J. DIFFERENTIAL GEOMETRY
22 (1985) 233-241

A REGULARITY THEOREM FOR HARMONIC MAPS WITH SMALL ENERGY

GUOJUN LIAO

1. Introduction

This paper studies the regularity problem of harmonic maps in higher dimensions. We consider maps from the unit ball B in $\mathbf{R}^n$ ($n > 2$) equipped with a metric g into a compact submanifold N^m of $\mathbf{R}^k$. We say that $u \in L_1^2(B, N)$ if $u \in L_1^2(B, \mathbf{R}^k)$ and $u(x) \in N$ a.e $x \in B$. The energy $E(u)$ of u is defined as $E(u) = \int_B |\nabla u|^2 \, dv$. A weakly harmonic map is defined to be the weak solution to the formal Euler-Lagrange equations, which form a nonlinear elliptic system. The equations are

$$\Delta u^i(x) = g^{\alpha\beta}(x) A^i\left(\frac{\partial y}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta}\right), \qquad i = 1, 2, \cdots, K, \tag{1.1}$$

where $A_u(X, Y) \in (T_u N)^\perp$ is the second fundamental form of N given by $A_u(X, Y) = (D_X Y)^\perp$. X, Y are vector fields on N in a neighborhood of $u \in N$.

It is easy to see that u is harmonic if and only if $(d/dt)E(u_t)|_{t=0} = 0$, where u_t is a 1-parameter family of maps defined by $u_t(x) = \Pi(u(x) + t\eta(x))$ $\forall \eta \in C_0^\infty(B, \mathbf{R}^k)$. Π is the nearest point projection of $\mathbf{R}^k$ into N.

There is another type of variation that one may consider. One takes $u_t = u \circ \varphi_t$ for φ_t a 1-parameter family of compactly supported C^1 diffeomorphisms of B with $\varphi_0 = \mathrm{Id}$. $E(u_t)$ is differentiable in t. If u is always critical for this type of variations and if u is harmonic, then u is called a stationary map.

So far not much is known about the regularity of weak harmonic maps. For $n = 2$ it is proved in [6] that a harmonic map with finite energy does not have isolated singularity. A theorem of [7] says that u has no interior singularity if u is stationary and $n = 2$.

Received November 10, 1984 and, in revised form, April 29, 1985.

In this paper we generalize the result of [6] to higher dimensions. For $n > 2$ we cannot expect the finiteness of total energy to be sufficient for the removability of isolated singularity. For example, take any harmonic map w from S^{n-1} into N with finite energy $E(w)$. Define map u: $B \to N$ by $u(x) = w(x/|x|)$. u is again harmonic with finite energy since $E(u) = E(w)/(n-2)$. u has singularity at 0 unless w is constant.

We will assume the smallness of the total energy and show the apparent isolated singularity is removable. Our main result is

Main Theorem. *Let B be the unit ball $B(0) \subset \mathbf{R}^n$ with a smooth Riemannian metric g. Let u be any harmonic map belonging to $C^\infty(B \setminus \{0\}, N)$. There exists a constant $\varepsilon > 0$ independent of u such that $u \in C^\infty(B, N)$ provided $E(u) = \int_B |\nabla u|^2 \, dv \leqslant \varepsilon$.*

Our proof is based on the a priori estimates of C^2 harmonic maps obtained by R. Schoen and K. Uhlenbeck and a monotonicity inequality.

We will present some preliminary results in the next section. In §3 we will prove the theorem assuming that u is stationary. In §4 we will prove that the monotonicity inequality is true for harmonic maps of finite energy with isolated singularity. This result then enables us to complete the proof of the theorem.

2. Preliminary results

Lemma 1 (*monotonicity inequality*). *Suppose u is a stationary map from B into $N \subset \mathbf{R}^k$. For $n > 2$ we have for $0 < \sigma < \rho < \operatorname{dist}(x_0, \partial B)$*

$$
\begin{aligned}
&e^{C\Lambda\rho}\rho^{2-n}\int_{B_\rho(x_0)} |\nabla u|^2 \, dx - e^{C\Lambda\sigma}\sigma^{2-n}\int_{B_\sigma(x_0)} |\nabla u|^2 \\
&\quad \geqslant 2\int_{B_\rho(x_0) - B_\sigma(x_0)} e^{C\Lambda\sigma}|x - x_0|^{2-n}\left|\frac{\partial u}{\partial r}\right|^2 dx,
\end{aligned} \tag{2.1}
$$

where Δ and C are constants, $B_\rho(x_0)$ (and $B_\sigma(x_0)$) is the geodesic ball of radius ρ (and σ) centered at x_0, respectively. For a proof one can read [5].

