J. DIFFERENTIAL GEOMETRY
23 (1986) 207–240

THE WITTEN COMPLEX AND THE DEGENERATE MORSE INEQUALITIES

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Abstract

In this paper, we use the complex introduced by Witten to prove the Morse inequalities, in the case where the critical points of the Morse function are isolated, and also the degenerate Morse inequalities of Bott. The method is based on a natural extension of the heat equation method for the proof of the index theorem. In the degenerate case, the de Rham complex is twisted using a nontrivial transformation of the exterior algebra on the neighborhood of the critical points. The cohomology of a manifold is compared to the L^2 cohomology of certain fiber bundles over the critical submanifolds. A L^2 version of the Thom isomorphism is then proved, from which the degenerate Morse inequalities follow.

0. Introduction

In a very interesting paper [19], Witten has shown how to prove analytically the Morse inequalities for a Morse function h with isolated critical points. The proof involves the construction of a family of complexes indexed by $t > 0$ associated with the operator $d^{h/t}$ defined by

$$d^{h/t} = e^{-h/t} d e^{h/t}. \tag{0.1}$$

By studying the lower part of the spectrum of the corresponding Laplacian $\square^{h/t}$ as $t \downarrow\downarrow 0$, Witten proved the Morse inequalities. Also, Witten suggested a method of proof of the degenerate Morse inequalities of Bott [9] when h has critical submanifolds.

In [7], we gave a probabilistic proof of the Atiyah-Singer index theorem and of the corresponding Lefschetz fixed point formulas, based on the heat

Received October 2, 1984. This work was partially supported by National Science Foundation Grant MCS-8108814(A02) while the author was visiting the Institute for Advanced Study, Princeton.

equation method developed by Atiyah, Bott and Patodi [1] who identified the index of an operator to the difference of traces of certain heat kernels.

After the completion of [7], we noted that when the critical points of h are isolated, the heat equation method could be adapted to the proof of the Morse inequalities, the main difference being that the equality in the proof of the index theorem should be replaced by an inequality. This is briefly done in §1.

When checking routinely the possible extension of this method to the degenerate case, we immediately noticed that several obstacles made this extension difficult. Here we list some of these difficulties, and indicate how they are solved in this paper.

(1) Replacing h by h/t would be insufficient to produce the localization of the lowest eigenforms on the critical submanifolds, because this procedure would precisely be insufficient on the critical submanifolds themselves.

(2) A natural idea is then to expand the metric by the factor $1/t$ in directions transversal to the critical submanifolds, while keeping invariant the metric on the critical submanifolds. In the case of isolated critical points this is equivalent to replacing $\square^{h/t}$ by $t\square^{h/t}$. However we noticed that if there are nontrivial critical submanifolds, if $\nabla \cdot dh$ is not parallel on the critical submanifolds, the localization of the eigenfunctions (i.e. of the 0 forms which are eigenvectors) does not take place.

We were then led to adapt the choice of the metric in M to the given function h, this by using the Morse lemma in the degenerate case [13]. If M_1 is a critical submanifold, then (f, E) a tubular neighborhood of M_1 exists such that E splits into

$$E = E^+ \oplus E^- \tag{0.2}$$

where E^+, E^- are Euclidean bundles, and moreover

$$h \circ f(y^+, y^-) = \frac{1}{2}\left(|y^+|^2 - |y^-|^2\right). \tag{0.3}$$

A natural metric then exists on the Riemannian manifold E which can be extended to the whole manifold M. This is done in §2(a).

(3) With such a choice of the metric in M, we produced the localization of the eigenfunctions when expanding the metric in the directions E by the factor $1/t$. However, much to our dismay, we noticed that localization of the eigenforms of degree: $\geqslant 1$ did not take place in general, because in general the bundles E^+, E^- are not flat. This is of course a basic difference with the case where the critical points are isolated. After studying carefully the simple case where M is exactly the Riemannian manifold E, and where h is given by (0.3), we noticed that in order to produce the localization of eigenforms, the

transformation of the de Rham complex (0.1) was not adequate. Instead (0.1) has to be replaced by

$$(0.4) \qquad e^{-h/t}\tau_{\sqrt{t}}de^{h/t}[\tau_{\sqrt{t}}]^{-1},$$

where $\tau_{\sqrt{t}}$ is a linear transformation of the forms which acts according to their degree in the vertical directions.

The expansion of the metric in the vertical directions was then no longer needed, but of course on 0 forms, the new Laplacian associated with (0.4) acts in the same way as the Laplacian constructed by expanding the metric. The transformation (0.4) has a natural interpretation in terms of the dilations acting on E^+, E^-. It is introduced in §2(d).

(4) After obtaining the adequate localization of eigenforms on the critical submanifolds, we prove inequalities comparing the Betti numbers of M to the Betti numbers of the L^2 cohomology of E for the operator $\bar{d}^\alpha$ given by

$$(0.5) \quad \bar{d}^\alpha = \exp\left(-\alpha\left(\frac{|y^+|^2}{2} - \frac{|y^-|^2}{2}\right)\right) d \exp\left(\alpha\left(\frac{|y^+|^2}{2} - \frac{|y^-|^2}{2}\right)\right).$$

These inequalities are proved in §2(g). This is an unpleasant feature of the method. In fact we then have to study the Hodge theory for the operator $\bar{d}^\alpha$ on a fiber bundle. Clearly Hodge theory is not exactly the adequate tool to study the cohomology of a fiber bundle!

We were then led to prove that for α large enough, the L^2 cohomology in E associated with $\bar{d}^\alpha$ is exactly the ordinary cohomology on E^+ combined with the compactly supported cohomology in E^-. This is done in §2(h) via a functional version of the Thom isomorphism on E^-, and of the retraction of E^+ on M, and also the Mayer-Vietoris argument.

We have not addressed the question of proving analytically that an explicit model for the cohomology of the manifold can be constructed using the critical points, as first shown by Smale [17] (see Witten [19]). This question has recently received a complete analytical solution by Helffer and Sjöstrand [12], when the critical points are isolated.

Our treatment of the localization on the critical submanifolds is probabilistic, and uses our result in [4], as well as estimates on certain heat kernels which are obtained using the Malliavin calculus [4], [6], [16], [18]. In the degenerate case only the main steps are indicated. We have refrained from giving motivation for the introduction of such and such objects like $\tau_{\sqrt{t}}$ in the hope that the presentation of the problem given here is explicit enough.

Although our method is certainly not the simplest for the proof of the degenerate Morse inequalities, we hope it sheds some more light on the strange interplay of algebraic and analytical considerations which appears in the article of Witten [19].

For the probabilistic terminology used in this paper, we refer to [5], [8], [6, Chapter 2], [7, §2]. In particular if X is a continuous semimartingale, then dX denotes its differential in the sense of Stratonovitch and δX its differential in the sense of Itô.

1. Morse inequalities and the Witten complex: the nondegenerate case

In this section, we prove the Morse inequalities for a Morse function h whose critical points are isolated.

In (a), the main assumptions and notations are introduced, and the Witten complex [19] associated with the operator $e^{-h}de^{h}$ is described.

In (b), the basic inequalities on the traces of certain heat kernels are proved. These inequalities are the natural extension of the corresponding equalities for the proof of the index theorem using heat equation methods [1].

In (c), the Morse inequalities are proved by studying the asymptotics (for small time) of the traces of certain heat kernels.

In §2, we will adapt these techniques to the case where the critical points of h are degenerate. However the analysis will be much more difficult.

(a) Assumptions and notations. M is a C^∞ compact connected Riemannian manifold of dimension n.

∇ denotes the covariant differentiation operator with respect to the Levi-Civita connection on M.

$\Lambda^p(M)$ denotes the set of p-differential forms on M, and $\Lambda(M)$ is the exterior algebra

$$\Lambda(M) = \bigoplus_{0}^{n} \Lambda^p(M). \tag{1.1}$$

The operators d, δ acting on the C^∞ sections of $\Lambda(M)$ are defined in the usual way. The Laplacian $\square$ is defined by

$$\square = (d+\delta)^2. \tag{1.2}$$

B_p $(0 \leqslant p \leqslant n)$ are the Betti numbers of M. h is a C^∞ function on M with values in R.

Recall that if A is a (n, n) tensor, A acts as a derivation on $\Lambda(M)$, so that if $\alpha \in \Lambda^1(M)$ and $X \in TM$, then

$$(A\alpha)(X) = -\alpha(AX). \tag{1.3}$$

Following Witten [19], we now define the operators

$$d^h = e^{-h}de^h, \quad \delta^h = e^h\delta e^{-h}, \quad \square^h = (d^h + \delta^h)^2. \tag{1.4}$$

Clearly, $(d^h)^2 = (\delta^h)^2 = 0$. As observed by Witten [19], the complex associated with the operator d^h has the same Betti numbers as the original de Rham complex associated with d.

An easy computation shows that

$$\square^h = \square + |dh|^2 - \Delta h - 2\nabla \cdot dh. \tag{1.5}$$

Remark 1. If α, β are C^∞ p-forms, set

$$\langle \alpha, \beta\rangle_h = \int_M \langle \alpha, \beta\rangle(x) e^{-2h(x)}\, dx. \tag{1.6}$$

The adjoint operator of d for the scalar product $\langle\ ,\ \rangle_h$ is the operator δ'^h given by

$$\delta'^h = e^{2h}\delta e^{-2h} \tag{1.7}$$

which also writes as

$$\delta'^h = \delta + 2i_{dh}. \tag{1.8}$$

Set

$$\square'^h = (d + \delta'^h)^2. \tag{1.9}$$

Clearly,

$$\square'^h = \square + 2L_{dh}. \tag{1.10}$$

Moreover,

$$\square'^h = e^h \square^h e^{-h}. \tag{1.11}$$

As in Witten [19], we will now scale adequately the function h.

Definition 1.1. For $t > 0$, $\square_t$ and $\square'_t$ are the operators

$$\square_t = t\square^{h/t}, \qquad \square'_t = t\square'^{h/t}. \tag{1.12}$$

For $\alpha > 0$ and $t > 0$, $P_t^\alpha(x, y)$ denotes the C^∞ kernel associated with the operator $e^{-\alpha\square_t/2}$, and $P_t'^\alpha(x, y)$ denotes the C^∞ kernel associated with the operator $e^{-\alpha\square'_t/2}$.

Using (1.11), we see that

$$P_t^\alpha(x, x) = P_t'^\alpha(x, x). \tag{1.13}$$

In what follows we can then use indifferently P_t^α or $P_t'^\alpha$.

Definition 1.2. $J_p(\alpha, t, x)$ denotes the trace of the linear operator $P_t^\alpha(x, x)$ acting on $\Lambda_x^p(M)$. $K_p(\alpha, t)$ is defined by

$$K_p(\alpha, t) = \int_M J_p(\alpha, t, x)\, dx. \tag{1.14}$$

(b) The basic inequalities. We now have the following result, which is the adequate substitute for the well-known trace equality in the heat equation method for the proof of the index theorem in Atiyah, Bott and Patodi [1].

Theorem 1.3. *For any $\alpha > 0$, $t > 0$ and p such that $0 \leqslant p \leqslant n$, we have*

$$K_p(\alpha, t) - K_{p-1}(\alpha, t) + \cdots + (-1)^p K_0(\alpha, t) \geqslant B_p - B_{p-1} + \cdots + (-1)^p B_0, \tag{1.15}$$

where equality holds for $p = n$.

Proof. Note that the Witten complex has the same Betti numbers as the original de Rham complex. Moreover let λ be a positive eigenvalue of $\square_t$, F_λ the corresponding eigenspace in the set of C^∞ differential forms, which breaks into $F_\lambda^0, \cdots, F_\lambda^n$ (where F_λ^p is the set of p-forms which are eigenvectors for the eigenvalue λ). Since λ is nonzero, the sequence

$$0 \underset{d^{h/t}}{\rightarrow} F_\lambda^0 \underset{d^{h/t}}{\rightarrow} F_\lambda^1 \rightarrow \cdots \rightarrow F_\lambda^n \underset{d^{h/t}}{\rightarrow} 0 \tag{1.16}$$

is exact so that for any p, if R_λ^p is defined by

$$R_\lambda^p = \dim F_\lambda^p - \dim F_\lambda^{p-1} + \cdots + (-1)^p \dim F_\lambda^0,$$

then $R_\lambda^p \geqslant 0$ and $R_\lambda^n = 0$. Now the left-hand side of (1.15) is given by

$$B_p - B_{p-1} + \cdots + (-1)^p B_0 + \sum_{\lambda \neq 0} e^{-\lambda t} R_\lambda^p. \tag{1.17}$$

The theorem is proved.

