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HARMONIC MAPS OF THE TWO-SPHERE INTO THE COMPLEX HYPERQUADRIC

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Introduction

Let $G(k, n; \mathbf{C})$ denote the Grassmann manifold of all k-dimensional subspaces $\mathbf{C}^k$ of complex n-space $\mathbf{C}^n$. Let P_{n-1} denote complex projective $(n-1)$ space, $P_{n-1} = G(1, n; \mathbf{C})$ and let $Q_{n-2} \subset P_{n-1}$ denote the complex hyperquadric, that is, the complex hypersurface, of P_{n-1} defined by the equation

$$Z_0^2 + Z_1^2 + \cdots + Z_{n-1}^2 = 0,$$

where $\{Z_0, \cdots, Z_{n-1}\}$ are homogeneous coordinates of P_{n-1}. Q_{n-2} has a natural Kähler metric which it inherits as a complex submanifold of P_{n-1}. In this note we will study the minimal immersions or harmonic maps of the two-sphere S^2 into Q_{n-2}. Our result can be described as follows: To each harmonic map $f\colon S^2 \to Q_{n-2}$ we associate a *directrix curve* $\Delta_f\colon S^2 \to G(2, n; \mathbf{C})$ which is either a holomorphic curve or a degenerate harmonic map. (The degenerate harmonic maps arise in the study of harmonic maps $S^2 \to G(2, n; \mathbf{C})$. In [4] it is shown that they can be constructed from holomorphic curves $S^2 \to P_{n-1}$.) The directrix curve Δ_f will be shown to satisfy strong nullity conditions, in the sense that its lth osculating space is null for $0 \leqslant l \leqslant r$ (where $r \geqslant 0$ depends on f). The harmonic map f can be recovered from its directrix curve Δ_f via differentiation and the choice of holomorphic sections of P_1 bundles over S^2. This description and Calabi's description of minimal maps $S^2 \to S^N$ [1] are related. In fact, the nullity conditions on the directrix curves of harmonic maps $S^2 \to Q_{n-2}$ are similar to those on the directrix curves of minimal maps $S^2 \to S^N$.

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In §1 we will discuss the geometry of the spaces $G(k, n; \mathbf{C})$, $G(2, n; \mathbf{R})$, and Q_{n-2}. In §2 we will give an account of the basic results on harmonic maps $M^2 \to G(2, n; \mathbf{C})$ as developed in [4]. We will omit proofs and refer the reader to [3] or [4].

1. Some geometry

Let $V, W \in \mathbf{C}^n$,

$$V = (v_1, \cdots, v_n), \qquad W = (w_1, \cdots, w_n). \tag{1.1}$$

We equip $\mathbf{C}^n$ with two inner products. First, with the standard Hermitian inner product so that

$$\langle V, W \rangle = \sum v_A \overline{w}_A = \sum v_A w_{\overline{A}} \tag{1.2}$$

and, second, with the symmetric inner product so that

$$(V, W) = \sum v_A w_A. \tag{1.3}$$

Of course $\langle V, \overline{W} \rangle = (V, W)$.

A frame consists of an ordered set of n linearly independent vectors Z_A, so that

$$Z_1 \wedge \cdots \wedge Z_n \neq 0. \tag{1.4}$$

It is called unitary if

$$\langle Z_A, Z_B \rangle = \delta_{A\overline{B}}. \tag{1.5}$$

The space of unitary frames can be identified with the unitary group $U(n)$. Writing

$$dZ_A = \sum_B \omega_{A\overline{B}} Z_B, \tag{1.6}$$

the $\omega_{A\overline{B}}$ are the Maurer-Cartan forms of $U(n)$. They are skew-Hermitian, i.e. we have

$$\omega_{A\overline{B}} + \overline{\omega}_{B\overline{A}} = 0. \tag{1.7}$$

Taking the exterior derivative of (1.6), we get the Maurer-Cartan equations of $U(n)$:

$$d\omega_{A\overline{B}} = \sum_C \omega_{A\overline{C}} \wedge \omega_{C\overline{B}}. \tag{1.8}$$

We call equations (1.6) and (1.8) the *structure equations* of the frame.

An element of $G(k, n; \mathbf{C})$ can be given by a multivector $Z_1 \wedge \cdots \wedge Z_k \neq 0$, defined up to a factor. The vectors Z_α, $i \leqslant \alpha \leqslant k$, and their orthogonal vectors Z_i, $k + 1 \leqslant i \leqslant n$, are defined up to a transformation of $U(k)$ and $U(n-k)$, respectively. Thus, the form

$$ds^2 = \sum_{\alpha,i} \omega_{\alpha\bar{i}}\bar{\omega}_{\alpha\bar{i}} \tag{1.9}$$

is a positive definite Hermitian form on $G(k, n; \mathbf{C})$ and defines a Hermitian metric. Its Kähler form is

$$\Omega = \frac{\sqrt{-1}}{2} \sum_{\alpha,i} \omega_{\alpha\bar{i}} \wedge \bar{\omega}_{\alpha\bar{i}}. \tag{1.10}$$

By using (1.8) it can be verified immediately that Ω is closed, so that the metric ds^2 is Kählerian. When $k = 1$, $G(1, n; \mathbf{C})$ is complex projective $(n-1)$-space P_{n-1} and the metric (1.9) is called the Fubini-Study metric.

