THE SIMPLE LOOP CONJECTURE

DAVID GABAI

1. Introduction

The main result of this paper is the proof of the so-called Simple Loop Conjecture, Theorem 2.1. In §3 we prove analogous results for compact surfaces with boundary. In that setting simple arcs play the role of simple closed curves.

I wish to thank Allen Edmonds [see 2.2] for making me aware of the applicability of my work to this problem and to thank Joel Hass and Will Kazez for helpful conversations.

Notation. If \( E \subset S \), then \( N(E) \) denotes a tubular neighborhood of \( E \) in \( S \), \( \hat{E} \) denotes interior of \( E \), and \( |E| \) denotes the number of components of \( E \). See [1] or [3] for basic definitions regarding branched covers.

2. Closed surfaces

Theorem 2.1. If \( f: S \to T \) is a map of closed connected surfaces such that \( f_*: \pi_1(S) \to \pi_1(T) \) is not injective, then there exists a noncontractible simple closed curve \( \alpha \subset S \) such that \( f|\alpha \) is homotopically trivial.

Proof. We will assume that \( T \neq S^2 \).

Step 1. Either there exists a noncontractible simple closed curve \( \alpha \subset S \) such that \( f|\alpha \) is homotopically trivial or \( f \) is homotopic to a simple branched cover (i.e., if \( f \) is a branched cover of degree \( d \), then for every \( x \in T \) \(|f^{-1}(x)| \geq d - 1\)) or \( T = \mathbb{RP}^2 \) and there exists a simple branched cover \( f': S \to T \) such that \( \text{ker}f_* = \text{ker}f'_* \).

Proof of Step 1. Let \( D \) be a 2-disc in \( T \). Let \( \lambda_1, \ldots, \lambda_n \) be properly embedded arcs in \( T - \hat{D} \) such that \( T - (D \cup N) = E \) is a 2-disc where \( N \) is a product neighborhood in \( T - \hat{D} \), of \( \bigcup \lambda_i \).

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Let $g$ be a map homotopic to $f$ such that:

1. $g: g^{-1}(D) \to D$ is an immersion;
2. $g$ is transverse to $\cup \lambda_i$;

and which minimizes $c(g) = (|g^{-1}(D)|, |g^{-1}(\cup \lambda_i)|)$ where such pairs are lexicographically ordered.

Let $S' = S - g^{-1}(\hat{D})$. Note that $g^{-1}(\cup \lambda_i)$ is a union of pairwise disjoint properly embedded simple arcs and simple closed curves in $S'$. If $T \neq \mathbb{RP}^2$ and some component $C$ of $g^{-1}(\cup \lambda_i)$ is a simple closed curve, then $C$ is noncontractible in $S$ hence Step 1 holds. Otherwise $C$ bounds a disc in $S$ and one can find, using the fact $\pi_2(T) = 0$, a map $g_1$ homotopic to $f$ with $c(g_1) < c(g)$, contradicting minimality. If $T = \mathbb{RP}^2$ and $C$ bounds a disc $F$ in $S$ define $g': S \to T$ so that

$$g'|S - \hat{N}(F) = g|S - \hat{N}(F) \text{ and } (g(N(F))) \cap (D \cup (\cup \lambda_i)) = \emptyset.$$ 

Observe that $\ker g' \ast = \ker g \ast$ and $c(g') < c(g)$. $g'$ might not be homotopic to $g$. No component of $g^{-1}(\lambda_i)$ is an arc $C$ such that $g|C$ does not map onto $\lambda_i$. Otherwise $g$ is homotopic to a map $g_1$ satisfying $(\ast)$ such that $|g_1^{-1}(D)| = |g^{-1}(D)| - 2$, again contradicting minimality of $g$. We can therefore assume that either $f|f^{-1}(\cup D) \to \cup D$ is an immersion or we have found a simple loop in $\ker f \ast$.

