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A MODEL FOR CYCLIC HOMOLOGY AND ALGEBRAIC K-THEORY OF 1-CONNECTED TOPOLOGICAL SPACES

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It was proven in [2] and [5] that the cyclic homology of a connected topological space with coefficients in a field of characteristic zero[1] k, $\widetilde{HC}_*(X; k)$, is isomorphic to $H_*(ES^1 \times_{S^1} X^{S^1}; k)$; it was also proven in [1] that the reduced algebraic K-theory of the 1-connected topological space X, tensored by k, $\tilde{K}_{*+1}(X) \otimes k$, is isomorphic to the reduced cyclic homology $HC_*(X; k)$. In this paper we describe a Sullivan minimal model of $ES^1 \times_{S^1} X^{S^1}$ in terms of a Sullivan minimal model of X (Theorem A). Our result completes the results of [10] which provides a Sullivan minimal model of X^{S^1} by giving a description of the Λ-minimal extension (in the sense of Halperin) of the fibration $X^{S^1} \to ES^1 \times_{S^1} X^{S^1} \to BS^1$, where X is 1-connected.

The model is effective enough to permit new calculations of the algebraic K-theory of X (tensored by k), where $X = X_1 \times X_2$ with X_2 a "rational" H-space or co-H-space and X_1 a product of complex or quaternionic projective spaces (or even more general) (see Theorem B and its corollaries). The model is also effective enough to contradict a conjecture of [4, p. 376]. T. Goodwillie has also calculated $K_*(CP^n) \otimes k$ by a different method.

The model is explicit enough to deal with the more subtle structures of cyclic homology (or algebraic K-theory) providing a fast (alternative) proof of a result of T. Goodwillie (Corollary 4).

This paper is organized as follows: In §1, we review the basic definitions and state the results; in §2, we give the proof of Theorem A; and, in §3, we give the proofs of the remaining results. We thank S. Halperin for useful conversations he had with the first named author about Theorem A. The present proof was influenced by this discussion.

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[1]Actually in any commutative ring.

1.

In this paper, all spaces are ANR's and all differential graded algebras are commutative differential graded algebras with differential of degree $+1$ over the field of characteristic zero k.

Let X^{S^1} denote the "free loop space" $X^{S^1} = \{\alpha\colon S^1 \to X/\alpha \text{ continuous}\}$ equipped with the compact open topology and let $\mu_X\colon S^1 \times X^{S^1} \to X^{S^1}$, $\mu(\theta^{-1}, \alpha) = \alpha \circ T_\theta$, where T_θ denotes the translation $T_\theta\colon S^1 \to S^1$ corresponding to $\theta \in S^1$. μ_X is a continuous S^1-action whose fixed point set consists of the constant maps. We clearly have the following commutative diagram whose horizontal lines are fibrations:

$$\begin{array}{ccccc} X^{S^1} & \longrightarrow & ES^1 \times_{S^1} X^{S^1} & \longrightarrow & BS^1 \\ \uparrow & & \uparrow & & \| \\ X & \longrightarrow & BS^1 \times X & \longrightarrow & BS^1 \end{array} \tag{1.1}$$

The two fibrations are the bundles over BS^1 associated to the action μ_X, respectively, to the trivial S^1-action on X; the entire diagram is natural in X. We remember from [2] that Hochschild homology (cohomology) of X, $HH_*(X; k)$ ($HH^*(X; k)$), respectively cyclic homology (cohomology) of X, $HC_*(X; k)$ ($HC^*(X; k)$) identifies to $H_*(X^{S^1}; k)$ ($H^*(X^{S^1}; k)$), respectively $H_*(ES^1 \times_{S^1} X^{S^1}; k)$ ($H^*(ES^1 \times_{S^1} X^{S^1}; k)$) and Connes' exact sequence which connects the Hochschild and cyclic homology (1.2) and cohomology (1.2)′ to the Gysin sequence of the fibration (1.1).

$$\to HH_*(X;k) \xrightarrow{I} HC_*(X;k) \xrightarrow{S} HC_{*-2}(X;k) \xrightarrow{B} HH_{*-1}(X;k), \tag{1.2}$$

$$\leftarrow HH^*(X;k) \xleftarrow{I} HC^*(X;k) \xleftarrow{S} HC^{*-2}(X;k) \xleftarrow{B} HH^{*-1}(X;k). \tag{1.2$'$}$$

From now on, our discussion will refer only to cohomology. By duality the corresponding statements for homology remain true.

Let $k[\alpha]$ denote the graded free commutative algebra generated by α, $\deg \alpha = 2$; when regarded as a k-CDGA with differential zero, it will be denoted by $(\Lambda\alpha, 0)$. $HC^*(\mathrm{pt}; k) = k[\alpha]$ and therefore $HC^*(X; k)$ is a $k[\alpha]$ module with $\nabla\colon k[\alpha] \otimes HC^*(X;k) \to HC^*(X;k)$ given by $\nabla(\alpha^p \otimes x) = \mathcal{S}^p x$. More general, to give a $k[\alpha]$ module structure $\nabla\colon k[\alpha] \otimes P^* \to P^*$ on the k-graded vector space P^* is equivalent to giving a degree $+2$ k-linear map $S\colon P^* \to P^{*+2}$. The relation between ∇ and S is given by the formula $\nabla(\alpha^p \otimes x) = S^p(x)$.

Definition 1.1. The $k[\alpha]$ module (P^*, S) is called:

(a) free, if S is injective,

(b) trivial, if S is zero, and

(c) quasifree if it is a direct sum of a free and a trivial module.