It is easy to see that complex conjugation is an isometry of $G(k, n; \mathbf{C})$ with the metric (1.9). Its fixed point set is the real Grassmann manifold $G(k, n; \mathbf{R})$. Thus $G(k, n; \mathbf{R})$ lies totally geodesically in $G(k, n; \mathbf{C})$.

An element S of $G(k, n; \mathbf{C})$ is called null if

$$(V, W) = 0 \quad \text{for all vectors } V,\ W \in S$$

(equivalently $\langle V, \overline{W}\rangle = 0$). In particular an element $L \in P_{n-1}$ is null if $(Z, Z) = 0$ for any $Z \in L$. The manifold of all null lines is a complex hypersurface of P_{n-1} called the complex hyperquadric and denoted Q_{n-2}. The Fubini-Study metric induces a Kähler metric on Q_{n-2}. If Z is a homogeneous coordinate vector for a point on Q_{n-2}, then $(Z, Z) = 0$, so $\langle Z, \overline{Z}\rangle = 0$. That is, Z is orthogonal to $\overline{Z}$. Define a map

$$\phi\colon Q_{n-2} \to G(2, n; \mathbf{R})$$

as follows: Represent a point $p \in Q_{n-2}$ by a homogeneous coordinate vector Z and set

$$\phi(p) = \frac{\sqrt{-1}}{2} Z \wedge \overline{Z};$$

ϕ is clearly well defined. ϕ is one-to-one and onto. It follows easily using (1.6) and (1.9) that ϕ is an isometry. Using ϕ we will henceforth identify Q_{n-2} and $G(2, n; \mathbf{R})$ (for more details see [2]).

2. Harmonic maps of surfaces into $G(2, n; \mathbf{C})$

Let M be an oriented Riemannian surface and let $f\colon M \to G(2, n; \mathbf{C})$ be a nonconstant harmonic map. Denote the Riemannian metric on M by $ds_M^2 = \phi \cdot \bar{\phi}$, where ϕ is a $(1,0)$ form on M. For $x \in M$ the image $f(x) \in G(2, n; \mathbf{C})$ has an orthogonal space $f(x)^{\perp} \in G(n-2, n; \mathbf{C})$. If $Z \in f(x)$, we can write

$$dZ \equiv X\phi + Y\bar{\phi} \mod f(x), \tag{2.1}$$

where $X, Y \in f(x)^{\perp}$. If $Z \in \mathbf{C}^n - \{0\}$, we denote by $[Z]$ the point in P_{n-1} with Z as the homogeneous coordinate vector. Then

$$\partial\colon [Z] \to [X] \quad \text{and} \quad \bar{\partial}\colon [Z] \to [Y], \tag{2.2}$$

if not zero, are well-defined projective collineations of the projectivized space $[f(x)]$ into $[f(x)^{\perp}]$. These maps are called the *fundamental collineations*. The fundamental collineation $\bar{\partial}$ (resp. ∂) is zero if and only if f is holomorphic (resp. antiholomorphic).

Choose a unitary frame Z_A so that $\{Z_1, Z_2\}$ span $f(x)$. By (1.9) the one-forms $\omega_{\alpha\bar{i}}$, $\alpha = 1, 2$, $i = 3, \cdots, n$, form a unitary coframing of $G(2, n; \mathbf{C})$. We have

$$f^*\omega_{\alpha\bar{i}} = a_{\alpha\bar{i}}\phi + b_{\alpha\bar{i}}\bar{\phi}. \tag{2.3}$$

Set

$$\begin{aligned} Da_{\alpha\bar{i}} &= da_{\alpha\bar{i}} - a_{\beta\bar{i}}\omega_{\alpha\bar{\beta}} + a_{\alpha\bar{j}}\omega_{j\bar{i}} - \sqrt{-1}\, a_{\alpha\bar{i}}\eta, \\ Db_{\alpha\bar{i}} &= db_{\alpha\bar{i}} - b_{\beta\bar{i}}\omega_{\alpha\bar{\beta}} + b_{\alpha\bar{j}}\omega_{j\bar{i}} + \sqrt{-1}\, b_{\alpha\bar{i}}\eta, \end{aligned} \tag{2.4}$$

where η is the connection one-form of the metric ds^2_M.

Theorem 2.1. *The property that f is a harmonic map is expressed by one of the following equivalent conditions*:

(a) $Da_{\alpha\bar{i}} \equiv 0 \bmod \phi$, *or*

(b) $Db_{\alpha\bar{i}} \equiv 0 \bmod \bar{\phi}$.