Let $H = f^{-1}(E)$ and $K = S - \hat{H}$. $f|\partial H$ is an immersion into $\partial E$. Since $\pi_1(E) = 1$ and each component of $K$ is nonplanar if $T \neq \mathbb{RP}^2$, $H$ is a union of 2-discs or Step 1 holds. If $T = \mathbb{RP}^2$ and some component $c$ of $\partial H$ bounds a disc $F$ in $S$ but not in $H$, then we can find $f': S \to T$ such that $\ker f' \ast = \ker f \ast$ but $c(f') < c(f)$. Two maps $h_1, h_2: (D^2, \partial D^2) \to (E, \partial E)$ are homotopic if and only if $\deg h_1 = \deg h_2$. In particular, if $\deg h_1 = p \neq 0$ then $h_1$ is homotopic to the branched cover $h_3$ defined by $z \to z^p$ (viewing $F, E$ as unit discs in $C$) and by perturbing $h_3$ slightly we can obtain a simple branched cover. It follows that if $H$ is a union of 2-discs then $f|H$, hence $f$ is homotopic to a simple branched cover.

Remark. It was pointed out to me that an almost identical version of Step 1 and its proof is contained in the unpublished work of Tucker [5].

Step 2. Construct $g: S \to T \times I$ such that the following 3 conditions hold.

1) The diagram

$$
\begin{array}{ccc}
S & \xrightarrow{g} & T \\
\downarrow{f} & & \downarrow{p} \\
T \times I & \xrightarrow{T \times I} & T
\end{array}
$$

commutes where $p = \text{projection onto the first factor}$.
2) If $x \in T$ is a branch point there exists a disc $D_x \subset T$ such that $gf^{-1}(D_x)$ is a disjoint union of $n - 2$ horizontal (i.e., contained in $T_x$ point) embedded discs and one nearly horizontal branched disc, as in Figure 2.1.

![Figure 2.1](image)

For each branch point $x$, let $E_x$ be an open disc such that $E_x \subset D_x$. Let

$$S' = f^{-1}\left(T - \bigcup_{x \text{ branch pts.}} E_x\right).$$

3) $g|S'$: $S' \to N \times I$ is a general position immersion i.e., at most 3 distinct points of $S'$ map to the same point of $N \times I$ and if $D_1, D_2, D_3$ are pairwise disjoint discs in $S'$ such that $g|D_p, p = 1, 2, 3$ is an embedding, then $g(D_i)$ intersects $g(D_j)$, $g(D_j) \cap g(D_k)$ transversely for $i \neq j$ or $k$. q.e.d.

Observation. To each branch point $x$ in $T \times I$ there exists an immersed double arc in $T \times I$ with one endpoint on $x$ and another endpoint on $y$, $y$ another branch point in $T \times I$.

Step 3. $g$ can be homotoped to $h$: $S \to T \times I$ so that $h' = p \circ h$ is a branched cover. Step 2 holds with $h, h'$ in place of $g, f$ and each double arc (connecting branch points) is embedded in $T \times I$ and disjoint from all other double curves of $h(S)$ in $T \times I$. 
Proof of Step 3. Induction on the number of triple points of $g(S)$ in $T \times I$. If $J$ is an immersed double arc which is either not embedded or intersects other double curves, then $J$ must pass through triple points. In particular there exists a double arc $J' \subset J$ such that one endpoint of $J'$ is a branch point and the other end of $J'$ is a triple point (Figure 2.2(a)). Now homotope $g$ to $g'$ as in Figure 2.2(b). Note that $p \circ g'$ is a branched cover, $g'(S)$ has one fewer triple point than $g(S)$ and after a small isotopy (to satisfy 2) of Step 2) $g'$, $p \circ g'$ satisfy 1), 2), 3) of Step 2. Step 3 now follows by induction. q.e.d.

By homotoping $g$ further so that the images, in $T$, of the double curves connecting branch points are very short and disjoint one can find pairwise disjoint discs $D_1, \ldots, D_r \subset T$ (where $2r =$ number of branch points) such that $g f^{-1}(D_i)$ appears as in Figure 2.3(a). See Figure 3.1 for a view of Figure 2.3 chopped in half.
Since a branched cover without branch points is a covering map; hence, is injective on $\pi_1$, Theorem 2.1 follows by Step 4.

**Step 4.** $\alpha$ is homotopically trivial in $T$ and $g^{-1}(\alpha) = \lambda$ is either a homotopically nontrivial simple closed curve in $S$ or $T = \mathbb{R}P^2$ and there exists a map $f': S \to T$ such that $\ker f' = \ker f$ and $f'$ has fewer branch points than $f$.

**Proof.** $\alpha$ bounds a disc in $T \times I$ hence is homotopically trivial. We now suppose that $\lambda$ is homotopically trivial in $S$ for otherwise Step 4 has been completed.

