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INSTANTONS ON $\mathbf{CP}_2$

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0. Introduction

The purpose of this paper is to describe and classify the instantons on $\mathbf{CP}_2$ for the unitary and the classical compact simple Lie groups. The description closely resembles that given for the instantons on S^4 by Atiyah, Drinfeld, Hitchin and Manin [3], both being based on the bijective correspondence between instantons and holomorphic vector bundles on an associated complex manifold known as the Ward correspondence.

Although the problem of describing instantons can be converted into one in complex analysis and will be treated here strictly as such, its roots lie elsewhere and the background will now be briefly outlined.

Let X be an oriented 4-dimensional Riemannian manifold and G a compact Lie group. A *G-instanton* on X is a G-vector bundle F on X with G-connection ∇ such that the curvature F_∇ is *self-dual*: $*F_\nabla = F_\nabla$, where $*$ is the Hodge $*$-operator acting on 2-forms on X. In these circumstances, the *Yang-Mills equations* $\nabla * F_\nabla = 0$ are automatically satisfied by virtue of the Bianchi identity $\nabla F_\nabla = 0$, and solutions of the Yang-Mills equations are of considerable physical importance (see e.g. [1] or [17]).

The case $G = SU(2)$ is of particular interest from both a physical and a mathematical viewpoint. On the physical side, there is, for example, the well-known result that if X is spin, then the connection induced on the self-dual spin bundle by the Levi-Civita connection is self-dual iff X is Einstein. On the mathematical side, one has Donaldson's celebrated theorem [9] on the intersection forms of smooth compact 4-manifolds, the proof of which is based on topological properties of the space of $SU(2)$-instantons of second Chern class -1 on the 4-fold in question (endowed with a suitable metric). The existence of such instantons was proved by Taubes [19] amongst a number of other results; these will be returned to shortly.

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For certain manifolds X, the problem of describing its G-instantons can be converted into a problem in complex analysis, a process which is fully described in the paper of Atiyah, Hitchin and Singer [4]. These are the so-called *self-dual spaces*, namely those for which the anti-self-dual component of the Weyl curvature vanishes identically. For such spaces there is an associated complex 3-fold Z called the *twistor space*, which is fibered over X with fiber $\mathbf{CP}_1$. There is a 1-1 correspondence (called the *Ward correspondence*) between complex vector bundles on X with self-dual connection and holomorphic vector bundles satisfying certain conditions on Z; the imposition of a G-structure on the bundle and connection on X corresponds to the imposition of a certain holomorphic condition on the bundle on Z. There is an associated correspondence between solutions of certain differential equations coupled to an instanton and the analytic cohomology of the corresponding holomorphic bundle; this is usually called the *Penrose transform* [14].

An instanton is called *irreducible* if there are no subbundles preserved by the connection. In [4] it is shown that if X is compact and has positive scalar curvature and G is semisimple, then the space of irreducible G-instantons of fixed first Pontryagin class is either empty or a manifold of a specified dimension.

The standard examples of self-dual spaces are S^4 and $\mathbf{CP}_2$, each with its usual metric. Both are Einstein and have positive scalar curvature, the former being conformally flat. Hitchin [15] has in fact proved that these are the only self-dual Einstein manifolds with positive scalar curvature. The twistor space for S^4 is $\mathbf{CP}_3$, whilst that for $\mathbf{CP}_2$ is the flag manifold $\mathbf{F}_{1,2}$. More esoteric examples of self-dual spaces are provided by the $K3$ surfaces, each of which admits an (anti-) self-dual metric as a consequence of Yau's affirmative proof of the Calabi conjecture.

Returning to [4], the authors consider the particular case of G-instantons on $X = S^4$. Because of its elementary topology, the classification of G-instantons on S^4 for arbitrary compact G reduces to the case when G is connected, simply-connected and simple, and a specific condition is given under which, and only under which, a particular G-bundle admits an irreducible self-dual G-connection. These results form a part of Taubes' existence theorem [19] mentioned earlier, which can be stated as follows: If X is a compact, connected, oriented Riemannian 4-fold which has no nonzero anti-self-dual harmonic 2-forms and G is a compact, connected, simply-connected, simple Lie group, then a G-bundle F on X admits an irreducible self-dual G-connection if the G-bundle on S^4 with the same first Pontryagin class does (X is *not* required to be self-dual).

For the classical groups $G = SU(n)$, $\mathrm{Sp}(n)$, *and* $SO(n)$, the problem of describing the G-instantons on S^4 was solved by Atiyah, Drinfeld, Hitchin and Manin in [3]. Utilizing the Ward correspondence together with results and techniques from the classification theory of holomorphic vector bundles on complex projective spaces, they provide a description of instantons on S^4 in terms of *monads* on $\mathbf{CP}_3$, i.e. (essentially) in terms of linear algebra. These results are presented in detail in [2] and in [11], and parts of this paper are derived from the former. Indeed, a significant portion of this paper is aimed at replicating the ADHM construction for instantons on $\mathbf{CP}_2$ in such a way that the similarity between the two cases is clearly evident.

The plan of the remaining sections of this paper is as follows. In §1, relevant details of the construction of twistor spaces are reviewed, and a precise statement of the Ward correspondence is given. In §2, a variety of results and definitions are collected together in preparation for the description of instantons in the next section. In particular, the definition of monads and basic properties thereof are given in this section. In §3, the description of $U(n)$-instantons on $\mathbf{CP}_2$ in terms of *unitary monads* on the twistor space is presented; $\mathrm{Sp}(n)$- and $SO(n)$-instantons are described in terms of *self-dual monads*. The fourth section gives an outline of the Penrose transform in concrete terms, the purpose of which is to prove a technical result required for the monad descriptions. In §5, classifying spaces for the various G-instantons are constructed and precise topological conditions are given under which, and only under which, the subspaces corresponding to irreducible instantons are nonempty. The paper concludes with an example; namely, the construction of the moduli space of $SU(2)$-instantons on $\mathbf{CP}_2$ of second Chern class -1. (This space was not only predicted by Donaldson, but he also constructed it—unpublished but cited in [16].)

Throughout, a hermitian form ϕ on a complex vector space V is regarded as a *linear* map $\phi\colon V \to \overline{V}^*$ (satisfying $\bar{\phi}^* = \phi$), rather than an antilinear map $V \to V^*$. The associated inner product is $\langle u, v\rangle := \bar{u}^*\phi v$, with $\|v\|^2 := \langle v, v\rangle$ if $\langle\ ,\ \rangle$ is positive definite. Little or no distinction is made between a vector bundle and its locally free sheaf of sections.

Since completing the manuscript, I have learned that Donaldson has also given an almost identical description of the instantons on $\mathbf{CP}_2$, published in [10]. (His paper does not, however, include a proof that all instantons on $\mathbf{CP}_2$ are derived from the monad construction.)

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1. The Ward correspondence

This section gives an outline of the construction of twistor spaces for self-dual spaces and a precise statement of the correspondence between instantons and holomorphic vector bundles on the twistor space. For details the reader is referred to [4].

Let X be a compact, connected, oriented Riemannian 4-manifold. Although X may not possess a spin structure, the *projective* self-dual and anti-self-dual spin bundles $\mathbf{P}(V_+)$ and $\mathbf{P}(V_-)$ on X always exist. The conformal structure and orientation on X determine natural almost complex structures on each of these bundles, and the almost complex structure on $\mathbf{P}(V_\pm)$ is integrable iff $W_\pm \equiv 0$, where W_+ is the self-dual component of the Weyl curvature and W_- is the anti-self-dual component. If $W_- \equiv 0$, X is called a *self-dual space* and the complex manifold $Z := \mathbf{P}(V_-)$ is called the *twistor space* for X.

Each fiber of the projection $p\colon Z \to X$ is a complex projective line, called a *real line* in Z, and it has normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. There is an anti-holomorphic involution $\sigma\colon Z \to Z$ with no fixed points, given by the antipodal map on each real line.

Conversely, if Z is a complex 3-manifold and $L \hookrightarrow Z$ is a copy of $\mathbf{CP}_1$ with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$, the space M of deformations of L in Z is a complex 4-manifold (near L) possessing a complex conformal structure determined by the condition that two points in M are null-separated iff the corresponding lines in Z intersect. An anti-holomorphic involution $\sigma\colon Z \to Z$ with no fixed points but leaving L fixed induces an anti-holomorphic involution $\tilde{\sigma}\colon M \to M$, and the 4-real dimensional submanifold X of fixed points of $\tilde{\sigma}$ has a real conformal structure of definite signature. Z is fibered over X (near L) with fiber $\mathbf{CP}_1$, and an orientation for X is determined by the condition that the orientation class of the fibration $Z \to X$ restricted to L is the anti-holomorphic orientation.

Suppose now that X is a self-dual space with twistor space Z and F is a complex vector bundle on X with connection ∇. Then $E := p^*F$ is a complex vector bundle with connection on Z, and the condition that this connection induces a holomorphic structure on E is precisely that the anti-self-dual component of F_∇ be zero; i.e. that ∇ be a *self-dual* connection on F. The bundle E is then *holomorphically* trivial on every real line.

Conversely, if E is a holomorphic bundle on Z which is trivial on all real lines, the bundle F on X defined by $F_x := \Gamma(p^{-1}(x), \mathcal{O}(E))$ has an induced self-dual connection. This gives

Theorem 1 (*Ward correspondence*). *There is a bijective correspondence between complex vector bundles F on X with self-dual connection and holomorphic vector bundles E on Z which are trivial on all real lines.*

The assumption that X is compact is not used in the proof of this theorem, and there is a slightly more general formulation of it in which X is replaced by certain "nice" open subsets U of the space M of lines in Z. Although Z is not fibered over M, there is an associated double fibration

$$\begin{tikzcd} & W \arrow[dl, "\nu"'] \arrow[dr, "\mu"] & \\ M & & Z \end{tikzcd}$$

where $W := \{(m, z) \in M \times Z\colon\ z \in m\}$. In this context, the way in which a derived bundle on U acquires a self-dual connection is easily seen ([7], [12]) and there is an associated *Penrose transform* which is a linear isomorphism from $H^p(\mu(\nu^{-1}(U)), \mathcal{O}(E))$ into the pth cohomology of the complex

$$(1.1)\qquad 0 \to H^0(U, \mathcal{O}(F)) \xrightarrow{\nabla} H^0(U, \Omega^1(F)) \xrightarrow{\nabla_-} H^0(U, \Omega^2_-(F)) \to 0.$$

Here Ω^1 and Ω^2_- are respectively holomorphic 1-forms and anti-self-dual 2-forms on M. This transform will be explicitly described in §5 in the case $X = \mathbf{CP}_2$.

If X is spin, the tautological line bundle of $Z = \mathbf{P}(V_-)$ is holomorphic and its fourth power is the canonical bundle of Z. Denoting this line bundle by $\mathcal{O}(-1)$ and by $E(m)$ the bundle $\mathcal{O}(E) \otimes \mathcal{O}(m)$, the Penrose transform is actually defined on $H^p(\mu(\nu^{-1}(U)), \mathcal{O}(E(m)))$, with the complex (1.1) replaced by a different one (involving tensor products of F with powers of the spin bundles and with the differentials induced by the connection on F coupled to the spin connections). If X is not spin, the bundle $\mathcal{O}(-1)$ does not exist globally on Z, but its even powers do always exist; indeed $\mathcal{O}(-2)$ can be defined as a square root of the canonical bundle on Z.

The complex (1.1) is elliptic when restricted to the real submanifold $X \cap U$, and by taking direct limits over "nice" Stein U containing X one obtains the same Penrose transform as described by Hitchin in [14], but which he obtains by direct methods from the fibration $Z \to X$. Either way, the important result is that the *combined Penrose-Ward construction defines an equivalence of the category of holomorphic bundles on Z which are trivial on real lines with the category of complex vector bundles on X with self-dual connection*, where morphisms in the latter category are by definition bundle morphisms commuting with the connections. The equivalence is compatible with the standard operations of homological algebra: sums, duals, quotients, tensor products, etc.

If E is a holomorphic bundle on Z, then so too is $\sigma^*\bar{E}$ and moreover, the latter is trivial on every real line on which E is trivial. Hence one obtains a functor $E \mapsto \sigma^*\bar{E}$ on the category of holomorphic bundles on Z trivial on all real lines, where if ϕ: $E_1 \to E_2$ is a morphism, $\phi^\sigma := \sigma^*\bar{\phi}$: $\sigma^*\bar{E}_1 \to \sigma^*\bar{E}_2$ is the associated morphism. The corresponding functor on bundles on X with self-dual connection is simply complex conjugation.

Let F be a hermitian vector bundle on X with hermitian connection ∇ which is self-dual, and let ϕ: $F \to \bar{F}^*$ be the hermitian form (always positive definite). To say that ∇ is hermitian is to say that ϕ commutes with the connections of F and $\bar{F}^*$, so if E is the corresponding bundle on Z, ϕ corresponds via the Penrose transform to a holomorphic map ϕ: $E \to \sigma^*\bar{E}^*$ satisfying $\phi^{\sigma *} = \phi$. Moreover, ϕ induces a positive definite hermitian form on sections of E over real lines. Conversely, if F has no hermitian structure a priori and E possesses a map ϕ with these properties, then there is an induced hermitian form on F which is compatible with the connection. This is the twistor description of $U(n)$-instantons.

Since every compact Lie group G has an embedding in $U(n)$ for some n, the problem of describing the G-instantons on X is converted into that of describing all holomorphic bundles on Z corresponding to $U(n)$-instantons, and specifying the conditions under which the structure group can be reduced from $U(n)$ to G in terms of holomorphic conditions on the bundles on Z. For example, if E corresponds to a $U(n)$-instanton F, then F is an $SU(n)$-instanton iff $\det E$ is trivial.