Throughout this paper the criterion of Theorem 2.1 will be used repeatedly.

It follows from the harmonicity of f that the fundamental collineations have constant rank, except perhaps at isolated points. Denoting $\dim \partial[f(x)] = k_1 - 1$, we define the ∂-transform of f,

$$\partial f\colon M \to G(k_1, n; \mathbf{C}), \tag{2.5}$$

by $(\partial f)(x) = \partial[f(x)]$, $x \in M$. Similarly we define the $\bar{\partial}$-transform of f.

Theorem 2.2. *Let $f\colon M \to G(2, n; \mathbf{C})$ be a harmonic map. Then*

(a) *The maps ∂f, $\bar{\partial} f$ are harmonic.*

(b) *If $k_1 = 2$, $\bar{\partial}\partial f$ is f itself.*

Repeating the construction of the theorem we get two sequences of harmonic maps

$$\begin{aligned} &L_0(=f) \xrightarrow{\partial} L_1 \xrightarrow{\partial} L_2 \to \cdots, \\ &L_0 \xrightarrow{\bar{\partial}} L_{-1} \xrightarrow{\bar{\partial}} L_{-2} \to \cdots \end{aligned} \tag{2.6}$$

whose image spaces are connected by fundamental collineations. Such sequences are called *harmonic sequences*. When all the L_ρ's are two-dimensional we can combine the sequence into one:

$$\cdots L_{-2} \underset{\bar\partial}{\overset{\partial}{\rightleftarrows}} L_{-1} \underset{\bar\partial}{\overset{\partial}{\rightleftarrows}} L_0 \underset{\bar\partial}{\overset{\partial}{\rightleftarrows}} L_1 \cdots . \tag{2.7}$$

Two consecutive spaces $L_\rho(x)$ and $L_{\rho+1}(x)$, $x \in M$, of a harmonic sequence are orthogonal. If any two members of a harmonic sequence are orthogonal, then the sequence is called a *Frenet harmonic sequence*. A Frenet harmonic sequence whose members span the ambient space is called a *full Frenet harmonic sequence*.

Because the two-sphere has no nonzero holomorphic differential forms, we have

Theorem 2.3. *Let* L_λ: $S^2 \to G(2, n; \mathbf{C})$, $0 \leqslant \lambda \leqslant s-1$, $n \geqslant 2s$, *be harmonic maps which form a Frenet harmonic sequence*

$$L_0 \underset{\bar\partial}{\overset{\partial}{\rightleftarrows}} L_1 \underset{\bar\partial}{\overset{\partial}{\rightleftarrows}} \cdots \underset{\bar\partial}{\overset{\partial}{\rightleftarrows}} L_{s-1}. \tag{2.8}$$

Let π_+: $L_{s-1}^\perp \to L_0$ *and* π_-: $L_0^\perp \to L_{s-1}$ *be the orthogonal projections. Consider the maps*

$$\begin{aligned} D_+ &= \underbrace{\partial \circ \cdots \circ \partial}_{s-1} \circ (\pi_+ \circ \partial): L_{s-1} \to L_{s-1}, \\ D_- &= \underbrace{\bar\partial \circ \cdots \circ \bar\partial}_{s-1} \circ (\pi_- \circ \bar\partial): L_0 \to L_0. \end{aligned} \tag{2.9}$$

The trace and determinant of D_+ *and* D_- *vanish identically. In particular the maps* $\pi_+ \circ \partial$: $L_{s-1} \to L_0$ *and* $\pi_- \circ \bar\partial$: $L_0 \to L_{s-1}$ *are degenerate. When* $n = 2s$, ∂: $L_{s-1} \to L_0$ *and* $\bar\partial$: $L_0 \to L_{s-1}$; *so in this case the fundamental collineations are degenerate.*

Suppose that (2.8) is a full Frenet harmonic sequence (so that $n = 2s$). By Theorem 2.3 the fundamental collineation $\bar\partial$: $L_0 \to L_{s-1}$ is zero or has rank one. If the former, then L_0 is a holomorphic curve $S^2 \to G(2, n; \mathbf{C})$. If the latter, then by Theorem 2.2 the image of $\bar\partial$ describes a harmonic map $S^2 \to G(1, n; \mathbf{C}) = P_{n-1}$. The Din-Zakrzewski description of harmonic maps $S^2 \to P_{n-1}$ [5] gives that the image of $\bar\partial$ is an element of the classical Frenet frame of some holomorphic curve $S^2 \to P_{n-1}$. Thus we see that full Frenet harmonic sequences lead to holomorphic curves. The idea of [4] is to exploit this by associating to a given harmonic sequence a new harmonic sequence which is full and Frenet. This association is effected by an inductive construction called *crossing*. Crossing associates a Frenet harmonic sequence of length $l+1$ to a given Frenet harmonic sequence of length l.