(c) The Morse inequalities: the nondegenerate case. We now assume that h is a Morse function, i.e. h has a finite number of critical points $x_1, \cdots, x_l$ on which $\nabla \cdot dh$ is nondegenerate (recall that we use the convention (1.3)).

$\exp \alpha\nabla \cdot dh(x_i)$ acts on $\Lambda_{x_i}^p(M) \cdot \mathrm{Tr}_p \exp \alpha\nabla \cdot dh(x_i)$ denotes the corresponding trace.

We will now make $t \downarrow\downarrow 0$ and later $\alpha \uparrow\uparrow +\infty$ in (1.15). We first have:

Theorem 1.4. *As $t \downarrow\downarrow 0$, $K_p(\alpha, t)$ has a limit $K_p(\alpha)$ given by*

$$K_p(\alpha) = \sum_{i=1}^{l} \frac{\mathrm{Tr}_p[\exp \alpha(\nabla \cdot dh)(x_i)]}{|\det(I - \exp - \alpha(\nabla \cdot dh)(x_i))|}. \tag{1.18}$$

Proof. We give a proof based on the results in [4]. Weitzenböck's formula shows that:

$$\square = -\Delta^H + L, \tag{1.19}$$

where Δ^H is the horizontal Laplacian, and L is a bounded matrix valued operator of order 0.

We now proceed as in [7, Theorem 2.15]. We also use the notations of [6]–[7]. Let N be the bundle of orthonormal frames on M, $Y_1, \cdots, Y_n$ be the standard horizontal vector fields on N [6, Chapter 2] and π be the canonical projection $N \to M$.

If (W, P) is the probability space of the standard Brownian notion $w_t = (w_t^1, \cdots, w_t^n)$ in R^n, consider the Stratonovitch differential equation on N,

$$du = \sum_{i=1}^{n} Y_i(u) \cdot \sqrt{t\alpha}\, dw^i, \qquad u(0) = u_0, \tag{1.20}$$

and the associated flow of diffeomorphisms of N, $\psi_s(\sqrt{t\alpha}\, dw, \cdot)$, which depends smoothly on $\sqrt{t\alpha}$ (see [3, Chapter I], [6, Chapter II]).

Take $x_0 \in M$ and $u_0 \in N$ such that $\pi u_0 = x_0$. Set

$$u_s^{t\alpha} = \psi_s\left(\sqrt{t\alpha}\, dw, u_0\right), \qquad x_s^{t\alpha} = \pi u_s^{t\alpha}, \tag{1.21}$$

τ_0^s denotes the parallel transportation operator along $x_\cdot^{t\alpha}$ from fibers over $x_s^{t\alpha}$ to fibers over x_0 [3, Chapter VIII], [6, Chapter II]. Let $U_s^{t,\alpha}$ be the process of linear operators acting on $\Lambda_{x_0}(M)$ defined by the differential equation

$$\begin{aligned} dU_s^{t,\alpha} &= U_s^{t,\alpha}\left[\alpha\tau_0^s\nabla \cdot dh\left(x_s^{t\alpha}\right) - \tfrac{1}{2}t\alpha\tau_0^s L\left(x_s^{t\alpha}\right)\right] ds, \\ U_0^{t,\alpha} &= I. \end{aligned} \tag{1.22}$$

If $E_{x_0,x_0}^{t\alpha}$ is the expectation operator for the (time scaled) Brownian motion $x_\cdot^{t\alpha}$ conditional on $x_1^{t\alpha} = x_0$ (see [4, Chapter II]), we find using (1.5) that if $p_t(x, y)$ is the heat kernel for the Laplace Beltrami operator on M, then

$$\begin{aligned} P_t^\alpha(x_0, x_0) = p_{\alpha t}(x_0, x_0) E_{x_0,x_0}^{t\alpha}\Bigg[\exp\Bigg\{-\alpha\int_0^1 \frac{|dh|^2\left(x_s^{t\alpha}\right) ds}{2t} \\ + \alpha\int_0^1 \frac{\Delta h\left(x_s^{t\alpha}\right) ds}{2}\Bigg\} U_1^{t,\alpha}\tau_0^1\Bigg]. \end{aligned} \tag{1.23}$$

Take $\varepsilon > 0$. Set

$$H_\varepsilon = \left\{x_0 \in M; \inf d(x, x_i) \leqslant \varepsilon\right\}, \qquad 1 \leqslant i \leqslant l. \tag{1.24}$$

By [6, Theorem 3.12], we know that as $t \downarrow\downarrow 0$, for any $k \in N$

$$P_{x_0,x_0}^{\alpha t}\left[\sup_{0 \leqslant s \leqslant 1} d\left(x_0, x_s^{t\alpha}\right) \geqslant \varepsilon/2\right] = o(t^k) \tag{1.25}$$

uniformly on M. Moreover by the estimate of Azencott [2, VIII, Proposition 4.4] we know that for N large enough, for any $x_0 \in M$

$$p_{\alpha t}(x_0, x_0) \leqslant C/(\alpha t)^N. \tag{1.26}$$

Since $|dh| \geqslant \eta > 0$ on ${}^cH_{\varepsilon/2}$, we see from (1.25), (1.26) that as $t \downarrow\downarrow 0$

$$\int_M J_p(\alpha, t, x)\, ds \sim \int_{H_\varepsilon} J_p(\alpha, t, x)\, dx. \tag{1.27}$$

We can then use the asymptotic representation of the conditional Brownian bridge measure given in [6, Theorem 4.16]. Using the notations of [6], we know that if P_1 is the law of the standard Brownian bridge w^1 in R^n, with $w_0^1 = w_1^1 = 0$, then for any $k \in N$, as $t \downarrow\downarrow 0$

$$\begin{aligned} P_t^\alpha(x_0, x_0) = \int & \frac{1}{(\sqrt{2\pi\alpha t})^n} \exp\left\{-\alpha \int_0^1 \frac{|dh|^2}{2t}(x_s^{t\alpha})\,ds + \frac{\alpha}{2}\int_0^1 (\Delta h)(x_s^{t\alpha})\,ds\right\} \\ & \cdot \exp\left\{-\frac{\int_0^1 \left|v^2(\sqrt{t\alpha}\,dw^1, x_0)\right|^2}{2t}\,ds\right\} \\ & \cdot \left[U_1^{t\alpha}\tau_0^1\right] \frac{G(\sqrt{t\alpha}\,dw^1) g\left(\left|q(\sqrt{t\alpha}\,dw^1, x_0)\right|\right)}{\det C'(\sqrt{t\alpha}\,dw^1, x_0)}\,dP_1(w^1) + o(t^k), \end{aligned} \tag{1.28}$$

where $o(t^k)$ is uniform on M.

Using (1.27) and (1.28), we are essentially back to the situation we had considered in [7, Theorem 4.9] for the proof of the Lefschetz fixed point formulas of Atiyah-Singer. In fact we evaluate $\int_{d(x_i, x) \leq \varepsilon} J_p(\alpha, t, x)\,dx$ using geodesic coordinates centered at x_i, so that

$$\int_{d(x_i, x) \leq \varepsilon} J_p(\alpha, t, x)\,dx = \int_{|X| \leq \varepsilon} J_p(\alpha, t, X) k(X)\,dX \tag{1.29}$$

with $k(0) = 1$. By doing the change of variables $X = \sqrt{t}X'$, using (1.28) and proceeding as in [7, Theorem 4.9], we find that if $u_1, \cdots, u_l$ are orthogonal frames at $x_1, \cdots, x_l$, then

$$\begin{aligned} K_p(\alpha) = \sum_{i=1}^{l} \int_{W \times T_{x_i}M} & \exp\left\{\frac{\alpha}{2}\Delta h(x_i) - \frac{\alpha^2}{2}\int_0^1 \left|\nabla_{c_i + u_i w_s^1} dh(x_i)\right|^2 ds\right\} \\ & \cdot \operatorname{Tr}_p \exp\{\alpha \nabla \cdot dh(x_i)\}\,dP_1(w^1) \frac{dc_i}{(\sqrt{2\pi})^n}. \end{aligned} \tag{1.30}$$

Now $c_i + u_i w_s^1$ is a Brownian bridge in $T_{x_i}M$ starting and ending at c_i. Recall that by a well-known formula [15, p. 206], if b is a one-dimensional Brownian bridge starting and ending at x, then

$$E \exp -\frac{\beta^2}{2}\int_0^1 |b|^2\,ds = \sqrt{\frac{\beta}{\operatorname{sh}\beta}} \exp\left\{-\beta x^2 \operatorname{th}\frac{\beta}{2}\right\}. \tag{1.31}$$

By putting $\nabla \cdot dh(x_i)$ in diagonal form and using (1.31), (1.18) immediately follows.

Remark 2. There is eventually a much simpler argument using (1.10) and (1.13) which would make the proof strictly identical to the proof of the Lefschetz fixed point formulas.

Let M_p be the number of critical points of h whose index is equal to p (i.e., the matrix $\nabla \cdot dh$ has exactly p negative eigenvalues).

We now have

Theorem 1.5. *As* $\alpha \uparrow +\infty$, $K_p(\alpha)$ *converges to* M_p.

Proof. The argument is closely related to Witten [19]. If $\lambda_1^i \cdots \lambda_n^i$ are the eigenvalues of $\nabla \cdot dh(x_i)$, the eigenvalues of $\nabla \cdot dh(x_i)$ on $\Lambda^p_{x_i(M)}$ are all the possible sums $-\sum_{k=1}^p \lambda^i_{j_k}$, where the j_k are distinct. Each term in the right-hand side of (1.18) has a limit, which is 1 if the index of x_i is p, and 0 if not. q.e.d.

By collecting the results of Theorems 1.3, 1.4 and 1.5 we get

Theorem 1.6. *For any* p $(0 \leqslant p \leqslant n)$

$$M_p - M_{p-1} + \cdots + (-1)^p M_0 \geqslant B_p - B_{p-1} + \cdots + (-1)^p B_0. \tag{1.32}$$

For $p = n$, *there is equality in* (1.32).

2. The Morse inequalities: the degenerate case

In this section, we prove the Morse inequalities of Bott [9] in the case where the critical points of h are degenerate.

In (a), we construct a special metric on the manifold M. In fact, contrary to the case where the critical points are nondegenerate, we need to adapt the metric to the particular function h. Using the Morse lemma in the degenerate case, we embed each critical submanifold M_i in a tubular neighborhood $(f, E^+ \oplus E^-)$ so that

$$h \circ f(y^+, y^-) = \frac{|y^+|^2}{2} - \frac{|y^-|^2}{2}. \tag{2.1}$$

We extend the natural metric on the bundle $E = E^+ \oplus E^-$ to a metric on M.

In (b), a natural connection $\overline{\nabla}$ on TE is constructed which has in general a nontrivial torsion.

In (c), the operators d, δ on the Riemannian manifold E are briefly described using the connection $\overline{\nabla}$.

In (d), the de Rham complex of M is transformed as in Witten [19]. However contrary to what is done in Witten [19] and in §1, the new complex is obtained by means of a nonscalar operator which acts on forms on a neighborhood of the critical submanifolds according to their degree in the vertical directions.

In (e), we give a probabilistic construction of the heat kernel in the vector bundle E associated with the twisted Laplacian.

In (f), the Witten complex in the fiber bundle E associated with the operator

$$\bar{d}^{\alpha} = \exp - \alpha\left(\frac{|y^+|^2}{2} - \frac{|y^-|^2}{2}\right) d \exp\left(\alpha\left(\frac{|y^+|^2}{2} - \frac{|y^-|^2}{2}\right)\right) \tag{2.2}$$

is briefly described.

In (g), by computing the asymptotics of traces of certain heat kernels, the cohomology of M is compared with the L^2 cohomology of the fiber bundles E for the operator $\bar{d}^{\alpha}$. The proof is in principle very long, since there are many terms whose behavior must be exactly estimated. We only verify the main points of the proof.

In (h), we prove the essential result that the L^2 cohomology for $\bar{d}^{\alpha}$ on E is the ordinary cohomology of E^+ and the compactly supported cohomology in E^-. This is proved via a L^2 version of the Thom isomorphism, which is of a more functional nature here. The proof of this fact necessitates a Mayer-Vietoris argument.