Each free $k[\alpha]$ module P^* can be written as $k[\alpha] \otimes_k N^*$, N^* a graded vector space and $S(\alpha^p \otimes n) = \alpha^{p+1} \otimes n$; in this case, N^* is called a *base*. Therefore a quasifree $k[\alpha]$ module can be written as $k[\alpha] \otimes N^* + W^*$ with S acting trivially on W^*.

A minimal k-CDGA is a free commutative differential graded algebra (A, d) so that $A = \Lambda Z$, $Z = \oplus_{i \geqslant 1} Z^i$ a graded vector space so that $dz \in \Lambda(Z^{<p})$ for any $z \in Z^p$ with $Z^{<p} = \oplus_{i<p} Z^i$.[2] Given a graded vector space $Z = \oplus_{i \geqslant 2} Z^i$ (i.e., $Z^1 = 0$), we define $\overline{Z} = \oplus_{\bar{i} \geqslant 1} \overline{Z}^i$ as $\overline{Z}^i = Z^{i+1}$. Given a 1-connected minimal k-CDGA $(\Lambda Z, d)$, $Z = \oplus_{i \geqslant 2} Z^i$, one defines β: $\Lambda Z \otimes \Lambda \overline{Z} \to \Lambda Z \otimes \Lambda \overline{Z}$ as the unique degree (–1) derivation with the property $\beta(z) = \bar{z}$ (hence $\beta(\bar{z}) = 0$), as well as the k-CDGA's $(\Lambda \overline{Z} \otimes \Lambda Z, \delta)$ and $(\Lambda\alpha \otimes \Lambda Z \otimes \Lambda \overline{Z}, \mathscr{D})$ with $\delta z = dz$, $\delta \bar{z} = -\beta dz$ (hence $\delta\beta + \beta\delta = 0$), $\mathscr{D}\alpha = 0$, $\mathscr{D}u = \delta u + \alpha\beta u$ if $u \in \Lambda Z \otimes \Lambda \overline{Z}$.

Clearly, we have the extension:

$$(1.3) \qquad (\Lambda\alpha, 0) \xrightarrow{\tilde{p}} (\Lambda\alpha \otimes \Lambda Z \otimes \Lambda \overline{Z}, \mathscr{D}) \xrightarrow{\tilde{i}} (\Lambda Z \otimes \Lambda \overline{Z}, \delta)$$

with $\tilde{p}(\alpha) = \alpha \otimes 1 \otimes 1$, $\tilde{i}(\alpha) = 0$, $\tilde{i}(z) = z$, $\tilde{i}(\bar{z}) = \bar{z}$.

Sullivan theory allows us to associate with each simple space[3] X a unique (up to isomorphism) minimal k-CDGA $(\Lambda Z, d)$ and to each continuous map, f: $X \to X^1$, a morphism of k-CDGA's f^*: $(\Lambda Z^1, d^1) \to (\Lambda Z, d)$ which describes the k-homotopy theory of X and f. $(\Lambda Z, d)$ is called a Sullivan minimal model of X and f^* a minimal model of f. It has been proved in [10] that if X is 1-connected and $(\Lambda Z, d)$ is the Sullivan minimal model of X, then $(\Lambda Z \otimes \Lambda \overline{Z}, \delta)$ is the Sullivan model of X^{S^1}. The following theorem completes this result.

Theorem A. *If X is a* 1*-connected space with* $\dim \pi_i(X) \otimes k < \infty$ *for any i and $(\Lambda Z, d)$ the Sullivan minimal model of X, then in the extension* (1.3), *$(\Lambda\alpha \otimes \Lambda Z \otimes \Lambda \overline{Z}, \mathscr{D})$ is the Sullivan minimal model of $ES^1 \times_{S^1} X^{S^1}$ and $\tilde{p}$, respectively $\tilde{i}$, are models for p, respectively i, in the fibration*

$$(1.1)' \qquad X^{S^1} \xrightarrow{i} ES^1 \times_{S^1} X^{S^1} \xrightarrow{p} BS^1.$$

Corollary 1. *With the same hypothesis,*

(1) $HH^*(X; k) = H^*(\Lambda Z \otimes \Lambda \overline{Z}, \delta)$,

(2) $HC^*(X; k) = H^*(\Lambda\alpha \otimes \Lambda Z \otimes \Lambda \overline{Z}, \mathscr{D})$,

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[2] Actually the usual definition is slightly more general; a k-CDGA which verifies the above requirements is, however, 1-minimal in the sense of [9] or [8].

[3] Simple means the fundamental group is abelian and acts trivially on $H_i(X; k)$.

(3) *S is induced by multiplication with α, and*

(4) $\operatorname{Hom}(\tilde{K}_{*+1}(X); k) + H^*(BS^1; k) = H^*(\Lambda\alpha \otimes \Lambda Z \otimes \Lambda\bar{Z}, \mathscr{D})$.

