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RADON TRANSFORMS ON HIGHER RANK GRASSMANNIANS

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Abstract

We define a Radon transform R from functions $\mathrm{Gr}(k, n)$, the Grassmannian of projective k-planes in $\mathbf{C}P^n$ to functions on $\mathrm{Gr}(l, n)$. If $f \in C^\infty(\mathrm{Gr}(k, n))$ and $L \in \mathrm{Gr}(l, n)$, then $Rf(L)$ is the integral of $f(H)$ over all k-planes H which lie in L. If R' is the dual transform, we show under suitable assumptions on k and l that $R'R$ is invertible by a polynomial in the Casimir operators of $U(n+1)$, the group of isometries $\mathbf{C}P^n$. We also treat the real and quaternionic cases. Finally, we indicate some possible variations and generalizations to flag manifolds.

0. Introduction

Let P^n be a projective space over a real division ring, say $\mathbf{C}P^n$. The projective hyperplane transform, or *Radon transform* R, associates to a suitable function f on the projective space P^n a function Rf on P^{n*}, the space of projective hyperplanes in P^n, by integration: if H is a hyperplane in P^n, then

$$Rf(H) = \int_H f\,d\mu,$$

where $d\mu$ is normalized invariant measure. These transforms were first considered by S. Helgason. In [6] Helgason gave inversion formulas for R defined over real, complex, quaternionic, or octonionic projective spaces. A natural generalization of R is the k-plane transform R_k. This associates to f a function $R_k f$ on $\mathrm{Gr}(k, n)$, the Grassmann manifold of projective k-planes in P^n, again by integration. Helgason also gave inversion formulas for R_k defined

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over real projective spaces (see [7]); corresponding formulas exist for projective spaces over other division rings (see Grinberg [5]). All these formulas have the following form: Let R^t be the dual Radon transform (defined explicitly below) and let Δ be the Laplace-Beltrami operator on P^n. Then we have the formula

$$P_k(\Delta) \circ R_k^t \circ R_k = I,$$

where P_k is a polynomial which can be given explicitly. (In the real case such a formula exists only for k even.)

In this paper we consider a further generalization of R: $R_{k,l}$. This transform associated a function $R_{k,l}$ on $\mathrm{Gr}(l, n)$ to a function f on $\mathrm{Gr}(k, n)$, once more by integration: Let H denote a projective k-plane and L a projective l-plane. Then

$$R_{k,l}f(L) = \int_{\{H \mid H \subset L\}} f \, d\mu,$$

where $d\mu$ is again normalized invariant measure. The integration is over all k-planes H lying in L. In analogy with the k-plane transform, we expect R^tR to be invertible by an invariant differential operator (we will often abuse notation and write R for $R_{k,l}$). Since P^n is a rank one symmetric space, the algebra of differential operators on it is generated by the Laplace-Beltrami operator (Helgason [7]). However, for $0 < k \leqslant n - k - 1$ the space $\mathrm{Gr}(k, n)$ is of higher rank and hence its algebra of differential operators has $k + 1$ generators, say $\Box_0, \cdots, \Box_k$, which are called Casimir operators. We shall show that (for suitable k and l) there is a polynomial $P_{k,l}$ (P for short) such that the following formula is valid:

$$P(\Box_0, \cdots, \Box_k) R^tR = I.$$

This expression is of the same form as Helgason's original inversion formula (indeed the latter can be read off from the former).

Our principle tool is representation theory, including highest weights and Casimir operators. We are grateful to E. Bolker who suggested doing the analysis 'one rank-step at a time,' i.e. $l = k + 1$. This makes the calculations simpler.

The reader is referred to the recent papers of Gonzalez [2] and Strichartz [12] for related Radon tansforms on affine (i.e. nonprojective) Grassmannians. In the concluding section of our paper we briefly discuss possible generalizations of these transforms to flag manifolds, including cases considered in these two papers.

1. Harmonic analysis on $\mathrm{Gr}(k, n)$

Following [5] we view the Grassmannian $\mathrm{Gr}(k,n)$ as the set of $(k+1)$-dimensional (complex) vector subspaces of $\mathbf{C}^{n+1}$, or equivalently, as the set of projective k-planes in $\mathbf{C}P^n$. A $(k+1)$-plane H can be represented by the following $(n+1)\times(k+1)$ matrix whose columns give an orthonormal frame for H:

$$\left\| \begin{matrix} Z_0^0 & \cdots & Z_k^0 \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ Z_0^n & & Z_k^n \end{matrix} \right\| . \tag{1}$$

The set of such matrices forms the *Stiefel manifold*

$$\mathrm{St}(k,n) = U(n+1)/U(n-k).$$

Change of basis for a plane is realized by right multiplication of (1) by a matrix in $U(k+1)$ so

$$\mathrm{Gr}(k,n) = \frac{\mathrm{St}(k,n)}{U(k+1)} = \frac{U(n+1)}{U(k+1)\times U(n-k)}.$$

A function on $\mathrm{Gr}(k,n)$ can be viewed as a function on $\mathrm{St}(k,n)$ which is right-$U(k+1)$ invariant. We will often use the following functions:

$$\langle Z^i, Z^j\rangle = \sum_a Z_a^i \cdot \overline{Z_a^j}.$$

For simplicity, we will use the notation $\langle i, j\rangle \equiv \langle Z^i, Z^j\rangle$.