Now suppose that L_0: $S^2 \to G(2,n;\ \mathbf{R}) \subseteq G(2,n;\ \mathbf{C})$ is harmonic, so in particular $L_0 = \overline{L}_0$. In fact the harmonic sequences of (2.6) are, in this case, conjugates of one another. If all the L_i's are projective lines we have the harmonic sequence

$$(2.10) \qquad L_{-s} \xrightarrow{\partial} L_{-(s-1)} \xrightarrow{\partial} \cdots \xrightarrow{\partial} L_{-1} \xrightarrow{\partial} L_0 \xrightarrow{\partial} L_1 \to \cdots \xrightarrow{\partial} L_s,$$

where each L_ρ, $-s \leqslant \rho \leqslant s$, is the image of the previous one under a fundamental collineation and where $L_{-\rho} = \overline{L}_\rho$, $-s \leqslant \rho \leqslant s$. Such a harmonic sequence will be called a *real harmonic sequence*. A harmonic sequence of the form (2.10) with L_0 deleted and satisfying $L_{-\rho} = \overline{L}_\rho$ will also be called a *real harmonic sequence*. In §3 we will show that the construction of crossing takes real harmonic sequences to real harmonic sequences.

3. Crossing

Consider the real Frenet harmonic sequence

$$(3.1) \qquad L_{-s} \xrightarrow{\partial} \cdots \xrightarrow{\partial} L_{-1} \xrightarrow{\partial} L_0 \xrightarrow{\partial} L_1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} L_s.$$

Denote the fundamental collineation ∂: $L_\rho \to L_{\rho+1}$ (resp. $\overline{\partial}$: $L_\rho \to L_{\rho-1}$) by ∂_ρ (resp. $\overline{\partial}_\rho$) for $-s \leqslant \rho \leqslant s$. Then because (3.1) is a real harmonic sequence

$$(3.2) \qquad \sigma(\partial_\rho) = \overline{\partial}_{-\rho}, \qquad \sigma(\overline{\partial}_\rho) = \partial_{-\rho},$$

where σ denotes complex conjugation. By $\sigma(\partial_\rho) = \overline{\partial}_{-\rho}$ we mean that if $V \in L_\rho$, then

$$\overline{\partial_\rho(V)} = \overline{\partial}_{-\rho}(\overline{V}).$$

This follows immediately from (2.1) and its conjugate. Because the elements of (3.1) are mutually orthogonal

$$(3.3) \qquad \overline{\partial}_\rho = -\sigma\,{}^t(\partial_{\rho-1}),$$

where ^{t} denotes the adjoint with respect to the symmetric inner product.

Theorem 3.1. *The sequence obtained from* (3.1) *by adding* ∂_s: $L_s \to L_{s+1}$ (*or by adding* $\overline{\partial}_{-s}$: $L_{-s} \to L_{-(s+1)}$) *is a Frenet harmonic sequence.*

Proof. The structure equations associated to (3.1) show that L_{s+1} is orthogonal to $L_{-(s-1)}, \cdots, L_0, \cdots, L_s$ and that $L_{-(s+1)}$ is orthogonal to $L_{-s}, \cdots, L_0, \cdots, L_{s-1}$. Let π_+: $L_{s+1} \to L_{-s}$ and π_-: $L_{-(s+1)} \to L_s$ denote the orthogonal projections. Consider the maps $\pi_+ \circ \partial_s$: $L_s \to L_{-s}$ and $\pi_- \circ \overline{\partial}_{-s}$: $L_{-s} \to L_s$. Clearly

$$(3.4) \qquad \pi_+ \circ \partial_s = \sigma(\pi_- \circ \overline{\partial}_{-s}).$$

From the structure equations it follows that

$$\pi_+ \circ \partial_s = -\sigma\,{}^t(\pi_- \circ \bar{\partial}_{-s}). \tag{3.5}$$

Thus,

$$\pi_+ \circ \partial_s = -{}^t(\pi_+ \circ \partial_s). \tag{3.6}$$

By Theorem 2.3 the map $L_s \to L_s$ given by the composition $\partial_{s-1} \circ \cdots \circ \partial_{-s} \circ (\pi_+ \circ \partial_s)$ is degenerate. By assumption however the maps ∂_ρ: $L_\rho \to L_{\rho+1}$, $-s \leqslant \rho \leqslant s$, are nondegenerate and so $\det(\pi_+ \circ \partial_s) = 0$. This together with (3.6) implies that $\pi_+ \circ \partial_s = 0$ and therefore L_{s+1} is orthogonal to L_{-s}. q.e.d.

