}\circ d"] & & \mathcal{H}M/\mathcal{C}M \\
& & Lin(\mathbf{R}^n, LG) \arrow[u, "\tilde{\lambda}^n"'] \arrow[ull, "\mu^n"]
\end{tikzcd}$$

commutes. Here $\tilde{\lambda}^n = \mathrm{pr} \circ \lambda^n$ with $\mathrm{pr}: \mathcal{H}M \to \mathcal{H}M/\mathcal{C}M$ the natural projection. μ and μ^n can also be considered as maps

$$\mu : M \to Lin(LG, \mathbf{R}^n),$$

$$\mu^n : M \to Lin(\mathbf{R}^n, LG)^* \simeq Lin(LG, \mathbf{R}^n).$$

Therefore μ and μ^n can be identified by the isomorphism 1.9. One obtains:

6.7. Proposition. 1. *μ and μ^n differ only by the isomorphism $Lin(\mathbf{R}^n, LG)^* \simeq Lin(LG, \mathbf{R}^n)$ and can therefore be identified.*

2. *If $\mathbf{E}$ is a momentum-tensor map, then $\mu = \mathrm{tr}\,\mathbf{E}$ is a momentum map.*

3. *A polysymplectic action is strongly polysymplectic iff it has a momentum map.*

Using the concept of a momentum map, the Noether theorem can be formulated in a more elegant way:

6.8. Proposition (*Noether*). *Let $\Lambda : G \times M \to M$ be a polysymplectic group action with momentum μ, $H \in \mathcal{F}M$ a G-invariant function, and $\psi : U \to M$ a solution of $X \in \mathfrak{X}_H \subset \mathcal{H}M$, i.e.* $\mathrm{tr}(X \lrcorner \Omega) = dH$, $T\psi(u) = X(\psi(u))$. *Then*

$$\boxed{\mathrm{tr}\, d(\mu \circ \psi) = 0}\,.$$

Proof.

$$\mathrm{tr}(d(\mu \circ \psi)) = \mathrm{tr}(d\mu \circ T\psi) = \mathrm{tr}\big(X \lrcorner \Omega^b(\lambda)\big) = \lambda \lrcorner dH = 0.$$

Comment. As in 6.3 this result gives conserved currents, therefore 6.9 is a generalization of the Noether theorem. In the case $\Omega = -d\Theta$ (exact) and for actions leaving Θ invariant (e.g. point transformations in the classical case $M = I^*Q$, $\Lambda_g = I^*\tilde{\Lambda}_g$, $\tilde{\Lambda}: G \times Q \to Q$ action) one obtains:

6.9. Proposition. *Let $\Omega = -d\Theta$, $\Lambda: G \times M \to M$ be an action leaving Θ invariant. Then $\Theta \circ \lambda: M \to Lin(LG, \mathbf{R}^n)$ is a momentum map.*

Proof.

$$d(\Theta \circ \lambda(v)) = d(\lambda(v) \lrcorner \Theta) = \mathbf{L}_{\lambda(v)}\Theta \lrcorner \lambda(v) \lrcorner d\Theta$$
$$= \lambda(v) \lrcorner \Omega \quad \text{since } \mathbf{L}_{\lambda(v)}\Theta = 0.$$

This leads to the common version of the *field theoretic Noether* theorem:

6.10. Theorem. *Let $\Lambda: G \times M \to M$ be a strongly polysymplectic action on (M, Ω). Assume $\Omega = -d\Theta$ and Θ is G-invariant. Let $H \in \mathscr{F}M$ be a G-invariant function and ψ a solution of $\Omega^{\#-1}(dH)$. Then*

$$\operatorname{tr} d(\Theta \circ \lambda(v)) \circ \psi = 0 \quad \textit{for all } v \in LG.$$

Thus $\Theta \circ \lambda$ allows one to calculate in this case the constants of motion explicitly.

7. Reduction and polycoadjoint orbits

The reduction procedure by Marsden and Weinstein and the classification of symplectic homogeneous spaces by coadjoint orbits by Kostant and Souriau belong to the major achievements in Hamiltonian mechanics.

