J. DIFFERENTIAL GEOMETRY
21 (1985) 109–127

A COMPLETE EMBEDDED MINIMAL SURFACE IN $\mathbf{R}^3$ WITH GENUS ONE AND THREE ENDS

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1. Introduction

It has been a longstanding conjecture that the only complete embedded minimal surfaces in $\mathbf{R}^3$ of finite topological type are the plane, the catenoid, and the helicoid. This conjecture is false. We will exhibit a complete minimal surface, conformally the square torus $\mathbf{C}/\mathbf{Z}^2$ with three points removed, which is embedded in $\mathbf{R}^3$. It has the following interesting geometric properties.

- Its Gauss map composed with stereographic projection is given by a constant divided by the derivative of the Weierstrass P-function.
- It contains two straight lines which meet at right angles.
- It can be decomposed into eight congruent pieces, each of which lies in a different octant and each of which is a graph.
- It is invariant under the group of motions of $\mathbf{R}^3$ generated by

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(which is the dihedral group with eight elements).

This surface was first written down by Costa in his thesis [3]. He established that it was complete, and of genus one with three ends (see Theorem 1). We computed the coordinates of the surface and drew computer pictures of it. Observing that it looked embedded and it had dihedral symmetry, we were

Received April 3, 1985.
[1] The first author was partially supported by NSF Grant MCS-83-01936, and the second author partially supported by NSF Grant DMS-84-14330.

motivated to prove these observations mathematically. The symmetry of the surface is established in Theorem 2 of §4 and its embeddedness is proved in Theorem 3 of §5. Some computer graphics pictures of the surface appear in §6.

Acknowledgements.

The computer graphics programming used in this research was done using VPL, a system developed by J. Hoffman at the University of Massachusetts. We are grateful for the use of the Research Computing Facility of the Computer Science Department at the University of Massachusetts.

For useful conversations in the preparation of this paper we wish to thank R. Osserman and R. Schoen.

2. Embedded minimal surfaces

A complete (not necessarily embedded) minimal surface M in $\mathbf{R}^3$ with finite total curvature is conformally equivalent to a compact Riemann surface $\overline{M}$ from which r points, $p_1,\cdots,p_r$ have been removed, $r \geqslant 1$. Moreover the Gauss mapping, defined on M as a meromorphic function, extends to $\overline{M}$. This is a theorem of Osserman [6] who also established that the total curvature of M satisfies

$$(1) \qquad \int_M K\,dA \leqslant 4\pi(1-g-r) = 2\pi\big(\chi(\overline{M}) - 2r\big).$$

For each j, $j = 1,\cdots,r$, let E_j be the image in $\mathbf{R}^3$ of a punctured disk in $\overline{M}$ centered at p_j. This is called the "jth end" of M. Jorge and Meeks [5] proved that $((1/t)E_j) \cap S^2$ converges smoothly, as $t \to \infty$, to a geodesic on S^2.

This geodesic is covered with multiplicity d_j, for some positive integer d_j. Using these multiplicities it can be shown that

$$(2) \qquad \int_M K\,dA = 4\pi\left(1-g-r-\sum_{j=1}^{r}(d_j-1)\right),$$

giving an interpretation of difference between the right- and left-hand sides of (1). It is easy to see that each end E_j is embedded if and only if $d_j = 1$, from which it follows that a necessary condition for a complete minimal surface of finite total curvature to be embedded is that equality holds in (1):

$$(3) \qquad \int K\,dA = 4\pi(1-g-r) = 2\pi\big(\chi(\overline{M}) - 2r\big).$$

Another necessary condition is that the ends do not intersect each other, which implies that the values of the extended Gauss map agree (up to sign) at the ends $p_1,\cdots,p_r$.

We summarize these results as

Theorem 0. *Suppose $M = M - (p_1,\cdots,p_r)$ is a complete minimal surface of* *finite total curvature on which* (3) *holds and which satisfies the following property*: *if* $g\colon M \to \mathbf{C} \cup (\infty)$ *is the stereographic projection of the extended Gauss map, then after a rotation of* $\mathbf{R}^3$, $g(p_j) = 0$ *or* $\infty, j = 1, \cdots, r$. *Then the ends of* M *are embedded and parallel.*

The principal method for constructing complete minimal surfaces is the formula of Enneper-Weierstrass:

The Weierstrass representation formula

Let D be a Riemann surface, f an analytic function on D, and g a meromorphic function on D. Suppose further that g has a pole of order m at $z \in D$ if and only if f has a zero of order $2m$ at z. Form the $\mathbf{C}^3$-valued function

$$\phi(z) = (f/2)\big(1 - g^2, i(1 + g^2), 2g\big).$$

Then

$$X(z) = \operatorname{Re} \int_{z_0}^{z} \phi(z)\, dz \tag{4}$$

is a regular conformal minimal immersion, well-defined on the universal covering space of D. (If the components of $\phi\, dz$ have no real periods then X is well defined on D.) Furthermore g is the stereographic projection of the Gauss normal mapping of X.

Every simply-connected minimal surface may be represented in the form (4).

EXAMPLES. Consider the following three examples

1. $D = \mathbf{C}, f = 1, g = 0$
2. $D = \mathbf{C}, f = 1, g = z$
3. $D = \mathbf{C} - \{0\}, f = 1, g = 1/z$.

From (4) it can be derived that the metric on D is given by $\lambda^2 = |f|^2(1 + |g|^2)^2/2$. To show that each of these examples is a complete surface it is enough to establish that for every divergent curve γ on D,

$$\int_\gamma |f|\big(1 + |g|^2\big)\, dt = \infty.$$

This computation is straightforward in each case. In Example 3, one must verify that the components of ϕ have no real periods, but this is obvious. We conclude that each example represents a regular conformally immersed complete surface. Example 1 is easily seen to be the plane. Since the total curvature is the area of the Gaussian image, it follows that both Examples 2 and 3 have total curvature -4π. Since $r = 1$ in Example 2, its total curvature is too small for it to be embedded; in fact this surface is Enneper's minimal surface. Example 3 satisfies the necessary conditions of Theorem 0. In fact it is the catenoid, an embedded minimal surface of rotation. While this is not hard to prove, it is not immediately obvious from the Weierstrass representation. Prior knowledge of the symmetry of the surface suggests how to go about proving it is embedded.