Finally the degenerate Morse inequalities are proved in (i).

(a) The choice of a metric. h is a C^∞ function defined on M with values in R, whose critical points form a union of disjoint connected submanifolds $M_1, \cdots, M_r$ of dimension $n_1, \cdots, n_r$.

We also assume that h is nondegenerate, i.e. if $x \in M_i$, then $d^2h(x)$ is nondegenerate on any subspace of T_xM which intersects T_xM_i transversally. Let n_i^- be the index of d^2h on M_i. Set $n_i^+ = n - n_i - n_i^-$.

To simplify the exposition, we will talk about one specific M_i, say M_1, but everything will have to be done on all the M_i simultaneously.

By the generalized Morse lemma [13, Chapter 6], we know that M_1 possesses a tubular neighborhood (f, E) such that:

• E is a Euclidean bundle over M_1, which is endowed with the scalar product g_E. Moreover E, which has dimensions $n - n_1$, splits into two orthogonal subbundles

$$E = E^+ \oplus E^-, \tag{2.3}$$

where the dimension of E^+ is n_1^+, and the dimension of E^- is n_1^-.

• f embeds E into M. Moreover there is an open neighborhood $\mathscr{V}$ of M_1 in E such that if $y = (y^+, y^-) \in \mathscr{V}$, then

$$h \circ f(y) = \frac{|y^+|^2}{2} - \frac{|y^-|^2}{2}. \tag{2.4}$$

In the sequel, we will identify E and $f(E)$.

Let ρ be the projection $E \to M_1$. We first construct a metric on E.

Since E^+, E^- are Euclidean bundles, they can be endowed with Euclidean connections. E is then naturally endowed with a Euclidean connection, for which the splitting (2.3) is parallel. Let ∇ be the corresponding covariant differentiation operator. L is the curvature tensor of E, which splits into the curvature tensors L^+, L^- of E^+, E^-, i.e.,

$$L = L^+ \oplus L^-. \tag{2.5}$$

Then TE splits naturally into

$$TE = T^H E \oplus T^V E, \tag{2.6}$$

where $T^V E$ are the vectors of TE which lie in the fibers E (i.e. the vertical vectors), and $T^H E$ are the horizontal vectors in TE.

If $y \in Y$, then ρ_* identifies $T_y^H E$ with $T_{\rho(y)} M_1$. Moreover $T_y^V E$ and E can be naturally identified.

In the sequel, if $Y \in T_y E$, then (Y^H, Y^V) will denote its components with respect to the splitting (2.6).

Let g_1 be any Riemannian metric on M_1. Then $T_y^H E$ and $T_y^V E$ are both naturally endowed with a scalar product. We can assume as well that they are orthogonal for the scalar product $\bar{g}$ in TE which splits into $\bar{g} = g_1 \oplus g_E$.

Take $\varepsilon > 0$ such that if $y \in E$ and $|y| \leqslant 2\varepsilon$, then $y \in \mathscr{V}$.

Let $f(r)$ be a C^∞ function defined on R^+ with values in $[0,1]$ which is decreasing, such that if $r \leqslant \varepsilon^2/32$ then $f(r) = 1$, and if $r \geqslant \varepsilon^2/16$ then $f(r) = 0$. We also assume that on a left neighborhood of $\varepsilon^2/16$, $f(r)$ is given by

$$f(r) = \exp\left\{-\frac{1}{(\varepsilon^2/16 - r)}\right\}. \tag{2.7}$$

Let φ be the C^∞ function E,

$$\varphi(y) = f(|y|^2). \tag{2.8}$$

Let g' be any Riemannian structure on M which coincides with $\bar{g}$ on $\{y \in E; |y| \leqslant \varepsilon\}$.

We define a new metric g on M by setting

$$g = \varphi \bar{g} + (1 - \varphi) g'. \tag{2.9}$$

In particular on $\{y \in E; |y| \leqslant \varepsilon/4\sqrt{2}\}$, g and $\bar{g}$ coincide.

Remark 1. M_1 is totally geodesic in E for the metric $\bar{g}$, and in M for the metric g.

(b) A connection in TE. We now briefly construct a natural connection on TE.

First note that TM_1 is naturally endowed with the Levi-Civita connection associated with the metric g_1. We still note $\nabla.$ as the corresponding covariant differentiation operator and R is the curvature tensor of TM_1. $TM_1 \oplus E$ is then endowed with a Euclidean connection.

By identifying $T_y^H E$ with $T_{\rho(y)}M_1$, and $T_y^V E$ with E, it is clear that TE is also naturally endowed with a connection.

In particular if y_t is a C^1 curve with values in E and if $Y_0 = (Y_0^H, Y_0^V) \in T_y E$, then the parallel transportation of Y_0 along $y\cdot$ is obtained by taking the parallel transportation along $\rho(y.)$ of Y_0 which is identified to an element of $T_{\rho(Y_0)}M_1 \oplus E_{y_0}$.

Let $\overline{\nabla}$ be the covariant differentiation operator on TE and let $\overline{T}$ and $\overline{L}$ denote the torsion and the curvature of $\overline{\nabla}$. We now have

Theorem 2.1. *The metric $\bar{g}$ is parallel for $\overline{\nabla}$. Moreover if $X, Y, Z \in T_y E$, then*

$$\begin{aligned} \overline{T}(X,Y) &= \left[L(X^H,Y^H)y\right]V, \\ \overline{L}(X,Y)Z &= \left[R(X^H,Y^H)Z^H\right]^H + \left[L(X^H,Y^H)Z^V\right]^V. \end{aligned} \tag{2.10}$$

Proof. The first part of the theorem is obvious. The second part is very easy, and is left to the reader (for more details see Yano-Ishihara [20]).

(c) The operators d, δ on E.

We now briefly construct the operator d and its formal adjoint δ on the Riemannian manifold E.

Take $x \in M_1$. Let e_j $(1 \leqslant j \leqslant n_1)$, $f_k^{\pm}$ $(1 \leqslant k \leqslant n_1^{\pm})$ be orthogonal bases of $T_x M_1$, $E_x^{\pm}$. Let dx^j, $dx^{\pm,k}$ be the corresponding dual bases. Take $y \in E_x$. Using the decomposition (2.6) and the identifications which follow, we can lift e_j $(1 \leqslant j \leqslant n_1)$, $f_k^{\pm}$ $(1 \leqslant k \leqslant n_1^{\pm})$ to TE. Since there is no risk of confusion, we can assume as well that e_j $(1 \leqslant j \leqslant n_1)$ is a base of $T_y^H E$ and $f_k^{\pm}$ $(1 \leqslant k \leqslant n_1^{\pm})$ is a base of $T_y^V E$. Similarly, we lift dx^j, $dx^{\pm,k}$ in $T_y^* E$ so that they form an orthogonal base in $T_y^* E$.

Let d be the exterior differentiation operator acting on the C^∞ sections of the exterior algebra $\Lambda(E)$ of the manifold E. Let δ be its (formal) adjoint for the metric $\bar{g}$.

We will now briefly describe the operators d, δ by means of the connection $\overline{\nabla}$ on E.

Proposition 2.2. *If $x \in M_1$ and if $y = (y^+, y^-) \in E_x$, we have*

$$\begin{aligned} d &= dx^j \wedge \overline{\nabla}_{e_j} + dx^{\pm,j} \wedge \overline{\nabla}_{f_j^\pm} + \frac{1}{2} dx^k \wedge dx^l i_{[L^\pm(e_k,e_l)y^\pm]^V}, \\ \delta &= -i_{e_j}\overline{\nabla}_{e_j} - i_{f_j^\pm}\overline{\nabla}_{f_j^\pm} - \frac{1}{2}\left[L^\pm(e_k,e_l)y^\pm\right]^V \wedge i_{e_k} i_{e_l}. \end{aligned} \tag{2.11}$$

Proof. The first line in (2.11) is an obvious consequence of the expression for $\overline{T}$ given in Theorem 2.1.

Clearly the adjoint of the operator $\frac{1}{2} dx^k \wedge dx^l i_{[L^\pm(e_k,e_l)y^\pm]^V}$ is $\frac{1}{2}[L^\pm(e_k,e_l)y]^V i_{e_l} i_{e_k}$.

So we must only consider the adjoint of the first two terms in the expression of d.

Let $\overline{\nabla}'$ be the Levi-Civita connection on E. Let S be the tensor defined by

$$S = \overline{\nabla}' - \overline{\nabla}. \tag{2.12}$$

It is essentially trivial to prove that the adjoint of the operator

$$dx^j \wedge \overline{\nabla}_{e_j} + dx^{\pm,j} \wedge \overline{\nabla}_{f_j^\pm} \tag{2.13}$$

is given by

$$-i_{e_j}\overline{\nabla}_{e_j} - i_{f_j^\pm}\overline{\nabla}_{f_j^\pm} + i_{S(e_j)e_j + S(f_j^\pm)f_j^\pm}. \tag{2.14}$$

Now using the expression of the torsion $\overline{T}$ given in Theorem 2.1, it follows from some obvious computations that for any j

$$S(e_j)e_j = 0, \qquad S\left(f_j^\pm\right)f_j^\pm = 0. \tag{2.15}$$

The theorem is proved.

(d) The modified de Rham complex. We will now produce a family of complexes depending on $t > 0$, which extends what has been done by Witten in [19].

α is a positive number, which will later be chosen to be large enough.

$\Lambda(M)$ is the exterior algebra of M, which decomposes into

$$\Lambda(M) = \bigoplus_0^n \Lambda^p(M), \tag{2.16}$$

where $\Lambda^p(M)$ is the set of p-forms on M.

For $s > 0$, let σ_s be the mapping from $\Lambda(M)$ into itself, which sends $\omega \in \Lambda^p(M)$ into $s^p\omega$.

The exterior algebra $\Lambda(E)$ also has a more complex grading. Namely for $y \in E$ and $p, q \in N$ with $p + q \leqslant n$, we define $\Lambda^{p,q}(E)$ to be the subspace of $\Lambda^{p+q}(E)$ spanned by

$$dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dx^{\pm,j_1} \wedge \cdots \wedge dx^{\pm,j_q}. \tag{2.17}$$

Note that $\Lambda^{p,q}(E)$ depends on the connection $\overline{\nabla}$.

For $s > 0$, let τ_s be the mapping from $\Lambda(E)$ into itself which sends $\omega \in \Lambda^{p,q}(E)$ into $s^q\omega$.

Recall that φ has been defined in §2(a). Of course if $\varphi(y) \neq 0$, we can identify $\Lambda_y(M)$ and $\Lambda_y(E)$.

Definition 2.3. For $t > 0$, $\alpha > 0$ the operators $d^{t,\alpha}, \delta^{t,\alpha}, \bar{\square}^{t,\alpha}$ are defined by

$$\begin{aligned} & d^{t,\alpha} = e^{-\alpha h/t}\big(\varphi\tau_{\sqrt{t}} + (1-\varphi)\sigma_{\sqrt{t}}\big) d \big(\varphi\tau_{\sqrt{t}} + (1-\varphi)\sigma_{\sqrt{t}}\big)^{-1} e^{\alpha h/t}, \\ (2.18)\quad & \delta^{t,\alpha} = e^{\alpha h/t}\big(\varphi\tau_{\sqrt{t}} + (1-\varphi)\sigma_{\sqrt{t}}\big)^{-1} \delta \big(\varphi_{\sqrt{t}} + (1-\varphi)\sigma_{\sqrt{t}}\big) e^{-\alpha h/t}, \\ & \bar{\square}^{t,\alpha} = d^{t,\alpha}\delta^{t,\alpha} + \delta^{t,\alpha} d^{t,\alpha}. \end{aligned}$$

To better understand (2.18) we now analyse the two extreme cases where $\varphi = 0$ and $\varphi = 1$.

First on $(\varphi = 0)$ (i.e. out of a neighborhood containing M_1), it is obvious that (using the notations of §1)

$$(2.19)\qquad \bar{\square}^{t,\alpha} = t\square^{\alpha h/t}.$$

When $\varphi = 0$, we are back to the situation considered in §1.