$\lambda$ and $\lambda' = g\circ \theta(\alpha')$ (Figure 2.3(a)) bound an annulus in $S$ and individually bound discs $E, E'$ such that (after possibly changing the names of $\lambda, \lambda'$) $E \supset E'$. It follows that there exist branched covers

$$k' = g \circ f: S^2 \to T, \quad f' = g \circ f_1: S \to T,$$

such that

$$S^2 = E' \cup F', \quad F' \text{ a 2-disc} \quad \text{and} \quad k|E' = g|E'.$$

$k(F')$ is a horizontal disc (Figure 2.3(b)) such that $k(\partial F') = \alpha'$.

$$f_1|S - E = g|S - E,$$

$f_1(E)$ is a horizontal disc (Figure 2.3(b)) such that $f_1(\partial E) = \alpha$.

If $z \in \pi_1(S)$, then $z$ can be represented by a curve $\gamma \subset S$ such that $\gamma \cap E = \emptyset$, therefore $f_*'(z) = f_*'(z)$ hence $\ker f_*' = \ker f_*$. $f'$ has at least 2
fewer branch points than \( f \). Finally if \( T \neq \mathbb{RP}^2 \) the following Euler characteristic calculation yields, (where \( b \) is the number of branch points of \( k^' \) and \( r \) is the degree of \( k' \))

\[
2 = \chi(S^2) = r(\chi(T)) - b \leq 0.
\]

**Remarks.** Partial results on this problem were obtained by Berstein and Edmonds in [3] and [2].

**Acknowledgement 2.2.** The author is grateful to Allen Edmonds for pointing out that the simple loop conjecture follows as a corollary from the remark stated without proof on page 502 of [4] (the remark claims that a stronger theorem than the one proven in [4] is in fact true).

The remark implies that if \( f: S \rightarrow T \) is a continuous map of closed surfaces, then either there exists a simple loop in \( \ker f_*: \pi_1(S) \rightarrow \pi_1(T) \), or one can find \( g: S \rightarrow T \times I \) where \( g \) is an immersion, \( p \circ g \) is homotopic to \( f \) and either \( g(S) \) is transverse to the product fibration \( T \times I \), except along saddle tangencies, or \( g(S) \) is an immersion onto some fibre \( T \times pt \). The latter implies that \( p \circ g \) is a covering map, hence \( f \) is 1-1 on \( \pi_1 \) while the former could not occur. A point \( x \in g(S) \) which is maximal in the \( I \) factor of \( T \times I \) would correspond to a non saddle tangency between \( g(S) \) and the fibration.

### 3. Surfaces with boundary

One cannot find in general noncontractible simple loops in the kernel of a map of surfaces with boundary. The following example is due to Tom Tucker. If \( S = S^2 - 3 \) discs, \( T = S^1 \times I \) and \( f: S \rightarrow T \) is the 2 fold branched cover, branched over a single point, then \( \ker f_* \neq \emptyset \) but contains no simple loops.

For manifolds with boundary, simple non boundary parallel arcs play the role of simple loops.

**Theorem 3.1.** If \( f: S \rightarrow T \) is a map of bounded connected surfaces such that \( f_*: \pi_1(S) \rightarrow \pi_1(T) \) is not injective, then there exists an essential simple arc \( \alpha \subset S \) and a map \( g \) homotopic to \( f \) such that \( g(\alpha) \) is a boundary parallel arc.

This is an unpublished but known result. We indicate a proof along the lines of the proof of Theorem 2.1.

**Proof.** Apply the methods of Step 1 to conclude that either Theorem 3.1 holds or \( f \) is homotopic to a branched cover. Argue as in Steps 2 and 3 to homotope \( f \) so that double arcs in \( T \times I \) emanating from branched points appear either in pairs (Figure 2.3) or as singles (Figure 3.1). By homotoping \( f \) a bit further we can assume that all such double arcs appear as in Figure 3.1. If
$T = D^2$ the result is trivial. If $T \neq D^2$ then arguing as in Step 4 shows that $\alpha$ (Figure 3.1) is the desired simple arc.

**Question 3.2.** Let $f: S \to T$ be a map between surfaces with boundary. When does there exist an essential simple closed curve $C \subset S$ such that $f|C$ is homotopically trivial?

**References**