In this section, we extend these results to polysymplectic manifolds. Most preparations for such an extension were already done in the last two sections. Therefore, we will be able to prove the desired results by a direct generalization of the ideas of Marsden-Weinstein and Kostant-Souriau and obtain a reduction procedure for polysymplectic group actions and an orbit classification of polysymplectic homogeneous spaces.

Let $\mathrm{Ad}: G \times LG \to LG$ be the adjoint action. We denote by Ad^n the induced action on $Lin(\mathbf{R}^n, LG)$:

$$\mathrm{Ad}^n: G \times Lin(\mathbf{R}^n, LG) \to Lin(\mathbf{R}^n, LG),$$
$$\mathrm{Ad}^n_g(f)(x) = \mathrm{Ad}_g(f(x)), \qquad f \in Lin(\mathbf{R}^n, LG),\ x \in \mathbf{R}^n,\ g \in G.$$

The dual of Ad^n is denoted by $\mathrm{Ad}^{\#}$:

$$\mathrm{Ad}^{\#}: G \times LG^* \otimes \mathbf{R}^n \to LG^* \otimes \mathbf{R}^n, \qquad \mathrm{Ad}^{\#}_g(\alpha) = \alpha \circ \mathrm{Ad}^n_g.$$

From the standard formula

$$\lambda(\mathrm{Ad}_g u) = \Lambda_g^*(\lambda(u)) \quad \left(\text{or simply } \underline{\mathrm{Ad}_g u} = \Lambda_g^* \underline{u}\right)$$

(using the notation $\lambda(u) = \underline{u}$, $u \in LG$, $g \in G$, λ inf. morphism of Λ) one obtains

$$\Lambda_g^* \lambda^n(f) = \lambda^n(\mathrm{Ad}_g^n f) \quad \text{for all } g \in G,\ f \in Lin(\mathbf{R}^n, LG).$$

7.1. Proposition. *Let $\Lambda : G \times M \to M$ be a strongly polysymplectic group action with momentum map $\mu : M \to Lin(LG, \mathbf{R}^n) = LG^* \otimes \mathbf{R}^n$. Assume M is connected. Then the map*

$$M \to LG^* \otimes \mathbf{R}^n, \qquad m \mapsto \mu(\Lambda_g m) - \mathrm{Ad}_g^{\#}(\mu(m)),$$

is a constant on M for all $g \in G$.

Proof. Let $g \in G$, $u \in LG$, and $\underline{u} = \lambda(u) \in \mathscr{X}^H M$. Then

$$\Lambda_g^*(\underline{u}) \lrcorner \Omega = \Lambda_g^*(\underline{u} \lrcorner \Omega) = -\Lambda_g^*(d\mu(u))$$
$$= -d\left(\Lambda_g^* \mu(u)\right) = -d\left(\mu(u) \circ \Lambda_g\right).$$

Since $\Lambda_g^* \underline{u} = \underline{\mathrm{Ad}_g u}$ it follows by the definition of μ that

$$d\left(\mu(\mathrm{Ad}_g u)\right) = \underline{\mathrm{Ad}_g u} \lrcorner \Omega = -d\left(\mu(u) \circ \Lambda_g\right) = d\left((\mu \circ \Lambda_g) \cdot u\right),$$

which is the desired result.

7.2. Corollary. *There is a smooth map χ,*

$$\chi : G \to LG^* \otimes \mathbf{R}^n, \qquad \chi(g) = \mu(\Lambda_g m) - \mathrm{Ad}_g^{\#}(\mu(m)),$$

with the following properties:

1. *χ is a 1-cocycle, i.e. for all $g, h \in G$,*

$$\chi(gh) = \mathrm{Ad}_h^{\#}(\chi(g)) + \chi(h).$$

2. *The bilinear map φ on LG, $\varphi := L_\chi : LG \to LG^* \otimes \mathbf{R}^n$, $\varphi : LG \times LG \to \mathbf{R}^n$, is a 2-cocycle, i.e. $\varphi(u, [v, w]) + \varphi(v, [w, u]) + \varphi(w, [u, v]) = 0$ for all $u, v, w \in LG$.*

3. *Moreover,*

$$L_\chi(u) = T\mu(\underline{u}(m)) + \mu(m) \circ ad_u,$$
$$\underline{v} \lrcorner \underline{u} \Omega = \mu([u, v]) - L_\chi(u) \cdot v,$$
$$\varphi(u, v) = \mu([u, v]) - \{\mu(u), \mu(v)\}$$

for all $m \in M$, $u, v \in LG$, $\mu(u), \mu(v) \in \mathscr{F} M$.