In the next section we present the new embedded example. It is produced by using the Weierstrass $\wp$-function in the Weierstrass representation formula. To put this example in the proper context we mention some recent progress on embeddedness. In [7], R. Schoen proved that a complete embedded minimal surface of finite total curvature with two ends must be the catenoid; there is no assumption about the genus. In [5], Jorge and Meeks were able to establish that a complete minimal surface of finite total curvature with genus 0 could not be embedded if it had three, four, or five ends. Indeed, there are no known complete and embedded minimal surfaces of genus 0 except the plane, the catenoid and the helicoid. Recently, the authors have been able to show that for every genus greater than zero, there exist complete embedded minimal surfaces of finite total curvature, [2].

3. Existence of the surface: the $\wp$-function

In [1], Costa established the following result.

Theorem 1. *Let $P(z)$ be the Weierstrass $\wp$-function for the square lattice, L, generated by 1 and i. Let $T^2 = \mathbf{C}/L$ and p_0, p_1, and p_2 the points in T^2 corresponding to 0, $\omega_1 = \frac{1}{2}$ and $\omega_2 = \frac{i}{2}$. Let $a = 2\sqrt{2\pi}\,P(\frac{1}{2})$ and $D = T^2 - \{p_0, p_1, p_2\}$. The conformal minimal immersion $X\colon D \to \mathbf{R}^3$ defined by the Weierstrass representation formula* (4) *with $f = P$ and $g = a/P'$ is regular and the surface $M = X(D) \subset \mathbf{R}^3$ is complete with total curvature equal to -12π.*

We will not prove this theorem here. However, we wish to describe in a series of remarks some of the features of the proof, emphasizing those properties which are needed for our proof of embeddedness.

Remarks. 1. The choice of the constant a is forced by the requirement that the components of ϕ have no real periods.

2. The Weierstrass $\wp$-function for a lattice L is defined to be the function

$$P(z) = \frac{1}{z^2} + \sum_{\Omega \in L - \{0\}} \left(\frac{1}{z - \Omega^2} - \frac{1}{\Omega^2} \right). \tag{5}$$

It is the unique (up to a multiplicative constant) elliptic function on $\mathbf{C}$ with poles of order two exactly at the lattice points of L. In the next section we will use various properties of P to prove that M is highly symmetric. For complete details, see [1].

3. The derivative of P, P', is elliptic with poles of order 3 on L. Thus $g = a/P'$ covers the extended complex plane three times (considered as a map from T^2), so the total curvature of M is -12π.

We may consider $F = \{u + iv | 0 \leqslant u, v \leqslant 1\}$ to be identified with T^2 once we identify opposite edges and $D = T^2 - \{p_0, p_1, p_2\}$ to be identified with $F - \{0, \omega_1, \omega_2\}$ where $\omega_1 = \frac{1}{2}$, $\omega_2 = \frac{i}{2}$. It can be shown that $e_1 = P(\omega_1) = -P(\omega_2)$ and that e_1 is real and positive. In fact, to six decimal places, $e_1 = 6.875185$ (see [1]).

4. The only zero of P occurs at $\omega_3 = \frac{1}{2} + \frac{i}{2}$, and it is of order two. The zeros of P' in F occur at ω_1, ω_2, and ω_3 and are all simple. Thus the only pole in D of $g = a/P'$ occurs at ω_3 and it is simple. Since $f = P$ has a double zero at ω_3, the conformal minimal immersion given by (4) with $f = P$, $g = a/P'$ is regular on $D = F - \{0, \omega_1, \omega_2\}$.

Moreover, the Weierstrass 1-forms (see (4))

$$(6)\quad \phi_1\,dz = \frac{P}{2}\left(1 - \frac{a^2}{P'^2}\right)dz,\quad \phi_2\,dz = \frac{iP}{2}\left(1 + \frac{a^2}{P'^2}\right)dz,\quad \phi_3\,dz = \frac{aP}{P'}dz$$

have the property that at each end point $0, \omega_1, \omega_2$, at least one ϕ_i has a pole of order two or more. This essentially implies that $X(D) = M$ is complete.

Proposition 1.

(i) *Each of the ends of D is embedded and parallel.*

(ii) *Outside of a sufficiently large compact set $K \subset D$, X is an embedding.*

(iii) *The third coordinate function $X_3(z)$ has the following behavior as one approaches the ends of M:*

$$As\ z \to \omega_1,\ X_3(z) \to -\infty;$$

$$As\ z \to \omega_2,\ X_3(z) \to +\infty.$$

As $z \to 0$, $X_3(z) \to 0$ and the end E_0 is asymptotic to a plane $x_3 =$ constant.

Proof. (i) D has genus 1 with $r = 3$ and total curvature -12π; therefore (3) is satisfied. From Remark 4, we have $g(0) = \infty$, and $g(\omega_1) = g(\omega_2) = 0$. Statement (i) now follows from Theorem 0.

(ii) This follows from (i) and (iii): Statement (i) says each end is embedded and (iii) implies that the ends are mutually disjoint.

(iii) To prove this statement, we will compute $X_3(z)$. We start with the well-known identity

$$P'^2 = 4(P - e_1)(P - e_2)(P - e_3),$$

where $P(\omega_i) = e_i$, $i = 1, 2, 3$. In our case $e_1 = -e_2$ and $e_3 = 0$. Therefore

$$P'^2 = 4P(P - e_1)(P + e_1).$$

Thus using (6), we have

$$\phi(z)\,dz = \frac{1}{2}\left(P - \frac{a^2P}{P'^2}, i\left(P + \frac{a^2P}{P'^2}\right), \frac{2aP}{P'}\right)dz$$

$$(7) \qquad = \frac{1}{2}\left(P - \frac{a^2}{4(P - e_1)(P + e_1)}, i\left(P + \frac{a^2}{4(P - e_1)(P + e_1)}\right), \frac{aP'}{2(P - e_1)(P + e_1)}\right)dz.$$

It follows that

$$X_3(z) = \operatorname{Re}\int_{z_0}^{z} \phi_3(\zeta)\,d\zeta = \frac{\tilde{a}}{4}\operatorname{Re}\int_{z_0}^{z} \frac{P'}{2}\left[\frac{1}{P - e_1} - \frac{1}{P + e_1}\right]d\zeta$$

$$(8) \qquad = \frac{\tilde{a}}{8}\left[\ln\left|\frac{P(z) - e_1}{P(z) + e_1}\right| - \text{constant}\right], \qquad \tilde{a} = a/e_1.$$

As $z \to 0$, $P(z) \to \infty$, so $X_3(z)$ converges to a constant value. As $z \to \omega_1$ (resp. ω_2), $P(z) \to e_1$ (resp. $e_2 = -e_1$), so $X_3(z) \to -\infty$ (resp. $+\infty$).