Theorem B. *Let X be a 1-connected space such that $H^*(X; k)$ is a truncated algebra in one generator. (For instance, $X = S^n$, or CP^n, HP^n, CaP^2.) Then*

(1) *$HC^*(X; k)$ is quasifree and $HC^*(X; k) = k[\alpha] + W^*$ with W^* the k-graded vector spaces whose Poincaré series[4] is given by:*

(a) *If $H^*(X; k) = \Lambda u$ and $\deg u = 2p + 1$, then $P_{W^*}(t) = t^{2p}/(1 - t^{2p})$.*

(b) *If $H^*(X; k) = \Lambda u/u^{n+1}$ and $\deg u = 2p$, $n \geqslant 1$, then*

$$P_{W^*}(t) = \frac{t^{2p-1}(1 - t^{2pn})}{(1 - t^{2p})(1 - t^{2(pn+p-1)})}.$$

(2) *$HH^*(X; k)$ is the k-graded vector space whose Poincaré series is given by:*

(a) *If $H^*(X; k) = \Lambda u$ with $\deg u = 2p + 1$, then*

$$P_{HH_*}(t) = \frac{1 + t^{2p+1}}{1 - t^{2p}}.$$

(b) *If $H^*(X; k) = \Lambda u/(u^{n+1})$ and $\deg u = 2p$, then*

$$P_{HH_*}(t) = \frac{t^{2p-1}(1 - t^{2pn})}{(1 - t^{2p})(1 - t^{2(pn+p-1)})} + \frac{1}{1 - t^{2p}} - \frac{t^{2p(n+1)}(1 - t^{2(p-1)})}{(1 - t^{2p})(1 - t^{2(pn+p-1)})}.$$

The same formulae remain true for $HC_*(\ )$ and $HH_*(\ )$. The arguments of [2] permit us to calculate HC_* and HH_* (respectively HC^* and HH^*) for $X = X_1 \times X_2 \times X_3$ with X_1 an H-space, X_2 a product of co-H-spaces, and X_3 a product of spaces as in Theorem B.

The following is an equivariant version of the main result of [10].

Corollary 2. *Let X be a 1-connected space such that $\dim H^*(X; k) < \infty$ and $\tilde{H}^*(X; k) \neq 0$. The following conditions are equivalent:*

(i) *The cohomology algebra $H^*(X; k)$ cannot be generated by one element*

(ii) *The sequence $\dim HC_i(X; k)$ is not bounded*

(iii) *The sequence $\dim HH_i(X; k)$ is not bounded.*

Corollary 3. *If X is a 1-connected space, then $H_{*+1}(X; k)$ is a direct summand in $HC_*(X; k)$.*

Corollary 4. *If $PHC^*(X; k) = \varinjlim_S HC^{*+2n}(X; k)$ and X is 1-connected, then $PHC^*(X; k) = PHC^*(\mathrm{pt}; k)$.*

Corollary 4 is a particular case of a result of T. Goodwillie [5], which claims that $PHC^*(X; k) = PHC^*(k[\pi_1(x)])$. As the reader will see in the proof of

[4] For a graded k-vector space W^* the Poincaré series $P_{W^*}(t) = \sum_i \dim W^i t^i$.

Corollary 4, Theorem A provides an immediate verification of Goodwillie's result when X is 1-connected. With similar arguments, Goodwillie's result can be concluded in full generality (but they will not be given here).

2.

Proof of Theorem A. Suppose X is a 1-connected space with minimal model $(\Lambda Z, d)$ and $\dim Z^i < \infty$ for each i. If μ: $S^1 \times X^{S^1} \to X^{S^1}$ is an S^1-action, the associated fibration (1.1)$'$ has by [8] a model, the extension

$$(1.3)' \qquad (\Lambda\alpha, 0) \xrightarrow{\tilde{p}_\mu} \left(\Lambda\alpha \otimes \Lambda Z \otimes \Lambda \overline{Z}, \Delta_\mu\right) \xrightarrow{\tilde{\iota}} \left(\Lambda Z \otimes \Lambda \overline{Z}, \delta\right)$$

with $\Delta_\mu(\alpha) = 0$ and $\Delta_\mu(\omega) = \delta\omega + \Sigma_{i \geqslant 1}\alpha^i\theta_i(\omega)$ for $\omega \in \Lambda Z \otimes \Lambda\overline{Z}$; each term represents a minimal model for the corresponding term in (1.1)$'$.

Let F_μ: $\Lambda\alpha \otimes \Lambda Z \otimes \Lambda\overline{Z} \to \Lambda\alpha \otimes \Lambda Z \otimes \Lambda\overline{Z}$ be the unique morphism of graded algebras defined by $F_\mu(\alpha) = \alpha$, $F_\mu(z) = z$ and $F_\mu(\bar{z}) = \Sigma_{i \geqslant 1}\alpha^{i-1}\theta_i(z)$, $z \in Z$ and $\bar{z} \in \overline{Z}$.

Proposition 2.1. *If F_μ is an isomorphism of graded algebras, then* (1.3)$'$ *is isomorphic to* (1.3).

Proof. If $D_\mu = F_\mu^{-1}\Delta_\mu F_\mu$, then $\mathscr{F}_\mu$: $(\Lambda\alpha \otimes \Lambda Z \otimes \Lambda\overline{Z}, \Delta_\mu) \to (\Lambda\alpha \otimes \Lambda Z \otimes \Lambda\overline{Z}, D_\mu)$ is an isomorphism of differential graded algebras so that $F_\mu\tilde{p} = \tilde{p}$. But $D_\mu = \mathscr{D}$. Indeed

$$D_\mu(\alpha) = \left(F_\mu^{-1}\Delta_\mu F_\mu\right)(\alpha) = F_\mu^{-1}\Delta_\mu(\alpha) = 0,$$

$$D_\mu(z) = F_\mu^{-1}\Delta_\mu F_\mu(z) = F_\mu^{-1}\Delta_\mu(z) = F_\mu^{-1}\left(\delta z + \alpha F_\mu(\bar{z})\right) = \delta z + \alpha\bar{z}.$$

It remains to verify that $D_\mu\bar{z} = -\beta dz$ or equivalently that $\alpha D_\mu\bar{z} + \alpha\beta dz = 0$. Since we have already verified that $\mathscr{D}$ and D_μ agree on ΛZ we have

$$0 = D_\mu^2 z = D_\mu\left(dz + \alpha\beta(z)\right) = ddz + \alpha\beta dz + D_\mu\alpha\beta(z) = \alpha\beta dz + \alpha D_\mu\beta z.$$

q.e.d.