Consider the real Frenet harmonic sequence

$$L_{-s} \xrightarrow{\partial_{-s}} \cdots \xrightarrow{\partial_{-2}} L_{-1} \xrightarrow{\partial_{-1}} L_1 \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_{s-1}} L_s. \tag{3.7}$$

The proof of Theorem 3.1 gives

Theorem 3.2. *The sequence obtained from* (3.7) *by adding* ∂_s: $L_s \to L_{s+1}$ *(or by adding* $\bar{\partial}_{-s}$: $L_{-s} \to L_{-(s+1)}$*) is a Frenet harmonic sequence.*

Consider the Frenet harmonic sequence

$$L_{-s} \xrightarrow{\partial_{-s}} \cdots \xrightarrow{\partial_{-1}} L_0 \xrightarrow{\partial_0} \cdots \to L_s \xrightarrow{\partial_s} L_{s+1} \tag{3.8}$$

obtained from (3.1) by adding ∂_s: $L_s \to L_{s+1}$. Suppose that the harmonic sequence obtained from (3.8) by adding $\bar{\partial}_{-s}$: $L_{-s} \to L_{-(s+1)}$ is not Frenet, i.e., $L_{-(s+1)} = \bar{L}_{s+1}$ is not orthogonal to L_{s+1}. Let π: $L_{-(s+1)} \to L_{s+1}$ denote the orthogonal projection and consider the map $\pi \circ \bar{\partial}_{-s}$: $L_{-s} \to L_{s+1}$. By Theorem 2.3 $\pi \circ \bar{\partial}_{-s}$ has rank one. Let $Z_{2s+2} \in L_{s+1}$ be a vector of unit length whose span is the image of $\pi \circ \bar{\partial}_{-s}$ and choose $Z_{2s+1} \in L_{s+1}$ so that $\{Z_{2s+1}, Z_{2s+2}\}$ is a unitary framing of L_{s+1}. Now inductively choose a unitary framing $\{Z_{2\rho-1}, Z_{2\rho}\}$ of L_ρ, $\rho > 0$, by setting

$$\partial_\rho(Z_{2\rho-1}) = \alpha_{2\rho+1} Z_{2\rho+1} \tag{3.9}$$

(where $\alpha_{2\rho+1}$ is a scalar) and choosing $Z_{2\rho}$ orthogonal to $Z_{2\rho-1}$ in L_ρ. With respect to the bases $\{Z_{2\rho-1}, Z_{2\rho}\}$ and $\{Z_{2\rho+1}, Z_{2\rho+2}\}$ of L_ρ and $L_{\rho+1}$, respectively, the matrix A_ρ of ∂_ρ has the form

$$A_\rho = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}, \qquad 1 \leqslant \rho \leqslant s. \tag{3.10}$$

The conjugated frames $\{\bar{Z}_{2\rho-1}, \bar{Z}_{2\rho}\}$ are unitary frames of the spaces $\bar{L}_\rho = L_{-\rho}$. Let $A_{-\rho}$ denote the matrix representation of $\partial_{-\rho}$: $L_{-\rho} \to L_{-\rho+1}$. Then with respect to the bases $\{\bar{Z}_{2\rho}, \bar{Z}_{2\rho-1}\}$ and $\{\bar{Z}_{2\rho-2}, \bar{Z}_{2\rho-3}\}$ of $L_{-\rho}$ and $L_{-\rho+1}$, respectively, $A_{-\rho}$ has the form

$$A_{-\rho} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}, \qquad -2 \geqslant -\rho \geqslant -s. \tag{3.11}$$

(Note the ordering of the frames.) (3.11) follows from (3.2), (3.3), and (3.9).

Since Z_{2s+1} is orthogonal to the image of $\pi \circ \bar{\partial}_{-s}$, Z_{2s+1} is orthogonal to $L_{-(s+1)}$. In particular Z_{2s+1} is orthogonal to $\overline{Z}_{2s+1}$. Let C denote the matrix representation of $\pi \circ \bar{\partial}_{-s}$ with respect to the bases $\{\overline{Z}_{2s}, \overline{Z}_{2s-1}\}$ and $\{Z_{2s+1}, Z_{2s+2}\}$ of L_{-s} and L_{s+1}, respectively. Then since $\partial_s(Z_{2s-1}) = \alpha_{2s+1}Z_{2s+1}$, it follows that

$$\bar{\partial}_{-s}(\overline{Z}_{2s-1}) = \bar{\alpha}_{2s+1}\overline{Z}_{2s+1}.$$

Thus, $\pi \circ \bar{\partial}_{-s}(\overline{Z}_{2s-1}) = 0$. C has the form

$$C = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}. \tag{3.12}$$

We need to choose a framing of L_0. Let $Z_0 \in L_0$ be a vector of unit length satisfying

$$\partial_{-1}(\overline{Z}_2) = \bar{\alpha}_0\overline{Z}_0. \tag{3.13}$$

Now (3.3) and (3.9) imply that

$$\bar{\partial}_\rho(Z_{2\rho}) = (\ )Z_{2\rho-2} \tag{3.14}$$

and (3.13) implies that $\bar{\partial}_1(Z_2) = \alpha_0 Z_0$. So

$$\bar{\partial}_1 \circ \bar{\partial}_2 \circ \cdots \circ \bar{\partial}_s \circ (\pi \circ \bar{\partial}_{-s})(\overline{Z}_{2s}) = (\ )Z_0.$$