We now calculate the operators $d^{t,\alpha}$, $\delta^{t,\alpha}$, $\bar{\square}^{t,\alpha}$ on $(\varphi = 1)$. More generally, by setting $\varphi = 1$ in (2.18), we will calculate $d^{t,\alpha}$, $\delta^{t,\alpha}$ acting on the C^∞ sections of $\Lambda(E)$.

Theorem 2.4. *The action of $d^{t,\alpha}$, $\delta^{t,\alpha}$ on the C^∞ sections of $\Lambda(E)$ is given by*

$$\begin{aligned} d^{t,\alpha} &= dx^j \wedge \overline{\nabla}_{e_j} + \sqrt{t}\, dx^{\pm,j} \wedge \overline{\nabla}_{f_j^\pm} \\ &\quad + \frac{1}{2\sqrt{t}} dx^k \wedge dx^l \wedge i_{[L(e_k,e_l)y]^V} \\ (2.20)\qquad &\quad + \frac{\alpha}{\sqrt{t}}[y^+\wedge - y^-\wedge], \\ \delta^{t,\alpha} &= -i_{e_j}\overline{\nabla}_{e_j} - \sqrt{t}\, i_{f_j^\pm}\overline{\nabla}_{f_j^\pm} - \frac{1}{2\sqrt{t}}[L(e_k,e_l)y]^V \wedge i_{e_k} i_{e_l} \\ &\quad + \frac{\alpha}{\sqrt{t}}\big[i_{y^+} - i_{y^-}\big]. \end{aligned}$$

Proof. Note that since ∇ is a Euclidean connection on E^+, E^-, then $\overline{\nabla}_{e_j}|y^+|^2 = \overline{\nabla}_{e_j}|y^-|^2 = 0$. (2.20) is then an obvious extension of Proposition 2.2. q.e.d.

We will now give some indications on the structure of $\bar{\square}^{t,\alpha}$.

We briefly show how to compute $\bar{\square}^{t,\alpha}$ on $(\varphi = 1)$, and more generally, we compute $\bar{\square}^{t,\alpha}$ on E.

Definition 2.5. Δ^H, Δ^V denote the second order differential operator on E

$$(2.21)\qquad \Delta^H = \sum_1^{n_1} \overline{\nabla}_{e_i}^2, \qquad \Delta^V = \sum \overline{\nabla}_{f_j^\pm}^2,$$

where Δ^H is the horizontal Laplacian. If $e_1(x) \cdots e_{n_1}(x)$ is an orthogonal frame at $x \in M_1$ depending smoothly on x which varies in a small ball, then by definition

$$\Delta^H = \sum_1^{n_1} \overline{\nabla}_{e_i}\overline{\nabla}_{e_i} - \overline{\nabla}_{\sum_{j=1}^{n_1}\nabla_{e_j}e_j}.$$

Δ^V is the standard vertical Laplacian along the fibers of $E_+ \oplus E_-$.

Recall that the curvature tensor $\overline{L}$ has been computed in Theorem 2.1. We also recall the convention (1.3).

Finally, let H be the operator acting on $\Lambda(E)$ which sends

$$\omega = dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dx^{+,j_1} \wedge \cdots \wedge dx^{+,j_q} \wedge dx^{-,k_1} \wedge \cdots \wedge dx^{-,k_{q'}}$$

into $(q - q')\omega$.

$\overline{\square}_E^{t,\alpha}$ denotes the operator $\overline{\square}^{t,\alpha}$ calculated with $\varphi = 1$ and extended to the whole bundle E.

To obtain the adequate estimates, we are forced to compute explicitly $\overline{\square}_E^{t,\alpha}$. We have

Proposition 2.6. *For $t > 0$, $\overline{\square}_E^{t,\alpha}$ is given by*

$$\begin{aligned}
\overline{\square}_E^{t,\alpha} = {} & -\overline{\nabla}_{e_i}^2 - t\overline{\nabla}_{f_j^\pm}^2 + dx^j \wedge i_{e_k}\Big(\overline{L}(e_k,e_j) - \nabla_{[L(e_k,e_j)y]^V}\Big) \\
& - \frac{1}{2\sqrt{t}}\Big[dx^j \wedge \big[\overline{\nabla}_{e_j}L(e_k,e_l)y\big]^V \wedge i_{e_k}i_{e_l} \\
& \qquad + i_{e_j}dx^k \wedge dx^l \wedge i_{[\overline{\nabla}_{e_j}L(e_k,e_l)y]^V}\Big] \\
& - \frac{1}{\sqrt{t}}\Big[\big(L(e_k,e_j)y\big)^V \wedge i_{e_k}\overline{\nabla}_{e_j} + dx^k \wedge i_{[L(e_j,e_k)y]^V}\overline{\nabla}_{e_j}\Big] \\
& - \frac{1}{2}\Big[dx^{\pm,j} \wedge \big(L^\pm(e_k,e_l)f_j^\pm\big)^V \wedge i_{e_k}i_{e_l} + i_{f_j^\pm}i_{[L^\pm(e_k,e_l)f_j^\pm]^V}dx^k \wedge dx^l\Big] \\
& - \frac{1}{4t}\Big[dx^k \wedge dx^l \wedge i_{[L(e_k,e_l)y]^V}\big[L(e_{k'},e_{l'})y\big]^V \wedge i_{e_{k'}}i_{e_{l'}} \\
& \qquad + \big[L(e_{k'},e_{l'})y\big]^V \wedge i_{e_{k'}}i_{e_{l'}}dx^k \wedge dx^l \wedge i_{(L(e_k,e_l)y)^V}\Big] \\
& + \frac{\alpha^2}{t}\Big(|y^+|^2 + |y^-|^2\Big) + 2\alpha H - \alpha\big(n_1^+ - n_1^-\big).
\end{aligned} \tag{2.22}$$

Proof. The proof of (2.22) follows from lengthy calculations which are left to the reader.

Remark 2. On E, the transformation (2.18) has another very simple interpretation. Namely, for $s > 0$ let r_s be the transformation acting on the sections of $\Lambda(E)$ by

$$\begin{aligned}
& k(x,y)dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dx^{\pm,j_1} \wedge \cdots \wedge dx^{\pm,j_q} \\
& \quad \xrightarrow[r_s]{} k(x,sy)dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dx^{\pm,j_1} \wedge \cdots \wedge dx^{\pm,j_q}.
\end{aligned} \tag{2.23}$$

Then it is trivial to verify that on E

$$
\begin{aligned}
d^{t,\alpha} &= [r_{\sqrt{t}}]^{-1} d^{1,\alpha} r_{\sqrt{t}}, \\
\delta^{t,\alpha} &= [r_{\sqrt{t}}]^{-1} \delta^{1,\alpha} r_{\sqrt{t}}, \\
\bar{\square}_E^{t,\alpha} &= [r_{\sqrt{t}}]^{-1} \square^{1,\alpha} r_{\sqrt{t}}.
\end{aligned} \tag{2.24}
$$

Formulas (2.24) show that $\bar{\square}_E^{t,\alpha}$ has the same spectrum as $\bar{\square}_E^{1,\alpha}$. Moreover if k is an eigenform for $\bar{\square}_E^{1,\alpha}$ with L_2 norm equal to 1, the corresponding eigenform for $\bar{\square}_E^{t,\alpha}$ is

$$\frac{k}{t(n_1^+ + n_1^-)/2}\left(x, \frac{y}{\sqrt{t}}\right).$$

This shows that since, as we shall see in (2.69), for α large enough, k decays faster than $e^{-\delta|y|^2}$ (for $\delta > 0$), as $t \downarrow\downarrow 0$, the eigenforms of $\bar{\square}_E^{t,\alpha}$ localize on M_1.

On E, (2.24) is more explicit than (2.18). However it is difficult to piece (2.24) with an adequate transformation of $d^{1,\alpha}$, $\delta^{1,\alpha}$ on the whole manifold M_1. This is why we chose the description (2.18), which is given in purely local terms.

We now give a few trivial but useful facts on $d^{t,\alpha}$, $\bar{d}^{t,\alpha}$, $\bar{\square}^{t,\alpha}$ calculated in the region $(0 < \varphi < 1)$. Recall that $(0 < \varphi < 1) \subset \mathscr{V}$, so that we are working on the fiber bundle E.

On $(0 < \varphi < 1)$, the action of $\bar{d}^{t,\alpha}$ on C^∞ sections of $\Lambda^{p,q}(E)$ is given by

$$
\begin{aligned}
\bar{d}^{t,\alpha} = {} & \frac{\varphi t^{q/2} + (1-\varphi) t^{(p+q+1)/2}}{\varphi t^{q/2} + (1-\varphi) t^{(p+q)/2}}\, dx^j \wedge \bar{\nabla}_{e_j} + \sqrt{t}\, dx^{\pm,j} \wedge \bar{\nabla}_{f_j^\pm} \\
& + \frac{\varphi t^{(q-1)/2} + (1-\varphi) t^{(p+q+1)/2}}{2\left(\varphi t^{q/2} + (1-\varphi) t^{(p+q)/2}\right)}\, dx^k \wedge dx^l \wedge i_{[L^\pm(e_k,e_l) y^\pm]^V} \\
& - \frac{\left(t^{(q+1)/2} - t^{(p+q+1)/2}\right) d\varphi \wedge}{\varphi t^{q/2} + (1-\varphi) t^{(p+q)/2}} + \frac{\alpha}{\sqrt{t}}(y^+ - y^-)
\end{aligned} \tag{2.25}
$$

essentially because by (2.8), $d\varphi$ is a vertical form. As $t \downarrow\downarrow 0$, the first terms have the same or milder singularities than in (2.20). We now claim that as $t \downarrow\downarrow 0$

$$\left|\frac{\left(t^{(q+1)/2} - t^{(p+q+1)/2}\right) d\varphi}{\varphi t^{q/2} + (1-\varphi) t^{(p+q)/2}}\right| = \left|\frac{\sqrt{t}\,(1 - t^{p/2})\, d\varphi}{\varphi + (1-\varphi) t^{p/2}}\right| \tag{2.26}$$

converges to 0 uniformly. For $\eta > 0$, this is clear in the region $(\varphi \geqslant \eta)$. Because of the assumption we have on f and φ in (2.7), (2.8) we can concentrate on what happens in a neighborhood of the region $|y| = \varepsilon/4$. Because of (2.7), (2.8), in such a neighborhood which is small enough, we have

$$d\varphi = -\frac{2\varphi y^V}{\left[\varepsilon^2/16 - |y|^2\right]^2}, \tag{2.27}$$

and so

$$|d\varphi| \leqslant \exp\left\{-\frac{1}{2\left(\varepsilon^2/16 - |y|^2\right)}\right\}. \tag{2.28}$$

Take $A > 0$, and assume that

$$\left||y|^2 - \frac{\varepsilon^2}{16}\right| \leqslant \frac{A}{|\operatorname{Log} t|}. \tag{2.29}$$

Using (2.28) and (2.29), we find that (2.26) is dominated by

$$2t^{(1-p)/2+1/2A}. \tag{2.30}$$

If A is small enough, (2.30) tends to 0 as $t \downarrow\downarrow 0$. Moreover if $-|y|^2 + \varepsilon^2/16 \geqslant A/|\operatorname{Log} t|$, using (2.27) we find that (2.26) is dominated by

$$Ct^{1/2}|\operatorname{Log} t|^2/A^2$$

which also converges to 0 as $t \downarrow\downarrow 0$.

We have then proved that (2.26) converges to 0 uniformly as $t \downarrow\downarrow 0$.

Of course what really interests us is the operator $\bar{\square}^{t,\alpha}$. The analysis which we have done in (2.25)–(2.30) incorporates the typical features of the region $(0 < \varphi < 1)$.

(e) A probabilistic construction of the heat kernel in E. We now briefly give a probabilistic construction of the heat kernel $e^{-s\bar{\square}_E^{t,\alpha}/2}$ acting of $\Lambda(E)$. Recall that here $\bar{\square}_E^{t,\alpha}$ is defined on the whole E by (2.22).

We first describe the Brownian motion in the Riemannian manifold E, and the heat kernel $e^{-s\bar{\square}_E^{t,\alpha}/2}$ acting on functions.

We use the notations and the results in [6, Chapters 2 and 3].