Lemma 2 [8]. *Suppose $u \in C^2(B_r, N)$ is harmonic with respect to a metric g on B_r. Suppose*

$$\Lambda^{-1}j(\delta_{\alpha\beta}) \leqslant g_{\alpha\beta} \leqslant \Lambda(\delta_{\alpha\beta}), \ |\partial_\nu g_{\alpha\beta}| \leqslant \Lambda r^{-1}.$$

There exists $\varepsilon = \varepsilon(\Lambda, n, N) > 0$ such that if $r^{2-n}\int_{B_r} |\nabla u|^2 \leqslant \varepsilon$, then

$$r^2 \sup_{B_{r/2}} |\nabla u|^2 \leqslant C r^{2-n} \int_{B_r} |\nabla u|^2. \tag{2.2}$$

The proof of Lemma 2 makes use of Lemma 1, noticing that "C^2 harmonic" implies "stationary". The smallness of the energy is used to ensure that a rescaled version v of u satisfying $e(v)(0) = 1$ and $\sup e(v) \leqslant 4$ is defined in a ball B_{r_0} with $r_0 \leqslant 1$. The boundedness of $e(v)$ then enables one to use the linear elliptic estimates. Here $e(v)$ denotes the energy density. For details see [7].

Lemma 3 (*First variation formula*). *For a smooth family φ_t of diffeomorphisms which are the identity near ∂B we let $u_t = u \circ \varphi_t$. We then have*

$$\text{(2.3)} \qquad \frac{d}{dt}E(u_t)\Big|_{t=0} = -\int_B \left[|du|^2 \operatorname{div} X - 2\left\langle du(\nabla_{e_i} X), du(e_i)\right\rangle\right],$$

where X = variation vector field = $(d/dt)\varphi_t|_{t=0}$, e_i, $i = 1, 2, \cdots, n$, form an orthonormal basis on B.

This is a standard result. One can prove it by a change of coordinates. We mention one more result.

Lemma 4 [4]. *If the image of a harmonic map u lies in a local strictly convex coordinate chart on N, then u is regular.*

3. The regularity of stationary maps

In this section we prove the following result:

Proposition 1. *If u is a stationary map from B^n into $N \subset \mathbf{R}^k$, $n > 2$, with respect to a metric g, then there exists a constant $\varepsilon > 0$ such that $u \in C^\infty(B, N)$ provided that*

$$E(u) \leqslant \varepsilon \quad \textit{and} \quad u \in C^\infty(B - \{0\}, N).$$

Proof. By a change of scale we can reduce to the case that g is close to the standard metric g_0. Thus we assume without loss of generality that

$$g_{\alpha\beta} = \delta_{\alpha\beta} + O(\varepsilon), \qquad \partial_\nu g_{\alpha\beta} = O(\varepsilon) \quad \text{in } B.$$

Apply Lemma 2 to the ball $B_r(x)$ where $x \neq 0$, $r = |x|/2$. We then get

$$r^2|du|^2(x) \leqslant Cr^{2-n}\int_{B_r(x)} |\nabla u|^2 \leqslant CE(u).$$

In the second inequality we have used the montonicity lemma. Assume $E(u) \leqslant \varepsilon$. We have the estimate

$$\text{(3.1)} \qquad |du|(x) \leqslant \frac{C\varepsilon^{1/2}}{|x|}.$$

Take a ball of radius $r > 0$ centered at 0. Let $x, y \in \partial B_r(0)$. Then

$$\text{(3.2)} \qquad |u(x) - u(y)| \leqslant \int_\Gamma |du| \leqslant \frac{\varepsilon^{1/2}C}{r} \cdot \Lambda 2\pi r = C\varepsilon^{1/2},$$

where Γ is a geodesic on $\partial B_r(0)$ connecting x to y with length $\leqslant \Lambda 2\pi r$. This shows that $\mathrm{Osc}(u, \partial B_r) \leqslant C\varepsilon^{1/2}$.