(3.11) and (3.13) imply that

$$\partial_{-1} \circ \cdots \circ \partial_{-s}(\overline{Z}_{2s}) = (\ )\overline{Z}_0.$$

We claim that $\langle \overline{Z}_0, Z_0 \rangle = 0$. To see this note that

$$\begin{aligned} &\langle \partial_{-1} \circ \cdots \circ \partial_{-s}(\overline{Z}_{2s}), \bar{\partial}_1 \circ \cdots \circ \bar{\partial}_s \circ (\pi \circ \bar{\partial}_{-s})(\overline{Z}_{2s}) \rangle \\ &\quad = \langle \overline{Z}_{2s}, \bar{\partial}_{-(s-1)} \circ \cdots \circ \bar{\partial}_0 \circ \bar{\partial}_1 \circ \cdots \circ \bar{\partial}_s \circ (\pi \circ \bar{\partial}_{-s})(\overline{Z}_{2s}) \rangle \\ &\quad = \langle \overline{Z}_{2s}, D(\overline{Z}_{2s}) \rangle, \end{aligned} \tag{3.15}$$

where $D = \bar{\partial}_{-(s-1)} \circ \cdots \circ \bar{\partial}_s \circ (\pi \circ \bar{\partial}_{-s})$.

By Theorem 2.3 D has zero trace. As $D(\overline{Z}_{2s-1}) = 0$, $\langle \overline{Z}_{2s}, D(\overline{Z}_{2s}) \rangle = 0$, and therefore $\langle \overline{Z}_0, Z_0 \rangle = 0$. Thus $\{\overline{Z}_0, Z_0\}$ is a unitary framing of L_0.

Define the projective lines

$$\lambda_\rho = Z_{2\rho-2} \wedge Z_{2\rho-1}, \quad \lambda_{-\rho} = \bar{\lambda}_\rho, \qquad 1 \leqslant \rho \leqslant s+1. \tag{3.16}$$

Theorem 3.3. *The projective lines* $\lambda_{-\rho}$ *and* λ_ρ, $1 \leqslant \rho \leqslant s+1$, *form a Frenet harmonic sequence*

$$\lambda_{-(s+1)} \xrightarrow{\partial} \lambda_{-s} \xrightarrow{\partial} \cdots \to \lambda_{-1} \xrightarrow{\partial} \lambda_1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \lambda_{s+1}. \tag{3.17}$$

Proof. Let A_{-1} (resp. A_0) denote the matrix representative of ∂_{-1} (resp. ∂_0) with respect to the bases $\{\bar{Z}_2, \bar{Z}_1\}$ and $\{\bar{Z}_0, Z_0\}$ of L_{-1} and L_0 (resp. $\{\bar{Z}_0, Z_0\}$ and $\{Z_1, Z_2\}$ of L_0 and L_1). Then A_i, $i = 0, -1$, has the form

$$A_i = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}.$$

Hence

$$A_\rho = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}, \qquad -s \leqslant \rho \leqslant s. \tag{3.18}$$

We have

$$d\begin{pmatrix} Z_{2\rho-1} \\ Z_{2\rho} \end{pmatrix} = -\,{}^t\bar{A}_{\rho-1}\bar{\phi}\begin{pmatrix} Z_{2\rho-3} \\ Z_{2\rho-2} \end{pmatrix} + \pi_\rho\begin{pmatrix} Z_{2\rho-1} \\ Z_{2\rho} \end{pmatrix} + A_\rho\phi\begin{pmatrix} Z_{2\rho+1} \\ Z_{2\rho+2} \end{pmatrix} \qquad \text{for } 1 \leqslant \rho \leqslant s,$$

$$d\begin{pmatrix} \bar{Z}_0 \\ Z_0 \end{pmatrix} = -\,{}^t\bar{A}_{-1}\bar{\phi}\begin{pmatrix} \bar{Z}_2 \\ \bar{Z}_1 \end{pmatrix} + \pi_0\begin{pmatrix} \bar{Z}_0 \\ Z_0 \end{pmatrix} + A_0\phi\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \tag{3.19}$$

$$d\begin{pmatrix} \bar{Z}_{2\rho} \\ \bar{Z}_{2\rho-1} \end{pmatrix} = -\,{}^t\bar{A}_{-(\rho+1)}\bar{\phi}\begin{pmatrix} \bar{Z}_{2\rho+2} \\ \bar{Z}_{2\rho+1} \end{pmatrix} + \pi_{-\rho}\begin{pmatrix} \bar{Z}_{2\rho} \\ \bar{Z}_{2\rho-1} \end{pmatrix} + A_{-\rho}\phi\begin{pmatrix} \bar{Z}_{2\rho-2} \\ \bar{Z}_{2\rho-3} \end{pmatrix} \qquad \text{for } 1 \leqslant \rho \leqslant s-1.$$