Proof. 1. For $g, h \in G$ we have

$$\begin{aligned}\chi(hg) &= \mu \circ \Lambda_{hg}(m) - \mathrm{Ad}^{\#}_{hg}\mu(m)\\ &= \mu \circ \Lambda_g(\Lambda_h m) - \mathrm{Ad}^{\#}_g \circ \mu(\Lambda_h m) + \mathrm{Ad}^{\#}_g \circ \mu(\Lambda_h m) - \mathrm{Ad}^{\#}_g \mathrm{Ad}^{\#}_h \circ \mu(m)\\ &= \chi(g) + \mathrm{Ad}^{\#}_g(\chi(h)).\end{aligned}$$

3. $$\begin{aligned}L\chi(u) &= T\chi(u_3) = T(\mu \circ \Lambda_{\bullet}(m)) - \mathrm{Ad}^{\#}(\cdot) \circ \mu(m) \cdot u_e\\ &= T_m\mu \circ T_e A_{\bullet}(m) \cdot u - T_e \mathrm{Ad}^{\#}_{(\bullet)} \circ \mu(m) \cdot u\end{aligned}$$

(since $\mu(m)$ is linear on LG)

$$= T_m\mu \circ \underline{u}(m) + \mu(m) \circ ad_u$$

which implies the first formula of 3.

$$\begin{aligned}\underline{v} \lrcorner \underline{u} \lrcorner \Omega &= \underline{v} \lrcorner -d(\mu(u)) = -T\mu(v) \cdot v = \mu(ad_u \cdot v) - L\chi(u) \cdot v\\ &= \mu([u, v]) - L\chi(u) \cdot v.\end{aligned}$$

Finally, by definition of μ, $\{\mu(u), \mu(v)\} = -\underline{u} \lrcorner \underline{v} \lrcorner \Omega$, which implies the last expression.

2. Follows simply from 3.

7.3. Theorem. *Let $\Lambda : G \times M \to M$ be a polysymplectic action with momentum map $\mu : M \to LG^* \otimes \mathbf{R}^n$. Then the map*

$$\Xi : G \times LG^* \otimes \mathbf{R}^n \to G \times LG^* \otimes \mathbf{R}^n, \qquad \Xi(g, \eta) = \chi(g) + \mathrm{Ad}^{\#}_g \eta,$$

is an affine operation of G on $LG^* \otimes \mathbf{R}^n$ such that

$$\begin{array}{ccc} M & \xrightarrow{\mu} & LG^* \otimes \mathbf{R}^n \\ {\scriptstyle \Lambda_g}\downarrow & & \downarrow{\scriptstyle \Xi_g} \\ M & \xrightarrow{\mu} & LG^* \otimes \mathbf{R}^n \end{array}$$

commutes for all $g \in G$, i.e. μ is G-equivariant.

Proof. Since

$$\begin{aligned}\Xi(gh, \eta) &= \chi(gh) + \mathrm{Ad}^{\#}_{gh}\eta = \chi(h) + \chi(g) \circ \mathrm{Ad}_h + \mathrm{Ad}^{\#}_h \circ \mathrm{Ad}^{\#}_g \eta\\ &= \chi(h) + \mathrm{Ad}^{\#}_h\big(\chi(g) + \mathrm{Ad}^{*}_g h\big) = \Xi(h, \Xi(g, \eta)),\end{aligned}$$

Ξ is an action.

In addition,

$$\begin{aligned}\Xi_g \circ \mu(m) &= \chi(g) + \mathrm{Ad}^{\#}_g \circ \mu(m)\\ &= \mu(\Lambda_g m) - \mathrm{Ad}^{\#}_g(\mu(m)) + \mathrm{Ad}^{\#}_g \mu(m) = \mu \circ \Lambda_g(m).\end{aligned}$$

7.4. Remark. As in the conventional case $n = 1$, for semisimple G or if $\Omega = -d\Theta$ and Θ is G-invariant χ will be zero and therefore, χ will be zero and therefore $\Xi = \mathrm{Ad}^{\#}_{g}$. In these cases μ is a Lie algebra morphism.