The groups in addition to $U(n)$ which are considered here are the classical simple groups $SU(n)$, $\mathrm{Sp}(n)$, and $SO(n)$. An $\mathrm{Sp}(n)$-instanton is a $U(2n)$-instanton F with a compatible symplectic structure; that is, a linear isomorphism α: $F \to F^*$ commuting with the connections and satisfying $\alpha^* = -\alpha$ and $\bar{\alpha}^*\bar{\phi}^{-1}\alpha = \phi$. Similarly, the complexification of an $SO(n)$-instanton is a $U(n)$-instanton F with a compatible orthogonal structure; i.e. a linear isomorphism α: $F \to F^*$ commuting with the connections and satisfying $\alpha^* = +\alpha$ and $\bar{\alpha}^*\bar{\phi}^{-1}\alpha = \phi$. When these conditions are included in the Ward correspondence, the following is obtained:

Theorem 2. *For $G = U(n)$, $SU(n)$, $\mathrm{Sp}(n)$, there is a bijective correspondence between G-instantons on X and holomorphic vector bundles E on Z which are trivial on real lines and for which*

(a) *for $G = U(n)$: E has rank n and there is an isomorphism ϕ: $E \to \sigma^*\bar{E}^*$*

with $\phi^{\sigma *} = \phi$ *which induces a positive hermitian form on sections of* E *over real lines*;

(b) *for* $G = SU(n)$: *the same as* (a) *with the additional constraint that* $\det E$ *is trivial*;

(c) *for* $G = \mathrm{Sp}(n)$: *the same as* (a) *except that* E *has rank* $2n$ *and there is an isomorphism* α: $E \to E^*$ *satisfying* $\alpha^* = -\alpha$ *and* $\alpha^{\sigma *}\phi^{\sigma^{-1}}\alpha = \phi$;

(d) *for* $G = SO(n)$: *the same as* (a) *and there exists an isomorphism* α: $E \to E^*$ *satisfying* $\alpha^* = +\alpha$ *and* $\alpha^{\sigma *}\phi^{\sigma^{-1}}\alpha = \phi$.

An isomorphism ϕ: $E \to \sigma^*\overline{E}^*$ satisfying $\phi^{\sigma *} = \phi$ will be called a *unitary structure* on E; it is certainly *not* a hermitian form, but there should be no confusion as all morphisms on Z are required to be holomorphic. The structure will be called *positive* if it induces a positive definite hermitian form on sections of E over all real lines. A map α: $E \to E^*$ satisfying $\alpha^* = -\alpha$ (resp. $+\alpha$) will be called a *symplectic structure* (resp. *orthogonal structure*) on E, and it is *compatible* with a unitary structure ϕ if it satisfies $\alpha^{\sigma *}\phi^{\sigma^{-1}}\alpha = \phi$.

As mentioned earlier, an *irreducible* instanton F is one which has no subbundles preserved by the connection. It follows easily using the Penrose transform that *F is irreducible iff the corresponding bundle E on Z is simple*, i.e. its only endomorphisms are scalar multiples of the identity. By decomposing a general instanton into a sum of irreducibles, the following is then a straightforward application of the Function Calculus:

Lemma 1. *The structures listed in Theorem* 2 *are unique up to bundle isomorphism.*

2. Preliminaries

This section commences with the basic definitions and properties of *monads*, the objects in terms of which instantons will subsequently be described; much of this material is taken directly from [18]. Following this is a discussion of the twistor space for $\mathbf{CP}_2$, together with a collection of basic results central to the description. The section concludes with a discussion of the topological classification of instantons.

Let Z be a compact complex manifold. A *monad* M on Z is a complex of (holomorphic) vector bundles on Z of the form

$$M: 0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$$

which is exact at A and C and such that the image of a is a subbundle of B. The bundle $E := \ker b/\operatorname{im} a$ is called the *cohomology* of M, denoted by $E(M)$.

A *morphism* m: $M \to M'$ of monads is a triple $m = (\mu, \nu, \rho)$ of bundle morphisms such that

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "a"] \arrow[d, "\mu"'] & B \arrow[r, "b"] \arrow[d, "\nu"'] & C \arrow[r] \arrow[d, "\rho"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r] & 0
\end{tikzcd}$$

commutes, and with composition defined in the obvious way, the set of monads on Z forms a category.

For each morphism m: $M \to M'$ there is an associated morphism $e(m)$: $E(M) \to E(M')$ induced by taking cohomology, and this gives a functor from the category of monads to the category of (holomorphic) vector bundles on Z. The following lemma and its accompanying corollary are taken directly from [18].

Lemma 2. *Let M, M' be monads on Z and $E := E(M)$, $E' := E(M')$. The map e: $\mathrm{Hom}(M, M') \to \mathrm{Hom}(E, E')$ is bijective if the following cohomology groups vanish*: $\mathrm{Hom}(B, A')$, $\mathrm{Hom}(C, B')$, $H^1(Z, C^* \otimes A')$, $H^1(Z, B^* \otimes A')$, $H^1(Z, C^* \otimes B')$, $H^2(Z, C^* \otimes A')$.

Corollary 2. *If the hypotheses of Lemma 2 are satisfied for the pairs (M, M'), (M, M), (M', M'), and (M', M), then the isomorphisms of the monads M, M' correspond bijectively under e to the isomorphisms of the associated bundles E, E'.*

The proof of Lemma 2 is straightforward albeit tedious, requiring the writing-out of the *displays* for the monads M, M'. The *display* associated to the monad M is the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & E \arrow[r] \arrow[u] & Q \arrow[r] \arrow[u] & C \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[u] & B \arrow[r, "b"] \arrow[u] & C \arrow[r] \arrow[u, equal] & 0 \\
 & A \arrow[u] \arrow[r, equal] & A \arrow[u, "a"'] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

where $Q :=$ coker a and $K :=$ ker b. This display determines the monad M and from it one can read off the rank and total Chern class of $E = E(M)$: $\mathrm{rk}\, E = \mathrm{rk}\, B - \mathrm{rk}\, A - \mathrm{rk}\, C$ and $c(E) = c(B)c(A)^{-1}c(C)^{-1}$.

The monads describing instantons will have certain additional structures: the *dual* M^* of a monad M is the monad

$$M^*: 0 \to C^* \xrightarrow{b^*} B^* \xrightarrow{a^*} A^* \to 0,$$

and M is said to possess a *self-dual structure* if there is an isomorphism α: $M \to M^*$ such that $\alpha^* = \pm\alpha$. When $\alpha^* = +\alpha$ the structure is called *orthogonal*, and when $\alpha^* = -\alpha$ it is called *symplectic*. M itself is called self-dual (orthogonal, symplectic) if $C = A^*$ and $b = a^*\alpha$ for some self-dual (orthogonal, symplectic) structure α: $B \to B^*$. Note that if $\alpha = (\mu, \alpha, \nu)$: $M \to M^*$ is a self-dual structure, the monad M': $0 \to A \xrightarrow{a} B \xrightarrow{\nu b} A^* \to 0$ has cohomology $E(M)$ and is self-dual since $\nu b = a^*\alpha$. The self-dual structure on M' is $(\pm 1, \alpha, 1)$ as $\alpha^* = \pm\alpha$.

A further type of structure which can be imposed on a monad in some circumstances is a *unitary structure*: if Z is the twistor space of a self-dual space X, a unitary structure on a monad M is an isomorphism ϕ: $M \to \sigma^*\overline{M}^*$ satisfying $\phi^{\sigma *} = \phi$. Here $\sigma^*\overline{M}^*$ is the monad

$$\sigma^*\overline{M}^*: 0 \to \sigma^*\overline{C}^* \xrightarrow{b^{\sigma *}} \sigma^*\overline{B}^* \xrightarrow{a^{\sigma *}} \sigma^*\overline{A}^* \to 0.$$

By definition, a unitary structure on M incorporates a unitary structure on B, and the former will be called *positive* if the latter is positive. A unitary structure ϕ and a self-dual structure α are *compatible* if $\alpha^{\sigma *}\phi^{\sigma -1}\alpha = \phi$. The monad M itself is called *unitary* if $c = \sigma^*\overline{A}^*$ and $b = a^{\sigma *}\phi$ for some unitary structure ϕ on B, and the unitary structure on M in this case is $(1, \phi, 1)$. As in the self-dual case, the cohomology of a monad with unitary structure is always the cohomology of a uniquely determined unitary monad.

Morphisms of unitary (resp. self-dual) monads are monad morphisms preserving unitary (resp. self-dual) structures, i.e. p: $M \to M'$ satisfies $p^{\sigma *}\phi' p = \phi$ (resp. $p^*\alpha' p = \alpha$).

This completes the introduction to monads, and the next task is to look at the space Z on which the monads of interest to this paper are defined.

Let V be a 3-dimensional complex vector space, which will remain fixed hereafter. Denote by $\mathbf{F}$ the set of pairs (L_1, L_2) such that L_i is an i-dimensional linear subspace of V with $L_1 \subset L_2$, given the structure of a complex manifold by the transitive action of $\mathrm{GL}(V)$: $\mathbf{F} \simeq \mathrm{GL}(V)/$isotropy group of a point. Similarly, let $\mathbf{F}_1$ and $\mathbf{F}_2$ respectively denote the spaces of 1- and 2-dimensional linear subspaces of V, so $\mathbf{F}$ is a hypersurface in $\mathbf{F}_1 \times \mathbf{F}_2$. If $\mathbf{F}_1$ is identified with $\mathbf{P}(V)$ and $\mathbf{F}_2$ with $\mathbf{P}(V^*)$, then $\mathbf{F} = \{(z, w) \in \mathbf{P}(V) \times \mathbf{P}(V^*): wz = 0\}$. Moreover, there are canonical projections p_i: $\mathbf{F} \to \mathbf{F}_i$ which exhibit $\mathbf{F}$ as a locally trivial holomorphic fibration over $\mathbf{F}_i$ with fiber $\mathbf{CP}_1$.

Denote by $\mathcal{O}(p,q)$ the sheaf $p_1^*\mathcal{O}_{\mathbf{F}_1}(p) \otimes p_2^*\mathcal{O}_{\mathbf{F}_2}(q)$, so the normal bundle of $\mathbf{F}$ in $\mathbf{F}_1 \times \mathbf{F}_2$ is $\mathcal{O}(1,1)$ and the canonical bundle is $\mathcal{O}(-2,-2)$. If $x := c_1(\mathcal{O}(1,0))$ and $y = c_1(\mathcal{O}(0,1))$ are the first Chern classes of these basic line bundles, the Leray-Hirsch theorem gives

$$H^*(\mathbf{F},\mathbf{Z}) = \mathbf{Z}[x,y]/x^3,\ y^3,\ x^2+y^2-xy$$

for the cohomology ring of $\mathbf{F}$. The fundamental class of $\mathbf{F}$ is $x^2y = xy^2 \in H^6(\mathbf{F},\mathbf{Z})$.

Let ϕ_0 be a *fixed* positive definite hermitian form on V, and define σ: $\mathbf{F} \to \mathbf{F}$ by $\sigma(L_1, L_2) := (L_2^\perp, L_1^\perp)$, where $\perp$ denotes orthogonal complement with respect to ϕ_0. σ is anti-holomorphic, has no fixed points and its fixed lines are precisely the fibers of the surjection p_0: $\mathbf{F} \to \mathbf{P}(V)$ defined by $p_0(L_1, L_2) := L_1^\perp \cap L_2$. These are the *real lines* in $\mathbf{F}$, and when restricted to each such line, σ is the antipodal map.

The space $\mathbf{M}$ of deformations of some real line $L \hookrightarrow \mathbf{F}$ can be identified with $\mathbf{F}_1 \times \mathbf{F}_2 \setminus \mathbf{F}$, where $(L_1', L_2') \in \mathbf{M}$ corresponds to the line $\{(L_1, L_1 + L_1'):$ $L_1 \subset L_2'\}$ in $\mathbf{F}$. The involution $\tilde{\sigma}$ on $\mathbf{M}$ induced by σ is given by $\tilde{\sigma}(L_1', L_2') = (L_2'^{\perp}, L_1'^{\perp})$, and the subspace $\mathbf{P}(V)$ of real lines in $\mathbf{F}$ is embedded in $\mathbf{M}$ as the anti-holomorphic diagonal $\{(L_1', L_1'^{\perp})\}$. The lines in $\mathbf{F}$ corresponding to the points of $\mathbf{M}$ will be called *complex lines*.

As mentioned in the first section, $\mathbf{M}$ has a natural (holomorphic) conformal structure determined by the condition that two points lie on a common null geodesic iff the corresponding lines in $\mathbf{F}$ intersect. The restriction of this structure to the real submanifold $\mathbf{P}(V) \hookrightarrow \mathbf{M}$ gives a definite real conformal structure on $\mathbf{P}(V)$, this being precisely the conformal class of the Fubini-Study metric induced by ϕ_0.

If $z := c_1(\mathcal{O}(1)) \in H^2(\mathbf{P}(V),\mathbf{Z})$ denotes the canonical generator of $H^*(\mathbf{P}(V),\mathbf{Z})$, then the homomorphism $H^*(\mathbf{P}(V),\mathbf{Z}) \to H^*(\mathbf{F},\mathbf{Z})$ induced by p_0 is generated by $z \mapsto y - x$. If $L \hookrightarrow \mathbf{F}$ is a real (or indeed complex) line and $h \in H^2(L,\mathbf{Z})$ is its fundamental class, the map $H^*(\mathbf{F},\mathbf{Z}) \to H^*(L,\mathbf{Z})$ induced by inclusion is generated by $x \mapsto h$, $y \mapsto h$; that is, $\mathcal{O}(p,q)|_L = \mathcal{O}_L(p+q)$. Since $-x(y-x)^2 = x^2y$, the orientation acquired by $\mathbf{P}(V)$ as the space of real lines in $\mathbf{F}$ agrees with its standard orientation as a compact complex manifold. In this way, $\mathbf{F}$ is realized as the twistor space for $\mathbf{CP}_2 \simeq \mathbf{P}(V)$.

The fiber of $\mathcal{O}(-1,0)$ (resp. $\mathcal{O}(0,-1)$) at $(L_1, L_2) \in \mathbf{F}$ is the 1-dimensional vector space L_1 (resp. $(V/L_2)^*$). Hence the fiber of $\sigma^*\overline{\mathcal{O}(-1,0)}$ at (L_1, L_2) is $L_2^\perp \simeq (V/L_2)^*$; i.e., $\sigma^*\overline{\mathcal{O}(-1,0)} \simeq \mathcal{O}(0,-1)$. It follows that $\sigma^*\overline{\mathcal{O}(p,q)} \simeq \mathcal{O}(q,p)$ for any p, q; a particular choice of isomorphism will be made later.

