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THE LINEARITY OF PROPER HOLOMORPHIC MAPS BETWEEN BALLS IN THE LOW CODIMENSION CASE

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Let $B^n = \{z \in C^n: \|z\| < 1\}$ and let $f\colon B^n \to B^k$ be a proper holomorphic map. We shall always take $n > 2$. Cima and Suffridge [1] have conjectured that if f extends to a twice continuously differentiable function on the closure of B^n and $k \leqslant 2n - 2$, then f is linear fractional. The purpose of this note is to show

Theorem. *If $f\colon B^n \to B^k$ is a proper holomorphic map which extends holomorphically to a neighborhood of $\overline{B^n}$ and $k \leqslant 2n-2$, then f is linear fractional.*

(It should be remarked that the map $(z_1,\cdots,z_n) \to (z_1,\cdots,z_{n-1}, z_1z_n,\cdots,z_{n-1}z_n, z_n^2)$ shows that the theorem is false if $k \geqslant 2n-1$; see [1].)

So, let $f\colon B^n \to B^k$ be a proper map, holomorphic in a neighborhood of $\overline{B^n}$, $k \leqslant 2n-2$. Let $\langle z,w\rangle = \sum_{j=1}^{p} z_j\overline{w}_j$ be the hermitian inner product in C^p. Let $z' = f(z)$. Applying the Hopf lemma to the function $r' = \langle z',z'\rangle - 1$ on B^n, we see that

$$\langle z',z'\rangle - 1 = u(z,\bar z)(1 - \langle z,z\rangle) \tag{1}$$

for some real analytic function $u(z,\bar z)$, nonzero in a neighborhood of ∂B^n. Complexifying, (1) becomes

$$\langle z',w'\rangle - 1 = u(z,\overline{w})(1 - \langle z,w\rangle), \tag{2}$$

where $w' = f(w)$.

Let $z_0 \in \partial B^n$. (2) is valid for $(z,w) \in U \times U$ for some open neighborhood U of z_0. Thus if z is a point on the hyperplane $Q_w = \{\zeta\colon 1 - \langle \zeta,w\rangle = 0\}$, $(z,w) \in U \times U$, then $z' = f(z)$ is on the hyperplane $Q'_{w_0} = \{\zeta'\colon 1 - \langle \zeta',w'\rangle = 0\}$, $w' = f(w)$. Thus f maps points lying in a complex hyperplane to points lying in a complex hyperplane. Let $\phi_n\colon P^n \to P^{n^*}$ be the antiholomorphic map

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sending a point w to its reflection Q_w. (P^{n*} = the projective space of hyperplanes in P^n.) ϕ_n is an antiholomorphic isomorphism, so we may define a map f^* by the commutative diagram

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\phi_n"'] & P^k \arrow[d, "\phi_k"] \\ \phi_n(U) \arrow[r, "f^*"] & P^{k*} \end{tikzcd}$$

i.e., $f^*(Q_w) = Q_{w'}$. The point of the remarks above is that if $Q_w \cap U \neq \varnothing$, then $f(Q_w \cap U) \subset Q_{w'} = f^*(Q_w)$. (Note that $z_0 \in Q_{z_0}$ so $\{w \in U: Q_w \cap U \neq \varnothing\}$ is open and nonempty.)

In the sequel we shall let P^r stand for an r-dimensional linear subspace of projective space, constant, variable, arbitrary, etc., depending on context. For convenience we write $f(P^r)$ for $f(P^r \cap U)$.

Let $G(1, U) = \{P^1 \subset P^n: P^1 \cap U \neq \varnothing \text{ and } P^1 \subset Q_w, \text{ some } w \in U\}$. For $P^1 \in G(1, U)$ define $d(P^1)$ = the dimension of the smallest linear subspace containing $f(P^1)$ and define $d = \max_{P^1 \subset G(1,U)}\{d(P^1)\}$. Note that $d(P^1)$ is the rank of the $k \times \infty$ matrix whose columns are derivatives of f along P^1. Thus $\{P^1: d(P^1) < d\}$ is given by the vanishing of a collection of $d \times d$ determinants, hence is a proper subvariety of $G(1, U)$. We now have a number of cases to look at.