Let Δ be the Laplace-Beltrami operator on M_1. Take $x_0 \in M_1$ and $y_0 \in E_x$. Let x_s be the Brownian motion on the Riemannian manifold M_1 with $x(0) = x_0$ (for a construction see [6, Chapter II] and the proof of Theorem 1.4). Let τ_s^0 be the continuous process of parallel transportation operators along x_u $(0 \leqslant u \leqslant s)$ from fibers over x_0 into fibers over x_s. Set

$$\tau_0^s = \left[\tau_s^0\right]^{-1}. \tag{2.31}$$

For the precise definitions of τ_s^0 and τ_0^s, see [6, Chapter 2].

$z_s = (z_s^+, z_s^-)$ denotes a Brownian motion with values in the Euclidean space $(E^+ \oplus E^-)_{x_0}$ independent of x, and such that $z_0 = 0$.

We now have

Theorem 2.7. *Set*

$$y_s = \tau_s^0\left(y_0 + \sqrt{t}\, z_s\right). \tag{2.32}$$

Then (x_s, y_s) is exactly the Brownian motion on E starting at (x_0, y_0) when E is endowed with the metric $g_1 \oplus g_E/t$. Moreover if $f \in C_b^\infty(E)$, set

(2.33)

$$(P_s f)(x_0, y_0) = E\left[\exp\left\{-\frac{\alpha^2}{2t}\int_0^s |y|^2\, dv + \frac{\alpha(n_1^+ - n_1^-)s}{2}\right\} f(x_s, y_s)\right].$$

Then

$$P_s f = e^{-s\bar{\Box}_E^{t,\alpha}/2} f. \tag{2.34}$$

Proof. Observe that the restriction of $\bar{\Box}_E^{t,\alpha}$ to the C^∞ sections of $\Lambda^0(E)$ is given by

$$-\bar{\nabla}_{e_j}^2 - t\bar{\nabla}_{f_j^\pm}^2 + \frac{\alpha^2}{t}|y|^2 - \alpha(n_1^+ - n_1^-). \tag{2.35}$$

$-\bar{\Box}_E^{t,0}$ is exactly the Laplace-Beltrami operator of E endowed with the metric $g_1 \oplus g_E/t$.

Using the Feynman-Kac formula, the theorem follows.

We now briefly show how to construct the semigroup $e^{s\bar{\Box}_E^{t,\alpha}/2}$ on the C^∞ sections of $\Lambda(E)$. To do this we need a more explicit construction of x.

By a well-known construction of Malliavin, Eells-Elworthy, we know that the development $\beta.$ of $x.$ in $T_{x_0}M_1$ is a Euclidean Brownian motion in $T_{x_0}M_1$ (see [6, Chapter 2]).

Let $u_0 = (e_1, \cdots, e_{n_1})$ be an orthogonal base of $T_{x_0}M_1$ and $v_0^\pm = (f_1^\pm, \cdots, f_{n_1^\pm}^\pm)$ an orthogonal base of $E_{x_0}^\pm$. By identifying u_0, $v_0^\pm$ with linear isometries from the Euclidean spaces R^{n_1}, $R^{n_1^\pm}$ into $T_{x_0}M_1$, $E_{x_0}^\pm$, set

$$w_s = u_0^{-1}\beta_s, \qquad w_s^\pm = \left[v_0^\pm\right]^{-1} z_s^\pm. \tag{2.36}$$

Then $w.$, $w.^\pm$ are independent standard Brownian motions.

Let P be the probability law of w on $\mathscr{C}(R^+; R^{n_1})$ and $P'^\pm$ the probability law of $w^\pm$ on $\mathscr{C}(R^+; R^{n_1^\pm})$. E denotes the expectation operator with respect to $P \otimes P'^+ \otimes P'^-$.

$e_{i,s}$, $f_{j,s}^\pm$ denote the parallel transports of e_i, $f_j^\pm$ along x_v $(0 \leqslant v \leqslant s)$. For simplicity, the subscript s is omitted in what follows. Similarly $dx^i, dx^{\pm,j}$ denote the corresponding dual bases.

Definition 2.8. U_s denotes the process of linear mappings from $\Lambda_{y_0}(E)$ into itself defined by the stochastic differential equation

$$
\begin{aligned}
dU_s = \frac{U_s}{2}\tau_0^s\Big\{ & dx^j \wedge i_{e_k}\bar{L}(e_j, e_k)\, ds \\
& + \frac{1}{2\sqrt{t}}\Big[dx^j \wedge \big(\nabla_{e_j} L(e_k, e_l) y\big)^V \wedge i_{e_k} i_{e_l} \\
& \qquad + i_{e_j} dx^k \wedge dx^l \wedge i_{(\nabla_{e_j} L(e_k, e_l) y)^V}\Big]\, ds \\
& + \frac{1}{2}\Big[dx^{\pm, j} \wedge \big(L^{\pm}(e_k, e_l) f_j^{\pm}\big)^V \wedge i_{e_k} i_{e_l} \\
& \qquad + i_{f_j^{\pm}} i_{(L^{\pm}(e_k, e_l) f_j^{\pm})^V}\, dx^k \wedge dx^l\Big]\, ds \qquad (2.37)\\
& + \frac{1}{4t}\Big[dx^k \wedge dx^l \wedge i_{(L(e_k, e_l) y)^V} \big(L(e_{k'}, e_{l'}) y\big)^V i_{e_{k'}} i_{e_{l'}} \\
& \qquad + \big(L(e_{k'}, e_{l'}) y\big)^V \wedge i_{e_{k'}} i_{e_{l'}}\, dx^k \wedge dx^l i_{(L(e_k, e_l) y)^V}\Big]\, ds \\
& + \frac{1}{\sqrt{t}}\, dx^j \wedge i_{e_k}\big\langle \tau_0^s L^{\pm}(e_k, e_j) y^{\pm}, \delta z_s^{\pm}\big\rangle \\
& + \frac{1}{\sqrt{t}}\Big[\big(L(e_k, \tau_s^0 u_0 \delta w) y\big)^V \wedge i_{e_k} + dx^k \wedge i_{[L(\tau_s^0 u_0 \delta w, e_k) y]^V}\Big]\Big\} - 2\alpha H\, ds,
\end{aligned}
$$

$$U(0) = I.$$

We now have

Theorem 2.9. *There are constants $C, C' > 0$ such that for any $\alpha, t, s > 0$*

$$|U_s| \leqslant \exp\left\{C(1+\alpha)s + C'\int_0^s \frac{|y_h|^2}{t}\, dh\right\}. \tag{2.38}$$

If $\alpha^2 > 2C'$ and if k is a C^∞ bounded section of $\Lambda(E)$, set

$$(P_s k)(y_0) = E\left[\exp\left\{-\frac{\alpha^2}{2t}\int_0^s |y_h|^2\, dh + \frac{\alpha(n_1^+ - n_1^-)}{2} s\right\} U_s \tau_0^s k(y_s)\right]. \tag{2.39}$$

Then

$$P_s = e^{-s\Box_E^{t,\alpha}/2}. \tag{2.40}$$

Proof. First note that the operators

$$\big(dx^j \wedge i_{e_k}\big)\big\langle L(e_k, e_j) y, e_l\big\rangle,$$
$$\big(L(e_k, e_j) y\big)^V \wedge i_{e_k} + dx^k \wedge i_{[L(e_j, e_k) y]^V}$$

are antisymmetric on $\Lambda(E)$.

Consider first the stochastic differential equation

$$dV = \frac{V}{2\sqrt{t}} d\Big\{ \int_0^s \tau_0^h \Big[dx^j \wedge i_{e_k} \big\langle \tau_0^h L^{\pm}(e_k, e_j) y^{\pm}, \delta z_h^{\pm} \big\rangle$$

$$\tag{2.41} + \big(L(e_k, \tau_h^0 u_0 \delta w) \big)^V \wedge i_{e_k} + dx^k \wedge i_{[L(\tau_h^0 u_0 \delta w, e_k) y]^V} \Big] \Big\},$$

$$V(0) = I,$$

where $\int_0^s \cdots \langle \cdots \rangle \delta z_h^{\pm} \rangle$ is an Itô integral, and d denotes the Stratonovitch differential of the process which follows.

For any $s \geqslant 0$ V_s is an isometry on $\Lambda_{y_0}(E)$.

We now rewrite the part of equation (2.37) which contains δz, δw as an equation in the sense of Stratonovitch. We get [8]

$$dU = \frac{U}{2} \Big\{ d \int_0^s \tau_0^h \Big[\frac{1}{\sqrt{t}} dx^j \wedge i_{e_k} \big\langle \tau_0^h L^{\pm}(e_k, e_j) y^{\pm}, \delta z^{\pm} \big\rangle$$

$$+ \frac{1}{\sqrt{t}} \big(L(e_k, \tau_h^0 u_0 \delta w) y \big)^V \wedge i_{e_k} + \frac{1}{\sqrt{t}} dx^k \wedge i_{[L(\tau_h^0 u_0 \delta w, e_k) y]^V} \Big]$$

$$\tag{2.42} + \frac{U}{2} \tau_0^s \Big[-\frac{1}{4t} dx^{j'} \wedge i_{e_{k'}} \wedge dx^j$$

$$\wedge i_{e_k} \big\langle L^{\pm}(e_{k'}, e_{j'}) y^{\pm}, L^{\pm}(e_k, e_j) y^{\pm} \big\rangle ds$$

$$- \frac{1}{4t} \big(L(e_{k'}, e_l) y \big)^V \wedge i_{e_{k'}} \big(L(e_k, e_l) y \big)^V \wedge i_{e_k} ds \cdots \Big] \Big\}.$$

In (2.42) we only wrote the first terms of the Stratonovitch corrections, which are however easily controllable.

Set

$$\tag{2.43} U' = UV^{-1}.$$

Then U' is itself the solution of a standard differential equation, where all the terms containing δz, δw have disappeared, which is of the type

$$\tag{2.44} \begin{aligned} dU' &= \frac{U'}{2} \Big[V_s \tau_0^s \big(dx^j \wedge i_{e_k} \bar{L}(e_j, e_k) \big) V_s^{-1} + \cdots) \Big] \\ U'(0) &= I. \end{aligned}$$

Since V is an isometry, the introduction of the operator $\operatorname{ad} V_s$ in the right-hand side of (2.44) does not change the size of the various terms.

It immediately follows that

$$\tag{2.45} |U_s'| \leqslant \exp\Big\{ C(1+\alpha)s + C' \int_0^s \frac{|y_v|^2}{2t} dv \Big\}.$$

From (2.43), (2.45), we find that (2.38) holds.

We now prove (2.40). Using [3, IX, Theorem 1.1], we know that

$$
(2.46)\qquad \begin{aligned} \tau_0^s k(x_s, y_s) = k(x_0, y_0) &+ \int_0^s \frac{\tau_0^v}{2}(\Delta^H + t\Delta^V) k\, dv \\ &+ \int_0^s \tau_0^v\left(\nabla_{e_i} k \delta w^i + \nabla_{f_j^{\pm}} k\sqrt{t}\,\delta w^{\pm j}\right). \end{aligned}
$$

By using Itô's formula, and (2.37), we find that if $\overline{U}_s$ is given by

$$
(2.47)\qquad \overline{U}_s = \exp\left\{-\frac{\alpha^2}{2t}\int_0^s |y_v|^2\, dv + \frac{\alpha}{2}(n_1^+ - n_1^-)s\right\} U_s,
$$

then

$$
\overline{U}_s \tau_0^s k(x_s, y_s) = k(x_0, y_0) - \int_0^s \overline{U}_v \tau_0^v \left[\frac{\overline{\square}_E^{t,\alpha} k}{2}\right] dv + M_s,
$$

where M_s is a local martingale with respect to the filtration of $(w, w^{\pm})$. Using (2.38), we find that if $\alpha^2 \geqslant 2C'$, M is a martingale, and that it is feasible to take expectations in (2.47), so that

$$
(2.48)\qquad E\left[\overline{U}_s \tau_0^s k(x_s, y_s)\right] = k(x_0, y_0) - \int_0^s E\left[\overline{U}_v \tau_0^v \frac{\overline{\square}_E^{t,\alpha_k} k}{2}\right] dv.
$$

From (2.48), (2.40) follows immediately.