For a degenerate vector valued two-form Ω on a manifold N denote by $CN : \mathrm{Ker}\,\Omega$ the characteristic bundle of Ω. For closed Ω, CN is involutive.

7.5. Lemma. *Let $\Lambda : G \times M \to M$ be a polysymplectic action with momentum $\mu : M \to LG^* \otimes \mathbf{R}^n$, let $\eta \in LG^* \otimes \mathbf{R}^n$ be a (weakly) regular value of μ (cf. Marsden and Weinstein* [35]), *and let $G_\eta : \{ g \in G \mid \Xi_g \eta = \eta \}$ be the isotropy group of η. Then for all $m \in \mu^{-1}(\eta)$ the following holds:*

1. $T_m(\mu^{-1}(\eta)) = \mathrm{orth}^{\Omega}(T_m(G \cdot m))$.
2. $T_m(G_\eta \cdot m) = C_m(G \cdot m) = C_m(\mu^{-1}(\eta))$.

The proof is the same as for the symplectic case.

7.6. Definition (The reduced phase space). Let $\Lambda : G \times M \to M$ be polysymplectic action with momentum μ. For $\eta \in LG^* \otimes \mathbf{R}^n$ let G_η be the isotropy group of η with respect to the action $\Xi : G \times LG^* \otimes \mathbf{R}^n \to LG^* \otimes \mathbf{R}^n$. Then, by 7.3, $\mu^{-1}(\eta)$ is G_η-invariant and one can define

$$M_\eta := \mu^{-1}(\eta)/G_\eta.$$

M_η is called the *reduced phase space*.

7.7. Theorem. *Let $\Lambda : G \times M \to M$ be a polysymplectic action with momentum map $\mu : M \to LG^* \otimes \mathbf{R}^n$, and let $\eta \in LG^* \otimes \mathbf{R}^n$ be a (weakly) regular value of μ. Then there exists uniquely a polysymplectic form Ω_η on $M_\eta = \mu^{-1}(\eta)/G_\eta$ with $p_\eta^* \Omega_\eta = i_\eta^* \Omega$, where $p_\eta : \mu^{-1}(\eta) \to G_\eta$ is the natural projection and $i_\eta : \mu^{-1}(\eta) \hookrightarrow M$ is the natural injection.*

Proof. Since G_η acts freely and regularly on $\mu^{-1}(\eta)$, $\mu^{-1}(\eta)$ is a submanifold of M and p_η is a submersion. Define $\Omega_\eta(Tp_\eta(v_m), Tp_\eta(w_m)) = \Omega(v_m, w_m)$. Then Ω_η is well defined by Lemma 7.5(2). Because of $dp_\eta^* \Omega_\eta = p_\eta^* d\Omega_\eta = 0$, Ω_η is closed. Because of 7.5, $T_{p_\eta(m)} M_\eta \simeq T_m \mu^{-1}(\eta)/T_m(G_\eta \cdot m) = T_m \mu^{-1}(\eta)/C_m(G \cdot m)$ and $C_{p_\eta(m)} M_\eta = C_m(G \cdot m)/C_m(G \cdot m) = 0$. Therefore Ω_η is nondegenerate, which proves the theorem.

7.8. Example. Let G be a Lie group and $L : G \times G \to G$ the left translations action. $I^*G = \mathrm{Hom}(TG, \mathbf{R}^n) \simeq G \times (LG^* \otimes \mathbf{R}^n)$ since TG is trivial. Let $I^*L : G \times I^*G \to I^*G$ be the induced action. The momentum map corresponding to I^*L is the projection

$$\mu : I^*G \to LG^* \otimes \mathbf{R}^n.$$

Therefore, as in the symplectic case;

$$\mu^{-1}(\eta)/G_\eta \cong G/G_\eta \cong G \cdot \eta \subset LG^* \otimes \mathbf{R}^n,$$

and $G \cdot \eta$ is a polysymplectic manifold.

Similar to the symplectic case the polysymplectic form Ω_η on $G \cdot \eta$ can be explicitly computed (cf. Abraham and Marsden [1]):

$$\Omega_\eta(\xi(v)_\eta, \xi(u)_\eta) = -\eta([u, v])$$

for $\eta \in LG^* \otimes \mathbf{R}^n$ and $\xi: LG \to \mathscr{X}(G \cdot \eta)$ the infinitesimal action of G on $G \cdot \eta \subset LG^* \otimes \mathbf{R}^n$.