The Lie algebra of $U(n+1)$ is $u(n+1)$, the set of skew-Hermitian matrices. A basis for $u(n+1)$ over $\mathbf{R}$ is given by

$$\begin{aligned} a_{ij} &= e_{ij} - e_{ji} && (0 \leqslant i,\ j \leqslant n), \\ b_j &= \sqrt{-1}\,(e_{ij} + e_{ji}) && (0 \leqslant i,\ j \leqslant n), \\ c_i &= \sqrt{-1}\,e_{ii} && (0 \leqslant i \leqslant n). \end{aligned}$$

The Lie algebra $u(n+1)$ acts by differential operators on $C^\infty(\mathrm{Gr}(k,n))$. On the level of polynomial functions in the products $\langle Z^i, Z^j\rangle$ the above elements act as certain first order operators A_{ij}, B_{ij}, and C_i, respectively. The operator $D_{ij} \equiv (A_{ij} - \sqrt{-1}\,B_{ij})/2$ acts as

$$\sum_a \left\{ Z_a^i \partial/\partial Z_a^j - \overline{Z_a^j}\,\partial/\partial \overline{Z_a^i} \right\}.$$

The collection $\{D_{ii}\} = \{C_i\}$ is a (maximal) commuting family of operators in $u(n+1)$. A function $f(Z)$ in $C^\infty(\mathrm{Gr}(k,n))$ is a *weight vector* if f is a simultaneous eigenvector of $\{C_i\}$. The *weight* of f is $(m_0,\cdots,m_n)$ if

$$C_i f = m_i f \qquad (i = 0,\cdots,n).$$

The weight vector $f \neq 0$ is a *highest weight vector* if f is annihilated by all *raising operators* $\{D_{ij}\}_{i<j}$:

$$D_{ij} f = 0 \quad (\text{all } i < j).$$

A consequence of this is that $m_0 \geqslant m_1 \geqslant \cdots \geqslant m_n$, hence we say that $(m_0,\cdots,m_n)$ is a *highest weight.*

Lemma 1.1. *The highest weights occurring in $C^\infty(\mathrm{Gr}(k,n))$ are precisely those of the form $(m_0, m_1,\cdots,m_k,\ 0,\cdots,0,-m_k,-m_{k-1},\cdots,-m_0)$, with $m_0 \geqslant m_1 \geqslant \cdots \geqslant m_k$ and where all the m_j's are integers.*

A proof can be found in [10]. In [5] we construct explicit weight vectors. These vectors will play a crucial role in our inversion of the Radon transform so we review their construction.

Let $m = m_0 + \cdots + m_k$. We will consider multi-indices I of total order m and of order m_0 in the symbol '0',$\cdots$, order m_k in the symbol 'k'. We can view I as a map from $\{0,1,\cdots,m-1\}$ to $\{0,1,\cdots,k\}$. Let J be the standard such multi-index corresponding to the sequence

$$0,\cdots,0,1,\cdots,1,\cdots,k,\cdots,k.$$

Finally, for any multi-index I, let ε^I be the sign of a permutation of least order in the symmetric group on m symbols which takes I to the standard multi-index J. Put

$$f_{m_0,\cdots,m_k} = \sum_I \varepsilon^I \langle I(0), n - J(0)\rangle \cdots \langle I(m-1), n - J(m-1)\rangle.$$

Here the summation is over all multi-indices of the multi-order described above. Each multi-index is viewed as a map from $\{0,\ldots,m-1\}$ to $\{0,\cdots,k\}$, hence the notation $I(0)$, $I(1)$, etc.

Theorem 1.2 [5]. *For each highest weight $(m_0,\cdots,m_k,\ 0,\cdots,0,\ -m_k,\cdots,-m_0)$ the vector $f_{m_0,\cdots,m_k}$ is a highest weight vector in $C^\infty(\mathrm{Gr}(k,n))$ with this weight.*

It will be useful to have an alternative description (quite similar to an expression due to R. S. Strichartz) of the highest weight vectors. We will write $Z^0,\cdots,Z^n$ to denote the rows of the Stiefel matrix (1). Then the highest weight vector $f_{m_0, \dots, m_k}$ can be written as

$$\det\left\{\|Z^0\| \cdot \|Z^n\|^*\right\}^{m_0-m_1} \times \det\left\{\left\|\begin{matrix} Z^0 \\ Z^1 \end{matrix}\right\| \cdot \left\|\begin{matrix} Z^n \\ Z^{n-1} \end{matrix}\right\|^*\right\}^{m_1-m_2}$$

$$\times \cdots \times \det\left\{\left\|\begin{matrix} Z^0 \\ \vdots \\ Z^{k-1} \end{matrix}\right\| \cdot \left\|\begin{matrix} Z^n \\ \vdots \\ Z^{n-k+1} \end{matrix}\right\|^*\right\}^{m_{k-1}-m_k} \times \det\left\{\left\|\begin{matrix} Z^0 \\ \vdots \\ Z^{k} \end{matrix}\right\| \cdot \left\|\begin{matrix} Z^n \\ \vdots \\ Z^{n-k} \end{matrix}\right\|^*\right\}^{m_k}.$$