Remark 5. We will choose $z_0 = \omega_3$. With that choice,

$$(9) \qquad X_3(z) = \frac{\tilde{a}}{8}\ln\left|\frac{P(z) - e_1}{P(z) + e_1}\right|,$$

$$(10) \qquad X_i(z) = \operatorname{Re}\int_{\omega_3}^{z} \phi_i(z)\,dz, \qquad i = 1, 2.$$

Symmetries of the 𝔓-function. The symmetries of M are a consequence of the symmetries of P. The ones we need are collected in Lemma 1 below. For completeness we give a construction of the 𝔓-function from which these symmetry properties follow easily.

The 𝔓-function may be constructed as follows. Consider the triangular domain $\mathrm{I} \subset F$ with vertices ω_3, ω_1, and 1 (see Figure 1.1). Map I onto the positive quadrant by a Möbius transformation, say $f(z)$, with $f(\omega_3) = 0$, $f(\omega_1) = e_1$, and $f(1) = \infty$. On the diagonal $\overline{\omega_3, 1}$, $f(z)$ is purely imaginary with $|f(z)|$ increasing, and $f(z)$ is real and increasing on the segments $\overrightarrow{\omega_3\omega_1}$ and $\overrightarrow{\omega_1 1}$. Extend $f(z)$ to triangle II by reflection across $\overrightarrow{\omega_3, 1}$; for $z \in \mathrm{II}$, $f(z) = \overline{-f(\mu(z))}$ where $\mu(\omega_3 + z) = \omega_3 - i\bar{z}$. Now $f(z)$ is real on the line segment $\overrightarrow{\omega_3, \omega_2 + 1}$. Therefore, it may be extended by reflection to $\mathrm{III} \cup \mathrm{IV}$, the square with vertices ω_3, $\omega_2 + 1$, $1 + i$, and $\omega_1 + i$; $f(z) = \overline{f(\beta(z))}$ where $\beta(\omega_3 + z)$

$= \omega_3 + \bar{z}$. The function $f(z)$ is now defined on the right half of F and is real on $\overrightarrow{\omega_1, \omega_1 + i}$. Therefore it may be extended to F by reflection across this line. Now extend f to $\mathbf{C}$ by making it periodic with periods 1 and i. The result is a meromorphic function with poles of order two at the lattice points L, and zeros of order two precisely at the points congruent to ω_3. Thus $f(z)$ is equal to $P(z)$ up to a multiplicative constant. Since $f(\omega_1) = P(\omega_1)$, $f(z) = P(z)$. From this construction, the following lemma is easily proved.

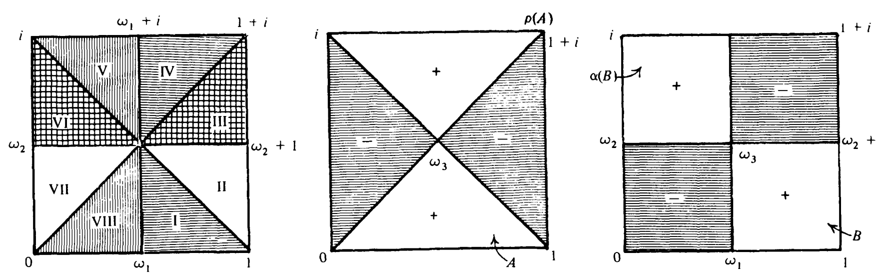
Lemma 1. *Let $P(z)$ be the Weierstrass $\mathfrak{P}$-function for the unit-square lattice*:

(i) $P(\mu(\omega_3 + z)) = -\overline{P(\omega_3 + z)}$, where $\mu(\omega_3 + z) = \omega_3 - i\bar{z}$;
$P(\rho(\omega_3 + z)) = -P(\omega_3 + z)$, where $\rho(\omega_3 + z) = \omega_3 + iz$;
$P(\beta(\omega_3 + z)) = \overline{P}(\omega_3 + z)$, where $\beta(\omega_3 + z) = \omega_3 + \bar{z}$;

(ii) $P(z)$ restricted to the vertical line segment $\overrightarrow{\omega_3, \omega_1}$ is a monotonically increasing map onto the interval $[0, e_1]$; $P(z)$ restricted to the horizontal line segment $\overrightarrow{\omega_1, 1}$ is a monotonically increasing map onto $[e_1, \infty]$; $P(z)$ restricted to the diagonal line segment $\overline{\omega_3, 1}$ maps this line segment monotonically onto the nonnegative imaginary axis.

(iii) $\mathrm{Re}(P(z)) > 0$ on the interior of the two triangles A and $\beta(A)$ where A is the triangle with vertices 0, ω_3, and 1; $\mathrm{Re}(P(z)) < 0$ on the interior of $F - (A \cup \beta(A))$ (see Figure 1.2).

(iv) $\mathrm{Im}(P(z)) > 0$ on the interior of the two squares B and $\alpha(B)$ where $\alpha(\omega_3 + z) = \omega_3 + i\bar{z} = \rho(\beta(\omega_3 + z))$ and B is the square with vertices ω_3, ω_1, 1, and $\omega_2 + 1$ (see Figure 1.3).



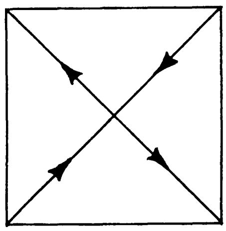
1.1

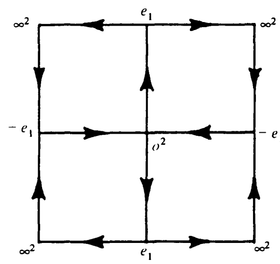
1.2 The sign of $\mathrm{Re}(P(z))$

1.3 The sign of $\mathrm{Im}(P(z))$

FIGURE 1

Remark 6. It follows from parts (i) and (ii) of Lemma 1 that $P(z)$ is purely imaginary on the diagonals through ω_3. Moving away from ω_3, $|P(z)|$ is increasing from 0 to ∞ as one approaches the corners of F. Furthermore the diagonal with positive slope is mapped onto the nonpositive imaginary axis, the one with negative slope is mapped onto the nonnegative imaginary axis. On





$P(z)$ real on these lines. Arrow indicates direction of increasing $P(z)$.
Values of $P(z)$ given at vertices.