The analytic cohomology of the bundles $\mathcal{O}(p,q)$ can be determined from the embedding $\mathbf{F} \hookrightarrow \mathbf{F}_1 \times \mathbf{F}_2$ and the Bott Rules for $\mathbf{CP}_n$ [18], or more directly from Bott's original paper [5]. The result is $H^r(\mathbf{F}, \mathcal{O}(p,q)) = 0$ unless r is the minimum number of transpositions needed to arrange the sequence $(0, p+1, p+q+2)$ in increasing order, and in this case

$$\dim H^r(\mathbf{F}, \mathcal{O}(p,q)) = (-1)^r(p+1)(q+1)(p+q+2).$$

Since $H^1(\mathbf{F}, \mathcal{O}) = 0 = H^2(\mathbf{F}, \mathcal{O})$, the bundles $\mathcal{O}(p,q)$ represent all holomorphic (and topological) line bundles on $\mathbf{F}$. Classifying all holomorphic bundles on $\mathbf{F}$ of rank greater than 1 is of course more involved, as it is in the case of $\mathbf{CP}_n$. In the latter case, an important stepping-stone in the classification process is a theorem of Beilinson, of which the following lemma is an analogue for the current situation. The proof is a modification of the proof of Beilinson's theorem in [18].

Lemma 3. *Let E be a holomorphic vector bundle on* $\mathbf{F}$. *Then there is a spectral sequence $E_1^{p,q}$ converging to*

$$E_\infty^r = \begin{cases} E & \text{if } r = 0, \\ 0 & \text{otherwise} \end{cases}$$

with

$$E_1^{p,q} = 0 \text{ if } p < -3 \text{ or } p > 0,$$
$$E_1^{0,q} = H^q(E) \otimes_{\mathbf{C}} \mathcal{O},$$
$$E_1^{-3,q} = H^q(E(-1,-1)) \otimes_{\mathbf{C}} \mathcal{O}(-1,-1),$$

and exact sequences

(2.1)

$$\cdots \to H^q(E(-1,0)) \otimes_{\mathbf{C}} \mathcal{O}(0,-1) \to E^{-1,q} \to \begin{matrix} H^q(E(-1,0)) \otimes_{\mathbf{C}} \mathcal{O}(-1,1) \\ \oplus \\ H^q(E(1,-1)) \otimes_{\mathbf{C}} \mathcal{O}(0,-1) \end{matrix} \to \cdots,$$

(2.2)

$$\cdots \to \begin{matrix} H^q(E(0,-1)) \otimes_{\mathbf{C}} \mathcal{O}(0,-2) \\ \oplus \\ H^q(E(-2,0)) \otimes_{\mathbf{C}} \mathcal{O}(-1,0) \end{matrix} \to E_1^{-2,q} \to H^q(E(0,-1)) \otimes_{\mathbf{C}} \mathcal{O}(-1,0) \to \cdots,$$

(*where* $H^q(E(a,b)) := H^q(\mathbf{F}, \mathcal{O}(E) \otimes \mathcal{O}(a,b))$).

Proof. Denote by π_1 and π_2 the projections $\mathbf{F} \times \mathbf{F} \to \mathbf{F}$ onto first and second factors respectively, and let $\mathcal{O}(p,q)(r,s)' := \pi_1^*\mathcal{O}(p,q) \otimes \pi_2^*\mathcal{O}(r,s)$. From the Bott Rules,

$$H^1\big(\mathbf{F} \times \mathbf{F},\ \mathcal{O}(0,0)(1,-2)'\big) = \mathbf{C} = H^1\big(\mathbf{F} \times \mathbf{F},\ \mathcal{O}(-2,1)(0,0)'\big),$$

so let R be the extension

$$0 \to \mathcal{O}(1,0)(1,-1)' \oplus \mathcal{O}(-1,1)(0,1)' \to R \to \mathcal{O}(1,0)(0,1)' \to 0$$

corresponding to $1 \oplus 1 \in H^1(\mathbf{F} \times \mathbf{F}, \mathcal{O}(0,0)(1,-2)' \oplus \mathcal{O}(-2,1)(0,0)')$. Then $H^0(\mathbf{F} \times \mathbf{F}, R) = V^* \otimes V$, so R has a canonical section s corresponding to $1 \in \operatorname{End} V = V^* \otimes V$. It is straightforward to check that the zero set of s is precisely the diagonal Δ in $\mathbf{F} \times \mathbf{F}$, giving the Koszul resolution of $\mathcal{O}_\Delta$

$$0 \to \Lambda^3 R^* \to \Lambda^2 R^* \to R^* \xrightarrow{s} \mathcal{O} \to \mathcal{O}_\Delta \to 0. \tag{2.3}$$

If E is a holomorphic vector bundle on $\mathbf{F}$, tensor through (2.3) by $\pi_1^* E$, delete the last term on the right and take direct images under π_2. This gives the spectral sequence $E_1^{p,q} = \pi_{2*}^q(\pi_1^* E \otimes \Lambda^{-p} R^*)$ converging to

$$E_\infty^r = \pi_{2*}^r\left(\pi_1^* E \otimes \mathcal{O}_\Delta\right) = \begin{cases} E & \text{if } r = 0, \\ 0 & \text{otherwise.} \end{cases}$$

One has

$$\begin{aligned} E^{0,q} &= \pi_{2*}^q\left(\pi_1^* E\right) = H^q(E) ß \mathcal{O}, \\ E_1^{-3,q} &= \pi_{2*}^q\left(\pi_1^* E \otimes \det R^*\right) = \pi_{2*}^q\left(\pi_1^* E(-1,-1)(-1,-1)'\right) \\ &= H^q(E(-1,-1)) \otimes \mathcal{O}(-1,-1), \end{aligned}$$

and the sequences (2.1), (2.2) follow similarly using the definition of R and the identification $\Lambda^2 R^* = R \otimes \det R^*$. q.e.d.

As in [18], there are other versions of this result (e.g. replace $\mathcal{O}(p,q)$ by $\mathcal{O}(q,p)$ throughout), but the above suffices for current purposes.

Applications of Lemma 3 are simplified whenever certain cohomology groups are known to vanish, and to this end, several tools are available.

First is a pair of exact sequences on $\mathbf{F}$:

$$\begin{aligned} &\text{(a)} \quad 0 \to \mathcal{O}(-3,0) \to V(-2,0) \to \Omega_1^1 \to 0, \\ &\text{(b)} \quad 0 \to \mathcal{O}(-1,-1) \to \Omega_1^1 \to \mathcal{O}(-2,1) \to 0, \end{aligned} \tag{2.4}$$

where $\Omega_1^1 := p_1^* \Omega_{\mathbf{F}_1}^1$. (2.4)(a) is the pull-back of the Euler sequence on $\mathbf{F}_1$ twisted by $\mathcal{O}(-3,0)$ with the identification $(\Omega_1^1)^* = \Omega_1^1 \otimes (\det \Omega_1^1)^* = \Omega_1^1(3,0)$; (2.4)(b) arises from the fact that $\mathbf{F}$ is essentially the projectivized holomorphic tangent bundle on $\mathbf{F}_1$.

The second tool is Serre Duality, which in this context states that $H^p(E) \simeq H^{3-p}(E^*(-2,-2))^*$ for a holomorphic bundle E on $\mathbf{F}$.

Third is the Riemann-Roch formula, and for current purposes the following less general expression suffices:

$$\begin{aligned} &\sum(-1)^i h^i(E(p,q)) \\ &\qquad = \tfrac{1}{2}(p+q+2)\left[n(p+1)(q+1) - l(p-q) - l^2 - 2k\right] \end{aligned}$$

if $c(E) = 1 + l(x - y) + kxy$, $\mathrm{rk}(E) = n$, where

$$h^i(E(p,q)) := \dim H^i(E(p,q)).$$

The remaining pieces of information are cohomology vanishing statements, specific to instanton bundles. If E is trivial on a real line L, then $H^0(L, E(p,q)|_L) = 0$ if $p + q < 0$ since $\mathcal{O}(p,q)|_L = \mathcal{O}_L(p+q)$. Thus if E is trivial on *every* real line in $\mathbf{F}$, $H^0(\mathbf{F}, E(p,q)) = 0$ for $p + q < 0$, since any section of $E(p,q)$ must vanish at every point. A somewhat deeper result is the Atiyah-Drinfeld-Hitchin-Manin vanishing theorem for instanton bundles, which plays a crucial role in the classification of instantons on S^4. If E corresponds to a $U(n)$-instanton F on $\mathbf{P}(V)$, the Penrose transform identifies $H^1(\mathbf{F}, E(-1,-1))$ with solutions of the conformally invariant Laplace equation $(\nabla^*\nabla + \frac{1}{6}R)s = 0$ on $\mathbf{P}(V)$, where R is the scalar curvature of the metric and $*$ denotes formal adjoint. Since $R > 0$ the only global solution of this equation is $s \equiv 0$ (for details, see [14]). To summarize these results for future reference:

Lemma 4. *If E corresponds to a $U(n)$-instanton on $\mathbf{P}(V)$, $H^0(\mathbf{F}, E(p,q)) = 0$ if $p + q < 0$ and $H^1(\mathbf{F}, E(-1,-1)) = 0$.*

To complete this section, a brief discussion of the topological classification of instantons will now be given.

Let X be a compact, connected orientable 4-manifold. The complex vector bundles F on X are classified up to topological isomorphism by their rank and their first and second Chern classes $c_i(F) \in H^{2i}(X, \mathbf{Z})$, $i = 1,2$. Thus the $U(n)$-bundles F on X are classified by $c_1(F)$ and $c_2(F)$, and for the simply-connected groups $G = SU(n)$, $\mathrm{Sp}(n)$, the second Chern class along classifies the G-bundles on X (the standard representation of G being assumed in each case).

In the case of $SO(n)$, its double-covering group $\mathrm{Spin}(n)$ is simply-connected for $n > 2$ and the structure group of an $SO(n)$-bundle F on X can be lifted to $\mathrm{Spin}(n)$ iff its second Stiefel-Whitney class $w_2(F) \in H^2(X, \mathbf{Z}_2)$ vanishes. If $n > 2$ and $n \neq 4$, $SO(n)$ is simple and the $SO(n)$-bundles F on X are classified in these cases by $w_2(F)$ and the first Pontryagin class $p_1(F) := -c_2(F \otimes_{\mathbf{R}} \mathbf{C})$ [8]. The group $SO(4)$ is not simple and the $SO(4)$-bundles F on X are classified by $p_1(F)$, $w_2(F)$ and the 4th Stiefel-Whitney class $w_4(F) \in H^4(X, \mathbf{Z})$.

A choice of orientation for X determines an isomorphism $H^4(X, \mathbf{Z}) = \mathbf{Z}$ via evaluation on the orientation class, and by means of this the characteristic classes $c_2(F)$, $p_1(F)$, and $w_4(F)$ are identified with integers (or integers mod 2 in the last case). Not all integers are necessarily realized in this way however; there is an identity $w_2^2 \equiv p_1 \pmod 2$, which implies for example, that every

$SO(n)$-bundle on S^4 has even first Pontryagin class. In fact, p_1 is actually determined mod 4 by w_2, w_4 using cohomology operations [8] which implies for example that for an $SO(3)$-bundle F on S^4 or $\mathbf{CP}_2$, $p_1(F) \equiv 0 \pmod 4$ in the former case and $p_1(F) \equiv 0$ or $1 \pmod 4$ in the latter. Indeed, on both these spaces the relationship between p_1, w_2, and w_4 means that $SO(n)$-bundles are classified topologically by p_1 alone for any $n > 2$. Every integer of the form $4m$ or $4m + 1$ (resp. $4m$) occurs as the first Pontryagin class of an $SO(3)$-bundle on $\mathbf{CP}_2$ (resp. S^4), and every integer (resp. even integer) occurs as the first Pontryagin class of an $SO(4)$-bundle on $\mathbf{CP}_2$ (resp. S^4). Every pair of integers (r, s) (resp. $(0, s)$) occurs as the first and second Chern classes of a $U(2)$-bundle on $\mathbf{CP}_2$ (resp. S^4) [18].

The following terminology for describing the topological type of a G-instanton (F, ∇) on $X = S^4$ or $\mathbf{CP}_2$ will be adopted here: for $G = SU(n)$ or $\mathrm{Sp}(n)$, the *index* of the instanton will be the integer (corresponding to) $-c_2(F)$; for $G = SO(n)$, the index will be the integer $p_1(F)$. A $\mathrm{Spin}(n)$-instanton will be defined to be an $SO(n)$-instanton of even index and the index of a $\mathrm{Spin}(n)$-instanton (F, ∇) will be $\frac{1}{2}p_1(F)$. For $X = \mathbf{CP}_2$, the index of a $U(n)$-instanton (F, ∇) will be the pair of integers (k, l) such that $c(F) = 1 - lz - kz^2$; i.e. $c_1(F) = -lz$ and $c_2(F) = -kz^2$. These definitions are consistent with those in [4] except in the case of $\mathrm{Spin}(n)$ for $n < 7$, where the authors of [4] use the isomorphism $\mathrm{Spin}(3) \simeq SU(2)$, $\mathrm{Spin}(4) \simeq SU(2) \times SU(2)$, $\mathrm{Spin}(5) \simeq \mathrm{Sp}(2)$, and $\mathrm{Spin}(6) \simeq SU(4)$.

3. Description of instantons

In this section, three different unitary monads are canonically constructed from a bundle E corresponding to a $U(n)$-instanton, each having cohomology E and being of a particularly simple form. It is shown that the isomorphism classes of such bundles correspond bijectively to the isomorphism classes of such monads. The description of $\mathrm{Sp}(n)$- and $SO(n)$-instantons is obtained by imposing self-dual structures on the monads.

The development is along the lines of Atiyah's presentations in [2].