This expression is clearly right-$U(k+1)$ invariant. It is also easy to see that it gives a highest weight vector. In fact, if $0 < i \leqslant k$, then the raising operator $D_{i-1,i}$ applied to any matrix in the above expression results in either the zero matrix (if the holomorphic row Z_i does not occur originally) or in a matrix with two equal rows (Z_{i-1}) and hence zero determinant, so $D_{i-1,i}f = 0$. Similar considerations show that $D_{ij}f = 0$ for any raising operator D_{ij}. Moreover, f is clearly not identically zero (to see this, evaluate f on a Stiefel matrix whose upper $(k+1) \times (k+1)$ part is diagonal), so f is a highest weight vector.

Since $\mathrm{Gr}(k, n)$ is a symmetric space, each irreducible representation in $L^2(\mathrm{Gr}(k, n))$ occurs with multiplicity 1, hence the above weight vectors are unique up to scalar multiplication.

2. Diagonalization of the Radon transform

We now consider the Radon transform R:

$$C^\infty(\mathrm{Gr}(k, n)) \to C^\infty(\mathrm{Gr}(k+1, n)).$$

We will assume throughout that $(k+2) \leqslant (n-k-1)$. If $f[H]$ is a smooth function of (projective) k-planes, we define a function $Rf(L)$ of (projective) $(k+1)$-planes via

$$Rf(L) = \int_{H \subset L} f(H)\, d\mu(H).$$

Here the integration is over all k-planes H contained in L and the measure $d\mu$ is the unique normalized $U(L)$-invariant measure. Clearly R commutes with the action of $U(n+1)$ on k-planes and on $(k+1)$-planes. If W is an irreducible subrepresentation of $U(n+1)$ in $C^\infty(\mathrm{Gr}(k, n))$, then there is a (unique) copy of W in $C^\infty(\mathrm{Gr}(k+1, n))$. If we identify the two copies, then R must reduce to scalar multiplication on W. To compute this scalar, we select a particular element $f \in W$ (a highest weight vector) and compute its Radon transform Rf at a particular $(k+1)$-plane L_0. Let L_0 be the projective $(k+1)$-plane whose Stiefel matrix is

$$\left\|\begin{matrix} 1/\sqrt{2} & & 0 \\ & \ddots & \\ & & 1/\sqrt{2} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ & & 1/\sqrt{2} \\ & \cdot^{\cdot^{\cdot}} & \\ 1/\sqrt{2} & & 0 \end{matrix}\right\|.$$

We now consider a k-plane H contained in L_0. If $[Z]$ is a Stiefel matrix for H we can add a column to $[Z]$, say $(w_0, \cdots, w_n)^T$, and obtain a Stiefel matrix for L_0. This column is unique up to a scalar. Let $\langle i, j\rangle$ be a function on $\mathrm{Gr}(k, n)$. We emphasize that all functions involving the $\langle i, j\rangle$ products are viewed in this discussion as functions of k-planes, not $(k+1)$-planes. We can express these functions in terms of the added column $(w_0, \cdots, w_n)^T$ above:

$$\begin{aligned} \langle i, j\rangle &= -w_i\overline{w_j}, \quad (i \neq j), \\ \langle i, i\rangle &= \tfrac{1}{2} - |w_i|^2. \end{aligned} \tag{2}$$

These relations follow from the fact that a square matrix is unitary by rows if and only if it is unitary by columns, together with the observation that the top square submatrix of a Stiefel matrix of L_0 is unitary, except for a scalar factor. Finally, from the definition of L_0 we have the relations $\langle i, n-j\rangle = \langle i, j\rangle$ for all i, j and all k-planes H in L_0.

Lemma 2.1. *Let u, v be complex $p \times 1$ column vectors. Then*

$$\det(I + uv^*) = (1 + v^*u).$$

Proof. The equality is easy to verify if u is a multiple of $e_1 = (1\,0 \cdots 0)^t$. For general u, there is a unitary $p \times p$ (Householder) matrix P with Pu a multiple of e_1. Then

$$\begin{aligned} \det(I + uv^*) &= \det\big(P[I + uv^*]P^*\big) = \det\big(I + (Pu)(Pv)^*\big) \\ &= 1 + (Pv)^*(Pu) = 1 + v^*u. \end{aligned}$$