The structure equations and the harmonicity of the sequence (3.8) imply that the matrices of 1-forms π_ρ have the form

$$\begin{aligned} \pi_\rho &= \begin{pmatrix} - & q_\rho\phi \\ -\bar{q}_\rho\bar{\phi} & - \end{pmatrix}, \qquad 1 \leqslant \rho \leqslant s, \\ \pi_0 &= \begin{pmatrix} \omega_{\bar{0}\bar{0}} & 0 \\ 0 & \omega_{0\bar{0}} \end{pmatrix}, \\ \pi_{-\rho} &= \begin{pmatrix} - & q_{-\rho}\phi \\ -\bar{q}_{-\rho}\bar{\phi} & - \end{pmatrix}, \qquad 1 \leqslant \rho \leqslant s-1. \end{aligned} \tag{3.20}$$

The theorem now follows easily from the structure equations of the unitary frame $\{\bar{Z}_{2s+1}, \bar{Z}_{2s}, \cdots, \bar{Z}_1, \bar{Z}_0, Z_0, Z_1, Z_2, \cdots, Z_{2s}, Z_{2s+1}\}$ using (3.12), (3.18), (3.19), (3.20), and Theorem 2.1. q.e.d.

The operation of passing from the Frenet harmonic sequence (3.8) to the Frenet harmonic sequence (3.17) is called *crossing*. Note that Theorem 3.2 ensures that the harmonic sequence obtained from (3.17) by adding ∂: $\lambda_{s+1} \to \lambda_{s+2}$ (or by adding $\bar{\partial}$: $\lambda_{-(s+1)} \to \lambda_{-(s+2)}$) is a Frenet harmonic sequence.

Theorem 3.3 shows that we must consider Frenet harmonic sequences of the form

$$(3.21)\qquad \lambda_{-s} \xrightarrow{\partial} \lambda_{-(s-1)} \xrightarrow{\partial} \cdots \to \lambda_{-1} \xrightarrow{\partial} \lambda_1 \xrightarrow{\partial} \lambda_2 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \lambda_s,$$

where $\lambda_{-\rho} = \bar{\lambda}_\rho$, $1 \leqslant \rho \leqslant s$. Consider the fundamental collineations ∂_{-1}: $\lambda_{-1} \to \lambda_1$ and $\bar{\partial}_1$: $\lambda_1 \to \lambda_{-1}$. By (3.2), $\sigma(\partial_{-1}) = \bar{\partial}_1$ and by (3.3), $\bar{\partial}_1 = -\sigma\, '(\partial_{-1})$. Thus, $\partial_{-1} = -\, '\partial_{-1}$, i.e., ∂_{-1} is skew-symmetric. If $\{Z_0, Z_1\}$ is a unitary framing of λ_1, then with respect to the framings $\{\bar{Z}_1, \bar{Z}_0\}$ and $\{Z_0, Z_1\}$ of λ_{-1} and λ_1, respectively, ∂_{-1} has the form

$$\partial_{-1} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

With this observation the techniques discussed previous to Theorem 3.3 can be applied to the sequence (3.21). We have

Theorem 3.4. *The operation of crossing can be applied to the sequence* (3.21) *to construct a Frenet harmonic sequence*

$$(3.22)\qquad l_{-s} \xrightarrow{\partial} l_{-s-1} \xrightarrow{\partial} \cdots\ l_{-1} \xrightarrow{\partial} l_0 \xrightarrow{\partial} l_1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} l_s,$$

where $\bar{l}_\rho = l_{-\rho}$, $0 \leqslant \rho \leqslant s$.

The process inverse to crossing, called *recrossing*, comes in two types: one inverting the construction of Theorem 3.3, the other inverting the construction of Theorem 3.4. The inversion of the construction of Theorem 3.3 involves an arbitrariness as it passes from a real Frenet harmonic sequence of the type of (3.17) to a real Frenet harmonic sequence of the type of (3.8). This operation proceeds as follows:

Consider the real Frenet harmonic sequence (3.17). The projective line λ_1 describes a P_1-bundle over S^2. λ_1 is a subbundle of the trivial $\mathbf{C}^n$-bundle over S^2. Using the standard connection on the trivial $\mathbf{C}^n$-bundle and the Newlander-Nirenberg theorem, λ_1 admits a natural holomorphic structure (cf. [4]). The arbitrariness of recrossing involves choosing a holomorphic section, σ, of the bundle λ_1. Adapt a unitary framing to (3.17) by setting

$$(3.23)\qquad \begin{aligned} &Z_1 = \sigma, \\ &\underbrace{\partial \circ \cdots \circ \partial}_{\rho}(Z_1) = (\ )Z_{2\rho+1}, \qquad 1 \leqslant \rho \leqslant s, \end{aligned}$$

and letting $Z_{2\rho+2}$ be a point on $\lambda_{\rho+1}$ of unit length and orthogonal to $Z_{2\rho+1}$. As above $\{\bar{Z}_{2\rho+2}, \bar{Z}_{2\rho+1}\}$ will be a framing of the line $\lambda_{-(\rho+1)}$. Set

$$(3.24)\qquad \begin{aligned} L_{-\rho} &= \bar{Z}_{2\rho+1} \wedge \bar{Z}_{2\rho}, \\ L_0 &= \bar{Z}_1 \wedge Z_1, \qquad 1 \leqslant \rho \leqslant s. \\ L_\rho &= Z_{2\rho} \wedge Z_{2\rho+1}, \end{aligned}$$