$P(z)$ imaginary or diagonal lines, with "negative" values on the positively sloped diagonal, positive values on the other.

FIGURE 2

each of the horizontal and vertical line segments bounding B, $\alpha(B)$, and $F - (B \cup \alpha(B))$, $P(z)$ is real and monotonic (see Figure 2.)

Lemma 2. *Let $X_3(z)$ denote the third component of $X(z)$, $z \in D$. Then $X_3(z) > 0$ if and only if z lies in the interior of $D \setminus (A \cup \beta(A))$, and $X_3(z) = 0$ if and only if z lies on $\mathscr{L}$ where*

$$\mathscr{L} = \overrightarrow{0, 1+i} \cup \overrightarrow{i, 1}, \tag{11}$$

the union of the two diagonals of F. Moreover, X maps the diagonal with positive slope one to one and onto the line $x_1 - x_2 = x_3 = 0$, and the diagonal with negative slope one to one and onto the line $x_1 + x_2 = x_3 = 0$.

Proof. By (9),

$$X_3(\tilde{z}) = \frac{a}{8} \ln \left| \frac{P(z) - e_1}{P(z) + e_1} \right|,$$

where $\tilde{a}$ is real and positive. Since e_1 is real and positive, $X_3(z) > 0$ if and only if $P(z)$ is closer to $-e_1$ than to e_1; i.e. if and only if $\operatorname{Re}(P(z)) < 0$. Also $X_3(z) = 0$ if and only if $\operatorname{Re}(P(z)) = 0$. By virtue of Lemma 1, parts (ii) and (iii), we have that $X_3(z) \leqslant 0$ if and only if $z \in A \cup \beta(A)$ and $X_3(z) = 0$ if and only if $z \subset \mathscr{L}$. All that is left to prove is the last statement of the lemma.

Consider the diagonal line segment $\overline{\omega_3, 1+i} \subset \mathscr{L}$. Parametrize this segment by $\eta(t) = (\frac{1}{2} + t) + (\frac{1}{2} + t)i$, $0 \leqslant t \leqslant \frac{1}{2}$. By Remark 6 following Lemma 1, $P(\eta(t)) = i\lambda(t)$ for a real-valued function $\lambda(t)$ with $\lambda(0) = 0$, $\lambda(\frac{1}{2}) = -\infty$ and $\lambda'(t) < 0$, $0 < t < \frac{1}{2}$. From (7), we have

$$\phi_1(\eta(t))\, d\eta(t) = \frac{1}{2}\left[i\lambda(t) + \frac{a^2}{4} \frac{1}{\lambda^2(t) + e_1^2} \right](dt + i\, dt),$$

$$\phi_2(\eta(t))\, d\eta(t) = \frac{i}{2}\left[i\lambda(t) - \frac{a^2}{4} \frac{1}{\lambda^2(t) + e_1^2} \right](dt + i\, dt),$$

which implies

$$\text{(12)} \qquad \begin{aligned} \operatorname{Re}(\phi_1(\eta(t))\,d\eta(t)) &= \frac{1}{2}\left[-\lambda(t) + \frac{a^2}{4}\left(\frac{1}{\lambda^2(t) + e_1^2}\right)\right] dt \\ &= \operatorname{Re}(\phi_2(\eta(t))\,d\eta(t)). \end{aligned}$$

From (10) it follows that $X_1(\eta(t)) = X_2(\eta(t))$. Since

$$-\lambda(t) + \frac{a^2}{4}\frac{1}{\lambda^2(t) + e_1^2} > 0,$$

$X_i(\eta(t))$ is monotonic increasing, with $X_i(\eta(0)) = 0$, $i = 1, 2$. Clearly $\operatorname{limit}_{t\to 1/2} X_i(\eta(t)) = \infty$, $i = 1, 2$.

Using the properties of P established in Lemma 1, it is now straightforward to complete the proof of the Lemma.

4. The symmetries of M

Let G be the dihedral group with 8 elements. We may consider G as acting on the square F by reflections through the horizontal, vertical, and diagonal lines through ω_3 and rotation by integer multiples of $\pi/2$ about ω_3. In complex notation,

$\beta(\omega_3 + z) = \omega_3 + \bar{z}$	is reflection about the horizontal line,
$\rho^k(\omega_3 + z) = \omega_3 + (i)^k z$	is rotation by $k\pi/2$ about ω_3, $k = 1, 2, 3$,
$\alpha(\omega_3 + z) = \rho(\beta(\omega_3 + z)) = \omega_3 + i\bar{z}$	is reflection through the positive diagonal,
$\hat{\alpha}(\omega_3 + z) = \rho^2(\beta(\omega_3 + z)) = \omega_3 - \bar{z}$	is reflection through the vertical line, and
$\mu(\omega_3 + z) = \rho^3(\beta(\omega_3 + z)) = \omega_3 + i\bar{z}$	is reflection through the negative diagonal.

Clearly G is generated by β and ρ.

We may also consider G as acting on $\mathbf{R}^3$ by identifying the generators β and ρ with the orthogonal motions

$$\text{(13)} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

B is reflection in the (x_1, x_3)-plane and R is rotation by $\pi/2$ about the x_3-axis followed by reflection in the (x_1, x_2)-plane. It is easy to see that G, acting on F, induces an action of G on $D = T^2 - \{p_0, p_1, p_2\}$.

Theorem 2. *G, acting on $\mathbf{R}^3$, is a symmetry group of $M = X(D) \subset \mathbf{R}^3$. The immersion X: $D \to \mathbf{R}^3$ is compatible with the action of G on D and $\mathbf{R}^3$. Specifically* (i) $X \circ \rho = R \circ X$ *and* (ii) $X \circ \beta = B \circ X$. *In the metric on D induced by X, G acts by isometries.*

Proof. All that is necessary to establish the proof of the theorem is to prove (i) and (ii). By Lemma 1(i), $P(\rho(\omega_3 + z) = -P(\omega_3 + z))$. By formula (7) we have

$$\begin{aligned} \phi_1(\rho(\omega_3 + z)) &= \frac{1}{2}\left[-P(\omega_3 + z) - \frac{a^2}{4}\left(\frac{1}{P^2(\omega_3 + z) - e_1^2}\right)\right] \\ &= i\phi_2(\omega_3 + z). \end{aligned} \tag{14.1}$$