(f) The Witten complex on the fiber bundle E. For a given $\alpha > 0$, we consider the operators $\bar{d}^\alpha$, $\bar{\delta}^\alpha$ defined by

$$
(2.49)\qquad \begin{aligned} \bar{d}^\alpha &= \exp\left(-\alpha\left(\frac{|y^+|^2}{2} - \frac{|y^-|^2}{2}\right)\right) d \exp\left(\alpha\left(\frac{|y^+|^2}{2} - \frac{|y^-|^2}{2}\right)\right), \\ \bar{\delta}^\alpha &= \exp\left(\alpha\left(\frac{|y^+|^2}{2} - \frac{|y^-|^2}{2}\right)\right) \delta \exp\left(-\alpha\left(\frac{|y^+|^2}{2} - \frac{|y^-|^2}{2}\right)\right); \end{aligned}
$$

$\bar{d}^\alpha$, $\bar{\delta}^\alpha$ are exactly the operators $d^{1,\alpha}$ and $\delta^{1,\alpha}$ calculated in Theorem 2.4.

Similarly set

$$
(2.50)\qquad \overline{\square}_E^\alpha = \bar{d}^\alpha \bar{\delta}^\alpha + \bar{\delta}^\alpha \bar{d}^\alpha;
$$

$\overline{\square}_E^\alpha$ is the operator $\overline{\square}_E^{1,\alpha}$ calculated in Proposition 2.6.

Let Γ be the Hilbert space of the square integrable sections of $\Lambda(E)$ which splits into

$$
(2.51)\qquad \Gamma = \bigoplus_0^{n_1} \Gamma^p,
$$

where Γ^p is the subspace of square integrable p-forms.

Now using a general result on global pseudo-differentials in Hörmander [14, Theorem 3.4] (also see Helffer [11] for the general theory), we know that for $\alpha > 0$ large enough, $\bar{\square}_E^\alpha$ has a discrete spectrum, and that the corresponding eigenspaces in Γ have finite dimension. Note that such a result also follows from a careful study of the heat equation semigroup $e^{-s\bar{\square}_E^\alpha/2}$ which we constructed in Theorem 2.9, in particular by studying its behavior as $|y| \to +\infty$ (see in particular equation (2.69)).

Finally since clearly

$$(\bar{d}^\alpha)^2 = (\bar{\delta}^\alpha)^2 = 0 \tag{2.52}$$

for $\alpha > 0$ large enough, we can develop on Γ the usual Hodge theory on $(\bar{d}^\alpha, \bar{\delta}^\alpha)$.

$\alpha > 0$ is now chosen large enough so that the conditions in Theorem 2.9 are verified, and that $\bar{\square}_E^\alpha$ has a discrete spectrum.

For $0 \leqslant p \leqslant n$, set

$$K_\alpha^p = \left\{k \in \Gamma^p; \bar{\square}_E^\alpha k = 0\right\}, \qquad B_p^{\alpha,1} = \dim K_p^\alpha. \tag{2.53}$$

Using Hodge's theory, we see that $B_p^{\alpha,1}$ is exactly the pth Betti number for the operator $\bar{d}^\alpha$ acting on Γ. More generally, for each M_i, we can define $B_p^{\alpha,i}$.

As we will later see in §2(h), $B_p^{\alpha,1}$ can be very easily computed by studying the cohomology of M_1 twisted by the orientation of E^- and so does not depend on α.

(g) A basic inequality. We now prove basic inequalities which will allow us to compare the Betti numbers B_p of M to the $B_p^{\alpha,i}$.

We have the key result.

Theorem 2.10. *For α large enough, for any p $(1 \leqslant p \leqslant n)$,*

$$\sum_{i=1}^{r} \left[B_p^{\alpha,i} - B_{p-1}^{\alpha,i} + \cdots + (-1)^p B_0^{\alpha,i}\right] \geqslant B_p - B_{p-1} + \cdots + (-1)^p B_0. \tag{2.54}$$

For $p = n$, there is equality in (2.54).

Proof. Let $Q_s^t(x, x')$ be the C^∞ kernel of the operator $e^{-s\bar{\square}^{t,\alpha}/2}$. Let $J_p^t(s, x)$ be the trace of $Q_s^t(x, x)$ acting on $\Lambda_x^p(M)$. The proof of Theorem 1.3 shows that

$$\begin{aligned}\int_M \left[J_p^t(s,x)\,dx - J_{p-1}^t(s,x) + \cdots + (-1)^p J_0(s,x)\right] dx \\ \geqslant B_p - B_{p-1} + \cdots + (-1)^p B_0,\end{aligned} \tag{2.55}$$

with equality when $p = n$.

Let $P_s^1(y, y')$ be the kernel of $e^{-s\bar{\square}_E^\alpha/2}$ on the fiber bundle of E. More generally let $P_s^i(y, y')$ be the corresponding kernel on the fiber bundle E_i constructed over M_i. For $y \in E$ let $L_p^1(s, y)$ be the trace of $P_s^1(y, y)$ acting on

$\Lambda_y^p(E)$. Similarly $L_p^i(s, y)$ denotes the corresponding function calculated over M_i.

The proof will consist in proving that for $\alpha > 0$ large enough, $e^{-s\bar{\square}_E^\alpha/2}$ is trace class, and moreover that

$$\lim_{t \downarrow\downarrow 0} \int_M J_p^t(s, x) = \sum_{i=1}^r \int_{E_i} L_p^i(s, y)\, dy. \tag{2.56}$$

Before proving (2.56), we now show how to derive (2.54) from (2.55)–(2.56). In fact by (2.55), (2.56), we have that for any $s > 0$

$$\begin{aligned}\sum_{i=1}^r \int_{E_i} \left(L_p^i(s, y) - L_{p-1}^i(s, y) + \cdots + (-1)^p L_0^i(s, y)\right) dy \\ \geqslant B_p - B_{p-1} + \cdots + (-1)^p B_0,\end{aligned} \tag{2.57}$$

with equality when $p = n$.

Now standard spectral theory shows that

$$\lim_{s \to +\infty} \int_{E_i} L_j^i(s, y)\, dy = B_j^{\alpha, i} \tag{2.58}$$

so that (2.54) will follow.

Proof of (2.56). We now briefly indicate the principle of the proof, which is closely related to the proof of Theorem 1.4. To simplify the exposition, we will use the notation M_1 for any critical submanifold.

We first claim that for any neighborhood $\mathscr{V}'$ of M_1, as $t \downarrow\downarrow 0$

$$\int_M J_p^t(s, x)\, dx \sim \int_{\mathscr{V}'} J_p^t(s, x)\, dx. \tag{2.59}$$

We briefly indicate the principle of the proof of (2.59), which, if entirely written, is very long.

A. *The region* ($\varphi = 0$). We can first prove that as $t \downarrow\downarrow 0$

$$\int_{(\varphi = 0)} J_p^t(s, x)\, dx \to 0. \tag{2.60}$$

The situation on ($\varphi = 0$) is essentially the same as in (1.24), (1.25) except that a path starting at x with ($\varphi(x) = 0$) may well escape to a region close to M_1, where $\bar{\square}^{t,\alpha}$ has a singularity like $1/\sqrt{t}$ or $1/t$. First note that for α large enough, as shown in (2.30), the singularity of $\bar{\square}^{t,\alpha}$ is killed by

$$\exp\left\{-\frac{\alpha^2}{2t} \int_0^s |dh|^2(x_v)\, dv\right\}.$$

Moreover it can easily be proven that the contribution of these paths which go close to M_1 are dominated by exp $\{-\chi/t\}$, with $\chi > 0$: the argument being clearly related to what is done in (2.61)–(2.66), we refer to (2.61)–(2.66) for more details. Note that in (2.61)–(2.66) we use estimates on the heat kernel $e^{-s\bar{\square}^{\alpha,t}/2}$ acting on functions, which are easy consequences of the Malliavin calculus ([4], [5], [16], [18]).

B. *The region* ($\varphi > 0$). In the region ($\varphi > 0$), the projection $\rho(y)$ of y on M_1 is essentially free to move as a Brownian motion on the Riemannian manifold M_1.

Recall that $\varepsilon > 0$ has been defined in §2(a).

Take $y \in E$ such that $\varphi(y) > 0$. Then $|y| \leqslant \varepsilon/4$. Let $q_s^t(\cdot, \cdot)$ be the heat kernel of the operator $e^{-s\bar{\square}^{t,0}/2}$ acting on functions. Note that because of (2.25), when acting as functions $\bar{\square}^{t,0}$ contains also a 0 order term, which tends to 0 uniformly as $t \downarrow\downarrow 0$ by the arguments given in (2.26)–(2.30). Let R_y^t be the law of the associated Markov diffusion starting at y at time 0; note that since $\bar{\square}^{t,0}$ contains a 0 order term, we also incorporate a multiplicative functional term in the definition of R_y^t. Let $R_y^{s,t}$ be the probability law of the corresponding Brownian bridge with $y_0 = y_s = y$ (for the precise definition, see [6, Chapter II]). Set

$$T_\varepsilon = \inf\{t > 0; |y_t| \geqslant \varepsilon\}. \tag{2.61}$$

We claim that $\chi > 0$ exists such that for any $y \in E$ with $|y| \leqslant \varepsilon/4$

$$q_s^t(y, y) R_y^{s,t}[T_\varepsilon \leqslant s] \leqslant e^{-\chi/t}. \tag{2.62}$$

We first calculate

$$q_s^t(y, y) R_y^{s,t}[T_\varepsilon \leqslant s/2] = E^{R_y^t}\left[q_{s/2}^t(y_s, y) 1_{T_\varepsilon \leqslant s/2}\right]. \tag{2.63}$$

We claim that for N large enough, we have the uniform bound

$$q_{s/2}^t(y', y) \leqslant \frac{C(s)}{t^N}. \tag{2.64}$$

This estimate is an easy consequence of the Malliavin calculus as used by Stroock in [18, Part II]. Note that here, the localization techniques of [18] are not necessary since a global calculus of variations can be done on M (see [6, Chapter 2] for details).

Moreover using Theorem 2.7, and bounding uniformly the contributions of the term where φ appears as in (2.25)–(2.30), it is easy to see that

$$R_y^t(T_\varepsilon \leqslant s/2) \leqslant e^{-C\varepsilon^2/ts}. \tag{2.65}$$

By (2.64), (2.65) it follows that for one $\chi > 0$

$$q_s^t(y, y) R_y^{s,t}(T_\varepsilon \leqslant s/2) \leqslant e^{-\chi/t}. \tag{2.66}$$

(2.62) then follows from (2.66) by using time reversal as in [6, Chapter 3].

Using Theorem 2.9 and (2.62), we have then proved that if $\varphi(y) > 0$, the contribution to $J_p^t(s, y)$ of those paths which reach the region $|y| \geqslant \varepsilon$ is asymptotically zero.

Using (2.62), we now replace $\bar{\square}^{t,\alpha}$ everywhere by $\bar{\square}_E^{t,\alpha}$ (i.e. do $\varphi = 1$) and prove the corresponding estimates as if M was the manifold E itself.

Let $P_s^t(y, y')$ be the kernel of $e^{-s\bar{\square}_E^{t,\alpha}/2}$. Let $J_p'^t(s, y)$ be the trace of $P_s^t(y, y)$ acting on $\Lambda_y^p(E)$. We now prove that for $\delta > 0$.

$$\lim_{t \downarrow\downarrow 0} \int_{|y| \geqslant \delta} J_p'^t(s, y)\, dy = 0. \tag{2.67}$$

Using Theorem 2.7, the probabilistic description of the law $R_y^{s,t}$ of the Brownian bridge in E (associated to the operator $\bar{\square}_E^{t,0}$) starting at y time 0 and ending at y at time s is easy. Set $x = \rho(y)$.

Let $P_{x,x}^s$ be the law of the Brownian bridge x_0 on M_1, with $x_0 = x_s = x$ (for the precise definition, see [6, Chapter II]). Let $m_s(x, y)$ be the heat kernel on M_1, which acts on functions.

For one trajectory of $x_\cdot$, let τ_v^0 $(0 \leqslant v \leqslant s)$ be the parallel transportation operator along x from fibers over x_0 to fibers over x_v. Set $\tau_0^v = [\tau_v^0]^{-1}$.

Let r be a Brownian bridge in E_{x_0} (with $r_0 = r_s = 0$) independent of x. whose law is Q. Set

$$y_h = \tau_h^0\left(y + \frac{h}{s}(\tau_0^s - I)y + \sqrt{t}\, r_h\right), \qquad 0 \leqslant h \leqslant s, \tag{2.68}$$

Then it is trivial to check that under $P_{x,x}^s \otimes Q$, the law of $y_\cdot$ is exactly $R_y^{s,t}$ (note that $y_0 = y_s = y$!).