The simplest case, that of $U(1)$-instantons, will first be dealt with. If $\mathcal{O}(p, q)$ corresponds to a $U(1)$-instanton, then $q = -p$ on purely topological grounds. The bundle $\mathcal{O}(p, -p)$ is trivial on every real (indeed, complex) line in $\mathbf{F}$. An isomorphism h: $\sigma^*\overline{\mathcal{O}(-1,0)} \to \mathcal{O}(0, -1)$ gives an isomorphism

$$V = H^0(\mathbf{F}, \mathcal{O}(0,1)) \to H^0\big(\mathbf{F}, \sigma^* \overline{\mathcal{O}(1,0)}\big) = \overline{V}^*$$

which is a multiple of ϕ_0. Hence there is a canonical choice of isomorphism h inducing precisely ϕ_0, and with it, a canonical isomorphism h^σ: $\mathcal{O}(-1,0) \to \sigma^* \overline{\mathcal{O}(0,-1)}$; $h^\sigma \otimes h^*$: $\mathcal{O}(-1,1) \to \sigma^*\overline{\mathcal{O}(-1,1)}^*$ then induces a *negative* definite form on sections of $\mathcal{O}(-1,1)$ over real lines, as is easily checked. Thus the line bundles on $\mathbf{F}$ corresponding to $U(1)$-instantons on $\mathbf{P}(V)$ are precisely those of the form $\mathcal{O}(p,-p)$ for $p \in \mathbf{Z}$; i.e. the pth powers of the pull-back of the tautological bundle on $\mathbf{P}(V)$.

Henceforth the bundles $\mathcal{O}(q,p)$ and $\sigma^*\overline{\mathcal{O}(p,q)}$ will be identified by means of h, h^σ above.

Suppose now that (E,ϕ) corresponds to a $U(n)$-instanton of index (k,l) on $\mathbf{P}(V)$; (E,ϕ) will remain fixed throughout this section. The cohomology vanishing statements of Lemma 4 apply not only to E but also to E^* $(\simeq \sigma^*\overline{E})$ or either of these bundles twisted by a power of $\mathcal{O}(1,-1)$ since all of them correspond to $U(n)$-instantons; this fact will be exploited to great advantage subsequently.

By Lemma 4, $H^p(E(-1,-1))$ vanishes for $p = 0,1$ and by Serre duality it also vanishes for $p = 3,2$. Thus

$$H^*(E(p,q)) = 0 \quad \text{if } p+q+2 = 0. \tag{3.1}$$

From (2.4) $\otimes$ $E(1,-1)$, it now follows that

$$H^p(E(-1,0)) = H^p\big(\Omega_1^1 \otimes E(1,-1)\big) = H^{p+1}(E(-2,-1)),$$

and by Serre duality together with the isomorphism $E \simeq \sigma^*\overline{E}^*$ one has

$$\begin{aligned} H^{p+1}(E(-2,-1)) = H^{2-p}(E^*(0,-1))^* &= H^{2-p}\big(\sigma^*(\overline{E(-1,0)})\big)^* \\ &= \overline{H^{2-p}(E(-1,0))^*}. \end{aligned}$$

By Lemma 4, it follows that $H^p(E(-1,0))$ vanishes for $p \neq 1$, so the same is true for $H^p(E(0,-1)) = H^p(E(1,-1)(-1,0))$. From the Riemann-Roch formula it follows

$$\begin{aligned} &H^p(E(-1,0)) = 0 \quad \text{for } p \neq 1;\ h^1(E(-1,0)) = k + \tfrac{1}{2}l(l-1), \\ &H^p(E(0,-1)) = 0 \quad \text{for } p \neq 1;\ h^1(E(0,-1)) = k + \tfrac{1}{2}l(l+1). \end{aligned} \tag{3.2}$$

Let K_1, K_2 be the complex vector spaces defined by $\overline{K}_1^* := H^1(E(0,-1))$ and $\overline{K}_2^* := H^1(E(-1,0))$; although the isomorphism $E \simeq \sigma^*\overline{E}^*$ gives $K_i \simeq \overline{K}_i^*$ as above, it is useful to retain the distinction.

Let Q_1 be the extension of $\overline{K}_1^*(0,1)$ by E corresponding to $1 \in \mathrm{Hom}(\overline{K}_1^*, \overline{K}_1^*) = H^1(\mathrm{Hom}(\overline{K}_1^*(0,1), E))$, described by the exact sequence

$$0 \to E \to Q_1 \to \overline{K}_1^*(0,1) \to 0. \tag{3.3}$$

Using the isomorphism $\sigma^*\bar{E}^* \simeq E$, $\sigma^*\overline{(3.3)}^*$ gives a second exact sequence

$$0 \to K_1(-1,0) \to \tilde{Q}_1 \to E \to 0, \tag{3.4}$$

where $\tilde{Q}_1 := \sigma^*\bar{Q}_1^*$. Note that $\tilde{Q}_1$ is the extension of E by $K_1(-1,0)$ corresponding to the image of $1 \in \operatorname{End} K_1$ under the isomorphism

$$\begin{aligned}\operatorname{End} K_1 &= K_1^* \otimes K_1 = H^1\left(\sigma^*\left[\overline{E(0,-1)}\right]\right) \otimes K_1 \\ &\xrightarrow{\Phi^\sigma} H^1(E^*(-1,0)) \otimes K_1 = H^1(\operatorname{Hom}(E, K_1(-1,0))).\end{aligned}$$

Dualizing (3.3) and using the Bott Rules gives $H^p(Q_1^*(-1,0)) = H^p(E^*(-1,0))$ for all p. This implies that for each extension of E by $\mathcal{O}(-1,0)$ there is a unique and compatible extension of Q_1 by $\mathcal{O}(-1,0)$. If W_1 is the extension of Q_1 by $K_1(-1,0)$ corresponding to (3.4), the compatibility means that there is a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & K_1(-1,0) & \to & \tilde{Q}_1 & \to & E & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & K_1(-1,0) & \to & W_1 & \to & Q_1 & \to & 0 \end{array} \tag{3.5}$$

Combining (3.3), (3.5), and (3.5) gives the commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \uparrow & & \uparrow & & & & \\ 0 & \longrightarrow & E & \longrightarrow & Q_1 & \longrightarrow & \bar{K}_1^*(0,1) & \to & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & Q_1 & \longrightarrow & W_1 & \longrightarrow & \bar{K}_1^*(0,1) & \to & 0 \\ & & \uparrow & & \uparrow & & & & \\ & & K_1(-1,0) & = & K_1(-1,0) & & & & \\ & & \uparrow & & \uparrow & & & & \\ & & 0 & & 0 & & & & \end{array} \tag{3.6}$$

That is, the display for a monad M_1: $0 \to K_1(-1,0) \to W_1 \to \bar{K}_1^*(0,1) \to 0$ with cohomology E.

It remains now to identify the bundle W_1.

Applying Lemma 3 to $W_1(0,-1)$, one has the following: from the display $(3.6) \otimes \mathcal{O}(0,-1)$ and the Bott rules, $H^q(W_1(0,-1)) = H^q(Q_1(0,-1))$ for all q. By construction, $H^0(\bar{K}_1^*(0,0)) \to H^1(E(0,-1))$ is an isomorphism, so by (3.2) and the Bott Rules, $H^q(Q_1(0,-1)) = 0$ for all q. Hence $E_1^{0,q} = 0$ for all q.

Similarly, $H^q(W_1(-1,-2)) = H^q(\tilde{Q}_1(-1,-2)) = 0$ for all q, so $E_1^{-3,q}$ also vanishes for all q. Hence $E_2^{p,q} = E_\infty^{p,q}$.

For the $E_1^{-1,q}$ terms one needs to compute $H^q(W_1(-1,-1))$ and $H^q(W_1(1,-2))$, inserting these into the sequence (2.1). From (3.6), (3.1), and the Bott rules,

$$H^q(W_1(-1,-1)) = H^q(Q_1(-1,-1)) = 0,$$
$$H^q(W_1(1,-2)) = H^q(Q_1(1,-2)) = H^q(E(1,-2)),$$

for all q. Hence $E_1^{-1,q} = H^q(E(1,-2)) \otimes_{\mathbf{C}} \mathcal{O}(0,-1)$.

For the $E_1^{-2,q}$ terms, the relevant groups are $H^q(W_1(0,-2))$ and $H^q(W_1(-2,-1))$. As above, $H^q(W_1(0,-2)) = 0$ and $H^q(W_1(-2,-1)) = H^q(E(-2,-1))$ for all q, so $E_1^{-2,q} = H^q(E(-2,-1)) \otimes_{\mathbf{C}} \mathcal{O}(-1,0)$.

Since $H^0(\mathbf{F}, \mathcal{O}(1,-1)) = 0$, the differentials $E_1^{-2,q} \to E_1^{-1,q}$ are all zero, so $E_1^{p,q} = E_\infty^{p,q}$. Applying now the conclusion of Lemma 3, it follows that $H^q(E(1,-2))$ is nonzero only for $q = 1$, $H^q(E(-2,-1))$ is nonzero only for $q = 2$, and there is an exact sequence

$$0 \to H^1(E(1,-2)) \otimes \mathcal{O}(0,-1) \to W_1(0,-1)$$
$$\to H^2(E(-2,-1)) \otimes \mathcal{O}(-1,0) \to 0.$$

Since $H^*(\mathbf{F}, \mathcal{O}(1,-1)) = 0$, this sequence has a unique splitting, and with $N_1 := H^1(E(1,-2))$ and the identification $H^2(E(-2,-1)) = H^1(E(-1,0)) = \overline{K}_2^*$, the net result is the identification $W_1 = N_1 \oplus \overline{K}_2^*(-1,1)$. To summarize the results so far, E is the cohomology of a uniquely determined monad

$$(3.7) \qquad M_1 \colon 0 \to K_1(-1,0) \xrightarrow{a} N_1 \oplus \overline{K}_2^*(-1,1) \xrightarrow{b} K_1(0,1) \to 0,$$

where K_1, K_2, and N_1 are complex vector spaces of dimensions $k + \frac{1}{2}l(l+1)$, $k + \frac{1}{2}l(l-1)$, and $n + k + \frac{1}{2}l(l+3)$ respectively.

The pair $(M_1, \sigma^*\overline{M}_1^*)$ satisfies the hypotheses of Corollary 2, and it follows that there is a unique unitary structure ϕ_1: $M_1 \to \sigma^*M_1^*$ inducing ϕ on cohomology. If $\phi_1 = (\mu, \dot{\phi}_1, \nu)$, then $\bar{\nu}^* = \mu \in \operatorname{Aut} K_1$ and the unitary structure $\dot{\phi}_1$ on $N_1 \oplus \overline{K}_2^*(-1,1)$ is of the form $\dot{\phi}_1 = \phi_1 \oplus \chi_2 \otimes h^\sigma \otimes h^*$ for some hermitian forms ϕ_1, χ_2 on N_1, $\overline{K}_2^*$, respectively. Since $\nu b = a^{\sigma *}\phi_1$, b in (3.7) can be replaced by νb to give a uniquely determined unitary monad (M_1, ϕ_1) with $(E(M_1), e(\phi_1)) = (E, \phi)$.

In fact, it is not necessary to apply Corollary 2 at this stage, for if the construction is carefully traced through it is found that W_1 already possesses a canonical unitary structure ϕ_1 with $b = a^{\sigma *}\phi_1$. However, the corollary does imply the important result that (M_1, ϕ_1) is essentially unique: if (M_1', ϕ_1') is another unitary monad of the same form with $(E(M_1'), e(\phi_1')) \simeq (E, \phi)$, then the corollary implies $(M_1', \phi_1') \simeq (M_1, \phi_1)$.

The construction could equally well have commenced with $\bar{K}_2^*(1,0)$ rather than $\bar{K}_1^*(0,1)$, and the conclusion would then have been that E is the cohomology of a uniquely determined unitary monad

$$(3.8)\qquad M_2\colon 0 \to K_2(0,-1) \xrightarrow{a} N_2 \oplus \bar{K}_1^*(1,-1) \xrightarrow{a^{\sigma*}\phi_2} \bar{K}_2^*(1,0) \to 0,$$

where $N_2 := H^2(E(-1,0)) = H^1(E(-2,1))$ is a complex vector space of dimension $n + k + \frac{1}{2}l(l-3)$. (Lemma 3 is applied to $W_2(-1,0)$ for the fastest derivation.) As before, (M_2, ϕ_2) is unique up to isomorphism of unitary monads of this form. Note that this implies $M_2(E) \simeq M_1(E(-1,1)) \otimes \mathcal{O}(1,-1)$ and $M_2(\sigma^*\bar{E}) \simeq \sigma^*\overline{M_1(E)}$, where $M_i(E)$ denotes the unitary monads canonically constructed from E as above.

Neither of the pairs (M_1, M_1^*), (M_2, M_2^*) satisfies the hypotheses of Corollary 2 (except in the degenerate case $K_1 = 0$ or $K_2 = 0$) so that a symplectic or orthogonal structure on E is not induced by a corresponding structure on either of these monads (except when E is trivial). To bypass this difficulty, and also to dispense with the need to choose between M_1 or M_2 to describe E, both descriptions can be chosen simultaneously by commencing the construction with $\bar{K}_1^*(0,1) \oplus \bar{K}_2^*(1,0)$ and proceeding as before. The end result is that E is then described as the cohomology of a uniquely determined unitary monad

$$(3.9)\qquad \begin{gathered} M_3\colon 0 \to A \xrightarrow{a} W_3 \xrightarrow{a^{\sigma*}\phi_3} \sigma^*\bar{A}^* \to 0, \\ A := K_1(-1,0) \oplus K_2(0,-1), \end{gathered}$$

where W_3 is a vector space of dimension $n + 4k + 2l^2$ and ϕ_3 is a nondegenerate hermitian form. It then follows from Corollary 2 that a compatible self-dual structure on E is induced by a unique compatible self-dual structure on M_3, as desired.

The hypothesis that $\phi\colon E \to \sigma^*\bar{E}^*$ is a *positive* unitary structure on E has not been used ostensibly (although it is in fact used to prove $H^1(E(-1,-1)) = 0$). The hypothesis is manifested in the monad descriptions as the positivity of the induced unitary structures on M_1, M_2, M_3. This is a corollary of the following lemma, whose proof occupies the next section.