Clearly Proposition 2.1 and the following Proposition 2.2 imply Theorem A.

Proposition 2.2. *With the hypotheses in Theorem* A, *if* μ: $S^1 \times X^{S^1} \to X^{S^1}$ *is the action* μ_X *described in* §1, *then* F_μ *is an isomorphism.*

Proof. It suffices to show that

$$(2.1) \qquad \bar{z} = \lambda\theta_1(z) + z' + B(z)$$

with λ a nonzero number independent of z, $z' \in \Lambda Z$ and $B(z) \in$ "Decomposable part" of $\Lambda Z \otimes \Lambda\overline{Z} = \mathrm{Dec}(\Lambda Z \otimes \Lambda\overline{Z})$. If (2.1) is established, then it follows by induction on the degree of z that $z, \alpha, \bar{z} \in \mathrm{Im}\, F_\mu$; hence F_μ is surjective, and because $\dim(\Lambda\alpha \otimes \Lambda Z \otimes \Lambda\overline{Z})^r < \infty$, F_μ is an isomorphism.

In order to verify (2.1) we observe that if i: $X^{S^1} \to ES^1 \times_{S^1} X^{S^1}$ is "the inclusion of the fibre" for the fibration $X^{S^1} \to ES^1 \times_{S^1} X^{S^1} \to BS^1$ associated with the action μ and p_2: $S^1 \times X^{S^1} \to X^{S^1}$ is the second factor projection, then ip_2 and $i\mu$ are homotopic. If Λt is the minimal model of S^1 (hence $\deg t = 1$), $\tilde{\varepsilon}$: $(\Lambda Z \otimes \Lambda \bar{Z}, \delta) \to (\Lambda Z \otimes \Lambda \bar{Z}, \delta) \otimes \Lambda t$ is the canonical inclusion (which is the model of p_2) and $\tilde{\mu}$: $(\Lambda Z \otimes \Lambda \bar{Z}, \delta) \to (\Lambda Z \otimes \Lambda \bar{Z}, \delta) \otimes \Lambda t$ is a model of μ: $S^1 \times X^{S^1} \to X^{S^1}$, then $\tilde{\varepsilon}\tilde{\imath}$ and $\tilde{\mu}\tilde{\imath}$ are homotopic. If one uses the definition of homotopy as given in [8, Chapter 5], we have a morphism of differential graded algebras Φ: $(C^{\mathrm{I}}, \underline{\Delta}) \to (\Lambda Z \otimes \Lambda \bar{Z}, \delta) \otimes \Lambda t$, where $C = \Lambda Y$, $Y = \{\alpha\} + Z + \bar{Z}$, which satisfies:

(i) $\Phi | C = \tilde{\varepsilon}\tilde{\imath}$,

(ii) $\Phi(\mathrm{Id} + \Sigma(\underline{\Delta} j + j\underline{\Delta})^n/n!)|_C = \tilde{\mu}\tilde{\imath}$.

We recall from [8] that $C^{\mathrm{I}} = \Lambda Y \otimes \Lambda \check{Y} \otimes \Lambda \check{\check{Y}}$ and $\check{\check{Y}} = Y$ and $(\check{Y})^p = Y^{p+1}$, $\underline{\Delta}|Y = \Delta$, $\underline{\Delta}(\check{y}) = \check{\check{y}}$, $\underline{\Delta}(\check{\check{y}}) = 0$, and j is the unique derivation of degree-1 which extends $j(y) = \check{y}$, $j(\check{\check{y}}) = 0$. Note that (i) implies (a) $\Phi(\alpha) = 0$ and (b) $\Phi | \Lambda Z \otimes \Lambda \bar{Z} = \mathrm{id}$. Note that for dimensional reasons we have (c) $\Phi(\check{\alpha}) = \lambda t$, $\lambda \in Q$, and because $j\underline{\Delta} j | Y = 0$ (see [8]) (ii) implies (d) $\tilde{\mu}\tilde{\imath}(z) = \Phi(z + \underline{\Delta} jz + j\underline{\Delta} z + \Sigma_{n \geqslant 2}(j\underline{\Delta})^n z/n!)$. By expanding Δz as $dz + \Sigma_{i \geqslant 1} \alpha^i \theta_i(z)$ and by using (a), we conclude that $\tilde{\mu}\tilde{\imath}(z) = z + A_1(z) + tB_1(z) + \Phi(jdz) + \lambda t \theta_1(z)$ with $A_1(z), B_1(z) \in \mathrm{Dec}(\Lambda Z \otimes \Lambda \bar{Z})$.