By using the majoration (2.38), we find that for one $\eta > 0$ (which depends on α)

$$(2.69)\ J_p'^t(s, y) \leqslant m_s(x, x) \int \frac{\exp\left\{-\frac{1}{2t}|(I - \tau_0^s)y|^2 - \frac{\eta^2}{2t}\int_0^s |y_h|^2\, dh\right\} dP_{x,x}^s\, dQ(r)}{(\sqrt{2\pi s t})^{n_1^+ + n_1^-}}.$$

Now by a well-known formula in [15, p. 206], for one given $x_\cdot$ we know that

$$\begin{aligned} &\exp -\frac{|(I - \tau_0^s)y|^2}{2t} \int \exp\left\{-\frac{\eta^2}{2t}\int_0^s |y_h|^2\, dh\right\} dQ(r) \\ &\quad = \sqrt{\frac{\eta s}{\operatorname{sh} \eta s}} \exp\left\{\frac{-\eta s}{2\operatorname{sh}\eta s}\left[\frac{2|y|^2}{t}\operatorname{ch}\eta s - 2\frac{\langle y, \tau_0^s y\rangle}{t}\right]\right\} \\ &\quad \leqslant \exp\left\{-\frac{\eta s}{\operatorname{sh}\eta s}[\operatorname{ch}\eta s - 1]\frac{|y|^2}{t}\right\}. \end{aligned} \tag{2.70}$$

From (2.69), (2.70), we find that (2.67) holds.

We are now essentially back to the situation considered in §1.

Namely, we find that for $t \downarrow\downarrow 0$, if U_s^t is the solution of (2.37) for $y \in E_x$, then

$$\int_M J_p^t(s,x)\,dx \sim \int_{M_1} m_s(x,x)\,dx \int_{\substack{y\in E_x\\ |y|<\delta}} dy$$

$$\cdot (2\pi st)^{-\frac{1}{2}(n_1^+ + n_1^-)} \int \exp\left\{-\frac{|(I-\tau_0^s)y|^2}{2t} + \frac{\alpha}{2}(n_1^+ - n_1^-)s - \frac{\alpha^2}{2t}\int_0^s |y_h|^2\,dh\right\}$$

$$\cdot \mathrm{Tr}_p U_s^t \tau_0^s \, dP_{x,x}^s \, dQ(r), \tag{2.71}$$

(where $\mathrm{Tr}_p U_s^t \tau_0^s$ is the trace of $U_s^t \tau_0^s$ acting on $\Lambda_y^p(E)$).

We now do the change of variables $y = \sqrt{t}y'$ in (2.71). The process $y.$ is then given by

$$y_h = \sqrt{t}\,\tau_h^0\left[y' + \frac{h}{s}(\tau_0^s - I)y' + r_h\right].$$

We then note that all the singularities in the equation of $U_\cdot^t$ disappear. Moreover using (2.38) it easily follows that as $t \downarrow\downarrow 0$, the right-hand side of (2.70) converges to

$$\int_{M_1} m_s(x,x)\,dx \int_{y\in E_x} dy$$

$$\cdot (2\pi s)^{-\frac{1}{2}(n_1^+ + n_1^-)} \int \exp\left\{-\frac{|(I-\tau_0^s)y|^2}{2} + \frac{\alpha}{2}(n_1^+ - n_1^-)s - \frac{\alpha^2}{2}\int_0^s |y_h|^2\,dh\right\}$$

$$\cdot \mathrm{Tr}_p U_s^1 \tau_0^s \, dP_{x,x}^s \, dQ(r), \tag{2.72}$$

where now y, U^1 are calculated with $t = 1$. Using (2.38) and (2.69), we see that (2.72) is $< +\infty$. Now it is clear that (2.72) is also equal to $\int_E L_p^1(s,y)\,dy$. The theorem is proved.

(h) Cohomological interpretation of the Witten complex on E. We will now show the relationship of the Witten complex on E with the cohomology of M.

What follows is essentially a L_2 version of the Thom isomorphism on E^- (see Bott and Tu [10, Chapter 1]).

$\alpha > 0$ is chosen large enough so that the properties listed in §2(f) hold.

To simplify the discussion, we will assume in this section that E^- is orientable. If E^- is not orientable, we will have to twist all the considered objects by the orientation bundle of E^-.

Definition 2.11. For $0 \leqslant p \leqslant n$, $H^p_\alpha(E)$ denotes the pth cohomology group associated with the operator $\bar{d}^\alpha$ acting on Γ.

Recall that if

$$\begin{aligned} Z^p_\alpha(E) &= \{k \in \Gamma^p; \bar{d}^\alpha k = 0\}, \\ B^p_\alpha(E) &= \{k \in \Gamma^p; \exists k' \in \Gamma^{p-1}, k = \bar{d}^\alpha k'\}, \end{aligned} \tag{2.73}$$

(where $\bar{d}^\alpha k$, $\bar{d}^\alpha k'$ are taken in the sense of distributions), then

$$H^p_\alpha(E) = Z^p_\alpha(E)/B^p_\alpha(E). \tag{2.74}$$

The Hodge theory for the operator $\bar{d}^\alpha$ shows that $H^p_\alpha(E)$ is finite dimensional and that

$$B^{1,\alpha}_p = \dim H^p_\alpha(E). \tag{2.75}$$

Let Δ be the set of measurable sections of $\Lambda(M)$ which are square-integrable, Δ^p the set of p-forms in Δ. Set

$$\begin{aligned} &Z^p(M_1) = \{\omega \in \Delta^p; d\omega = 0\}, \\ &B^p(M_1) = \{\omega \in \Delta^p; \exists\omega' \in \Delta^{p-1}, \omega = d\omega'\}, \\ &H^p(M_1) = Z^p(M_1)/B^p(M_1), \\ &B^1_p = \dim H^p(M_1). \end{aligned} \tag{2.76}$$

Then ordinary Hodge theory for the compact manifold M_1 shows that $H^p(M_1)$ is the de Rham pth cohomology of M_1.

By Bott and Tu [10, §6] we know that a C^∞ closed n_1^- form Φ^- on the manifold E^- exists, which has compact support and represents the Thom class of the oriented fiber bundle E^- over M_1.

Definition 2.12. g denotes the linear mapping from Δ into Γ

$$\omega \to g(\omega) = e^{-\alpha(|y^+|^2 - |y^-|^2)/2}\omega \wedge \Phi^-. \tag{2.77}$$

Note that since Φ^- has compact support, if $\omega \in \Delta$, $g(\omega)$ is indeed in Γ.

We now have

Theorem 2.13. *For any $\omega \in \Delta$ such that $d\omega \in \Delta$, we have*

$$\bar{d}^\alpha g\omega = gd\omega. \tag{2.78}$$

g induces an isomorphism from $H^p(M_1)$ into $H^{p+n_1^-}_\alpha(E)$ so that for any p

$$B^1_p = B^{\alpha,1}_{p+n_1^-}. \tag{2.79}$$

Proof. (2.78) is a consequence of $d\Phi^- = 0$. It is then clear that g induces a homomorphism from $H^p(M_1)$ into $H^{p+n_1^-}(E)$. We now prove it is an isomorphism.

The proof is closely related with what is done in [10, Chapter 1]. However the treatment of E^+ is very different.

For any open set O in M_1, Δ_O (resp. Γ_O) denotes the set of square-integrable sections of $\Lambda(O)$ (resp. $\Gamma(E_O)$).

A. *The case of an open set.* Let O be an open set in M_1 which is diffeomorphic to the open ball $B(0,1)$ in R^{n_1} of center 0 and radius 1. The restrictions of the fibers bundles E^+, E^- to O are then trivial. If E_O^+, E_O^- denote these restrictions, then if $R^{n_1^+}$, $R^{n_1^-}$ are endowed with their canonical Euclidean structure and $R^{n_1^-}$ has its canonical orientation.

$$(2.80)\quad E_O^+ \sim O \times R^{n_1^+}, \quad E_O^- \sim O \times R^{n_1^-}, \quad E_O \sim O \times (R^{n_1^+} \oplus R^{n_1^-}),$$

g induces a natural homomorphism

$$H^p(O) \to H_\alpha^{p+n_1^-}(E_O).$$

We will prove it is one-to-one.

Let $b(r)$ be a compactly supported C^∞ function on R whose integral is equal to 1. Set

$$(2.81)\qquad b^* = b(y^{-,1})b(y^{-,2}) \cdots b(y^{-,n_1^-})\, dy^{-,1} \wedge \cdots \wedge dy^{-,n_1^-};$$

b^* is a representative of the Thom class of E_O^-. Moreover [10, Proposition 4.6] shows that a bounded C^∞ n_1^- form ψ on E_O^- exists such that

• ψ is compactly supported in the directions of $R^{n_1^-}$, i.e. the projection of the support of ψ on $R^{n_1^-}$ is bounded.

• On E_O^-, we have $\Phi - b^* = d\psi$. ψ is then in Γ_O. For $\omega \in \Delta_O$, set

$$(2.82)\qquad g_O'(\omega) = e^{-(\alpha/2)(|y^+|^2 - |y^-|^2)}\omega \wedge b^*.$$

Clearly, for $\omega \in \Delta_O^p$ such that $d\omega \in \Delta_O$

$$(2.83)\qquad g(\omega) - g_O'(\omega) = (-1)^p\Big[\bar{d}^\alpha\Big(e^{-(\alpha/2)(|y^+|^2 + |y^-|^2)}\omega \wedge \psi\Big) - e^{-(\alpha/2)(|y^+|^2 - |y^-|^2)}\, d\omega \wedge b^*\Big].$$

If $\omega \in \Delta_O$, $e^{-(\alpha/2)(|y^+|^2 - |y^-|^2)}\omega \wedge \psi \in \Gamma_O$. It is then clear from (2.83) that g and g_O' induce the same homomorphism from $H^p(O)$ into $H_\alpha^{p+n_1}(E_O)$.

We will then prove that g_O' is an isomorphism in cohomology.

A. *The case where* $n_1^- = 1$, $n_2^+ = 0$. We temporarily assume that $n_1^- = 1$, $n_1^+ = 0$. Let F be the linear mapping from Γ_O into Δ_O which is defined as follows:

• if $k = f(x, y^-)\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$, then $F(k) = 0$,

• if $k = k' \wedge dy^{1,-}$, then

$$(2.84)\qquad F(k)(x) = \int_R e^{-\alpha|z|^2/2}k'(x,z)\, dz.$$

If π_*^- is the operator of integration of forms along E^- (see Bott and Tu [10, §6]), then clearly

$$F(k) = \pi_*^- e^{-\alpha|y^-|^2/2} k.$$

Since $\pi_*^- d = d\pi_*^-$ by [10, Proposition 6.14.1], it is trivial to check that if k is C^∞ with a support whose projection on E^- is bounded:

$$F\bar{d}^\alpha k = dFk. \tag{2.85}$$

(2.85) immediately extends to any $k \in \Gamma_O$ such that $\bar{d}^\alpha k \in \Gamma_O$. F induces a homomorphism from $H_\alpha^p(E_O)$ into $H^{p-n_1^-}(O)$.

In this case g'_O is given by

$$\omega \in \Delta_O \to g'_O(\omega) = e^{(\alpha/2)|y^-|^2} \omega \wedge b(y^-)\, dy^- \in \Gamma_O.$$

Since the integral of b is 1, obviously

$$F g'_O(\omega) = \omega \tag{2.86}$$

so that g'_O is injective in cohomology.

We now prove it is onto.