From (14.1) it follows that $i\phi_2(\rho(\omega_3 + z)) = i\phi_2(\omega_3 + iz) = \phi_1(\omega_3 - z)$. But $\phi(\omega_3 + z) = \phi(\omega_3 - z)$ since $P(\omega_3 + z) = P(\omega_3 - z)$. Hence

$$\phi_2(\rho(\omega_3 + z)) = -i\phi_1(\omega_3 + z). \tag{14.2}$$

Using (14.1) and (14.2) and (10) we have

$$\begin{aligned} (X_1, X_2)(\rho(\omega_3 + z)) &= (X_1, X_2)(\omega_3 + iz) \\ &= \frac{1}{2}\operatorname{Re}\int_{\omega_3}^{z} (\phi_1, \phi_2)(\omega_3 + i\zeta)i\, d\zeta \\ &= \operatorname{Re}\int_{\omega_3}^{z} -(\phi_2, -\phi_1)(\omega_3 + \zeta)\, d\zeta \\ &= \operatorname{Re}\int_{\omega_3}^{z} (-\phi_2, \phi_1)(\omega_3 + \zeta)\, d\zeta \\ &= (-X_2, X_1)(\omega_3 + z). \end{aligned} \tag{15}$$

By (9),

$$X_3(\rho(\omega_3 + z)) = \frac{\tilde{a}}{8}\ln\left|\frac{-P(\omega_3 + z) - e_1}{-P(\omega_3 + z) + e_1}\right| = -X_3(\omega_3 + z).$$

This together with (15) is statement (i) of the Theorem. Statement (ii) is proved in a similar fashion. By Lemma 1(i), $P(\beta(\omega_3 + z)) = \overline{P}(\omega_3 + z)$. Using (7) and the fact that a is real, we have

$$\begin{aligned} \phi_1(\beta(\omega_3 + z)) &= \frac{1}{2}\left[\overline{P}(\omega_3 + z) - \left(\frac{a^2}{4}\right)\frac{1}{\overline{P}^2(\omega_3 + z) - e_1^2}\right] \\ &= \overline{\phi_1(\omega_3 + z)}, \end{aligned} \tag{16.1}$$

$$\begin{aligned} \phi_2(\beta(\omega_3 + z)) &= \phi_2(\omega_3 + \bar{z}) \\ &= \frac{i}{2}\left[\overline{P}(\omega_3 + z) + \frac{a^2}{4}\frac{1}{\overline{P}^2(\omega_3 + z) - e_1^2}\right] \\ &= \overline{-\phi_2(\omega_3 + z)}. \end{aligned} \tag{16.2}$$

Hence

$$
\begin{aligned}
(17) \qquad (X_1, X_2)(\beta(\omega_3 + z)) &= \frac{1}{2} \operatorname{Re} \int_{\omega_0}^{z} (\bar{\phi}_1, -\bar{\phi}_2)(\omega_z + \zeta)\, d\bar{\zeta} \\
&= (X_1, -X_2)(\omega_3 + z).
\end{aligned}
$$

By (9),

$$
\begin{aligned}
X_3(\beta(\omega_3 + z)) &= \frac{\tilde{a}}{8} \ln \left| \frac{\bar{P}(\omega_3 + z) - e_1}{\bar{P}(\omega_3 + z) + e_1} \right| \\
&= \frac{\tilde{a}}{8} \ln \left| \frac{P(\omega_3 + z) - e_1}{P(\omega_3 + z) + e_1} \right| \\
&= X_3(\omega_3 + z).
\end{aligned}
$$

This establishes statement (ii).

Corollary 1. (i) *The (x_1, x_3)-planes and the (x_2, x_3)-planes are planes of symmetry for $M = X(D)$. The line segments $\overrightarrow{\omega_2, 1 + \omega_2}$ and $\overrightarrow{0, 1}$ are mapped into the (x_1, x_3)-plane. The line segments $\overrightarrow{\omega_1, \omega_1 + i}$ and $\overrightarrow{0, i}$ are mapped into the (x_2, x_3)-plane.*

(ii) *The isometry $\alpha = \rho \circ \beta$ of D which is reflection across the positive diagonal of F, is compatible with the symmetry of M given by the Euclidean motion RB, which is rotation by π about the line $x_1 - x_2 = x_3 = 0$.*

Remark 7. If we decompose F into eight triangles I,$\cdots$,VIII as in Figure 1.1, it follows from Theorem 2 that each of the triangles is isometric to all of the others by an isometry in G.

From Theorem 2 and the above remark, the following corollary is immediate.

Corollary 2. *M is made up of eight congruent pieces. Each piece is isometric to $X(T)$, where $T \subset F$ is the triangle*

$$
(18) \qquad \{u + iv \mid \tfrac{1}{2} \leqslant u \leqslant 1, \tfrac{1}{2} \leqslant v \leqslant u\} - \{((\omega_2 + 1), (1 + i))\}
$$

and is produced from $X(T)$ by moving $X(T)$ by a symmetry in G.

Note. T is the triangle III in Figure 1.1.

5. M is embedded

In this section, we will prove that M is an embedded surface.

Theorem 3. *Let X: $D \to \mathbf{R}^3$ be as in Theorem* 1. *X is an embedding.*

Proof. Let $T \subset D$ be the triangle defined in (18). We will show in Proposition 3 below, that $X|T$ is an embedding, that $X(T)$ lies in the positive octant of $\mathbf{R}^3$, and X takes ∂T into the boundary of the positive octant. The eight elements of G, acting on $\mathbf{R}^3$, each moves the positive octant into a distinct octant of $\mathbf{R}^3$. Theorem 3 then follows from Corollary 2 to Theorem 2.

To prove Proposition 3, we will need some preliminary results. Loosely stated, the point of these four technical results is to show that $X(\partial T)$ lies on the boundary of the positive octant.

Lemma 3. *For $\delta > 0$ sufficiently small, X is an embedding on*

$$\{z \in D \mid |X_3(z)| < \delta\}.$$

Proof. We know by Proposition 1(ii), that X is an embedding outside of a compact set K and that $X(K)$ and $X(D \setminus K)$ are disjoint sets. We have established in Lemma 2 that the set $\mathscr{L} = \overrightarrow{0, 1+i} \cup \overrightarrow{i, 1}$ is equal to $X_3^{-1}(0)$ and that X is one-to-one on $\mathscr{L}$. Since X is an immersion it is locally one-to-one. Restricting our attention to K, the closed subset $\mathscr{L} \cap K$ is compact and therefore must have a neighborhood N in K on which X is one-to-one. For $\delta > 0$ sufficiently small, the set $\{z \in K \mid |X_3(z)| < \delta\}$ must lie in N. Since X is one-to-one on both N and $D \setminus K$, and $X(N)$, and $X(D \setminus K)$ are disjoint, it follows that X is one-to-one on

$$\{z \in \mid |X_3(z)| < \delta\}. \tag{19}$$

Note. It is possible to prove this lemma by arguing directly from the form of the third coordinate function $X_3(z)$, given in (10). We know that $X_3^{-1}(0)$ consists of the two diagonal lines through ω_3 and we choose K to contain ω_3. The level curves of X_3 (at nonzero values) are given by the inverse image under P of circles in the complex plane which are at constant distance from both e_1 and $-e_1$. Using these facts one can given an alternate proof of the lemma.