We want to compare u with a linear harmonic map h: $B \to \mathbf{R}^k$. To define h, we take $0' = \bar{u}(1)$ as the origin of $\mathbf{R}^k$, where $\bar{u}(r) = \fint_{\partial B_r} \mu$ is the average of u over ∂B_r. Let $\overline{C}$ be a constant so that $|A(\nabla u, \nabla u)| \leqslant \overline{C}|\nabla u|^2$. Let

$$\lambda = \tfrac{1}{2} \operatorname*{Min}_{Q \in N} \Big\{ \mathrm{Max}\big\{ \mu \,|\, \big(B_{2\mu}(Q) \cap N\big) \subset \text{a convex local coordinate chart of } N \big\} \Big\},$$

where $B_{2\mu}(Q)$ is the Euclidean ball of radius 2μ centered at Q. Let δ be the first point for which $\bar{u}(r)$ lies on $\partial B_\lambda(0')$, i.e., $\delta = \max\{r\colon |\bar{u}(r) - \bar{u}(1)| = \lambda\}$. We take the first coordinate axis in the direction $0'\bar{u}(\delta)$.

Clearly $\lambda > 0$. We claim that $\delta > 0$ if 0 is not removable and $E(u) \leqslant \varepsilon$ for an $\varepsilon > 0$ small. The reason is that $|u(x) - \bar{u}(|x|)| \leqslant C \cdot \varepsilon^{1/2}$ as a direct consequence of (3.2) and by definition $\bar{u}(|x|) \in B_\lambda(0')$ $\forall x \in B_1 \setminus B_\delta$. Thus we can choose ε small so that $u(x) \in B_{2\lambda}(\Pi(0'))$ $\forall x \in B_1 \setminus B_\delta$. Then $\delta = 0$ would imply that the image of u on B_1 lies in a convex local coordinate chart of N, hence u would be regular.

Also δ is uniformly away from 1 for ε small as a consequence of the a priori bound of the gradient (3.1). Indeed, we have

$$\lambda = |\bar{u}(\delta) - \bar{u}(1)| \leqslant \Big(\sup_{\delta \leqslant r \leqslant 1} |\nabla \bar{u}| \Big)(1 - \delta) \leqslant C\varepsilon^{1/2}/\delta,$$

hence $\delta \leqslant C\varepsilon^{1/2}\lambda^{-1}$.

Define h: $B \to \mathbf{R}^k$ by $h(x) = (h_1(x), 0, \cdots, 0)$, where

$$h_1(x) = h_1(|x|) = -\frac{\lambda}{\delta^{2-n} - 1} + \frac{\lambda}{\delta^{2-n} - 1}|x|^{2-n}.$$

Note that $\Delta_{g_0} h = 0$ and $h(x) = \bar{u}(\delta)$ on ∂B_δ, $h(x) = \bar{u}(1) = 0'$ on ∂B_1, where Δ_{g_0} is the Euclidean Laplacian.

Observe that for $n \geqslant 3$

$$\begin{aligned} \int_{B_1 \setminus B_\delta} \left|\frac{\partial h}{\partial r}\right|^2 r^{2-n} dv &= \int_{B_1 \setminus B_\delta} \left|\frac{\partial h}{\partial r}\right|^2 r^{2-n} \sqrt{g(x)}\, dx \geqslant C \int_\delta^1 |h_1'|^2 r\, dr \\ &= C\frac{\lambda^2(n-2)}{2} \frac{\delta^{2-n} + 1}{\delta^{2-n} - 1} \geqslant \frac{C\lambda^2}{2} > 0. \end{aligned} \tag{3.3}$$

On the other hand, by Lemma 1 we have

$$\int_{B_1 \setminus B_\delta} \left|\frac{\partial u}{\partial r}\right|^2 r^{2-n} dV \leqslant C \int_B |\nabla u|^2 dV = CE(u) \leqslant C\varepsilon. \tag{3.4}$$

Our plan is to show

$$\int_{B_1\setminus B_\delta} \left|\frac{\partial u}{\partial r} - \frac{\partial u}{\partial r}\right|^2 r^{2-n} dV \leqslant C\varepsilon^{1/2}, \tag{3.5}$$

which is a contradiction to (3.3) and (3.4) for ε small.