Proposition 2.2. *If $H \subset L_0$, then*

$$f_{m_0,\cdots,m_k}(H) = c_{k,m}\left(\tfrac{1}{2} - |w_0|^2\right)^{m_0-m_1}\left(\tfrac{1}{2} - |w_0|^2 - |w_1|^2\right)^{m_1-m_2}\cdots$$
$$\times\left(\tfrac{1}{2} - |w_0|^2 - |w_1|^2 - \cdots - |w_{k-1}|^2\right)^{m_{k-1}-m_k}$$
$$\times\left(\tfrac{1}{2} - |w_0|^2 - |w_1|^2 - \cdots - |w_{k-1}|^2 - |w_k|^2\right)^{m_k}.$$

Proof. We will ignore the constant in front for the moment. Let A be the top $(k+2) \times (k+1)$ part of the Stiefel matrix for H. We can adjoin a vector $W = (w_0, \cdots, w_{k+1})^t$ to A (on the right) to form a (partitioned) square matrix $B = (A|w)$ with $BB^* = I/2$. For each $j \leqslant k+1$ we denote the top $(j+1) \times (k+1)$ part of B by $B(j+1)$ (this is $(A(j+1)|W(j+1))$).

In this notation, the determinantal expression in §1 for f is

$$f_{m_0,\cdots,m_k} = \det\left(A(0)A(0)^*\right)^{m_0-m_1}\cdots\det\left(A(k)A(k)^*\right)^{m_k-0}.$$

But, since $2 \cdot B(j) \cdot B(j)^* = I_{j+1}$, the $(j+1) \times (j+1)$ identity matrix, we have

$$A(j) \cdot A(j)^* + W(j) \cdot W(j)^* = I/2,$$

hence

$$A(j) \cdot A(j)^* = I/2 - W(j) \cdot W(j)^*$$

and by Lemma 2 above, $2^j\det(A(j) \cdot A(j)^*) = \frac{1}{2} - W(j)^*W(j)$. This, combined with the above expression for f proves the proposition with $c_{k \cdot m} = c_k \cdot 2^{-m_1-\cdots-m_k}$. q.e.d.

We now compute the value of the Radon transform on L_0.

Proposition 2.3.

$$Rf_{m_0,\cdots,m_k}(L_0) = (k+1)!2^{-m_0-\cdots-m_k}$$
$$\times\frac{1}{(m_0+k+1)(m_1+k)\cdots(m_k+1)}.$$

Proof. Using the previous proposition, if $m = m_0 + \cdots + m_k$ then

$$Rf(L_0) = \int_{|W|^2=1} c_{k,m}\left(\tfrac{1}{2} - |w_0|^2\right)^{m_0-m_1}\left(\tfrac{1}{2} - |w_0|^2 - |w_1|^2\right)^{m_1-m_2}\cdots$$
$$\times\left(\tfrac{1}{2} - |w_0|^2 - |w_1|^2 - \cdots - |w_{k-1}|^2\right)^{m_{k-1}-m_k}$$
$$\times\left(\tfrac{1}{2} - |w_0|^2 - |w_1|^2 - \cdots - |w_{k-1}|^2 - |w_k|^2\right)^{m_k} d\mu(W),$$

where $W = (w_0, \cdots, w_{k+1})$ and $d\mu(W)$ is spherical measure. This integral can be rewritten as

$$\int_0^{[1/2]} \int_0^{[1/2-|w_0|^2]^{1/2}} \cdots \int_0^{[1/2-|w_0|^2-\cdots-|w_{k-1}|^2]^{1/2}}$$
$$\times c_{k,m}\left(\tfrac{1}{2}-|w_0|^2\right)^{m_0-m_1}\left(\tfrac{1}{2}-|w_0|^2-|w_1|^2\right)^{m_1-m_2}\cdots$$
$$\times\left(\tfrac{1}{2}-|w_0|^2-|w_1|^2-\cdots-|w_{k-1}|^2\right)^{m_{k-1}-m_k}$$
$$\times\left(\tfrac{1}{2}-|w_0|^2-|w_1|^2-\cdots-|w_{k-1}|^2-|w_k|^2\right)^{m_k} d\mu(w_0,\cdots,w_{k+1}).$$

The innermost integral is the integral of the function $|w_{k+1}|^2$ over the sphere in the (w_k, w_{k+1})-plane. The measure $d\mu(w_k, w_{k+1})$ must be a multiple of spherical measure on S^3. Taking $|w_{k+1}| = \rho_k \sin(\theta)$, where $\rho_k^2 = \frac{1}{2} - |w_0|^2 - \cdots - |w_{k-1}|^2$, the measure $d\mu$ becomes $(\rho_k \cos(\theta))(\rho_k \sin(\theta)) \cdot \rho_k \, d\theta$, and the integral is

$$\int_0^{\pi/2} (\rho_k \cos(\theta))(\rho_k \sin(\theta))^{2m_k}(\rho_k \sin(\theta))\rho_k \, d\theta$$
$$= (\rho_k)^{2m_k+3}\frac{1}{2(m_k+1)}.$$

The rest of the integrals can be evaluated similarly, and it is not hard to obtain the stated formula. q.e.d.