Lemma 4. *The line segment $\overrightarrow{\omega_3, \omega_3 + \frac{1}{2}}$ is mapped by X in a one-to-one manner onto a curve lying in the nonnegative quadrant of the (x_1, x_3)-plane. The curve meets an axis only at $X(\omega_3) = 0$.*

Proof. By Corollary 1(i), $\overrightarrow{\omega_3, \omega_3 + \frac{1}{2}}$ is mapped into the (x_1, x_3)-plane. Parametrize this line segment by $\gamma(t) = \omega_3 + t$, $0 \leqslant t \leqslant \frac{1}{2}$. By Lemma 1, parts (i) and (ii), $P(\gamma(t)) = \lambda(t)$ with $\lambda(0) = 0$, $\lambda(\frac{1}{2}) = -e_1$, and $\lambda'(t) < 0$ on $(0, \frac{1}{2})$. From (7) and (10) we have

$$X_1(\gamma(t)) = \frac{1}{2}\int_0^t \frac{\left(\lambda(t)\left(\lambda^2(t) - e_1^2\right) - a^2/4\right)}{\lambda^2(t) - e_1^2}\, dt, \tag{20}$$

$$X_3(\gamma(t)) = \frac{\tilde{a}}{8} \ln\left|\frac{\lambda(t) - e_1}{\lambda(t) + e_1}\right|. \tag{21}$$

Clearly $X_3(\gamma(t))$ is monotonically increasing on $[0, \frac{1}{2})$ with $X_3(\gamma(t)) = 0$ if and only if $t = 0$. Therefore $X(\gamma(t))$ is one-to-one and $X_3(\gamma(t)) \geqslant 0$ with equality if and only if $t = 0$. To complete the proof of the lemma we now show that $X_1(\gamma(t)) \geqslant 0$ with equality if and only if $t = 0$.

In fact, we will show that the integrand of (20) is strictly positive, from which the lemma follows. For $t \in [0, \frac{1}{2}]$, $\lambda(t) \in [-e_1, 0]$. Let $y(\lambda) = \lambda(\lambda^2 - e_1^2)$ and consider $y(\lambda)$ on $[-e_1, 0]$. The positivity of the integrand of (20) is equivalent to $y(\lambda) < a^2/4$ on $[-e_1, 0]$. We will now show this is the case.

Since $y(0) = y(-e_1) = 0$, $y(\lambda) \geqslant 0$ on $[-e_1, 0]$, and $y'(\lambda_0) = 0$ on $[-e_1, 0]$ only at $\lambda_0 = -e_1/\sqrt{3}$,

$$0 \leqslant y(\lambda) \leqslant y(\lambda_0) = 2e_1^3/3\sqrt{3}\,.$$

However $a^2/4 = 2\pi e_1^2$ and $e_1 < 6.9$, by Remark 3. Hence $y(\lambda) < a^2/4$. We are done.

Lemma 5. *The line segment $\overrightarrow{\omega_2, i}$ is mapped by X into the upper half of the (x_2, x_3)-plane. It is a graph over the positive x_3-axis and the value of X_2 tends to $+\infty$ as one approaches ω_2 or i.*

Proof. By Corollary 1(i), $X(\overrightarrow{\omega_2, i})$ lies in the (x_2, x_3)-plane. Parametrize $\overrightarrow{\omega_2, i}$ by ti, $\frac{1}{2} < t < 1$. By Lemma 1 and Remark 6, $P(ti) = K(t)$ for some real-valued function $K(t)$ with $K'(t) < 0$, $\text{limit}_{t \to 1/2} K(t) = -e_1$, and $\text{limit}_{t \to 1} K(t) = -\infty$.

Using these facts together with (7), (9) and (10) we have

$$(22) \qquad X_3(ti) = \frac{\tilde{a}}{8} \ln \left| \frac{K(t) - e_1}{K(t) + e_1} \right|,$$

$$(23) \qquad X_2(ti) = X\left(\frac{3}{4}i\right) - \int_{3/4}^{t} \left(\frac{1}{2}K(t) + \frac{a^2}{8} \left(\frac{1}{K^2(t) - e_1^2} \right) \right) dt,$$

$$(24) \qquad X_1(ti) \equiv 0.$$

Using (23) and the properties of $K(t)$ we also have

$$(25) \qquad \lim_{t \to 1/2} X_2'(ti) = \lim_{t \to 1} X_2'(ti) = \infty.$$

From the properties of $K(t)$ and equation (22) it follows that $X_3(ti)$ is positive, monotonically decreasing and

$$\lim_{t \to 1/2} X_3(ti) = +\infty, \qquad \lim_{t \to 1} X_3(ti) = 0.$$

Therefore $X(ti)$ one-to-one. Moreover, we can reparametrize by $s = X_3(ti)$, $0 < s < \infty$, and write the curve in question as

$$\Gamma(s) = X\big(X_3^{-1}(s)\big) = \big(0, X_2\big(X_3^{-1}(s)\big), s\big). \tag{26}$$

As $s \to 0$, $\Gamma(s)$ is a divergent curve in the embedded end E_0 and as $s \to \infty$, $\Gamma(s)$ diverges in the embedded end E_2. Since these ends are asymptotically parallel to the (x_1, x_2)-plane,

$$\lim_{s\to 0} |\Gamma_2(s)| = \lim_{s\to\infty} |\Gamma_2(s)| = \infty.$$

By (25) it follows that

$$\lim_{s\to 0} \Gamma_2(s) = \lim_{s\to\infty} \Gamma_2(s) = +\infty. \tag{27}$$

As a consequence of Proposition 2 below, $\overrightarrow{\omega_2, i}$ is actually mapped into the interior of the positive quadrant of the (x_2, x_3)-plane.