Applying Green's formula, we get (denoting $h_1 = h$, $u_1 = u$)

$$\begin{aligned}
&\int_{B_1\setminus B_\delta} r^{2-n}(h-u)\Delta(h-u) \\
&\quad = \int_{B_1\setminus B_\delta} \nabla\left[r^{2-n}(h-u)\right]\cdot\nabla(h-u) \\
&\qquad - \int_{\partial B_1\cup\partial B_\delta} r^{2-n}(h-u)\frac{\partial(h-u)}{\partial r} \\
&\quad = \frac{1}{2}\int_{B_1\setminus B_\delta} \nabla r^{2-n}\cdot\nabla|h-u|^2 + \int_{B_1\setminus B_\delta} r^{2-n}|\nabla(h-u)|^2 \\
&\qquad - \int_{\partial B_1\cup\partial B_\delta} r^{2-n}(h-u)\left(\frac{\partial h}{\partial r} - \frac{\partial u}{\partial r}\right) \\
&\quad = \frac{2-n}{2}\int_{\partial B_1\cup\partial B_\delta} |h-u|^2 r^{2-n} \int_{B_1\setminus B_\delta} r^{2-n}|\nabla(h-u)|^2 \\
&\qquad - \int_{\partial B_1\cup\partial B_\delta} r^{2-n}(h-u)\left(\frac{\partial h}{\partial r} - \frac{\partial u}{\partial r}\right) - \frac{1}{2}\int_{B_1\setminus B_\delta} |h-u|^2\Delta r^{2-n}.
\end{aligned}$$

We get from this

$$\begin{aligned}
&\int_{B_1\setminus B_\delta} r^{2-n}|\nabla(h-u)|^2 \\
&\quad = \frac{2-n}{2}\int_{\partial B_1\cup\partial B_\delta} |h-u|^2 r^{1-n} + \int_{\partial B_1\cup\partial B_\delta} r^{2-n}(h-u)\left(\frac{\partial h}{\partial r} - \frac{\partial u}{\partial r}\right) \\
&\qquad + \int_{B_1\setminus B_\delta} r^{2-n}(h-u)\Delta u - \int_{B_1\setminus B_\delta} r^{2-n}(h-u)\Delta h \\
&\qquad + \frac{1}{2}\int_{B_1 cdot\setminus B_\delta} |h-u|^2\Delta r^{2-n}.
\end{aligned} \tag{3.6}$$

Using the fact $\sup_{\partial B_\delta\cup\partial B_1}|h-u| \leqslant C\varepsilon^{1/2}$, we can estimate

$$\left|\int_{\partial B_\delta} r^{1-n}|h-u|^2\right| \leqslant C\varepsilon\int_{\partial B_1} \delta^{1-n}\delta^{n-1} \leqslant C\varepsilon, \tag{3.7}$$

$$\begin{aligned}
&\int_{\partial B_\delta} r^{2-n}|h-u|\left(\frac{\partial h}{\partial r}-\frac{\partial u}{\partial r}\right)\Big| \\
&\qquad \leqslant C\varepsilon^{1/2}\int_{\partial B_1}\delta^{2-n}\left[\lambda(n-2)\frac{\delta^{1-n}}{\delta^{2-n}-1}+\frac{\varepsilon^{1/2}}{\delta}\right]\delta^{n-1} \\
&\qquad \leqslant \lambda C\varepsilon^{1/2},
\end{aligned} \tag{3.8}$$

where we have used the fact δ small to assert

$$\frac{\delta^{1-n}}{\delta^{2-n}-1}=\frac{1}{\delta}\cdot\frac{1}{1-\delta^{2-n}}\leqslant C\delta^{-1}.$$

Similar estimates can be obtained at ∂B_1. To deal with the third term in (3.6), observe that

$$\begin{aligned}
\sup_{B_1\setminus B_\delta}|h-u| &\leqslant \sup|h|+\sup|u-\bar{u}|+\sup\bar{u} \\
&\leqslant \lambda+C\varepsilon^{1/2}+\lambda\leqslant 2\cdot\frac{1}{4\bar{C}}+C\varepsilon^{1/2}.
\end{aligned}$$

Choose ε small so that

$$\bar{C}\sup_{B_1\setminus B_\delta}|h-u|\leqslant\frac{1}{2}+\bar{C}\subset\varepsilon^{1/2}\leqslant 1. \tag{3.9}$$

We have

$$\left|\int_{B_1\setminus B_\delta} r^{2-n}(h-u)\Delta u\right|\leqslant\left(\int_{B_1\setminus B_\delta} r^{2-n}|\nabla u|^2\right)\bar{C}\sup_{B_1\setminus B_\delta}|h-u|. \tag{3.10}$$

The last two terms in (3.6) are bounded by $C\cdot\varepsilon^{1/2}$ as a consequence of our assumptions on $g_{\alpha\beta}$.