Because

$$dz = \sum c_{i_1 i_2 \cdots i_r} z_{i_1} z_{i_2} \cdots z_{i_r}, \qquad r \geqslant 2,\ c_{i_1 \cdots i_r} \in Q,$$

we know that

$$\Phi(jdz) = \sum c_{i_1 \cdots i_r} \left(\sum_{l=1}^{r} (-1)^{|z_{i_1}| + \cdots |z_{i_{l-1}}|} z_{i_1 \cdots i_{l-1}} \Phi(z_{i_l}) \cdots z_{i_r} \right)$$

belongs to $\mathrm{Dec}\Lambda Z \otimes \Lambda \bar{Z} + t(\mathrm{Dec}\Lambda Z \otimes \Lambda \bar{Z} + \Lambda Z)$; the contribution to $t(\Lambda Z)$ comes only from the monomials with at least one variable of degree 2. Consequently $\tilde{\mu}\tilde{\imath}(z) = z + A(z) + t(\lambda\theta_1(z) + z' + B(z))$ with $A(z), B(z) \in \mathrm{Dec}(\Lambda Z \otimes \Lambda \bar{Z})$, $z' \in \Lambda Z$. Then Lemmas 2.3 and 2.4 below will imply (2.1), completing the proof of Theorem A.

Lemma 2.3. $\tilde{\mu} = \mathrm{Id} + t\beta$.

Lemma 2.4. $\lambda \neq 0$.

Proof of Lemma 2.3. The adjoint map of the action μ: $S^1 \times X^{S^1} \to X^{S^1}$ is μ': $S^1 \times S^1 \times X^{S^1} \to X$, $\mu' = E \circ (m \times \mathrm{id})$ with m: $S^1 \times S^1 \to S^1$ given by multiplication and where E: $S^1 \times X^{S^1} \to X$ is the evaluation map $E(e^{i\theta}, \alpha) = \alpha(e^{i\theta})$. Clearly if $\tilde{\mu}'$: $(\Lambda Z, d) \to \Lambda t \otimes \Lambda t' \otimes (\Lambda Z \otimes \Lambda \bar{Z}, \delta)$ is a model of μ' given by $\tilde{\mu}'(\omega) = a(\omega) + tb(\omega) + t'(c(\omega) + th(\omega))$, $(\deg t = \deg t' = 1)$ with

$a(\omega)$, $b(\omega)$, $c(\omega)$, $h(\omega) \in \Lambda Z \otimes \Lambda \bar{Z}$, then $\tilde{\mu}$: $(\Lambda Z \otimes \Lambda \bar{Z}, \delta) \to \Lambda t \otimes (\Lambda Z \otimes \Lambda \bar{Z}, \delta)$, the model of μ is given by $\tilde{\mu}(z) = a(z) + tb(z)$, $\tilde{\mu}(\bar{z}) = c(z) + th(z)$. Conversely, if $\tilde{\mu}(\omega) = a(\omega) + tb(\omega)$ and $\tilde{\mu}(\beta(\omega)) = c(\omega) + th(\omega)$, then $\tilde{\mu}'(\omega) = a(\omega) + tb(\omega) + t'(c(\omega) + th(\omega))$. By [6] a model for $E, \tilde{E}$: $(\Lambda Z, d) \to \Lambda t \otimes (\Lambda Z \otimes \Lambda \bar{Z}, \delta)$ is given by $\tilde{E}(\omega) = \omega + t\beta(\omega)$ and because a model of $m, \tilde{m}$: $\Lambda\theta \to \Lambda t \otimes \Lambda t'$ is given by $\tilde{m}(\theta) = t \otimes 1 + 1 \otimes t'$ we conclude that $\tilde{\mu}\tilde{i}(\omega) = \omega + t\beta(\omega)$.

Proof of Lemma 2.4. If $\Phi(\check{\alpha}) = 0$, then $\operatorname{Im}\phi \subset \Lambda Z \otimes \Lambda \bar{Z}$ which is incompatible with the homotopy between $\tilde{\varepsilon}\tilde{i}$ and $\tilde{\mu}\tilde{i}$. Hence $\Phi(\check{\alpha}) \neq 0$ so $\lambda \neq 0$.

3.

Proof of Theorem B. Part (2) of this theorem is already contained in Addendum to [10].

(a) If $H^*(X; k) = \Lambda u$ with $|u| = 2p + 1$, then the minimal model of $ES^1 \times_{S^1} X^{S^1}$ is $(\Lambda(\alpha, u, \bar{u}), \mathscr{D})$ with $|\alpha| = 2$, $\alpha = 0$, $|u| = 2p + 1$, $|\bar{u}| = 2p$, $\mathscr{D}\bar{u} = 0$ and $\mathscr{D}u = \alpha\bar{u}$. Therefore,

$$HC^*(X; k) = H^*\left(ES^1 \times_{S^1} X^{S^1}; k\right) = \Lambda(\alpha, \bar{u})/\alpha\bar{u} = k[\alpha] + W^*,$$

where W^* is the vector space spanned by $\{\bar{u}^k,\ k \geqslant 1\}$ and we have $\alpha\bar{u}^k = 0$. Since the map S is the multiplication by α, W^* is a trivial $k[\alpha]$-module.

(b) If $H^*(X; k) = \Lambda(u)/(u^{n+1})$ with $\deg u = 2p$, then the minimal model of X is $(\Lambda(u, v), d)$, with $\deg v = 2p(n+1) - 1$, and $du = 0$, $dv = u^{n+1}$. Then by [10], respectively Theorem A, the minimal model of X^{S^1}, respectively $ES^1 \times_{S^1} X^{S^1}$, is $(\Lambda(u, \bar{u}, v, \bar{v}), \delta)$, respectively $(\Lambda(\alpha, u, \bar{u}, v, \bar{v}), \mathscr{D})$. Note that $\deg \bar{u} = 2p - 1$, $\deg \bar{v} = 2p(n+1) - 2$, $\deg \alpha = 2$ and $\delta u = 0$, $\delta \bar{u} = 0$, $\delta v = u^{n+1}$, $\delta \bar{v} = -(n+1)u^n\bar{u}$, $\mathscr{D}u = d\bar{u}$, $\mathscr{D}\bar{u} = 0$, $\mathscr{D}v = u^{n+1} + \alpha\bar{v}$, $\mathscr{D}\bar{v} = -(n+1)u^n\bar{v}$, $\mathscr{D}\alpha = 0$.