Let T be the triangle defined in (18) and let $R = T \cup \hat{\alpha}(T)$. In Figure 1.1, $R = \mathrm{III} \cup \mathrm{VI}$.

Proposition 2. *X maps $\mathring{R}$ into $E = \{(x_1, x_2, x_3) | x_2 > 0, x_3 > 0\}$.*

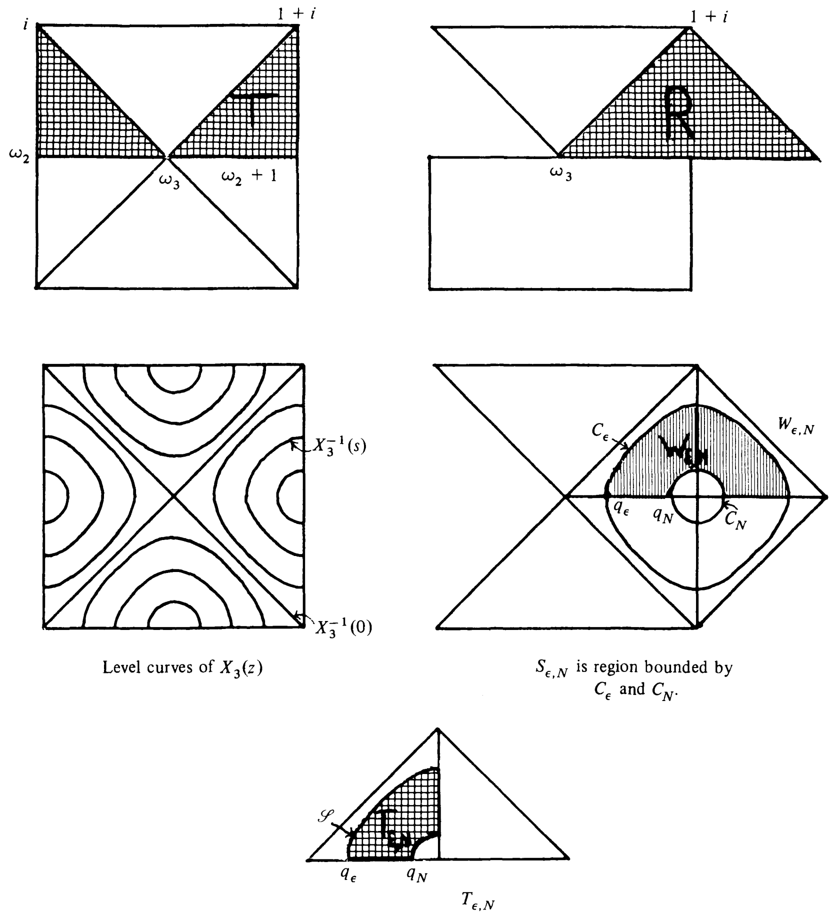
Proof. Let

$$S_{\varepsilon, N} = \left\{ z \in D | \varepsilon \leqslant X_3(z) \leqslant N \right\}$$

(see Figure 3). According to Proposition 1 and Lemma 3, if we chose $\varepsilon > 0$ small enough and N large enough then X is an embedding of the complement of $S_{\varepsilon,N}$ in $\{z \in D | X_3 > 0\}$. In particular X, restricted to $C_\varepsilon = X_3^{-1}(\varepsilon)$ and to $C_N = X_3^{-1}(N)$, is an embedding. By (9), these curves are level sets of $\ln|w(z)|$, where

$$w(z) = \frac{P(z) - e_1}{P(z) + e_1}$$

and are easily seen to be simple closed curves. Furthermore, C_ε and C_N are symmetric under reflection across $\overrightarrow{\omega_2, \omega_2 + 1}$ and their images under X, which we shall label C_ε' and C_N', are symmetric with respect to reflection through the (x_1, x_3)-plane and the (x_2, x_3)-plane. These facts are an immediate consequence of Theorem 2. Since they are simple, each of the arcs C_ε and C_N meet $\overrightarrow{\omega_2, \omega_2 + 1}$ exactly twice at points symmetric with respect to the midpoint, ω_3. We label these pairs of points q_ε, $\hat{\alpha}(q_\varepsilon)$ and q_N, $\hat{\alpha}(q_N)$, respectively.



Level curves of $X_3(z)$

$S_{\epsilon,N}$ is region bounded by C_ϵ and C_N.

FIGURE 3

The points q_ε and $\hat{\alpha}(q_\varepsilon)$ divide C_ε into two arcs one of which, C_ε^+, is mapped by X into the half-space $\{x_2 > 0\}$. It must be the arc which cuts $\overrightarrow{\omega_2, i}$ since we are free to choose ε small enough, by Lemma 5, so that C_ε must cut $\overrightarrow{\omega_2, i}$ at a point where X_2 is positive. Applying the same argument to C_N, we can assert that the subarc, C_N^+, of C_N on which $X_2 > 0$ is the one which crosses $\overrightarrow{\omega_2, i}$.

Let $W_{\varepsilon,N}$ be the intersection of R with $S_{\varepsilon,N}$. We now know that

$$X(\partial W_{\varepsilon,N}) = X\left(C_\varepsilon^+ \cup C_N^+ \cup \overrightarrow{q_\varepsilon q_N} \cup \hat{\alpha}\left(\overrightarrow{q_\varepsilon q_N}\right)\right)$$

$$\subset \partial\left\{ (x_1, x_2, x_3) | x_2 > 0, \varepsilon < x_3 < N \right\} \subset E.$$

By the convex hull property for minimal surfaces, $W_{\varepsilon,N}$ is contained in

$$\left\{ (x_1, x_2, x_3) | x_2 > 0, \varepsilon \leqslant x_3 \leqslant N \right\}.$$

Letting $\varepsilon \to 0$ and $N \to \infty$ finishes the proof.

Proposition 3. *Let T be the triangle defined in* (18)

$$T = \left\{ u + iv | \tfrac{1}{2} \leqslant u \leqslant 1, \tfrac{1}{2} \leqslant v \leqslant u \right\} - \left\{ \left(\omega_3 + \tfrac{1}{2} \right), (1 + i) \right\}.$$

Then $X|_T$ is an embedding of T into the nonnegative octant of $\mathbf{R}^3$, taking T into the boundary of the octant, and $\mathring{T}$ into the interior.