We now consider the dual Radon transform R^t. If $g(L)$ is a smooth function on $\mathrm{Gr}(k+1, n)$ we define a function on $\mathrm{Gr}(k, n)$ by

$$R^t g(H) = \int_{L \supset H} g(L)\, d\mu(L),$$

where the integration is over all $(k+1)$-planes L containing H, and $d\mu$ is the normalized invariant measure. Let H_0 be the projective k-plane whose Stiefel matrix is given by the left-most $(k+1)$ columns of the Stiefel matrix for L_0. Let $g_{m_0,\cdots,m_k}$ be the unique highest weight vector in $C^\infty(\mathrm{Gr}(k+1, n))$ with weight $(m_0, \cdots, m_k, 0, \cdots, 0, -m_k, \cdots, -m_0)$ and satisfying $g(L_0) = 2^{-m}$. Then a calculation similar to the above shows that

$$R^t g_{m_0,\cdots,m_k}(H_0)$$
$$= c_{k,n}\frac{2^{-m}}{(m_0+n-k-1)(m_1+n-k-2)\cdots(m_k+n-2k-1)}.$$

The complex codimension of H_0 in $\mathbf{C}P^n$ is $n-k$ and this accounts for the difference between this integral and the previous one. Combining these two integrals we have the following result:

Theorem 2.4.

$$R^tRf_{m_0,\cdots,m_k}$$
$$= c_{k,n}(m_0+n-k-1)^{-1}(m_1+n-k-2)^{-1}\cdots$$
$$\cdot(m_k+n-2k-1)^{-1}(m_0+k+1)^{-1}(m_1+k)^{-1}\cdots(m_k+1)^{-1},$$

where $c_{k,n}\neq 0$.

3. The inverse of R^tR

We wish to express the (left) inverse of R^tR as an invariant differential operator. It will prove useful to know the eigenvalues that such operators may have. Recall that a differential operator D on $C^\infty(U(n+1))$ is a *Casimir operator* if it is bi-invariant, or equivalently, if it lies in the center of the universal enveloping algebra of $U(n+1)$.

Theorem 3.1. *Let $P(m_0,\cdots,m_n)$ be a symmetric polynomial. Then there is a Casimir operator D whose eigenvalue on the irreducible $U(n+1)$ representation with highest weight $(m_0,\cdots,m_n)$ is $P(m_0+n, m_1+n-1,\cdots,m_{n-1}+1, m_n)$.*

A proof may be found in Zelobenko [14]. We now define a polynomial of $k+1$ variables

$$Q(m_0,\cdots,m_k)=(m_0+n-k-1)(m_1+n-k-2)\cdots$$
$$\cdot(m_k+n-2k-1)(m_0+k+1)(m_1+k)\cdots(m_k+1).$$

Lemma 3.2. *There is a symmetric polynomial $P(m_0,\cdots,m_n)$ such that*

$$P(m_0+n, m_1+n-1,\cdots,m_{n-1}+1, m_n)=Q(m_0,\cdots,m_k)$$

whenever $(m_0,\cdots,m_n)$ is of the form $(m_0,\cdots,m_k,0,\cdots,0,-m_k,\cdots,-m_0)$.

Proof. We can pair the linear factors of Q as follows:

$$\prod_{j=0}^{k}(m_j+n-k-1-j)(m_j+k+1-j)$$
$$=\prod_{j=0}^{k}\left[m_j+n-j-(k+1)\right](-1)\left[-m_j+j-(k+1)\right].$$

Let $S(m_0,\cdots,m_n)=\sum_j m_0\cdots\hat{m}_j\cdots m_n$, where $\hat{m}_j$ indicates that m_j is omitted. Put

$$P(m_0,\cdots,m_n)=c_{k,n}(-1)^kS(m_0-k-1,\cdots,m_n-k-1).$$

Assume now that $m_j = -m_{n-j}$ (for all j) and $m_j = 0$ for $k < j < n-k$ (and of course $m_0 \geqslant m_1 \geqslant \cdots \geqslant m_k \geqslant 0$). Since $m_{n-k-1} = 0$, among the summands in the expression $P(m_0 + n, m_1 + n - 1, \cdots, m_n)$ only the one with m_{n-k-1} omitted is nonzero. This summand is clearly of the form $c \cdot Q(m_0, \cdots, m_k)$, where $c = 1 \cdot 2 \cdots (n-k-1)$. Combining this with the diagonalization of R^tR in the previous section we obtain

Theorem 3.3. *Let Δ_{k+1} denote the Casimir operator of $U(n+1)$ whose eigenvalue on the irreducible representation with weight $(m_0, \cdots, m_n)$ is $P(m_0 + n, m_1 + n - 1, \cdots, m_n)$. Then*

$$c_{k,n} \Delta_{k+1} R^t R(f) = f$$

for all $f \in C^\infty(\mathrm{Gr}(k,n))$ and some constant $c_{k,n} \neq 0$.