Remark. In analogy with the symplectic case, $LG^* \otimes \mathbf{R}^n$ can be considered as a manifold with a *Polypoisson structure*.

Let $\Lambda: G \times M \to M$ be a transitive strongly polysymplectic action, i.e. (M, Ω) is a *poly-Hamiltonian G-space* and let $\mu: M \to LG^* \otimes \mathbf{R}^n$ be the momentum map. With the same kind of arguments as in symplectic geometry the following can be shown for $\eta \in LG^* \otimes \mathbf{R}^n$ (cf. Wallach [55]): The map $\mu: M \to G \subset \eta \subset LG^* \otimes \mathbf{R}^n$ has the properties:

1. $\mu(g \cdot m) = g \cdot \mu(m)$.
2. $\mu^*\Omega_\eta = \Omega$.
3. μ is a covering map.

This implies

7.9. Theorem. *Let (M, Ω) be a homogeneous poly-Hamiltonian G-space. Then there exists $\eta \in LG^* \otimes \mathbf{R}^n$ such that the momentum map $\mu: M \to G \cdot \eta$ is an equivariant covering map.*

This result has a similar meaning for field theories as the classical result for particles: it allows us to classify homogeneous polysymplectic G-spaces or, in physical terms, it gives a classification of free and elementary field systems.

8. Remarks on the global case

In the previous sections, the Hamiltonian formalism was developed for fields as functions, i.e. sections in a trivial bundle $E = U \times Q \to U$ over some open neighborhood $U \subset \mathbf{R}^n$. In the global case fields are considered to be sections in a fiber bundle (E, π, B) over some manifold B. Physically this case describes a general relativistic field theory. The fundamental object for the global Hamiltonian formalism is $J^*E := \operatorname{Hom}(\operatorname{Ver} E, \pi^*TB)$, the bundle over E of TB-valued one-forms on $\operatorname{Ver} E$. With the same construction as in §2, J^*E is shown to have a canonical TB-valued two-form

$$\Omega: \operatorname{Ver} J^*E \times \operatorname{Ver} J^*E \to TB.$$

The mechanism of §3 assigns to every function H on J^*E a section in the bundle $\operatorname{Hom}(\pi^*TB, \operatorname{Ver} J^*E)$ of linear maps $TB \to \operatorname{Ver} J^*E$.

But in contrast to the local case these sections cannot be interpreted as partial differential equations. In order to interpret sections in $\operatorname{Hom}(\pi^*TB, \operatorname{Ver} J^*E)$ as partial differential equations a connection on J^*E

has to be chosen. This allows us to identify $\mathrm{Hom}(\pi^*TB, \mathrm{Ver}\, J^*E)$ with the first jet bundle $J^1(J^*E)$ of J^*E and therefore allows us to assign to a function H on J^*E a section in $J^1(J^*E)$ which represents the desired Hamiltonian equations. *Thus in the global case the Hamiltonian formalism requires the choice of a connection in J^*E and leads therefore necessarily to a gauge theory.*

In the case of a regular Hamiltonian the so obtained gauge Hamiltonian equations are equivalent to the conventional Lagrange equations for gauge theories (see Bleeker [5]). The case of ordinary (nongauge) Lagrange systems appears then as a special case (with a particular connection chosen). The Hamilton-Cartan form can be easily expressed in terms of Ω, H, and a volume on B.

Symmetries are in the global case treated in the same spirit as in the local case (§§6 and 7). Since a symmetry group may operator nontrivially on the base B of E one obtains two types of conserved quantities: The conserved currents (see 6.2 and 6.3) from the "vertical part" of the operation on J^*E, and conserved quantities associated with the energy-momentum tensor from the "horizontal part" of the action on $J^*E \to B$. The classification of free elementary systems leads to orbit in the bundle $\mathrm{Hom}(LG, TB)$. See also the appendix.

The global case will be treated in detail in the forthcoming second part.