Lemma 5. *The hermitian form χ_2 on $\bar{K}_2^*$ in the monad M_1 is definite and of a sign independent of E.*

Corollary 5. *The unitary structures on M_1, M_2, M_3 are positive.*

Proof. It must be shown that the induced hermitian form on $\bar{K}_1^*$, $\bar{K}_2^*$, N_1, N_2, W_2 are all definite, being negative on $\bar{K}_1^*$, $\bar{K}_2^*$ and positive on N_1, N_2, W_3. (Recall that the unitary structure on $\bar{K}_2^*(-1,1)$ in M_1 is $\chi_2 \otimes h^\sigma \otimes h^*$ and $h^\sigma \otimes h^*\colon \mathcal{O}(-1,1) \to \sigma^*\overline{\mathcal{O}(-1,1)}^*$ induces a negative definite form on sections over real lines.)

Since $M_2(\sigma^*\bar{E}) \simeq \sigma^*\overline{M_1(E)}$, Lemma 5 implies that the form on $\bar{K}_1^*$ is definite and of the same sign as that on $\bar{K}_2^*$. Since $M_2(E) \simeq M_1(E(-1,1)) \otimes \mathcal{O}(1,-1)$ and the positive unitary structure on $E(-1,1)$ is $-\phi \otimes h^\sigma \otimes h^*$ the forms on N_1, N_2 are definite and of the opposite sign to that on $\bar{K}_2^*$. It follows that when either of the monads M_1, M_2 is restricted to a real line L and the middle term is trivialized, the result is a unitary monad of the form

$$0 \to K(-1) \xrightarrow{a} W \xrightarrow{a^{\sigma *}\phi} \bar{K}^*(1) \to 0, \tag{3.10}$$

where K, W are vector spaces and ϕ is a definite form. The cohomology of this monad is $E|_L$, and since the induced form on $\Gamma(L, E|_L)$ is positive, it follows that ϕ on W is positive. Hence the induce forms on $\bar{K}_1^*$, $\bar{K}_2^*$ are negative definite, and those on N_1, N_2 are positive definite, as claimed.

Now, if $K_2(0,-1)$ and $\bar{K}_2^*(1,0)$ are deleted from the monad M_3, a new unitary monad M_3': $0 \to K_1(-1,0) \to W_3 \to \bar{K}_1^*(0,1) \to 0$ is obtained. The cohomology $E(M_3')$ is itself the middle term in a unitary monad M_3'': $0 \to K_2(0,-1) \to E(M_3') \to \bar{K}_2^*(1,0) \to 0$, and if ϕ'' denotes the unitary structure on M_3'', then $(E(M_3''),\ e(\phi'')) = (E, \phi)$. Thus $(M_3'', \phi'') \simeq (M_2(E), \phi_2)$ and since $M_3' \simeq M_1(E(M_3'))$, it follows from the conclusion of the last paragraph that the unitary structure on W_3 is positive. q.e.d.

Granted Lemma 5, the first half of the description of $U(n)$-instantons is now complete: three unitary monads M_i, $i = 1,2,3$, of specific forms have been constructed from E, each possessing a positive unitary structure ϕ_i with $(E(M_i),\ e(\phi_i)) = (E, \phi)$, and each pair (M_i, ϕ_i) is unique up to isomorphism of unitary monads of this form. Moreover, it is clear from the canonical nature of the construction that each of the assignments $E \mapsto M_i(E)$ is functorial. The second half of the description is much easier: If M is a unitary monad of the form M_i with positive unitary structure ϕ, then $(E(M), e(\phi))$ corresponds to a $U(n)$-instanton of index (k,l) on $\mathbf{P}(V)$. All that needs to be shown is that $E(M)$ is trivial on all real lines and that $e(\phi)$ induces a positive definite hermitian form on sections of E over such lines.

Let $L \hookrightarrow \mathbf{F}$ be a real line, and consider $M|_L$. If M is of the form M_1 or M_2, the middle term is first trivialized over L and equipped with its induced form, as in the proof of Corollary 5. In all three cases therefore, $M|_L$ is of the form (3.10), with ϕ a positive definite form on W. (Recall that σ: $L \to L$ is the antipodal map.)

If $x \in L$, let $U_x := \operatorname{im} a(x)$. By definition of monads and their cohomology, U_x is a $(\dim K)$-dimensional subspace of W, $U_x \subset U_{\sigma x}^\perp$, and $E_x = U_{\sigma x}^\perp / U_x$. Here $E := E(M)|_L$ and $\perp$ denotes orthogonal complement with respect to ϕ.

A simple calculation shows that $U_x^\perp \cap U_{\sigma x}^\perp$ is independent of $x \in L$. Since ϕ is definite, $U_x \cap U_x^\perp = 0$, so

$$U_{\sigma x}^\perp = U_{\sigma x}^\perp \cap \left(U_x + U_x^\perp\right) = U_x + U_{\sigma x}^\perp \cap U_x^\perp = U_x + U_{\sigma y}^\perp \cap U_y^\perp$$

for some predetermined $y \in L$. Thus $E_x = U_{\sigma y}^\perp \cap U_y^\perp$ for every $x \in L$, implying that E is trivial, and moreover the induced form on $\Gamma(L, E) = U_{\sigma y}^\perp \cap U_y^\perp$ is positive definite since ϕ is. Thus $(E(M), e(\phi))$ corresponds to a $U(n)$-instanton.

If M is of the form M_i, then since M and $M_i(E(M))$ have the same cohomology and induce the same unitary structure on cohomology, it follows from Corollary 2 that there is an isomorphism $M \simeq M_i(E(M))$ preserving unitary structure. Indeed, if M' is an arbitrary monad with positive unitary structure of the form M_i such that $E(M') \simeq E(M)$, then by Lemma 1 there is an isomorphism $E(M') \simeq E(M)$ which preserves unitary structures, and this isomorphism lifts by Corollary 2 to an isomorphism $M' \simeq M$ preserving unitary structures.

To summarize,

Proposition 1. *Let $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ respectively denote the subcategory of monads on* $\mathbf{F}$ *of the form* (3.7), (3.8), (3.9) *and which possess a positive unitary structure, and let $\mathcal{E}$ denote the subcategory of holomorphic bundles of* $\mathbf{F}$ *corresponding to $U(n)$-instantons of index (k, l) on $\mathbf{P}(V)$. Then each of the functors $\mathcal{M}_i \ni M_i \mapsto E(M_i) \in \mathcal{E}$, $i = 1, 2, 3$, defines a bijective correspondence on isomorphism classes of objects, compatible with unitary structures.*

As an immediate corollary, the description of $SU(n)$-instantons of index k is the same as above with $l = 0$.

To deal with the symplectic and orthogonal cases, suppose now that $l = 0$ and E has a compatible self-dual structure α: $E \to E^*$ with $\alpha^* = \pm\alpha$. (If $\alpha^* = -\alpha$, $\operatorname{rk} E = 2n$ and $\dim W_3 = 2n + 4k$ instead of n and $n + 4k$ respectively.) By Corollary 2, α lifts to a unique and compatible self-dual structure on the monad $M_3(E)$. After unwinding the definitions, this self-dual structure gives an isomorphism $K_1 \simeq \overline{K}_2 =: K$ and $M_3(E)$ is canonically isomorphic to a monad

$$M\colon 0 \to \begin{matrix} K(-1,0) \\ \oplus \\ \overline{K}(0,-1) \end{matrix} \xrightarrow{a} W \xrightarrow{a^{\sigma *}\phi} \begin{matrix} \overline{K}^*(0,1) \\ \oplus \\ K^*(1,0) \end{matrix} \to 0. \tag{3.11}$$

The unitary structure on M is $(1, \phi, 1)$ and the self-dual structure is $(\mu, \alpha, \pm\mu^*)$, where $\mu = \left(\begin{smallmatrix} 0 & \pm 1 \\ 1 & 0 \end{smallmatrix}\right)$ and α is a self-dual structure on W compatible with ϕ. The map a in (3.11) is of the form $a = (a_1, \pm\alpha^{-1}\overline{\phi} a_1^\sigma)$, where a_1: $K(-1,0) \to W$.

Conversely, a monad M of the form (3.11) has cohomology $E(M)$ which satisfies the requirements of Theorem 2 for it to correspond to an $SO(n)$- or $\mathrm{Sp}(n)$-instanton of index k on $\mathbf{P}(V)$. If M' is another monad of the same form with $E(M') \simeq E(M)$, then by Lemma 1 there is an isomorphism p: $E(M) \to E(M')$ such that $p^{\sigma *}\phi' p = \phi$ and $p^*\alpha' p = \alpha$, and by Corollary 2 this lifts to an isomorphism p: $M \to M'$ with the same properties.

To summarize:

Proposition 2. *Let $\mathcal{M}$ be the subcategory of monads on F of the form* (3.11) *which possess compatible self-dual and unitary structures, and let $\mathcal{E}$ be the subcategory of bundles on* $\mathbf{F}$ *corresponding to $SO(n)$- or* $\mathrm{Sp}(n)$*-instantons of index k on $\mathbf{P}(V)$. Then the functor $\mathcal{M} \ni M \mapsto E(M) \in \mathcal{E}$ defines a bijection on equivalence classes of isomorphic objects, compatible with the self-dual and unitary structures.*

In the $\mathrm{Sp}(n)$ case, the monad (3.11) can be rewritten in a purely quaternionic way, as in the case of $\mathrm{Sp}(n)$-instantons on S^4. However, this reformulation does not appear to yield the same benefits such as ($\mathbf{CP}_2$ analogues of) the t'Hooft instantons.

4. The Penrose transform and proof of Lemma 5

The object of this section is to prove that the form χ_2 on $\overline{K}_2^*$ in the monad M_1 of §3 is definite and of a sign independent of E. The proof involves identifying χ_2 in terms of operations on the cohomology group $H^1(E(-1,0))$, reinterpreting these via the Penrose transform in terms of operations on the instanton bundle F corresponding to E, and showing that the latter gives a definite form. The Penrose transform presented here uses the method of double fibrations as in [7], [12], [13], in which further details can be found in the references (particularly [13]).

In the derivation of (3.2), the following isomorphisms were obtained:

$$\text{(a)}\quad \overline{K}_2^* = H^1(E(-1,0)) \xleftarrow{\sim} H^1\big(\Omega_1^1 \otimes E(1,-1)\big) \xrightarrow{\sim} H^2(E(-2,-1)),$$

(4.1)

$$\text{(b)}\quad H^2(E(-2,-1)) \simeq H^1(E^*(0,-1))^* \simeq H^1\big(\sigma^*\overline{E(-1,0)}\big)^* = K_2.$$

(4.1)(a) follows from (2.4) $\otimes\, E(1,-1)$ and the vanishing of $H^*(E(p,q))$ for $p+q+2=0$, and (4.1)(b) from Serre Duality and the unitary structure on E. If E is regarded as the cohomology of M_1: $0 \to K_1(-1,0) \to N_1 \oplus \overline{K}_2^*(-1,1) \to \overline{K}_1^*(0,1) \to 0$, then it is easy to check from the display that each operation in (4.1) "is" the same operation applied to $\overline{K}_2^*(-1,1)$ equipped with the unitary structure $\chi_2 \otimes h^\sigma \otimes h^*$. Thus χ_2 is identified with this sequence of operations on $H^1(E(-1,0))$, and these must be interpreted in terms of corresponding instanton (F, ∇); this requires considerable preparation.

Recall $\mathbf{M} = \mathbf{F}_1 \times \mathbf{F}_2 \setminus \mathbf{F}$ is the space of complex lines in $\mathbf{F}$, and there is the double fibration

$$(4.2)\qquad \begin{tikzcd} & \mathbf{G} \arrow[dl, "\nu"'] \arrow[dr, "\mu"] & \\ \mathbf{M} & & \mathbf{F} \end{tikzcd}$$

where $\mathbf{G} := \{(m, x) \in \mathbf{M} \times \mathbf{F}\colon\ x \in m\}$. In more concrete terms, choose a fixed orthonormal basis for V, and then with respect to this basis, homogeneous coordinates on $\mathbf{F}$ are (z^a, w_a), $a = 1, 2, 3$, with $z^a w_a = 0$ and homogeneous coordinates on $\mathbf{M}$ are (u^a, v_a) with $u^a v_a \neq 0$. (The summation convention is employed throughout; $z^a w_a$ and $u^a v_a$ will usually be denoted by $z \cdot w$ and $u \cdot v$, respectively.) The correspondence space $\mathbf{G}$ is then $\mathbf{G} = \{(u^a, v_a, z^a, w_a)\colon\ u \cdot v \neq 0,\ z \cdot w = u \cdot w = z \cdot v = 0\}$, and the maps μ, ν of (4.2) are the restrictions to $\mathbf{G}$ of the projections on $\mathbf{M} \times \mathbf{F}$.

The involution $\sigma\colon \mathbf{F} \to \mathbf{F}$ is $(z^a, w_a) \mapsto (\delta^{ab}\overline{w}_b,\ \delta_{ab}\overline{z^b})$, where δ_{ab}, δ^{ab} denotes the Kronecker delta, i.e. the form ϕ_0 in terms of the chosen basis. The corresponding involution σ on $\mathbf{M}$ is $(u^a, v_a) \mapsto (\delta^{ab}\overline{v}_b,\ \delta_{ab}\overline{u^b})$, and $\mathbf{P}(V) \hookrightarrow \mathbf{M}$ is $\{(u^a, v_a)\colon\ v_a = \delta_{ab}\overline{u^b}\}$; there is also an involution σ on $\mathbf{G}$, defined in the obvious way.

Denote by ε^{abc} a fixed volume form in $\Lambda^3 V$: it is totally skew-symmetric with $\varepsilon^{123} = 1$. Similarly $\varepsilon_{abc} \in \Lambda^3 V^*$ is totally skew with $\varepsilon_{123} = 1$.

Let $\mathcal{O}(a, b)(c, d)'$ denote the sheaf of germs of holomorphic functions on $\mathbf{G}$ homogeneous of degrees a, b, c, d in z, w, u, v, respectively, so $\mathcal{O}(a, b)(0, 0)' = \mu^*\mathcal{O}(a, b)$ and $u \cdot v$ is a nowhere zero section of $\mathcal{O}(0,0)(1,1)'$. By definition of $\mathbf{G}$, there are nowhere zero sections $\alpha \in \Gamma(\mathbf{G}, \mathcal{O}(1,-1)(0,-1)')$, $\beta \in \Gamma(\mathbf{G}, \mathcal{O}(-1,1)(-1,0)')$ such that

$$(4.3)\qquad z^a = \alpha\varepsilon^{abc} v_b w_c, \qquad w_a = \beta\varepsilon_{abc} u^b z^c,$$

and it follows that $\alpha\beta = -1/u \cdot v$. Note that $\beta = \alpha^\sigma$, and the fact that $\alpha\alpha^\sigma = -1/\|u\|^2$ over $\mathbf{P}(V)$ reflects the fact that $h^\sigma \otimes h^*\colon\ \mathcal{O}(-1,1) \to \sigma^*\overline{\mathcal{O}(-1,1)}^*$ induces a negative definite form on sections over real lines.