We now consider the Radon transform $R_{k,l}\colon C^\infty(k,n) \to C^\infty(l,n)$, where $k < l \leqslant n - l - 1$. $R_{k,l}$ and its dual $R^t_{k,l}$ are defined by integration as before. Until now we have been considering the case $l = k+1$. In fact, all these transforms are related:

$$R_{k,l} = c_{k,l,n} R_{l-1,l} R_{l-2,l-1} \cdots R_{k,k+1}.$$

This fact essentially asserts the uniqueness of invariant measures, since the right-hand side of the above equation is an integration operation with invariant measure. A corresponding fact holds true for R^t.

Theorem 3.4. *Let $R_{k,l}$ denote the Radon transform from functions on $\mathrm{Gr}(k,n)$ to functions on $\mathrm{Gr}(l,n)$. Let R^t denote its dual. Assume $l < k \leqslant n-k-1$. Then there are Casimir operators $\Delta_{k+1}, \Delta_{k+2}, \cdots, \Delta_l$ whose eigenvalues are given in Theorem 3.3 above such that*

$$\Delta_{k+1} \Delta_{k+2} \cdots \Delta_l R^t R = I.$$

The order of Δ_j is $2j$.

4. An inversion formula for the Penrose correspondence

Consider the double fibration

$$\begin{tikzcd} & F(0,1) \arrow[dl, "\pi"'] \arrow[dr, "\rho"] & \\ \mathbf{C}P^3 & & \mathrm{Gr}(1,3) \end{tikzcd}$$

Here $F(0,1)$ is the flag manifold of projective points in projective lines in $\mathbf{C}P^3$, while π and ρ are the standard fibrations. This correspondence comes up in twistor theory and is known as the Penrose correspondence (see Wells [13]). The Radon transform $R_{0,1}$ is a composition of the pull-back π^* with the push forward (integration over the fiber) ρ_*. Our calculations in §2 show that

$R_{0,1}^t R_{0,1}$ is inverted by the multiplier $(m_0 + 2)(m_0 + 1)$ (up to renormalization). On the other hand, the renormalized Laplace-Beltrami operator Δ on $\mathbf{C}P^3$ acts as the multiplier $m_0(m_0 + 3)$ (see Grinberg [4]). Thus if R is the Radon transform attached to the Penrose correspondence then we have the inversion formula

$$[\Delta + 2]R^t R = I.$$

One can consider other Radon-like transforms attached to the Penrose correspondence. For example, one can replace functions by sections of line bundles or vector bundles (see Gelfand [1] for some results in this direction). Indeed, it may be possible to extend the harmonic analysis arguments of the present paper to such situations.

5. Real Grassmannians

We now consider the analogous Radon transform for Grassmannians over the real numbers. Although much of the analysis is similar, there is a parity consideration (related to Huygens' principle) which makes the results look somewhat different.

Our group of isometries here is $SO(n + 1)$ and its Lie algebra $so(n+1)$ is the algebra of skew-symmetric $(n+1) \times (n+1)$ matrices. Let $\nu = [(n + 1)/2] - 1$. A Cartan subalgebra for $so(n)$ is spanned by $\{C_i\}_{0=1}^{\nu}$, where

$$C_i = e_{i,i+\nu} - e_{i+\nu,i} \qquad (i = 0,\cdots,\nu).$$

The highest weights occurring in $C^\infty(\mathrm{Gr}(k, n))$ for $k < \nu$ are those of the form $(2m_0,\cdots,2m_k)$ with $m_0 \geqslant m_2 \geqslant \cdots \geqslant m_k \geqslant 0$ and all m_j's are integers. Consider now a real Stiefel matrix

$$X = \begin{Vmatrix} X_0^0 & \cdots & X_k^0 \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ X_0^n & \cdots & X_k^n \end{Vmatrix},$$

whose columns represent an orthonormal basis for a projective k-plane H. Let X^j denote the jth row of X $(j = 0,\cdots,n)$. Finally, let $A(1)$ be the $(l + 1) \times (k + 1)$ matrix whose jth row is $X^{2j} + \sqrt{-1}\,X^{2j+1}$ $(j = 0,\cdots,1)$. Then a highest weight vector in $C^\infty(\mathrm{Gr}(k, n))$ with weight $(2m_0,\cdots,2m_k)$ is given by

$$f_{m_0,\cdots,m_k} = \det\left[A(0) \cdot A(0)^t\right]^{m_0 - m_1} \cdot \det\left[A(1) \cdot A(1)^t\right]^{m_1 - m_2} \cdots$$
$$\times \det\left[A(k) \cdot A(k)^t\right]^{m_k}.$$

This form of the highest weight is easily deduced from the form given in Strichartz [9].