From Proposition 2 of [10], we have

$$HC^*(X; k) = H^*\left(\Lambda(\bar{u}) \otimes \left(\Lambda(\alpha, u, \bar{v})/(u^{n+1} + \alpha\bar{v})\right), D\right)$$

with $D\alpha = 0$, $D\bar{u} = 0$, $Du = \alpha\bar{u}$ and $D\bar{v} = -(n+1)u^n\bar{u}$. The statement (b) is a consequence of the following two lemmas.

Lemma 3.1. $\operatorname{Ker} D$ *is the k-vector space spanned by the following monomials* (*of* $\Lambda\bar{u} \otimes (\Lambda(\alpha, u, \bar{v})/(u^{n+1} + \alpha\bar{v}))$):

(i) $1, (\alpha)^m, (u^a\bar{v}^b\bar{u})$, $m \geqslant 1, 0 \leqslant a \leqslant n-1, b \geqslant 0$.

(ii) $(u^n\bar{v}^b\bar{u}), (\alpha^s u^a\bar{v}^b\bar{u})$ *with* $0 \leqslant a \leqslant n,\ b \geqslant 0,\ s \geqslant 1$.

Lemma 3.2. *The vector space spanned by the monomials of* (ii) *is contained in* Im D *and the monomials of* (i) *are cohomologically independent.*

Proof of Lemma 3.1. Clearly the monomials of the type $(\alpha^s u^a \bar{v}^b \bar{u})$, with $s \geqslant 0$, $a \leqslant n$, $b \geqslant 0$, are all in $\operatorname{Ker} D$. We never consider a power of u greater than $(n+1)$ since we have $u^{n+1} = -\alpha\bar{v}$, and therefore $u^{k(n+1)+r} = u^r(-\alpha\bar{v})^k$ with $k \geqslant 0$, $0 \leqslant r \leqslant n$. Any element of even degree of $\Lambda\bar{u} \otimes (\Lambda(\alpha, u, \bar{v})/(u^{n+1} + \alpha\bar{v}))$ can be written $P = P_0(\alpha) + P_1(u, \bar{v}, \alpha)$ with $P_0(\alpha) \in \Lambda\alpha$, $P_1(\alpha, u, \bar{v}) \in \{u, \bar{v}\} \cdot (\Lambda(\alpha, u, \bar{v})/(u^{n+1} + \alpha\bar{v}))$, the ideal generated by the elements u and $\bar{v}$ in $\Lambda(\alpha, u, \bar{v})/(u^{n+1} + \alpha\bar{v})$. If $\deg P$ is odd, then $P = \bar{u}P_2(u, \bar{v}, \alpha)$ with $P_2(u, \bar{v}, \alpha) \in \{u, \bar{v}\} \cdot (\Lambda(\alpha, u, \bar{v})/(u^{n+1} + \alpha\bar{v}))$. If $P \in \operatorname{Ker} D$ we shall see that $P_1 = 0$ and this achieves the proof of Lemma 3.1. Indeed, if $DP = 0$, we have

$$0 = DP_1 = \frac{\partial P_1}{\partial u}\alpha\bar{u} - (n+1)\frac{\partial P_1}{\partial \bar{v}}u^n\bar{u}$$

and therefore

$$\alpha\frac{\partial P_1}{\partial u} - (n+1)u^n\frac{\partial P_1}{\partial \bar{v}} \in (u^{n+1} + \alpha\bar{v}).$$

Write $P_1 = \sum_{i=0}^{k} Q_i(\alpha, \bar{v})u^i + P_3$ with $Q_i \in \Lambda(\alpha, \bar{v})$, $k \leqslant n$ and $P_3 \in (u^{n+1} + \alpha\bar{v})$. Since

$$\begin{aligned}
&\alpha\frac{\partial P_1}{\partial u} - (n+1)u^n\frac{\partial P_1}{\partial \bar{v}} \\
&\quad = \left(\alpha Q_1 + 2\alpha Q_2 u + \cdots + k\alpha u^{k-1}Q_k\right) - (n+1)u^n\frac{\partial Q_0}{\partial \bar{v}} \\
&\qquad + (n+1)\alpha\bar{v}\frac{\partial Q_1}{\partial \bar{v}} + \cdots + (n+1)u^{k-1}\alpha\bar{v}\frac{\partial Q_k}{\partial \bar{v}} + P_4,
\end{aligned}$$

with $P_4 \in (u^{n+1} + \alpha\bar{v})$, then in $\Lambda(\alpha, \bar{v})$ we have $\partial Q_0/\partial\bar{v} = 0$ and $i\alpha Q_i + (n+1)\alpha\bar{v}\partial Q_i/\partial\bar{v} = 0$ for $1 \leqslant i \leqslant k \leqslant n$. Hence, $Q_0 \in \Lambda\alpha$ and $Q_i = 0$ for $i \geqslant 1$, hence $P_1 = 0$.