Appendix. Correspondences between classical mechanics, local and global field theory in the Hamiltonian formalism

Classical mechanics	Field theory (local)	Field theory (global)
Parameter space $\mathbf{R}$ (time)	$U \subset \mathbf{R}^n$ (local space time)	Bn – dim. manifold (space time)
dt time form	vol standard volume on U	vol volume on B
Configuration space Q	Space of field values Q	[(typical) fiber of values Q]
Configuration bundle $\mathbf{R} \times Q$	local field bundle $U \times Q$	Field bundle (E, ρ, B) (fiber Q)
Evolution space $\mathbf{R} \times T^*Q$	$U \times I^*Q$	J^*E
Phase space T^*Q	I^*Q	[typical fiber of J^*E over B]
Symplectic form ω_0 on T^*Q	polysymplectic form Ω_0 on I^*Q	[polysymplectic structure on fibers of J^*E]
Canonical one form ϑ_0	Canonical $\mathbf{R}^n$-valued one form Θ_0	[Canonical TB-valued one form Θ_0 on fibers]
General symplectic phase space (M, ω)	Polysymplectic manifold (M, Ω)	[typical fiber of a polysymplectic fiber bundle]
Presymplectic evolution space $(\mathbf{R} \times M, p^*\omega)$	$(U \times M, p^*\Omega)$	Polysymplectic fiber bundle $(\overline{E}, \Omega)$
Hamiltonian $H \in \mathscr{F}M$	$H \in \mathscr{F}M$	—

Classical mechanics	Field theory (local)	Field theory (global)
Time dependent Hamiltonian $H \in \mathscr{F}(\mathbf{R} \times M)$	$H \in \mathscr{F}(U \times M)$ space-time dependent	$H \in \mathscr{F}\bar{E}$
H (Energy)	Energy-momentum tensor $\mathbf{H}$	Energy-momentum tensor $\mathbf{H}$
Musical morphisms	Ω^b, $\Omega^\#$	Ω^b, $\Omega^\#$
Hamiltonian vector field $v \in \mathscr{X}^H M$	Hamiltonian system $X \in \mathscr{H}M \subset \mathscr{S}ec\,\mathrm{Hom}(\mathbf{R}^n, TM)$	Hamiltonian system $X \in \mathscr{H}\bar{E} \subset \mathscr{S}ec\,\mathrm{Hom}(\pi^* TB, \mathrm{Ver}\,\bar{E})$
Hamiltonian vector fields as infinitesimal generators	Infinitesimal canonical transformation $v \in \mathscr{X}^v M$	Infinitesimal canonical transformation $v \in \mathscr{V}er^H \bar{E}$
Hamiltonian Equations $v = \omega^\#(dH)$	Hamiltonian Equations $\Omega^{\#-1}(dH)$	Gauge Hamiltonian Equations $\bar{C}^{-1}\Omega^{\#-1}(d^v H)$
Observable $\mathscr{F}(M)$	$\mathscr{F}(M)$ or $\mathscr{F}h(M, \mathbf{R}^n)$ currents	—
Time dependent observables $\mathscr{F}(M \times \mathbf{R})$	$\mathscr{F}(U \times M)$ or $\mathscr{F}h(U \times M, \mathbf{R}^n)$	$\mathscr{F}\bar{E}$ or $\mathscr{F}h(\bar{E}, TB)$ currents
Poisson brackets $\{\cdot,\cdot\}$	$\{\cdot,\cdot\}$ (5.3.)	$\{\cdot,\cdot\}$ (11.4.)
Momentum map $\mu : M \to LG^*$	Momentum map $\mu : M \to LG^* \otimes \mathbf{R}^n$	Momentum map $\mu : \bar{E} \to \mathrm{Hom}(LG, TB)$
Coadjoint orbits in LG^*	Orbits in $LG^* \otimes \mathbf{R}^n$	Orbits in $\mathrm{Hom}(LG, TB)$
Conserved quantities $f \in \mathscr{F}M$ with $\mathbf{L}_V f = 0$	Currents $F \in \mathscr{F}h(M, \mathbf{R}^n)$ with $\mathrm{div}(F \circ \psi) = 0$ etc.	Currents $F \in \mathscr{F}h^{\mathscr{V}}(\bar{E}, TB)$ with $\mathrm{div}(F \circ \psi) = 0$

(The list can be continued to the Lagrange formalism.)

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