Let Ω^1_μ denote the sheaf of holomorphic relative 1-forms on $\mathbf{G}$: $\Omega^1_\mu = \operatorname{coker}(d\mu\colon\ \mu^*\Omega^1_{\mathbf{F}} \to \Omega^1_{\mathbf{G}}) = \mathcal{O}(1,0)(-1,0)' \oplus \mathcal{O}(0,1)(0,-1)'$, and let $d_\mu\colon \mathcal{O}_{\mathbf{G}} \to \Omega^1_\mu$ be the induced differential (differentiation along the fibers of μ). With $\nabla_a := \partial/\partial u^a$ and $\nabla^a := \partial/\partial v_a$, d_μ is expressed in homogeneous coordinates as

$$\mathcal{O}_{\mathbf{G}} \ni f \mapsto (z^a\nabla_a f,\ w_a\nabla^a f) \in \Omega^1_\mu.$$

Note that $d_\mu\alpha$, $d_\mu\beta$, $d_\mu u \cdot v$ are all zero, and $d_\mu u^a = (z^a, 0)$, $d_\mu v_a = (0, w_a)$.

The relative de Rham complex along the fibers of μ is

(4.4)

$$\begin{array}{ccccccc}
\Omega^0_\mu & \xrightarrow{d_\mu} & \Omega^1_\mu & \xrightarrow{d_\mu} & \Omega^2_\mu & \longrightarrow & 0 \\
\wr\wr & & \wr\wr & & \wr\wr & & \\
\mathcal{O}(0,0)(0,0)' & \longrightarrow & \mathcal{O}(1,0)(-1,0)' \oplus \mathcal{O}(0,1)(0,-1)' & \longrightarrow & \mathcal{O}(1,1)(-1,-1)' & & \\
\Psi & & \Psi & & \Psi & & \\
f & \longmapsto & (z^a\nabla_a f, w_a\nabla^a f) & & & & \\
 & & (g,h) & \longmapsto & z^a\nabla_a h - w_a\nabla^a g & &
\end{array}$$

and since μ is a surjective holomorphic mapping everywhere of maximal rank, this complex is a resolution of the topological inverse image of $\mathcal{O}_{\mathbf{F}}$, $\mu^{-1}\mathcal{O}_{\mathbf{F}}$ (i.e. holomorphic functions on $\mathbf{G}$ constant along the fibers of μ).

The direct image of (4.4) under ν is the complex

$$\begin{array}{ccccc}
\nu_*\Omega^0_\mu & \longrightarrow & \nu_*\Omega^1_\mu & \longrightarrow & \nu_*\Omega^2_\mu \\
\wr\wr & & \wr\wr & & \wr\wr \\
\mathcal{O}_{\mathbf{M}} & \xrightarrow{d} & \Omega^1_{\mathbf{M}} & \xrightarrow{d_-} & \Omega^2_-
\end{array} \tag{4.5}$$

where Ω^2_- is the sheaf of anti-self-dual holomorphic 2-forms on $\mathbf{M}$, and d_- is exterior differentiation followed by projection (cf. (1.1)).

All of the preceding is the exact replia of the $S^4/\mathbf{CP}_3$ case presented in detail in [13]. The Penrose transform itself must also be replicated, and this involves expressing the direct images $\nu_*{}^q\Omega^p_\mu(a,b)$ in terms of "known" bundles on $\mathbf{M}$ and identifying the induced differential operators $\nu_*{}^q d_\mu$: $\nu_*{}^q\Omega^p_\mu(a,b) \to \nu_*{}^q\Omega^{p+1}_\mu(a,b)$, as exemplified by (4.5). For current purposes, it is necessary to perform this procedure in only a few cases, namely $(a,b) = (-1,0), (-1,-1), (-2,-1)$ and $(-2,-2)$. Even in these cases, precise identifications will not be required.

Recall first that each fiber L of ν: $\mathbf{G} \to \mathbf{M}$ is a copy of $\mathbf{CP}_1$ (embedded by μ in $\mathbf{F}$ as the corresponding complex line), and that $\mathcal{O}(a,b)(c,d)'|_L \simeq \mathcal{O}_L(a+b)$. It follows that $\nu_*{}^q\Omega^p_\mu(a,b) = 0$ for all q if $a + b + p = -1$.

In the homogeneity $(-1,0)$ case, $\nu_*{}^q\Omega^p_\mu(-1,0)$ is nonzero only if $q = 0$ and $p = 1$ or 2, with $\nu_*\Omega^1_\mu(-1,0) = \mathcal{O}(-1,0)' \oplus \mathcal{O}(0,-2)' =: S'$ and $\nu_*\Omega^2_\mu(-1,0) = \nu_*\mathcal{O}(0,1)(-1,-1)' =: S$. The induced operator $S' \to S$ will be denoted by D_1^*. This notation is inspired by that [14]: locally, the restriction of D_1^* to $\mathbf{P}(V)$ is interpreted as the formal adjoint of the anti-self-dual Dirac operator. Since $\mathbf{P}(V)$ is not spin, this is purely a local interpretation and it is better simply to regard D_1^* as the first-order differential operator induced by d_μ.

The homogeneity $(-1,-1)$ case requires some identification: the only nonvanishing direct images $\nu_*^q\Omega^p_\mu(-1,-1)$ are $\nu_*^1\mathcal{O}(-1,-1) = \mathcal{O}_{\mathbf{M}}$ and $\nu_*\Omega^2_\mu(-1,-1) = \mathcal{O}_{\mathbf{M}}(-1,-1)'$. (The identification $\nu_*^1\mathcal{O}(-1,-1) = \mathcal{O}_{\mathbf{M}}$ can be achieved by using the monad $0 \to \mathcal{O}(-1,-1) \xrightarrow{z} V \otimes \mathcal{O}(0,-1) \xrightarrow{w} \mathcal{O}(0,0) \to 0$ pulled-back from $\mathbf{F}$. It has cohomology $\mathcal{O}(1,-2)$ and therefore can be regarded as exact when taking direct images; i.e. there is an induced "connecting homomorphism" $\nu_*\mathcal{O}(0,0) \to \nu_*^1\mathcal{O}(-1,-1)$ which is an isomorphism. This isomorphism will be exploited subsequently to convert H^1 cohomology on $\mathbf{G}$ into H^0 cohomology.) The differential operator D: $\mathcal{O}_{\mathbf{M}} \to \mathcal{O}_{\mathbf{M}}(-1,-1)'$ induced from (4.4) $\otimes\, \mathcal{O}(-1,-1)$ by taking direct images is of second order in this case. Now $D|_{\mathbf{P}(V)}$ must be a constant multiple of the operator $(d^*d + R/6)/\|u\|^2$ identified by Hitchin in [14], R being the scalar curvature of the Fubini-Study metric and $*$ denoting formal adjoint. It is easy to check that if f and s are holomorphic functions on an open subset of M, $D(fs) = -(\nabla_a\nabla^a f)s + fDs +$ 1st order derivatives of f, s. The symbol of $D|_{\mathbf{P}(V)}$ is therefore negative, so $D|_{\mathbf{P}(V)} = C(d^*d + R/y)/\|u\|^2$ for some *positive* constant C. (A little more work gives $Ds = -\nabla_a\nabla^a s + s/(u \cdot v)$.)

The $(-2,-1)$ case is similar to the $(-1,0)$ case: $\nu_*^q\Omega_\mu^p(-2,-1)$ is nonzero only if $q = 1$ and $p = 0$ or 1, with $\nu_*^1\mathcal{O}(-2,-1) = S$ and $\nu_*^1\Omega_\mu^1(-2,-1) = \nu_*\Omega_\mu^1(-1,0) = S'$. The induced operator $S \to S'$ will be denoted by D_1; as in the case of D_1^*, its precise identification is not required.

The $(-2,-2)$ case is similar to the $(0,0)$ case: $\nu_*[(4.4) \otimes \mathcal{O}(-2,-2)] = 0$, and $\nu_*^1[(4.4) \otimes \mathcal{O}(-2,-2)]$ is the complex:

$$\begin{array}{ccccc}
\nu_*^1\mathcal{O}(-2,-2) & \longrightarrow & \nu_*^1\Omega_\mu^1(-2,-2) & \longrightarrow & \nu_*^1\Omega_\mu^2(-2,-2) \\
\wr\| & & \wr\| & & \wr\| \\
\Omega_-^2 & \xrightarrow{\quad d \quad} & \Omega_\mu^3 & \xrightarrow{\quad d \quad} & \Omega_{\mathbf{M}}^4
\end{array}$$

The analogue of the Penrose transform as described in [13] can now be given: If $U \subset \mathbf{M}$ is an open subset, let $U' := \nu^{-1}(U)$ and $U'' := \mu(U')$. The set U will be assumed Stein and to possess the property that $\nu(\mu^{-1}(x)) \cap U$ is contractible for every $x \in \mathbf{F}$. The latter condition ensures that the canonical homomorphism $H^r(U'', \mathcal{S}) \to H^r(U', \mu^{-1}\mathcal{S})$ is an isomorphism for every r and locally free analytic sheaf $\mathcal{S}$ on U'' [6].

The Penrose transform for $\mathcal{O}(a,b)$ is comprised of the following operations: first the pull-back isomorphism $H^r(U'', \mathcal{O}(a,b)) \to H^r(U', \mu^{-1}\mathcal{O}(a,b))$ is applied. Then the latter cohomology group is expressed in terms of *analytic* cohomology on U' using the resolution $0 \to \mu^{-1}\mathcal{O}(a,b) \to \Omega_\mu^\cdot(a,b)$. This gives the spectral sequence $E_1^{p,q} = H^q(U', \Omega_\mu^p(a,b))$ converging to $E_\infty^{p+q} = H^{p+q}(U', \mu^{-1}\mathcal{O}(a,b))$. Finally, each term $H^q(U', \Omega_\mu^p(a,b))$ is expressed in terms of analytic cohomology on U using the Leray spectral sequence, and since U is Stein (and ν is proper), $H^q(U', \Omega_\mu^p(a,b)) = H^0(U, \nu_*^q\Omega_\mu^p(a,b))$.

Thus the complete transform is a spectral sequence

$$E_1^{p,q} = H^0\big(U, \nu^q \Omega_\mu^p(a,b)\big) \Rightarrow H^{p+q}\big(U'', \mathcal{O}(a,b)\big),$$

where the differentials are those induced by d_μ. For the homogeneities considered earlier, one obtains in particular the following:

$$\begin{aligned}
&H^1\big(U'', \mathcal{O}(-1,0)\big) \simeq \ker D_1^*\colon \Gamma\big(U, \mathcal{O}(-1,0)' \oplus \mathcal{O}(0,-2)'\big) \to \Gamma(U, S),\\
&H^2\big(U'', \mathcal{O}(-1,0)\big) \simeq \operatorname{coker} D_1^*,\\
&H^1\big(U'', \mathcal{O}(-1,-1)\big) \simeq \ker D\colon \Gamma(U, \mathcal{O}) \to \Gamma\big(U, \mathcal{O}(-1,-1)'\big),\\
&H^2\big(U'', \mathcal{O}(-1,-1)\big) \simeq \operatorname{coker} D,\\
&H^1\big(U'', \mathcal{O}(-2,-1)\big) \simeq \ker D_1\colon \Gamma(U, S) \to \Gamma\big(U, \mathcal{O}(-1,0)' \oplus \mathcal{O}(0,-2)'\big),\\
&H^2\big(U'', \mathcal{O}(-2,-1)\big) \simeq \operatorname{coker} D_1,\\
&H^3\big(U'', \mathcal{O}(-2,-2)\big) \simeq \Gamma(U, \Omega^4)/d\Gamma(U, \Omega^3).
\end{aligned}$$

Suppose now that E is a bundle on U'' which is trivial on every complex line in U''. Then μ^*E is trivial on each fiber of ν, and therefore $\mu^*E = \mu^*f$ for some bundle F on U, namely $F = \nu_*\mu^*E$. The bundle F has a holomorphic connection ∇ induced by d_μ via

$$\begin{array}{ccc}
\nu_*\mu^*E & \xrightarrow{\nu_* d_\mu} & \nu_0\Omega_\mu^1(\mu^*E)\\
\| & & \wr\|\\
F & \xrightarrow{\nabla} & \Omega_{\mathbf{M}}^1(F)
\end{array}$$

and this connection is self-dual since the composition $\nu_*\mu^*E \to \nu_*\Omega_\mu^1(\mu^*E) \to \nu_*\Omega_\mu^2(\mu^*E) \simeq \Omega_-^2(F)$ is zero. This is the "Ward transform" of the bundle E. For example, if $E = \mathcal{O}(-1,1)$, then $\mu^*E = \mathcal{O}(-1,1)(0,0)' \xrightarrow{\alpha} \mathcal{O}(0,0)(0,-1)'$, where α is as in (4.3). The induced connection is given by

$$\mathcal{O}(0,-1)' \ni s \mapsto \left(\nabla_a s, \nabla^a s + u^a s/u \cdot v\right) \in \Omega_{\mathbf{M}}^1(0,-1)',$$

as is easily checked. The unitary structure on $\mathcal{O}_{\mathbf{M}}(0,-1)'$ determined by $-h^\sigma \otimes h^*$ is given by $\langle s, s\rangle = u \cdot v s^\sigma s$.

Returning to the general case, the Penrose transform as described earlier can be repeated but with $\mathcal{O}(a,b)$ replaced by $E(a,b)$. Each operator $\nu_*^q\Omega_\mu^p(a,b) \to \nu_*^q\Omega_\mu^{p+1}(a,b)$ is then replaced by

$$\begin{array}{ccc}
\nu_*^q\left[\Omega_\mu^p(a,b) \otimes \mu^*E\right] & \to & \nu_*^q\left[\Omega_\mu^{p+1}(a,b) \otimes \mu^*E\right]\\
\simeq & & \simeq\\
\left[\nu^q\Omega_\mu^p(a,b)\right] \otimes F & \to & \left[\nu_*^q\Omega_\mu^{p+1}(a,b)\right] \otimes F
\end{array}$$

and the lower map is simply the original operator coupled to the connection on F.