The evaluation procedure for the real Radon transform $R_{k,k+1}f$ is quite similar to that of its complex counterpart. If L_0 is the $k+1$ plane consisting of all points $[p_0,\cdots,p_n]$ in $\mathbf{R}P^n$ with $p_1 = p_3 = \cdots = p_{2k+1} = 0$ and $p_{2k+2} = p_{2k+3} = \cdots = p_{2\nu} = p_n = 0$, then

$$Rf_{m_0,\cdots,m_k(L_0)} = \frac{c_k}{2^m} \cdot \frac{\Gamma(m_0+(k+1)/2)\Gamma(m_1+k/2)\cdots\Gamma(m_k+1/2)}{\Gamma(m_0+(k+2)/2)\Gamma(m_1+(k+1)/2)\cdots\Gamma(m_k+1)}.$$

This multiplier does not lead to an inversion formula involving a Casimir operator. However, if we consider the transform $R_{k,k+2} = R_{k+1,k+2} \circ R_{k,k+1}$ we find

$$R_{k,k+2}f_{m_0,\cdots,m_k}(L_0) = c_k 2^{-m} \frac{1}{(m_0+(k+1)/2)(m_1+k/2)\cdots(m_k+1/2)}.$$

A similar calculation can be done for $R^t_{k,k+2}$ and it leads to the following diagonalization:

$$R^t_{k,k+2}R_{k,k+2}f_{m_0,\cdots,m_k} = \left\{\prod_{j=0}^{k}(m_j+(n-j-2)/2)(m_j+(k+1-j)/2)\right\}^{-1} f_{m_0,\cdots,m_k}.$$

To obtain a useful left inverse to R^tR we need the following fact.

Lemma 5.1. *Let $P(t_0,\cdots,t_\nu)$ be a symmetric polynomial. For each vector of the form $m = (2m_0,\cdots,2m_\nu)$ let $t_j(m)$ denote the expression $(m_j + (n-1-j)/2)(m_j - j/2)$. Then there exists a Casimir operator Δ_p of the group $SO(n+1)$ which acts as the scalar $P(t_0(m),\cdots,t_\nu(m))$ on the irreducible representation with highest weight $(2m_0,\cdots,2m_\nu)$, where $\nu = [(n+1)/2]-1$.*

Again, a proof may be found in Zelobenko [14]. Now let $Q_k(m_0,\cdots,m_k)$ be the polynomial given by the expression

$$\begin{aligned}&\prod_{j=0}^{k}(m_j+(n-k-2-j)/2)(m_j+(k+1-j)/2)\\ &\quad= \prod_{j=0}^{k}\{(m_j+(n-1-j)/2)(m_j-j/2)+c_{n,k}\}\\ &\quad= \prod_{j=0}^{k}\{t_j(m)+c_{n,k}\},\end{aligned}$$

where $c_{n,k} = (n-k-2)(k+1)/4$; in particular, c_k does not depend on j (or m_j). As in the complex case, it is easy to find a symmetric polynomial $P_k(t_0,\cdots,t_\nu)$, so that

$$P_k(t_0(m),\cdots,t_\nu(m)) = Q_k(m_0,\cdots,m_k)$$

whenever $m_{k+1} = \cdots = m_\nu = 0$. Let Δ_k be the Casimir operator corresponding to P_k as in the above lemma.

Theorem 5.2. *Let $R_{k,l}$ denote the Radon transform on real Grassmannians (here $0 \leqslant k < l \leqslant (n+1)/2$). Let $R^t_{k,l}$ be the dual Radon transform and let $\Delta_k, \cdots, \Delta_l$ be Casimir operators as above. If $l-k$ is even then we have the inversion formula*

$$\Delta_{k+2}\Delta_{k+4} \cdots \Delta_l R^t_{k,l} R_{k,l} = I,$$

valid on, say, $C^\infty(\mathrm{Gr}(k,n))$. The degree of Δ_j is $2j$ $(j = 0, \cdots, \nu)$.

6. Quaternionic Grassmannians

We now consider the Radon transform on Grassmannians over the skew-field of quaternions. We will use the same notation as in the real and complex cases above for the various constructions involving the base field. Moreover, we will use the conventions for Cartan subalgebras and weights described in [5] (see also [10], [4]).

Theorem 6.1. *The highest weights occurring in $C^\infty(\mathrm{Gr}(k,n))$ for $k+1 \leqslant n-k$ are precisely those of the form $(m_0, m_0, \cdots, m_k m_k, 0, \cdots, 0)$. Each weight occurs with multiplicity one.*

As before, we refer to Strichartz [10]. The highest weight vectors can be constructed explicitly as before and the Radon transform can be diagonalized by the same general procedure. Thus we obtain the following result.

Theorem 6.2. *On the irreducible subrepresentation of $C^\infty(\mathrm{Gr}(k,n))$ with highest weight $(m_0, m_0, \cdots, m_k, m_k, 0, \cdots, 0)$ the operator $R^t_{k,k+1} R_{k,k+1}$ acts by the scalar*

$$\prod_{j=0}^{k} (m_j + 2k - 2j)(m_j + 2n - 2k - 2j + 1)(m_j + 2k - 2j + 1) \cdot (m_j + 2n - 2k - 2j).$$

As usual, we have a recipe for Casimir operators.