Since h is a radial map we can write the left side of (3.6) as

$$\int_{B_1\setminus B_\delta} r^{2-n}\left|\frac{\partial h}{\partial r}-\frac{\partial u}{\partial r}\right|^2+\int_{B_1\setminus B_\delta} r^{2-n}|D_T u|^2,$$

where $D_T u$ denotes the components tangential to ∂B_r. Absorbing the tangential term to the left, we get

$$\begin{aligned}
&\int_{B_1\setminus B_\delta} r^{2-n}\left|\frac{\partial h}{\partial r}-\frac{\partial u}{\partial r}\right|^2+\left(1-\bar{C}\sup_{B_1\setminus B_\delta}|h-u|\right)\int_{B_1\setminus B_\delta} r^{2-n}|D_T u|^2 \\
&\qquad\leqslant C\varepsilon^{1/2}+\bar{C}\sup_{B_1\setminus B_\delta}|h-u|\int_{B_1\setminus B_\delta} r^{2-n}\left|\frac{\partial u}{\partial r}\right|^2 \\
&\qquad\leqslant C\varepsilon^{1/2}+\varepsilon.
\end{aligned} \tag{3.11}$$

This inequality completes the proof.

4. Proof of the Main Theorem

We want to show the following extension of Lemma 1.

Proposition 2. *If harmonic map u is $C^\infty(B\setminus\{0\},N)$ and if $E(u)<\infty$, then we have for* $0<\rho_1<\rho_2\leqslant 1$

$$e^{C\Lambda\rho_2}\rho_2^{2-n}\int_{B_{\rho_2}(0)}|\nabla u|^2-e^{C\Lambda\rho_1}\rho_1^{2-n}\int_{B_{\rho_1}(0)}|\nabla u|^2\geqslant\int_{B_{\rho_2}(0)-B_{\rho_1}(0)}e^{C\Lambda r}r^{2-n}\left|\frac{\partial u}{\partial r}\right|^2,$$

where C and Λ are constants.

Proof. Take $X_\sigma(x)=\psi_\sigma(|x|)\cdot\eta_\tau(|x|)\cdot|x|\cdot(\partial/\partial r)(x)$ for $\sigma>0$, $\tau>0$ in the first variation formula, where $\eta_\tau\in C_0^\infty([0,1],\mathbf{R}^1)$ will be chosen later, ψ_σ is a cut-off function so that ψ_σ is smooth and nonnegative, $|\psi'_\sigma|\leqslant 2\sigma^{-1}$ and

$$\psi_\sigma(r)\begin{cases}=0 & \text{if } 0\leqslant r\leqslant\sigma,\\ =1 & \text{if } r\geqslant 2\sigma,\\ \leqslant 1 & \text{elsewhere.}\end{cases}\tag{4.2}$$

Define $u_{t,\sigma}\colon B\to N$ by

$$u_{t,\sigma}(x)=ku(x+tX_\sigma(x)).\tag{4.3}$$

Note that $u_{t,\sigma}$ is smooth. Thus we have

$$\frac{d}{dt}E(u_{t,\sigma})\bigg|_{t=0}=0\quad\forall\sigma>0$$

since u is harmonic.

Let $\phi\in C^\infty(\mathbf{R}^+,\mathbf{R}^1)$ so that $\phi(r)=1$ for $r\in[0,1]$; $\phi(r)=0$ for $r\in[1+\sigma_1,\infty)$; $\phi'(r)<0\,(\sigma_1>0=$ is fixed). Choose $\eta_\tau(r)=\phi(r/\tau)$ for $\tau\in[\rho_1,\rho_2]$ in X_σ. Choose an orthonormal basis $e_1,\cdots,e_{n-1},e_n=\partial/\partial r$. We have

$$\nabla_{e_i}X_\sigma=\psi_\sigma\eta_\tau\nabla_{re_i}\frac{\partial}{\partial r}\quad\text{for } i=1,2,\cdots,n-1,$$

$$\nabla_{\partial/\partial r}X_\sigma=(\psi_\sigma\eta r)'\frac{\partial}{\partial r},$$

where the derivative is taken with respect to r.