The stage is now set for the Penrose transform of (4.1). Let E be the bundle on $\mathbf{F}$ corresponding to the $U(n)$-instanton (F, ∇) of index (k, l) on $\mathbf{P}(V)$ as in §3. Since E is trivial on every real line, there is a neighborhood U of $\mathbf{P}(V)$ in $\mathbf{M}$ such that E is trivial on $\mu(\nu^{-1}(x))$ for every $x \in U$, giving an extension of (F, ∇) to a holomorphic bundle with holomorphic connection on U, also denoted (F, ∇). By restricting U if necessary, it can be assumed Stein, to satisfy $\sigma U = U$, and be such that $\nu(\mu^{-1}(x)) \cap U$ is contractible for each $x \in \mathbf{F} = U''$. A fixed Stein cover $\{U_i\}_{i \in I}$ of $\mathbf{F}$ is chosen, and with respect to this cover all cohomology on $\mathbf{F}$ will be computed. The covering is chosen to be σ-invariant; i.e. there is a map $\sigma\colon I \to I$ such that $\sigma(U_i) = U_{\sigma i}$. An isomorphism such as $H^1(\mathbf{F}, \mathcal{O}(a, b)) \simeq \overline{H^1(\mathbf{F}, \mathcal{O}(b, a))}$ is then given by

$$H^1(\mathbf{F}, \mathcal{O}(a, b)) \ni [f_{ij}] \mapsto [f_{ij}^\sigma] \in H^1(\mathbf{F}, \mathcal{O}(b, a)),$$

where $f_{ij}^\sigma(z, w) := \overline{f_{\sigma i \sigma j}(\overline{w}, \overline{z})}$. A compatible σ-invariant Stein cover of U' is then $\{\mu^{-1}(U_i) \cap U'\}_{i \in I}$, and with respect to this covering all (analytic) cohomology on U' will be computed.

In what follows, the majority of the calculations are performed on U' rather than U, and only at the very end will they be pushed down. To simplify the notation slightly, the symbol E will also be used to denote $\mu^*E|_{U'}$. All reference to the indexing set I will be dropped, and the cohomology homomorphism taking p-cochains into $(p+1)$-cochains will be denoted by δ. A p-cocycle will be referred to as an element of H^p in the obvious abuse of terminology. When the space on which cohomology is computed is not specified, it is taken to be $\mathbf{F}$, as before.

Let $f_2 \in H^2(E(-2, -1))$. Since $z^a f_2 \in H^2(V \otimes E(-1, -1))$ and $H^*(E(-1, -1)) = 0$, there is a unique 1-cochain f_1^a such that $\delta f_1^a = z^a f_2$. Then $f_1 := w_a f_1^a \in H^1(E(-1, 0))$ is the class corresponding to f_2 under the isomorphism of (4.1)(a).

The first isomorphism of (4.1)(b) is the isomorphism of Serre duality: it is determined by the cup product pairing

$$H^2(E(-2, -1)) \otimes_{\mathbf{C}} H^1(E^*(0, -1)) \xrightarrow{\cup} H^3(\mathbf{F}, \mathcal{O}(-2, -2)).$$

The transpose of the isomorphism of (4.1)(b) is $H^1(E(-1, 0)) \ni f_1 \mapsto (\phi f_1)^\sigma \in H^1(E^*(0, -1))$, ϕ being the unitary structure on E. Thus the hermitian form χ_2 on $\overline{K}_2^*$ which is the object of interest is now interpreted as $\langle f_1, f_1 \rangle = f_1^{\sigma *} \cup \phi f_2$.

Consider first the transform of $f_1 \in H^1(E(-1,0))$: $\mu^* f_1 = \delta f_0$ for some unique 0-cochain on U', giving $f := d_\mu f_0 \in H^0(U', \Omega^1_\mu \otimes E(-1,0))$ for the transform of f_1 (essentially, f should be pushed down to U to complete the procedure, but will be left as is for the moment).

Next consider the pull-back of the cochain f_1^a: since $\delta\mu^* f_1^a = z^a \mu^* f_2$,

$$q_1 := \left(v_a \mu^* f_1^a\right)/u \cdot v \in H^1\left(U', E(-1,-1)(-1,0)'\right).$$

The class q_1 is converted into an element of $H^0(U', E(-1,0)')$ as described earlier: $z^a q_1 = \delta q_0^a$ for some unique 0-cochain q_0^a, giving

$$q := w_a q_0^a \in H^0\left(U', E(-1,0)'\right)$$

as the corresponding section.

Since w_a and v_a are independent, there exists a 0-cochain with coefficients in $V \otimes E(-1,-1)$ on U', f_0^a say, such that $w_a f_0^a = f_0$ and $v_a f_0^a = 0$. Then $\mu^* f_1^a = (\mu^* f_1^a - u^a q_1 - \delta f_0^a) + u^a q_1 + \delta f_0^a$, and since the contraction of both w_a and v_a with the term in brackets is zero, it is necessarily of the form $z^a g_1$ for some 1-cochain g_1 with coefficients in $E(-2,-1)$. That is

$$\mu^* f_1^a = z^a g_1 + u^a q_1 + \delta f_0^a. \tag{4.6}$$

Applying δ to (4.6) gives $\mu^* z^a f_2 = z^a \delta g_1$, so the cochain g_1 can be used to give a representative for the transform of f_2. Explicitly, $d_\mu g_1 \in H^1(U', \Omega^1_\mu \otimes E(-2,-1))$, $z^a d_\mu g_1 = \delta g_0^a$ for some unique 0-cochain g_0^a, and finally $g := w_a g_0^a \in H^0(U', \Omega^1_\mu \otimes E(-1,0))$ is the section which, when pushed down to U, will represent the transform of f_2. The aim now is to express g in terms of f, thus giving the transform of the isomorphisms of (4.1)(a).

Applying d_μ to (4.6) gives $0 = d_\mu(\mu^* f_1^a) = z^a d_\mu g_1 + (z^a, 0) q_1 + u^a d_\mu q_1 + \delta d_\mu f_0^a$, using here $d_\mu u^a = (z^a, 0)$. Since $q_1 = v_a \mu^* f_1^a / u \cdot v$,

$$d_\mu q_1 = (0, w_a)\mu^* f_1^a / u \cdot v = \left(0, \mu^* f_1 / u \cdot v\right) = (0, \delta f_0 / u \cdot v),$$

and therefore

$$\begin{aligned} z^a d_\mu g_1 &= -\left(\delta q_0^a, 0\right) - \left(0, u^a \delta f_0 / u \cdot v\right) - \delta d_\mu f_0^a \\ &= -\delta\left[\left(q_0^a, u^a f_0 / u \cdot v\right) + d_\mu f_0^a\right]. \end{aligned}$$

It follows that $g = -(q, 0) - f$.

Now the class $q \in H^0(U', E(-1,0)')$ is related to f in the following way: since $d_\mu q_1 = (0, \delta f_0 / u \cdot v)$ and $d_\mu(0, f_0 / u \cdot v) = f \wedge (0, 1/u \cdot v)$, q is the (unique) section such that $Dq = h_0 / u \cdot v$, where $f = (h_0, h_1) \in \Gamma(U', E(0,0)(-1,0)' \oplus E(-1,1)(0,-1)')$. Hence the transform of (4.1)(a) is interpreted as

$$\begin{array}{lccc} & H^1(E(-1,0)) \simeq & & H^2(E(-2,-1)) \\ & \Cup & & \Cup \\ & f_1 & \leftrightarrow & f_2 \\ (4.7) & \updownarrow & & \updownarrow \\ & (h_0, h_1) & \leftrightarrow & -(q,0)-(h_0,h_1) \quad (\operatorname{mod} \operatorname{im} D_1) \\ & \Cap & & \Cap \\ & \ker D_1^* & & \operatorname{coker} D_1 \end{array}$$

where $Dq = h_0/u \cdot v$.

The transform of (4.1)(b) is far more straightforward: the isomorphism of Serre duality, regarded as a pairing, corresponds to the cup-wedge product

$$H^1\big(U', \Omega^1_\mu \otimes E(-2,-1)\big) \otimes H^0\big(U', \Omega^1_\mu \otimes E^*(0,-1)\big) \to H^1\big(U', \Omega^2_\mu(-2,-2)\big).$$

Under the isomorphisms $H^1(U', \Omega^1_\mu \otimes E(-2,-1)) \simeq H^0(U', \Omega^1_\mu \otimes E(-1,0))$ and $H^1(U', \Omega^2_\mu(-2,-2)) \simeq H^0(U', \Omega^2_\mu(-1,-1))$, this is simply the symplectic pairing

$$(4.8) \qquad \begin{array}{ccc} H^0\big(\Omega^1_\mu(-1,0)\big) \otimes H^0\big(\Omega^1_\mu(0,-1)\big) & \xrightarrow{\ \wedge\ } & H^0\big(\Omega^2_\mu(-1,-1)\big) \\ \Cup & & \Cup \\ (a,b) \otimes (c,d) & \longmapsto & ad - bc \end{array}$$

(E, E^* and U' suppressed).

The effect of the isomorphism $H^1(E(-1,0)) \to \overline{H^1\big(E^*(0,-1)\big)}$ is equally simple to determine: it is just

$$(4.9) \qquad H^0\big(\Omega^1_\mu \otimes E(-1,0)\big) \ni (h_0, h_1) \mapsto \big((\phi h_1)^\sigma, (\phi h_0)^\sigma\big) \in \overline{H^0\big(\Omega^1_\mu \otimes E^*(0,-1)\big)}.$$

Combining (4.7), (4.8), and (4.9) it follows that a representative section in $H^0(U', \Omega^2_\mu(-1,-1))$ for the transform of $f_1^{\sigma *} \cup \phi f_2$ is $s := h_0^{\sigma *}\phi h_0 + h_0^{\sigma *}\phi q - h_1^{\sigma *}\phi h_1$, where $Dq = h_0/u \cdot v$.

Now h_0 is an element of $H^0(U', E(0,0)(-1,0)')$ and therefore is simply the pull-back of a section $h_0 \in H^0(U, F(-1,0)')$ (after applying $\mu^* E \simeq \nu^* F$). However, $h_1 \in H^0(U', E(-1,1)(0,-1)')$ and it is the section $\alpha h_1 \in H^0(U', E(0,0)(0,-2)')$ which is a pull-back from U. Since $\alpha\alpha^\sigma = -1/u \cdot v$, this gives

$$s = h_0^{\sigma *}\phi h_0 + u \cdot v (Dq)^{\sigma *}\phi q + u \cdot v(\alpha h_1)^{\sigma *}\phi(\alpha h_1) \in H^0\big(U, \mathcal{O}(-1,-1)'\big).$$

Multiplying s by $u \cdot v$ to obtain a function and restricting to $\mathbf{P}(V)$ gives

$$u \cdot vs|_{\mathbf{P}(V)} = \|h_0\|^2 + \|\alpha h_1\|^2 + C\langle \nabla^* \nabla q, q \rangle + CR\|q\|^2/6$$

using the earlier identification of $D|_{\mathbf{P}(V)}$. Since C and R are positive, it follows that $\int_{\mathbf{P}(V)} s\, dV \geqslant 0$, with equality iff $s|_{\mathbf{P}(V)} = 0$. Since s is holomorphic, it can vanish on $\mathbf{P}(V)$ iff it vanishes on a neighbourhood of $\mathbf{P}(V)$ in $\mathbf{M}$, which would imply that f, and hence f_1, are also zero.

The hermitian form on $\overline{K}_2^*$ is thus definite. Its sign is independent of the bundle E since all choices in the transform procedure were independent of E. This completes the proof of Lemma 5.

5. Moduli spaces of instantons

The monad description of instantons given in the previous sections facilitates the explicit calculation of connection forms for the bundles on $\mathbf{P}(V)$ using the Penrose transform or methods similar to those in [2]. Of more concern to this paper, the description also gives a way to construct concrete topological spaces parametrizing the instantons of fixed index up to isomorphism (gauge equivalence); that is, the *moduli spaces* of instantons. This is the main objective of the section. The presentation is based on the construction of the moduli spaces of stable 2-bundles on $\mathbf{CP}_2$ in [18].

The construction of a moduli space relies on the existence of at least one instanton of the correct index, and unfortunately, the monad description is not well suited to answering questions of existence except in simple cases where dimensions are not large: the linear algebra rapidly gets out of hand. However, by a variety of different techniques, existence or nonexistence *can* be established in all cases and thereafter the results concerning moduli spaces become more meaningful. The relevant conclusions are listed in (5.4) below.

The section concludes with the construction of the moduli space of $SU(2)$-instantons of index 1.

To construct the moduli space of $U(n)$-instantons of index (k, l) on $\mathbf{P}(V)$, any one of the monads (3.7), (3.8), (3.9) can be used, but for simplicity, only the first will be used here; i.e. those of the form

$$(5.1) \qquad M\colon 0 \to K_1(-1,0) \xrightarrow{a} N_1 \oplus \overline{K}_2^*(-1,1) \xrightarrow{a^{\sigma *}\phi} \overline{K}_1^*(0,1) \to 0,$$

where K_1, K_2, and N_1 are complex vector spaces of dimensions $k + \frac{1}{2}l(l+1)$, $k + \frac{1}{2}l(l-1)$, and $n + k + \frac{1}{2}l(l+3)$ respectively, and $\phi = \psi \oplus -\chi \otimes h^\sigma \otimes h^*$ for some positive definite hermitian forms ψ on N and χ on $\overline{K}_2^*$.

Since all (hermitian) vector spaces of a given dimension are isomorphic, the spaces K_1, (K_2, χ), (N_1, ψ) can be fixed once and for all, so each monad of the form (5.1) is determined up to unitary isomorphism by

$$a = (a_1, a_2) \in \mathrm{Hom}(K_1 \otimes V, N_1) \oplus \mathrm{Hom}(K_1 \otimes V^*, \bar{K}_2^*) =: R.$$

For $a \in R$, let $M(a)$ denote the sequence (5.1) and let $P \subset R$ be the subset for which $M(a)$ is a monad. For $a \in P$, let $E(a) := E(M(a))$.