Theorem 6.3. *Let $P(t_0, \cdots, t_n)$ be a symmetric polynomial. Then there exists a Casimir operator Δ_p of the group $\mathrm{Sp}(n+1)$ whose eigenvalue on the irreducible representation with highest weight $(m_0, m_0, \cdots, m_n, m_n)$ is*

$$P((m_0 + n + 1)(m_0 + n), (m_1 + n)(m_1 + n - 1), \cdots, (m_n + 1)(m_n)).$$

Using this criterion, it is easy to see that there are Casimir operators Δ_{k+1} and

$\square_{k+1}$ *whose eigenvalues on the irreducible with highest weight* $(m_0, m_0, \cdots, m_k, m_k, 0, \cdots, 0)$ *are respectively*

$$(*)\qquad \begin{aligned} &\prod_{j=0}^{k} (m_j + 2k - 2j)(m_j + 2n - 2k - 2j + 1), \\ &\prod_{j=0}^{k} (m_j + 2k - 2j + 1)(m_j + 2n - 2k - 2j). \end{aligned}$$

Theorem 6.4. *Let* $R_{k,l}$ *denote the Radon transform from functions on* $\mathrm{Gr}(k, n)$ *to functions on* $\mathrm{Gr}(l, n)$. *Let* R^t *denote its dual. Assume* $k < l \leqslant n - l - 1$. *Then there are Casimir operators* $\Delta_{k+1}\square_{k+1}$, Δ_{k+2}, $\square_{l+2}, \cdots, \Delta_l, \square_l$ *whose eigenvalues are given in* $(*)$ *above such that the inversion formula*

$$\Delta_{k+1}\square_{k+1}\Delta_{k+2}\square_{k+2} \cdots \Delta_l\square_l R^t R = I$$

is valid on, say, $C^\infty(\mathrm{Gr}(k, n))$. *The order of both* Δ_j *and* $\square_j$ *is* $2j$.

7. Concluding remarks

We have considered Radon transforms involving pairs of Grassmannians $\mathrm{Gr}(k, n)$ and $\mathrm{Gr}(k, l)$ with the restriction $k < l \leqslant n - l - 1$. The general case can be easily reduced to this one using the duality $\mathrm{Gr}(k, n) \cong \mathrm{Gr}(n - k, n)$ and the harmonic analysis developed above.

Our discussion centered around the inversion problem for these Radon transforms. Another interesting question involves a range characterization: Which functions on $\mathrm{Gr}(l, n)$ are Radon transforms of functions on $\mathrm{Gr}(k, n)$? The answer can be given in terms of differential operators coming from the universal enveloping algebra of the group of isometries G of the corresponding projective space (see Grinberg [5]).

Finally, we observe that a further generalization of these Radon transforms is possible. If $\{n_j\}_{j=1}^{k}$ is a sequence of nonnegative integers, we have the flag manifold

$$F(n_1, \cdots, n_k) = \frac{U(n_1 + \cdots + n_k)}{U(n_1) \times \cdots \times U(n_k)}.$$

Geometrically, this is the space of all ascending chains of vector subspaces of $\mathbf{C}^n$, $V_1 \subset V_2 \subset \cdots \subset V_k$, where the $\mathbf{C}$-dimension of V_j is $n_1 + \cdots + n_j$ (or the corresponding projective analog). Given two such flag manifolds, say

$F(n_1,\cdots,n_k)$ and $F(m_1,\cdots,m_k)$ with $n_1+\cdots+n_k=m_1+\cdots+m_k$, one can define an incidence relation between them as follows: a flag $\{V_j\}_{j=1}^k\in F(n_1,\cdots,n_k)$ is said to be incident to a flag $\{W_j\}_{j=1}^k\in F(m_1,\cdots,m_k)$ if each pair (V_j,W_j) belongs to a preassigned orbit of the underlying general linear group. For example, if $n_j\leqslant m_j$ for $j<k$ one can require that V_j be a subspace of W_j for all j (this is trivial for $j=k$). For the case $k=2$, this gives the Grassmannians we have been studying. But further generalizations are possible. For instance, one can require that V_j intersect W_j in some subspace of fixed dimension d_j. For $d_j<\dim(V_j)$ the angle of intersection introduces some analytic problems (see Helgason [6]), so it seems desirable to require further that the vector spaces V_j and W_j intersect orthogonally (i.e. to require that the pair (V_j,W_j) lie in an orbit of the underlying orthogonal or unitary group). This approach (with $k=2$) is taken in Gonzalez [3] and Strichartz [12] (see also Petrov [8] and Gelfand et al. [1]). Another interesting variation is found in [2]. There the authors replace functions by sections of certain line bundles. The resulting Radon transform on real Grassmannians is intertwining for the larger group $SL_n(\mathbf{R})$. The authors give an inversion formula which is $SL_n(\mathbf{R})$ invariant.

Finally, one can even consider incidence relations between flag manifolds $F(m_1,\cdots,m_k)$ and $F(n_1,\cdots,n_l)$ with $k\neq l$, or indeed, between any two homogeneous spaces G/K and G/L over a Lie group G. It appears quite likely that the methods of this paper can be extended to the generalizations involving flag manifolds. Now flag manifolds which are not Grassmannians are not symmetric spaces, so their harmonic analysis is more complicated: for instance, we can no longer expect that irreducibles occur without multiplicity. Still, it seems plausible that highest weight vectors and their Radon transforms can be computed explicitly.

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