The proof is a L_2 version of the Poincaré Lemma for compactly supported cohomology [10, Proposition 4.6]. For $k \in \Gamma_O$, define: $Kk(x, y^-)$—which depends linearly on k—as follows:

- if k is horizontal (i.e. does not contain $dy^{-,1}$), then $Kk = 0$.
- if $k = k' \wedge dy^{-,1}$, then

$$\begin{aligned} Kk(x, y^-) = e^{\alpha|y^-|^2/2} \Bigg[& \left(\int_{-\infty}^{y^-} e^{-\alpha|z|^2/2} k'(x, z)\, dz \right) \int_{y^-}^{+\infty} b(z)\, dz \\ & - \int_{y^-}^{+\infty} e^{-\alpha|z|^2/2} k'(x, z)\, dz \int_{-\infty}^{y^-} b(z)\, dz \Bigg]. \end{aligned} \tag{2.87}$$

We now prove that $Kk \in \Gamma_O$. When $y^- \to +\infty$, since b has compact support, the first term in the right-hand side of (2.87) is 0. Moreover for $y^- \geqslant 1$

$$\begin{aligned} & e^{\alpha|y^-|^2} \left| \int_{y^-}^{+\infty} e^{-\alpha|z|^2/2} k'(x, z)\, dz \right|^2 \\ & \leqslant e^{\alpha|y^-|^2} \left[\int_{y^-}^{+\infty} e^{-\alpha|z|^2} z\, dz \right] \left[\int_{y^-}^{+\infty} \frac{|k'(x, z)|^2\, dz}{z} \right] \\ & = \frac{1}{2\alpha} \int_{y^-}^{+\infty} \frac{|k'(x, z)|^2\, dz}{z}, \end{aligned} \tag{2.88}$$

and moreover

$$\int_O dx \left[\int_1^{+\infty} dy^- \int_{y^-}^{+\infty} \frac{|k'(x, z)|^2\, dz}{z} \right] \leqslant \int_{O \times R} |k'(x, y^-)|^2\, dx\, dy^-. \tag{2.89}$$

The situation is identical for $y^- \to -\infty$. It is then not difficult to deduce that $Kk \in \Gamma_O$.

We now claim that if $k \in \Gamma_O^p$ is such that $\bar{d}^\alpha k \in \Gamma_O$, then

$$k - g'_O F(k) = (-1)^{p-1}[\bar{d}^\alpha Kk - K\bar{d}^\alpha k]. \tag{2.90}$$

If k is C^∞ and is compactly supported in E^-, this is exactly [10, Proposition 4.6]. The general case is obtained by an easy approximation agreement.

In particular if $k \in \Gamma_O$ is such that $\bar{d}^\alpha k = 0$, by (2.90), $\bar{d}^\alpha Kk \in \Gamma_O$. From (2.90), it is clear that g'_O is onto in cohomology.

B. *The case where* $n_1^+ = 1$, $n_1^- = 0$. We now assume $n_1^+ = 1$, $n_1^- = 0$. We define the linear mapping F' from Γ_O into Δ_O as follows:

• if $k = f(x, y^+)\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$, then

$$F'(k)(x) = \left(\frac{\alpha}{\pi}\right)^{1/2} \int_R e^{-\alpha|y^+|^2/2} k(x, y^+)\, dy^+, \tag{2.91}$$

• if $k = k' \wedge dy^+$, then $F'(k) = 0$.

Obviously if $k \in \Gamma_O$, $F'(k) \in \Delta_O$.

For $y^+ \in R$, let s^{y^+} be the mapping $x \in O \to (x, y^+) \in E_O^+$. Clearly

$$F'(k)(x) = \left(\frac{\alpha}{\pi}\right)^{1/2} \int_R e^{-\alpha|z|^2/2} (s^z)^* k\, dz. \tag{2.92}$$

Since $(s^z)^* d = d(s^z)^*$, it is trivial to check that if $k \in \Gamma_O$ is such that $\bar{d}^\alpha k \in \Gamma_O$, then

$$F'\bar{d}^\alpha k = dF'k, \tag{2.93}$$

and so F' defines a homomorphism from $H_\alpha^p(E_O)$ into $H^p(O)$.

In this case g'_O is given by

$$\omega \in \Delta_O \to g'_O(\omega) = e^{-(\alpha/2)|y^+|^2}\omega \in \Gamma_O. \tag{2.94}$$

Clearly if $\omega \in \Delta_O$

$$Fg'_O\omega = \omega \tag{2.95}$$

so that g'_O is injective in cohomology.

We now prove that g'_O is onto. For $k \in \Gamma_O$, we define $K'k$ (which depends linearly on k) as follows:

• if $k = f(x, y^+)\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$, then $K'k = 0$.

• if $k = k' \wedge dy^+$, then

$$\begin{aligned} K'(x, y^+) = \left(\frac{\alpha}{\pi}\right)^{1/2} e^{-\alpha|y^+|^2/2} \Bigg[&\int_{-\infty}^{y^+} k'(x, z) e^{\alpha|z|^2/2} \left[\int_{-\infty}^{z} e^{-\alpha u^2}\, du\right] dz \\ &- \int_{y^+}^{+\infty} k'(x, z) e^{\alpha|z|^2/2} \left[\int_{z}^{+\infty} e^{-\alpha u^2}\, du\right] dz \Bigg]. \end{aligned} \tag{2.96}$$

We first prove that $K'k \in \Gamma_O$. Obviously for $z > 0$

$$\int_z^{+\infty} e^{-\alpha u^2}\,du \leqslant \frac{e^{-\alpha z^2}}{2\alpha z}. \tag{2.97}$$

It is then trivial to control the second term of the right-hand side of (2.96) as $y^+ \to +\infty$. We now study the first term. We have

$$\begin{aligned} &\int_1^{+\infty} e^{-\alpha|y^+|^2}\left[\int_1^{y^+} |k'(x,z)|e^{\alpha|z|^2/2}\,dz\right]^2 dy^+ \\ &\leqslant \int_1^{+\infty} e^{-\alpha|y^+|^2}(y^+-1)\left[\int_1^{y^+} |k'(x,z)|^2 e^{\alpha|z|^2}\,dz\right] dy^+ \\ &\leqslant \frac{1}{2\alpha}\int_1^{+\infty} |k'(x,y^+)|^2\,dy^+. \end{aligned} \tag{2.98}$$

By (2.98) and a similar analysis as $y^+ \to -\infty$, we find that $K'k \in \Gamma_O$.

We now claim that if $k \in \Gamma_O^p$ is such that $\bar{d}^\alpha k \in \Gamma_O$, then

$$k - g'_O F'(k) = (-1)^{p-1}[\bar{d}^\alpha K'k - K'\bar{d}^\alpha k]. \tag{2.99}$$

When k is C^∞ with compact support, the proof of (2.99) is routine and left to the reader. (2.99) immediately extends under the given conditions on k.

If $k \in \Gamma_O$ is such that $\bar{d}^\alpha k = 0$, then by (2.99), $\bar{d}^\alpha K'k \in \Gamma_O$. By (2.99), it is now obvious that g'_O is onto in cohomology.

C. *The general case*. The general case does not directly follow from cases A and B.

Namely if $n_1^- > 1$, we consider $O \times R^{n_1^+} \times R^{n_1^-}$ as a fiber bundle of fiber R over $O \times R^{n_1^+} \times R^{n_1^- - 1}$, where the fiber carries the coordinate y^{-,n_1^-}. Now these two spaces both carry operators $\bar{d}^\alpha$. If $M = O \times R^{n_1^+}$, an argument strictly identical to what has been done in A shows that the mapping

$$k \in \Gamma_{M \times R^{n_1^- - 1}} \to \eta = e^{\alpha|y^{-,n_1^-}|^2/2}\eta \wedge b\left(y^{-,n_1^-}\right) dy^{-,n_1^-} \in \Gamma_{M \times R^{n_1^-}} \tag{2.100}$$

induces an isomorphism from $H_\alpha^p(M \times R^{n_1^- - 1})$ into $H_\alpha^{p+1}(M \times R^{n_1^-})$.

A similar argument applies to $R^{n_1^+}$. By recursion on n_1^-, n_1^+, it is then clear that g'_O is an isomorphism in cohomology.

The corresponding result on g is proved.

D. *The Mayer-Vietoris argument*

We will now use the Mayer-Vietoris argument as in [10, §§5, 6]. By [10, Theorem 5.1], M_1 can be covered by a finite family of open sets $O_1, \cdots, O_q$ such that all the intersections $O_{i_1} \cap \cdots \cap O_{i_p}$ are either empty, or diffeomorphic to $B(0,1)$.

For any open set O in M_1, let Δ'_O be defined by

$$\Delta'_O = \{\omega \in \Delta_O;\, d\omega \in \Delta_O\}.$$

Similarly Γ'_O is defined by

$$\Gamma'_O = \left\{ k \in \Gamma_O; \bar{d}^{\alpha} k \in \Gamma_O \right\}.$$

For two open sets O, O', as in [10, §2] we have the Mayer-Vietoris exact sequence

$$0 \to \Delta'_{O \cup O'} \to \Delta'_O \oplus \Delta'_{O'} \to \Delta'_{O \cap O'} \to O,$$
$$0 \to \Gamma'_{O \cup O'} \to \Gamma'_O \oplus \Gamma'_{O'} \to \Gamma'_{O \cap O'} \to O.$$

Note that in [10], C^{∞} forms are used, but the proof of the exactness of (3.1) is strictly identical to [10, §2].

From (c), we find that if O, O' are taken among $O_1, \cdots, O_q$, we have the long exact sequences with commutative diagrams:

$$\begin{array}{ccccccccc}
\to H^p(O \cup O') & \longrightarrow & H^p(O) \oplus H^p(O') & \longrightarrow & H^p(O \cap O') & \longrightarrow & H^{p+1}(O \cup O') & \to \\
\downarrow g & & \downarrow g & & \downarrow g & & \downarrow g & \\
\to H_{\alpha}^{p+n_1^-}(E_{O \cup O'}) & \to & H_{\alpha}^{p+n_1^-}(E_O) \oplus H_{\alpha}^{p+n_1^-}(E_{O'}) & \to & H_{\alpha}^{p+n_1^-}(E_{O \cap O'}) & \to & H_{\alpha}^{p+n_1^-+1}(E_{O \cup O'}) & \to
\end{array}$$

Using the results obtained in part 1 of the proof, and the Five lemma [10, §5] and proceeding as in [10, lemma 5.6], the theorem is now obvious.

Remark 3. If $E^+ = 0$, we could also prove by the same sort of argument that F (defined in (2.84)) is an isomorphism from $H^{p+n_1^-}(E^-)$ into $H^p(M_1)$. This would make the proof very similar to [10, §6], since F is a L_2 version of integration along the fiber (and is intrinsically defined). When E^+ is nonzero, the situation is different, essentially because we cannot use the usual restriction mapping s_* as in [10, §4] since we work with a L_2 cohomology. Moreover F' (defined in (2.92)) is not intrinsic. However by using a priori the fact that Φ^- exists, we have overcome these difficulties. Also note that K and K' (given in (2.87) and (2.96)) have been calculated from Green's functions of the harmonic oscillator.

Finally note that what we have done is essentially to prove that by piecing together various harmonic oscillators in the fiber bundles E^+, E^- over M_1, we have produced an adequate model for the cohomology of $E^+ \oplus E^-$.

(i) The degenerate Morse inequalities. For $0 \leqslant p \leqslant n_1$, let $B_p^{1,-}$ be the dimension of the pth cohomology of M_1 twisted by the orientation of E^-.

More generally, let $B_p^{i,-}$ be the corresponding number calculated for the manifold M_i. We have

Theorem 2.14. *For any p with $1 \leqslant p \leqslant n$, the following inequality holds*:

$$\sum_{i=1}^{r} \left[B_{p-n_i^-}^{i,-} - B_{p-n_i^- -1}^{i,-} + \cdots + (-1)^{p-n_i^-} B_0^{i,-} \right] \geqslant B_p - B_{p-1} + \cdots + (-1)^p B_0. \tag{2.101}$$

For $p = n$, equality holds in (2.101).

Proof. In the case where the negative bundles over $M_1 \cdots M_r$ are orientable, this is a consequence of Theorems 2.10 and 2.13. In the nonorientable case, any easy modification of the proof of Theorem 2.13 shows that $B_p^{\alpha,i} = B_{p-n_i^-}^{i,-}$ so that (2.101) still holds. q.e.d.

Let $P(t)$ be the Poincaré polynomial of M for the standard cohomology of M, $P_1^-(t)$ the Poincaré polynomial of M_1 for the cohomology of M_1 twisted by the orientation bundle of E^- and $P_i^-(t)$ the corresponding polynomial for M_i.

The generalized Morse inequalities of Bott [9] assert that a polynomial $Q(t)$ given by $Q(t) = Q_0 + Q_1 t + \cdots$ with $Q_0 \geqslant 0, \cdots Q_k \geqslant 0, \cdots$ exists, such that

$$\sum_1^r t^{n_i^-} P_i^-(t) - P(t) = Q(t)(1+t). \tag{2.102}$$

The reader will check that (2.101) and (2.102) are equivalent.

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