Denote $x=x(r,\theta)$ and

$$\varepsilon_{ij}(x)=\frac{\partial}{\partial r}\left(\left\langle\nabla_{re_i}\left(\frac{\partial}{\partial r}\right),e_j\right\rangle\right)(x),\qquad i,j=1,2,\cdots,n-1.$$

Take a constant $\Lambda>0$ so that $|\varepsilon_{ij}(x)|\leqslant\Lambda\ \forall x\in B$. Then we have for $i,j=1,2,\cdots,n-1$

$$\left\langle\nabla_{e_i}X_\sigma,e_j\right\rangle=\psi_\sigma\eta_\tau\delta_{ij}+\psi_\sigma\eta_\tau\int_0^r\varepsilon_{ij}(r',\varphi)\,dr'.$$

Thus

$$\operatorname{div} X_\sigma = \sum_{i=1}^{n-1} \langle \nabla e_i X_\sigma, e_i \rangle + \left\langle \nabla_{\partial/\partial r} X_\sigma, \frac{\partial}{\partial r} \right\rangle$$
$$\geqslant (\psi_\sigma \eta_\tau)' r + n \psi_\sigma \eta_\tau - (n-1) \psi_\sigma \eta_\tau \Lambda r.$$

Moreover, we have

$$\sum_{i=1}^{n} \left\langle du(\nabla_{e_i} X_\sigma), du(e_i) \right\rangle$$
$$= \sum_{i=1}^{n-1} \left\langle du\left(\psi_\sigma \eta_\tau \nabla_{re_i}\left(\frac{\partial}{\partial r} \right) \right), du(e_i) \right\rangle$$
$$+ \left\langle du(\nabla_{\partial/\partial r} X_\sigma), du\left(\frac{\partial}{\partial r} \right) \right\rangle$$
$$\leqslant \sum_{i=1}^{n-1} \psi_\sigma \eta_\tau |du(e_i)|^2 + (n-1) \psi_\sigma \eta_\tau \Lambda r |du|^2 + (\psi_\sigma \eta_\tau r)' \left| \frac{\partial u}{\partial r} \right|^2.$$

Thus the first variation formula gives

$$\begin{aligned} 0 \geqslant & \int_B \left[(\psi_\sigma \eta_\tau)' r + n \psi_\sigma \eta_\tau - (n-1) \psi_\sigma \eta_\tau \Lambda r \right] |du|^2 \, dv \\ & - 2 \int_B \psi_\sigma \eta_\tau |du|^2 \, dv - 2 \int_B (\psi_\sigma \eta_\tau)' r \left| \frac{\partial u}{\partial r} \right|^2 dv \\ & - 2(n-1) \int_B \psi_\sigma \eta \Lambda r |du|^2 \, dv. \end{aligned} \tag{4.4}$$

Claim.

$$\int_B |\psi_\sigma'| r |du|^2 \eta_\tau \, dv \to 0 \quad \text{as } \sigma \to 0.$$

To see this, use the estimate $|du|^2(x) \leqslant C_1 E(u) \sigma^{-2}$ for $x \in B_{2\sigma} \setminus B_\sigma$. Since $|\psi_\sigma'|(x) = 0$ for $x \in B_\sigma$ or $x \in B \setminus B_{2\sigma}$ and $|\psi_\sigma'| \leqslant 2\sigma^{-1}$, we get

$$\int_B |\psi_\sigma'| r |du|^2 \eta_\tau \, dv \leqslant C_2 E(u) \sigma^n \sigma^{-1} \sigma^{-2} \sigma = C_2 E(u) \sigma^{2-n}.$$