To describe P in more concrete terms, let S be the real vector space $\mathrm{Herm}(K_1 \otimes V, \overline{K_1 \otimes V^*})/\mathrm{Herm}(K_1, \overline{K_1}^*) \otimes \phi_0$ (where Herm denotes hermitian homomorphisms), and let f: $R \to S$ be given by $f(a) := \bar{a}_1^* \psi a_1 - 1 \otimes \phi_0 \bar{a}_2^* \chi a_2 1 \otimes \bar{\phi}_0$. Then $M(a)$ is a complex iff $f(a) = 0$. Explicitly, this condition means that there is a hermitian form λ on K_1 (which can be degenerate) such that

$$(5.2)\quad \langle a_1 k_0 \otimes v_0, a_1 k_1 \otimes v_1 \rangle - \langle a_2 k_0 \otimes \overline{\phi_0 v_1}, a_2 k_1 \otimes \overline{\phi_0 v_0} \rangle$$
$$= (\bar{k}_0^* \lambda k_1) \cdot \langle v_0, v_1 \rangle \quad \text{for all } k_0, k_1 \in K_1,\ v_0, v_1 \in V.$$

If $f(a) = 0$, the remaining condition that $a \in P$ is that $(a_1(z), a_2(w))$ have maximal rank at each $(z, w) \in \mathbf{F}$, and it follows that P is an open subset of $f^{-1}(0)$.

If E corresponds to a $U(n)$-instanton, then from the display (3.6) it follows that $H^2(E) = 0$, and since $\mathrm{End}\, E$ also corresponds to such, $H^2(\mathrm{End}\, E) = 0$. Monads of the form (5.1) satisfy the hypotheses of Lemma 4.1.7 of [18], and the arguments following that lemma can be modified in an obvious way to show that f has surjective differential at each point of P. (The lemma effectively identifies $H^2(\mathrm{End}\, E(a))$ with the complexified cokernel of $df(a)$.) Hence $P \subset f^{-1}(0)$ is a real submanifold of dimension $\dim R - \dim S = (2k + l^2 + l)(3n + 2k + l^2 + l)$.

To complete the construction, it remains only to factor out by the equivalence relation of isomorphism. By Lemma 1, $M(a) \simeq M(a')$ iff they are isomorphic as unitary monads. An isomorphism p: $M(a) \to M(a')$ is of the form $p = (\mu, \nu, \rho, \bar{\mu}^{*-1})$ for some $\mu \in \mathrm{GL}(K_1)$, $\nu \in U(\psi)$, $\rho \in U(\chi)$ with $(\nu a_1, \rho a_2) = (a_1' \mu, a_2' \mu)$. With $H := \mathrm{GL}(K_1) \times U(\psi) \times U(\chi)$, the group H acts on R by $(\mu, \nu, \rho) \cdot (a_1, a_2) = (\nu a_1 \mu^{-1}, \rho a_2 \mu^{-1})$, and P is an H-invariant subset. The dimension of H is $(2k + l + l^2)^2 + n(n + 2k + l + l^2) + 2l(l + n)$.

By Lemma 2, $\mathrm{End}\, E(a) = \mathrm{End}\, M(a)$, and by definition of morphisms of monads, $\mathrm{End}\, M(a) = \{(\mu, \nu, \rho, \sigma) \in \mathrm{End}\, K_1 \oplus \mathrm{End}\, N_1 \oplus \mathrm{End}\, \bar{K}_2^* \oplus \mathrm{End}\, \bar{K}_1^*$: $(a_1\mu, a_2\mu) = (\nu a_1, \rho a_2)$ and $(\psi a_1 \bar{\sigma}^*, \chi a_2 \bar{\sigma}^*) = (\bar{\nu}^* \psi a_1, \bar{\rho}^* \chi a_2)\}$. In particular, it is the kernel of a linear map depending linearly on $a \in R$. Thus the subset $P_0 \subset P$ of elements a such that $E(a)$ is simple is an open subset of P since it is determined by a rank condition on a. P_0 is an H-invariant subset, and the isotropy subgroup at $a \in P_0$ is just $U(1) \cdot (1_{K_1}, 1_{N_1}, 1_{\bar{K}_2^*}) = U(1)$. Thus if P_0 is not empty, the dimension of the manifold P_0/H is $\dim P_0 - \dim H + 1 = 4nk + 2l^2(n-1) - n^2 + 1$.

The moduli space of $U(n)$-instantons of fixed index is thus described as the quotient of a real submanifold of $\mathbf{C}^N$ by a matrix group. (To describe P/H as *the* moduli space is a slight abuse of terminology: to be in keeping with common usage, it would be necessary to show that the bijection $P/H \to [\mathscr{E}]$ has certain nice functorial properties. This is described in detail in [18] in the context of moduli of stable bundles over $\mathbf{CP}_2$: the arguments there are easily modified to show that P/H is a coarse moduli space for $[\mathscr{E}_0]$ ($:=$ isomorphism classes of simple bundles) if g.c.d. $(n, l, k + \frac{1}{2}l(l+1)) = 1$.)

Using either (3.8) or (3.9) in place of (3.7) results in a similar description, but if the latter is used, the description can be refined so that the moduli space is expressed as a quotient of a subspace of $U(n + 4k + 2l^2)$ by a closed subgroup.

The moduli spaces of $SO(n)$- and $\mathrm{Sp}(n)$-instantons are constructed in the same manner using (3.11), the main difference being in the dimension counts (see (5.4) below).

The boundary of the moduli space is contained in the set $a \in f^{-1}(0)$ for which $(a_1(z), a_2(w))$ fails to have maximal rank at some (but not all) $(z, w) \in \mathbf{F}$. From (5.2) it can be shown that if $a_1(z_0)k_0 = 0 = a_2(w_0)k_0$, then $a_1(z)k_0 = 0 = a_2(w)k_0$ for every (z, w) on the real line through (z_0, w_0). Moreover $\bar{k}_0^* \lambda k_1 = 0$ for all $k_1 \in K_1$ and there are vectors $\mu \in \bar{K}_2^*$, $n_1 \in N_1$ such that

$$a_1(z)\colon K_1/\langle k_0\rangle \to N_1/\langle n_1\rangle, \qquad a_2(w)\colon K_1/\langle k_0\rangle \to \bar{K}_2^*/\langle \mu\rangle$$

are well defined and satisfy (5.2) (with K_1 replaced by $K_1/\langle k_0\rangle$ etc.). If the new complex is nonsingular, it thus defines a monad corresponding to a $U(n)$-instanton of index $(k-1, l)$. This is the manifestation of the "bubbling-off" phenomenon occurring in the work of Uhlenbeck [20], [21] and Taubes and which plays a vital role in the work of Donaldson.

The above construction of moduli spaces has less relevance in lacking a knowledge of the *existence* of instantons of given topological type. In [4], this problem is resolved in the case of instantons on S^4. For the classical groups G the results are summarized in the following table, in which $\mathscr{M}_k$ denotes the moduli space of irreducible G-instantons of index k on $X = S^4$.

	G	$\dim \mathscr{M}_k$	$\mathscr{M}_k \neq \varnothing$ iff
	$SU(n)$	$4nk - n^2 + 1$	$n \leqslant 2k$
(5.3)	$\mathrm{Sp}(n)$	$4(n+1)k - n(2n+1)$	$n \leqslant k$
	$\mathrm{Spin}(n)$	$4(n-2)k - \frac{1}{2}n(n-1)$	$n \leqslant 4k$, $n \neq 4$

(The index here is as defined at the end of §2; a Spin(n)-instanton of index k was defined there to be an $SO(n)$-instanton of index $2k$. For Spin(3) it is required that k be even in order that there exist a bundle let alone a connection.)

The dimensions listed in (5.3) are also valid in the case $X = \mathbf{CP}_2$. Moreover, the existence theorem of Taubes [19] implies that for each of the groups G above, irreducible G-instantons of a given index exist on $\mathbf{CP}_2$ when they exist on S^4. Finally, from the display (3.6) it follows that for a bundle E corresponding to a G-instanton of index k, $H^0(E) \neq 0$ if the inequality in (5.3) is violated, implying that E is not simple. Thus (5.3) *is true for* $\mathbf{CP}_2$ *also.*

Taubes' existence results do not consider the case of $U(n)$-instantons with nonzero first Chern class or $SO(n)$-instantons of odd index. To deal with these cases, there are two viable methods. The first is to emulate the deformation argument of [4] proving existence on S^4, an approach which gives existence for $U(n)$ if $n > 2$ and $SO(n)$ if $n > 5$ (with the appropriate restrictions on the index). The second is to modify Taubes' method [19] so that instead of grafting a sequence of $SU(2)$-instantons onto a flat background, it is grafted onto a *self-dual* background. With a careful check of estimates of curvatures in various L^p norms, it is found that Taubes' principal existence theorems remain applicable in the new setting, and this method deals with those cases not covered by the deformation argument.

The final conclusions are summarized in the following table in which $\mathscr{M}_*$ denotes the moduli space of irreducible G-instantons of index k and of index (k, l) for $G = U(n)$.

	G	$\dim \mathscr{M}_*$	$\mathscr{M}_* \neq \varnothing$ iff
	$SU(n)$	$4nk - n^2 + 1$	$n \leqslant 2k$
	$\mathrm{Sp}(n)$	$4(n+1)k - n(2n+1)$	$n \leqslant k$
(5.4)	$\mathrm{Spin}(n)$	$4(n-2)k - \frac{1}{2}n(n-1)$	$n \leqslant 4k$
	$SO(n)$	$2(n-2)k - \frac{1}{2}n(n-1)$	$n \leqslant 2k$
	$U(n)$	$4nk + 2l^2(n-1) - n^2 + 1$	$n \leqslant 2k + l^2 - a(l+b)$

Here $l = an + b$, $|b| \leqslant \frac{1}{2}n$. For $SO(3)$ it is required that $k \equiv 0$ or $1 \pmod 4$, k must be even for Spin(3) $k \geqslant 4$ for $SO(4)$ and $k \geqslant 2$ for Spin(4); otherwise $\mathscr{M}_* = \varnothing$.

The fact that $\mathscr{M}_* = \varnothing$ if the inequalities in (5.4) are violated follows easily from the monad descriptions, as was earlier indicated.

To conclude, the moduli space of $SU(2)$-instantons of index 1 will be considered in more detail. As predicted by Donaldson [9], it is a cone over $\mathbf{CP}_2$.

From (5.1), the relevant monads are of the form

$$(5.5)\qquad M\colon 0 \to \mathcal{O}(-1,0) \xrightarrow{a} V \oplus \mathcal{O}(-1,1) \xrightarrow{a^{\sigma *}\phi} \mathcal{O}(0,1) \to 0,$$

where $\phi = \phi_0 \oplus - h^\sigma \otimes h^*$. The map a of (5.5) is given by a pair $(p, r) \in \operatorname{End} V \oplus V$, and the condition $a^{\sigma *}\phi a = 0$ is equivalent to

$$(5.6)\qquad \langle pv_0, pv_1\rangle - \langle v_0, r\rangle\langle r, v_1\rangle = \lambda\langle v_0, v_1\rangle, \qquad v_0, v_1 \in V,$$

for some constant λ.

Since ϕ_0 is positive definite, λ is necessarily real and nonnegative. The nondegeneracy condition ensuring that $\operatorname{im}(a)$ be a subbundle is simply that pz and wr never be simultaneously zero for $(z, w) \in \mathbf{F}$, from which it follows that $\lambda > 0$. The group of unitary automorphisms of M is $H = \mathbf{C}^* \times U(\phi_0) \times U(1)$, and the action of H on (p, r) is given by $(t, u, e^{i\theta}) \cdot (p, r) = (upt^{-1}, e^{i\theta}rt^{-1})$.

Multiplying p by a suitable element of $U(\phi_0)$, it can be assumed that r is an eigenvector of p, and from (5.6), the corresponding eigenvalue can be taken to be $(\lambda + \|r\|^2)^{1/2}$. Replacing p, r by $(\lambda + \|r\|^2)^{-1/2}p$, $(\lambda + \|r\|)^{-1/2}r$, one then has $pr = r$ and $\lambda = 1 - \|r\|^2$.

From (5.6) again, it follows that p can be multiplied by an element u of $U(\phi_0)$ with $ur = r$ and $up = \sqrt{\lambda}\,1$ on $r^\perp$, so when p is replaced by up it follows that

$$(5.7)\qquad pv = \left(1 - \|r\|^2\right)^{1/2} v + \frac{\langle r, v\rangle r}{1 + \left(1 - \|r\|^2\right)^{1/2}}, \qquad v \in V.$$

The only remaining freedom is the multiplicative action of $U(1)$ on r, so the moduli space of $SU(2)$-instantons of index 1 on $\mathbf{P}(V)$ is canonically identified with the open unit ball in V modulo the action of $U(1)$. The center, $r = 0$, corresponds to the bundle $\mathcal{O}(1, -1) \oplus \mathcal{O}(-1, 1)$ and (5.7) then gives the canonical monad (5.5) for this bundle. If $\|r\| = 1$, (5.7) implies that $pz = 0 = wr$ on $\{(z, w) \in \mathbf{F}\colon r^\perp z = 0 = wr\}$; i.e. the bundle is singular precisely on the real line corresponding to $[r] \in \mathbf{P}(V)$. Using the Penrose transform, an explicit calculation of the corresponding curvatures shows that as $\|r\| \to 1$, the curvatures become increasingly concentrated at $[r]$ and increasingly flat away from $[r]$.

An equally easy calculation is that of the moduli space of $U(2)$-instantons of index $(1, 1)$. This turns out to be the set of pairs (v_1, v_2) of linearly independent vectors in V modulo the action of $U(2)$ on the pair together with an overall scale factor. If coordinates are chosen so that the line joining $[v_1]$ and $[v_2]$ is not the line at infinity, then the instanton is "located" at a finite point $[v_1] \in \mathbf{P}(V)$ when $[v_2]$ is on the line at infinity, there being one remaining real parameter. This interpretation is consistent with the picture of such instantons arising by way of grafting $SU(2)$-instantons onto a nontrivial background field.

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