Proof of Lemma 3.2. We have $(a + 1 + bn + b)\alpha^s u^a \bar{v}^b \bar{u} = D(\alpha^{s-1}u^{a+1}\bar{v}^b)$ for $s \geqslant 1$, $a \leqslant n-1$, and $-(n+1)(b+1)\alpha^s u^n \bar{v}^b \bar{u} = D(\alpha^s \bar{v}^{b+1}\bar{u})$ for $s \geqslant 0$. This proves the first part. Suppose now we have a relation of the type $\sum \lambda_i u^{a_i}\bar{v}^{b_i}\bar{u} \in \operatorname{Im} D$ in $\Lambda(\bar{u}) \otimes \Lambda(\alpha, u, \bar{v})/(u^{n+1} + \alpha\bar{v})$, with $\lambda_i \in k$, $a_i \leqslant n - 1$; this means that there exists $\mu_j \in k$ and the positive integers d_i, t_i and c_i with $c_i \leqslant n$, so that

$$\sum_i \lambda_i u^{a_i}\bar{v}^{b_i}\bar{u} = D\Big(\sum_j \mu u^{c_j}\bar{v}^{d_j}\alpha^{t_j}\Big)$$

in $\Lambda(\bar{u}) \otimes \Lambda(\alpha, u, \bar{v})/(u^{n+1} + \alpha\bar{v})$. Then we have

$$\begin{aligned}
&\sum_i \lambda_i u^{a_i}\bar{v}^{b_i} - \sum_{c_j \geqslant 1} \mu_j c_j u^{c_j-1}\bar{v}^{d_j}\alpha^{t_j+1}\bar{u} + \sum \mu_j(n+1)d_j u^{n+c_j}\bar{v}^{d_j-1}\alpha^{t_j}\bar{u} \\
&\quad = \sum_i \lambda_i u^{a_i}\bar{v}^{b_i} - \alpha\left(\sum_{c_j \geqslant 1} \mu_j c_j \alpha^{t_j} u^{c_j-1}\bar{v}^{d_j}\bar{u}\right) \\
&\quad\quad -\alpha \sum_{c_j \geqslant 1} \mu_j(n+1)d_j u^{c_j-1}\bar{v}^{d_j}\alpha^{t_j}\bar{u} + \sum_{c_j=0} (n+1)\mu_j d_j u^n\bar{v}^{d_j-1}\alpha^{t_j}\bar{u} \\
&\quad \in (u^{n+1} + \alpha\bar{v}).
\end{aligned}$$

Since $a_i \leqslant n-1$, we have $\lambda_i = 0$ and $\mu_j = 0$ for every i and j; hence, Lemma 3.2 is proved.

Let W^* be the vector space generated by the monomials $(u^a\bar{v}^b\bar{u})$, $0 \leqslant a \leqslant n-1$, $b \geqslant 0$; then we have $\alpha \cdot w = 0$ for any $w \in W^*$ which shows that W^* is a trivial $k[\alpha]$ module. If $P_{W^*}(t) = \Sigma_i t^i \dim W^i$, then clearly

$$\begin{aligned}
P_{W^*} &= t^{2p-1}\frac{1}{1-t^{2(pn+p-1)}}(1 + t^{2p} + \cdots) \\
&= \frac{t^{2p-1}(1-t^{2pn})}{(1-t^{2p})(1-t^{2(pn+p-1)})},
\end{aligned}$$

which finishes the proof of Theorem B.

Note. For CP^n this calculation disproves the Conjecture of Dweyer, Hsiang & Staffeldt [4]; it was also done by T. Goodwillie.

In view of Theorem B, it is natural to ask if the cyclic homology of a 1-connected space whose cohomology is a polynomial algebra $k[u_1 \cdots u_p]$ truncated by a regular sequence of p homogeneous elements is *quasifree*. By S. Halperin [7], the hypotheses above are equivalent to $\dim H^*(X; k) < \infty$, $\dim \pi_*(X) \otimes k < \infty$ and $\Sigma_i (-1)^i \dim \pi_i(X) \otimes k = 0$.

Proof of Corollary 2. (ii) $\rightarrow$ (i) is implied by Theorem B. (i) $\rightarrow$ (ii) can be done using the same arguments as in [10]. The equivalence (i) $\leftrightarrow$ (iii) is done in [10].