Theorem 3.5. *The lines* (3.24) *form a real Frenet harmonic sequence*

$$L_{-s} \xrightarrow{\partial} \cdots \to L_0 \xrightarrow{\partial} \cdots \xrightarrow{\partial} L_s. \tag{3.25}$$

For appropriate choice of a holomorphic section, σ, the sequence (3.25) *is the original sequence* (3.8).

On the other hand the inversion of the construction of Theorem 3.4 involves no arbitrariness. Consider the real Frenet harmonic sequence (3.22). The projective line l_0 admits a unitary frame $\{\overline{Z}_0, Z_0\}$, where the vector Z_0 is unique up to multiplication by a factor of absolute value 1. Set $\partial(\overline{Z}_0) = (\ )Z_1$ and adapt a unitary frame to (3.22) as in (3.23). Recrossing now proceeds as above.

We conclude that to invert the construction (by crossing) of the real Frenet harmonic sequence

$$l_{-(s+1)} \xrightarrow{\partial} \cdots \xrightarrow{\partial} l_0 \xrightarrow{\partial} \cdots \xrightarrow{\partial} l_{s+1}$$

from the real Frenet harmonic sequence

$$L_{-s} \xrightarrow{\partial} \cdots \xrightarrow{\partial} L_0 \xrightarrow{\partial} \cdots \xrightarrow{\partial} L_s$$

requires the choice of one holomorphic section of a P_1-bundle over S^2.

4. Main theorem

Given a harmonic map L_0: $S^2 \to G(2, n; \mathbf{R})$ and its associated real harmonic sequence, by successively applying crossing we can construct longer and longer real Frenet harmonic sequences. However, crossing cannot be applied to a real Frenet harmonic sequence

$$\lambda_{-s} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \lambda_0 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \lambda_s, \tag{4.1}$$

when the fundamental collineation ∂_s: $\lambda_s \to \lambda_{s+1}$ is degenerate. There are two cases: (i) The fundamental collineation ∂_s: $\lambda_s \to \lambda_{s+1}$ is zero, or (ii) the fundamental collineation ∂_s: $\lambda_s \to \lambda_{s+1}$ has rank one. In case (i) the map λ_{-s}: $S^2 \to G(2, n; \mathbf{C})$ is holomorphic and the map λ_s: $S^2 \to G(2, n; \mathbf{C})$ is antiholomorphic. Moreover, for $0 \leqslant \rho \leqslant s-1$, the ρth osculating spaces of both λ_{-s} and λ_s are null. In case (ii) the harmonic maps λ_{-s} and λ_s are degenerate (i.e. one of their fundamental collineations is degenerate). Here, for $0 \leqslant \rho \leqslant s-1$, the ρth holomorphic (resp. antiholomorphic) osculating space of λ_{-s} (resp. λ_s) is null. A degenerate harmonic map can be constructed from holomorphic curves $S^2 \to P_{n-1}$ via the procedure of *returning* (cf. [4]). In fact, if λ_s is constructed from the curve δ: $S^2 \to P_{n-1}$, then $\lambda_{-s} = \bar{\lambda}_s$ is constructed from the curve $\bar{\delta}$: $S^2 \to P_{n-1}$. By iterating crossing until the ambient space is exhausted, one of case (i) or (ii) must occur. We will call a holomorphic or an antiholomorphic curve $M \to G(2, n; \mathbf{C})$ *r-null* if its ρth osculating spaces are null for $0 \leqslant \rho \leqslant r$. We will call a degenerate harmonic map $M \to G(2, n; \mathbf{C})$ *r-null* if its ρth osculating spaces in the nondegenerate direction are null for $0 \leqslant \rho \leqslant r$. We have

Theorem 4.1. *Let f: $S^2 \to G(2, n$: $\mathbf{C})$ be a real nondegenerate harmonic map. Then there is associated to f either* (i) *a unique r-null holomorphic curve Δ_f: $S^2 \to G(2, n; \mathbf{C})$, $r \geqslant 0$, or* (ii) *a unique r-null degenerate harmonic map Δ_f: $S^2 \to G(2, n; \mathbf{C})$, $r \geqslant 0$. f can be recovered from Δ_f via recrossing and the ∂ and $\bar{\partial}$ transforms.*

The curve Δ_f is called the *directrix curve* of f. Since degenerate harmonic maps can be constructed from holomorphic curves $S^2 \to P_{n-1}$ using returnings, Theorem 4.1 provides a description of the harmonic maps $S^2 \to Q_{n-2}$.

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