Case 0: $d = 0$. Then f is constant, hence improper.

Case 1: $d = 1$. Then the image of f is contained in the P^n spanned by the image of Df, and since f takes lines to lines, f is linear fractional.

Case 2: $d > 2$. Let P^{n-2*} be the $(n-2)$-dimensional space of hyperplanes in P^n containing $P^1 \in G(1 - U)$ and P^{k-d-1*} the $(k-d-1)$-dimensional space of hyperplanes in P^k containing P^d (= span of $f(P^1)$). If $P^{n-1} \supset P^1$, then the span of $f(P^{n-1}) \supset P^d$ so if also $P^{n-1} \in \phi_n(U)$, then $f^*(P^{n-1}) \in P^{k-d-1*}$, i.e., f^* maps P^{n-2*}'s (the set of hyperplanes containing a line) into P^{k-d-1*}'s. Since f and f^* are conjugate isomorphic, f maps P^{n-2}'s into P^{k-d-1}'s. (The exceptions would be those P^{n-2*}'s corresponding to P^1's with $d(P^1) < d$. This is an analytic subvariety. Since the dimension of the smallest linear subspace containing $f(P^{n-2})$ drops on subvarieties, every $f(P^{n-2})$ is contained in some P^{k-d-1}.)

Lemma. *f maps P^{n-1}'s into P^{k-d}'s.*

Proof. Pick a P^{n-1} near Q_{z_0} and an $x \in P^{n-1}$ such that $Df(x)$ has maximal rank. (There exist such (P^{n-1}, x) since f is proper. Indeed, the P^{n-1}'s for which we cannot do this form a subvariety. If $f(P^{n-1}) \subset P^{k-d}$ for all P^{n-1}'s off that subvariety, $f(P^{n-1}) \subset P^{k-d}$ since dimension can only drop along subvarieties). Suppose $f(P^{n-1})$ spans at least a P^{k-d+1}. Then there exist multi-indices $\alpha_1, \cdots, \alpha_{k-n-d+2}$ and directions $v_1, \cdots, v_{k-n-d+2}$ tangent to P^{n-1} so that the P^{k-d+1} is spanned by $f(x)$, the image of $Df(x)$ (restricted to P^{n-1}) and $\{D^{\alpha_j}f(x)(v_j^{|\alpha_j|})\}$. Note $k-n-d+2 < n-d < n-2$. So consider the P^{n-2} through x spanned by $x, v_1, \cdots, v_{k-n-d+2}$ (and enough other tangent directions w_p to make up a P^{n-2}). Then $Df(x)(v_j)$, $Df(x)(w_p)$ and $D^jf(x)(v_j^j)$ are contained in the span of $f(P^{n-2})$. Thus $f(P^{n-2})$ spans at least an $n-2+k-n-d+2 = k-d$ dimensional space, but $f(P^{n-2}) \subset P^{k-d-1}$. This contradiction proves $f(P^{n-1}) \subset P^{k-d}$ for P^{n-1} in an open set about Q_{z_0}. The lemma for all P^{n-1} follows by analytic continuation.

Let $f_1 = f|_{P^{n-1}}$. f_1: $P^{n-1} \to P^{k-d}$. Suppose $P^{n-1} \cap B^n \neq \varnothing$. Then $P^{n-1} \cap B^n$ will then be a B^{n-1}, $P^{k-d} \cap B^k$ will be a B^{k-d} and f_1: $B^{n-1} \to B^{k-d}$ will be proper. Note that the codimension has dropped by $d-1 \geqslant 1$. We can now proceed by induction.

Codimension 0. $k = n$, f: $P^n \to P^n$ taking hyperplanes into hyperplanes. The fundamental theorem of projective geometry then yields that f is linear fractional. (Or, f: $B^n \to B^n$ is proper, hence must be an automorphism.)

Codimension > 0. Assume the theorem is true for codimension less than $k-n$. Then if $d = 1$ we are done. If $d \geqslant 2$, the maps f_1 constructed above are linear by the induction hypothesis, hence f is linear fractional along every hyperplane intersecting B^n. It follows that f must be linear fractional.

References

[1] J. A. Cima & T. J. Suffridge, *A reflection principle with applications to proper holomorphic mappings*, preprint.

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