Proof of Corollary 3. Let us first observe that if $b = H_0 + \alpha H_1 + \alpha^2 H_2 + \cdots + \alpha^r H_r \in \Lambda\alpha \otimes \Lambda Z \otimes \Lambda\bar{Z}$ with $H_i \in \Lambda Z \otimes \Lambda\bar{Z}$ and $\mathcal{D}b = a \in \Lambda Z \otimes \Lambda\bar{Z}$, then $\delta H_0 = a$, $\delta H_1 + \beta H_0 = 0, \cdots, \delta H_r + \beta H_{r-1} = 0$, $\beta H_r = 0$. Moreover since $(\Lambda Z \otimes \Lambda\bar{Z}, \beta)$ is acyclic and $\delta\beta + \beta\delta = 0$, clearly the above equalities imply that there exists $C_0, \cdots, C_{r-1}, C_r \in \Lambda Z \otimes \Lambda\bar{Z}$ so that $\beta C_r = H_r$, $\beta C_{r-1} = H_{r-1} - \delta C_r, \cdots, \beta C_1 = H_1 - \delta C_2$, $\beta C_0 = H_0 - \delta C_1$, and $\delta H_0 = \delta\beta C_0 = a$. Moreover, if $a = \beta\omega$, then $\beta\omega = \beta\delta(-C_0)$. Suppose now $\beta\omega = \Sigma P_i(z)\beta(z_i) = \Sigma P_i(z)\bar{z}_i$ (for instance when $\omega \in \Lambda Z$). If C_0 constructed above satisfies $-C_0 = Q(z) + \Sigma Q_i(z)\bar{z}_i + \Sigma Q_{ij}(z)\bar{z}_i\bar{z}_j + \cdots$, then $\beta\delta(\Sigma Q_i(z)\bar{z}_i + \Sigma Q_{ij}(z)\bar{z}_i\bar{z}_j + \cdots)$ is a sum of monomials which contain at least two elements from $\bar{Z}$, which by our hypothesis on $\beta\omega$ is zero. Since β is injective when restricted to ΛZ, $\beta\omega = \beta\delta Q(z) = \beta dQ(z)$, which implies $\omega = dQ(z)$. Now to prove Corollary 3, we consider the cochain complex (C^*, d) with $C^n = (\Lambda Z)^{n+1}$ and let $B\colon C^* \to (\Lambda\alpha \otimes \Lambda Z \otimes \Lambda\bar{Z})$ be given by $B(\omega) = (-1)^{|\omega|}\beta(\omega)$. It is easy to verify that B is a morphism of cochain complexes; hence it induces $B^*\colon H^{n+1}(\Lambda Z, d) = H^n(C^*, d) \to H^n(\Lambda\alpha \otimes \Lambda Z \otimes \Lambda\bar{Z}, \mathscr{D})$. As we have seen, if $B\omega = b$, then $\omega = dQ(z)$ which shows that B^* is injective. q.e.d.

Proof of Corollary 4. Let $(\Lambda\alpha \otimes \Lambda Z \otimes \Lambda\bar{Z}, \mathscr{D})$ be as defined in §1, the minimal model of $X^{S^1} \times_{S^1} ES^1$. Note that $\mathscr{D} = d_{\mathrm{I}} + d_{\mathrm{II}}$ with d_{I} and d_{II} degree $+1$ differentials defined by $d_{\mathrm{I}}(z) = \delta z = dz$, $d_{\mathrm{I}}(\bar{z}) = \delta\bar{z} = -idz$, $d_{\mathrm{I}}\alpha = 0$ and $d_{\mathrm{II}}(z) = \alpha\bar{z}$, $d_{\mathrm{II}}(\bar{z}) = 0$, $d_{\mathrm{II}}(\alpha) = 0$, which satisfy $d_{\mathrm{I}}d_{\mathrm{II}} + d_{\mathrm{II}}d_{\mathrm{I}} = 0$. We regard $C(Z) = (\Lambda\alpha \otimes \Lambda Z \otimes \Lambda\bar{Z}, d_{\mathrm{I}}, d_{\mathrm{II}})$ as a bicomplex with $C^{p,q}(Z) = \{\omega \in C(Z) | \omega = \Sigma_{i=0}^{q} P_i(z,\bar{z})\alpha^i$, with $\deg\omega = p + q$, $P_i \in \Lambda Z \otimes \Lambda\bar{Z}$ and $P_q \neq 0\}$. Let $E_r^{p,q}(Z)$ be the associated spectral sequence which converges to $H^*(C(Z), \mathscr{D})$ and has $E_2^{p,q} = H^p_{d_{\mathrm{I}}}(H^q_{d_{\mathrm{II}}}(C(Z)))$. The multiplication by α defines a morphism of bicomplexes, $\hat{\alpha}\colon \Lambda\alpha \otimes \Lambda Z \otimes \Lambda\bar{Z} \to \Lambda\alpha \otimes \Lambda Z \otimes \Lambda\bar{Z}$, of degree $(0,2)$ which induces the morphism $\hat{\alpha}_{E,r}\colon E_r^{k*,*} \to E_r^{*,*+2}$ of spectral sequences and therefore we can consider a new spectral sequence

$$\underline{E}_r^{*,*} = \varinjlim \left(\cdots \to E_r^{*,*+2s} \to E_r^{*,*+2s+2} \to \cdots \right).$$

This new spectral sequence converges to $\varinjlim(\cdots H^{*+2s}(C(Z)) \to H^{*+2s}(C(Z)) \to \cdots)$. The morphism of differential graded algebras $k\colon (\Lambda(\phi), 0) \to (\Lambda Z, d)$, which corresponds to the map $X \to \mathrm{pt}$, induces the morphism $\{\pi_r^{p,q}\}\colon E_r^{p,q}(\phi) \to E_r^{p,q}(Z)$ and therefore the morphism $\{\underline{\pi}_r^{p,q}\}\colon \underline{E}_r^{p,q}(\phi) \to \underline{E}_r^{p,q}(Z)$ of spectral sequences; here ϕ is the empty vector space, hence $\Lambda(\phi) = k$. Notice that the acyclicity of $(\Lambda Z \otimes \Lambda\bar{Z}, \beta)$ implies $\varinjlim_r H^{n+2r}_{d_{\mathrm{II}}}(C(Z))$ equals zero if n is odd and equals k if n is even. Since $\underline{E}_1(Z) = \varinjlim(\cdots H_{d_{\mathrm{II}}}(C(Z)) \to H_{d_{\mathrm{II}}}(C(Z)) \to \cdots)$ we conclude that $\underline{\pi}_1^{*,*}$ is an isomorphism which implies $\mathrm{PHC}^*(\mathrm{pt}; k) = \mathrm{PHC}^*(X; k)$ for $k = Q$ and then for any k of characteristic zero.

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