Since $n \geqslant 3$, the conclusion follows. Similarly one can show that $\lim_{\sigma \to 0} \int_B \psi_\sigma \eta_\tau |du|^2 = \int_B \eta_\tau |du|^2$, etc. Letting $\sigma \to 0$, we get from (4.4)

$$0 \geqslant \int_B (\eta_\tau' r + n \eta_\tau - (n-1) \eta \Lambda r) |du|^2 - 2 \int_B \eta |du|^2$$
$$- 2 \int_B \eta' r \left| \frac{\partial u}{\partial r} \right|^2 - 2(n-1) \int_B \eta \Lambda r |du|^2.$$

It follows that

$$2\tau\frac{\partial}{\partial\tau}\int_B \eta\left|\frac{\partial u}{\partial r}\right|^2 \leqslant 3(n-1)\int_B \eta r\Lambda|du|^2 + \tau\frac{\partial}{\partial\tau}\int_B \eta_\tau|du|^2 + (2-n)\int_B \eta|du|^2$$
$$\leqslant 3(n-1)(1+\sigma_1)\int_B \eta\Lambda|du|^2 + \tau\frac{\partial}{\partial\tau}\int_B \eta_\tau|du|^2 + (2-n)\int_B \eta|du|^2.$$

Multiplying by $\tau^{1-n}e^{C\Lambda\tau}$ for $C = 3(n-1)$, we have

$$e^{C\Lambda\tau}2t^{2-n}\frac{\partial}{\partial\tau}\int_B \eta_\tau\left|\frac{\partial u}{\partial r}\right|^2 \leqslant \frac{\partial}{\partial\tau}\left(e^{C\Lambda\tau}\tau^{2-n}\int_B \eta_\tau|du|^2\right) + \sigma_1 e^{C\Lambda\tau}C\tau^{2-n}\int_B \eta_\tau|du|^2.$$

Integrate over $[\rho_1, \rho_2]$ and let $\sigma_1 \to 0$. We then get

$$e^{C\Lambda\rho_1}\rho_1^{2-n}\int_{B_{\rho_1}}|du|^2 - e^{C\Lambda\rho_2}\rho_2^{2-n}\int_{B_{\rho_2}}|du|^2 \geqslant 2\int_{B_{\rho_2}\setminus B_{\rho_1}} e^{C\Lambda r}r^{2-n}\left|\frac{\partial u}{\partial r}\right|^2.$$

In the above computation we denote $B = B_1(x_0)$ and $B_{\rho_2} = B_{\rho_2}(x_0)$, $B_{\rho_2} = B_{\rho_2}(x_0)$.

Proof of the Main Theorem. Under the assumption of the theorem we have for $x \in B_1 \setminus \{0\}$, $|x| < 1/2$,

$$\frac{|x|^{2-n}}{2}\int_{B_{|x|/2}(x)}|\nabla u|^2 \leqslant C(2|x|)^{2-n}\int_{B_{2|x|}(0)}|\nabla u|^2 \leqslant CE(u).$$

So the estimate (3.1) still holds. Here we have applied Proposition 2 with $\rho_1 = 2|x|$, $\rho_2 = 1$. Then the same argument in the proof of Proposition 1 can be carried through.

Acknowledgement. I am grateful to Professor R. Schoen for the valuable discussions on this work.

References

[1] J. Eells & L. Lemaire, *A report on harmonic maps*, Bull. London Math. Soc. **10** (1978) 1–68.

[2] J. Frehse, *A discontinuous solution to a mildly nonlinear elliptic system*, Math. Z. **134** (1973) 229–230.

[3] W. Garber, S. N. M. Ruijsenaars, E. Seiler & D. Bursn, *On finite action solutions of the nonlinear α-model*, Ann. Physics **119** (1979) 305–325.

[4] S. Hildebrandt & K. O. Widman, *On the Holder continuity of weak solutions of quasilinear elliptic systems of second order*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **4** (1977) 145–178.

[5] P. Price, *A monotonicity formula for Yang-Mills fields*, Manuscripta Math. **43** (1983) 131–166.

[6] J. Sacks & K. Uhlenbeck, *The existence of minimal immersions of 2-spheres*, Ann. of Math. (2) **113** (1981) 1–24.

[7] R. Schoen, *Analytic aspects of the harmonic map problem*, Preprint.

[8] R. Schoen & K. Uhlenbeck, *Regularity of minimizing harmonic maps into the spheres*, Invent. Math., to appear.

UNIVERSITY OF CALIFORNIA, BERKELEY