Proof. The boundary of the T consists of the line segments $\overrightarrow{\omega_3, \omega_3 + \frac{1}{2}}$, $\overrightarrow{\omega_3, 1 + i}$, and $\overrightarrow{\omega_2, i}$. $\overrightarrow{\omega_3, \omega_3 + \frac{1}{2}}$ is mapped monotonically onto a curve in the nonnegative quadrant of the (x_1, x_3)-plane, meeting the axes only at $X(\omega_3) = 0$ (see Lemma 4.) $\overrightarrow{\omega_3, 1 + i}$ is mapped monotonically onto the ray $x_1 - x_2 = x_3 = 0$, $x_1 \geqslant 0$ (see Lemma 2.) $\overrightarrow{\omega_2, i}$ is mapped monotonically into the interior of the positive quadrant of the (x_1, x_3)-plane. (This follows from Lemma 5 and Proposition 2.) Let Q denote the closure of the positive octant of $\mathbf{R}^3$. We have established that $X(\partial T) \subset \partial Q$.

Consider $T_{\varepsilon,N} = W_{\varepsilon,N} \cap T$. It is straightforward to restate the argument of Proposition 2 applied now to $T_{\varepsilon,N}$ to show that $X(T_{\varepsilon,N})$ is contained in Q. We conclude that X maps T into Q (and necessarily takes $\mathring{T}$ into $\mathring{Q}$).

To complete the proof we need to show that X is an embedding; we will do this by actually proving that $X(T)$ is a graph over the plane $P = \{ x \in \mathbf{R}^3 | x \cdot \vec{u} = 0 \}$, where $\vec{u} = (1, 1/2, 0)$. Let π denote the projection of $\mathbf{R}^3$ onto P. We assert that $\pi \circ X \colon T_{\varepsilon,N} \to P$ is actually a submersion which is one-to-one on the boundary of $T_{\varepsilon,N}$. Since $T_{\varepsilon,N}$ is simply connected this implies that $\pi \circ X$ is actually one-to-one on $T_{\varepsilon,N}$ and that $X(T_{\varepsilon,N})$ is a graph over P. Since the disks $T_{\varepsilon,N}$ exhaust T, T itself is embedded as a graph over P.

Proof of the assertion. Let $\sigma \colon S^2 \to \mathbf{C} \cup \{\infty\}$ denote stereographic projection. Recall that the Gauss map of X is given by $\sigma^{-1} \circ g$ where $g = a/P'$. From Lemma 1 which lists properties of P, it is easy to show that $g(T)$ is one of the regions in $\mathbf{C} \cup \{\infty\}$ bounded by the lines $\{ti | t \leqslant 0\}$ and $\{t(-1 + i) | t \geqslant 0\}$. By Remark 3 and Corollary 2 the total curvature of $X(T)$ is $-3\pi/2$. Since the total curvature of $X(T)$ is the area of the Gaussian image it follows that $g(T)$ is the "smaller sector" which contains the negative real axis. Therefore $\sigma^{-1} \circ g(T)$ is a region lying in the hemisphere $H = \{ n | n \cdot \vec{u} \leqslant 0 \}$ which meets $\partial H = P \cap S^2$ only at the points $(0, 0, \pm 1)$. Noting that $\sigma^{-1} \circ g(z) = (0, 0, \pm 1)$ on T only at ω_3 where $g(\omega_3) = 0$ and that $\omega_3 \notin T_{\varepsilon,N}$, we conclude that $\sigma^{-1} \circ g(T_{\varepsilon,N})$ is a compact subset of the interior of H. This means that projection of $X(T_{\varepsilon,N})$ onto P is regular: hence $\pi \circ X$ is a submersion of $T_{\varepsilon,N}$.

To complete the proof of the assertion we must show that X is one-to-one on $\partial T_{\varepsilon,N}$. This boundary consists of four arcs two of which are line segments. The line segment $\overrightarrow{q_\varepsilon q_N}$ is mapped by X one-to-one into the (x_1, x_3)-plane which projects one-to-one onto P. The other line segment bounding $T_{\varepsilon,N}$ lies on $\overrightarrow{\omega_2, i}$ which is mapped one-to-one into the (x_2, x_3)-plane and this plane also projects one-to-one onto P. Now let's look at the subarc $\mathscr{S}$ of C_ε^+ which bounds $T_{\varepsilon,N}$. Since X maps $\mathscr{S}$ one-to-one into the plane $x_3 = N$, $\pi \circ X$ maps $\mathscr{S}$ into the line $P \cap (x_3 = N)$. But the projection π is locally one-to-one along $X(\mathscr{S})$ so it must be globally one-to-one. A similar argument shows that $\pi \circ X$ is one-to-one on the subarc of C_N^+ which bounds $T_{\varepsilon,N}$.

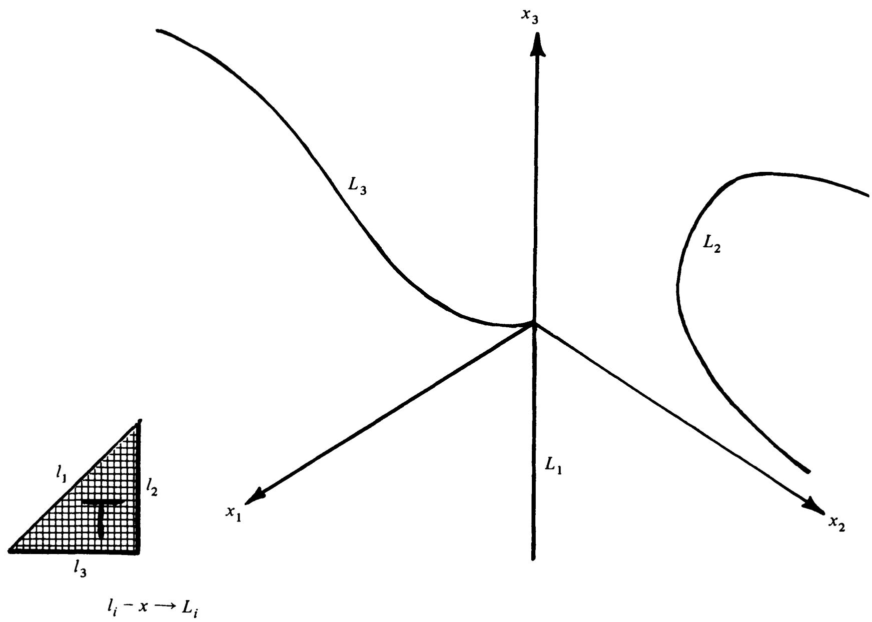
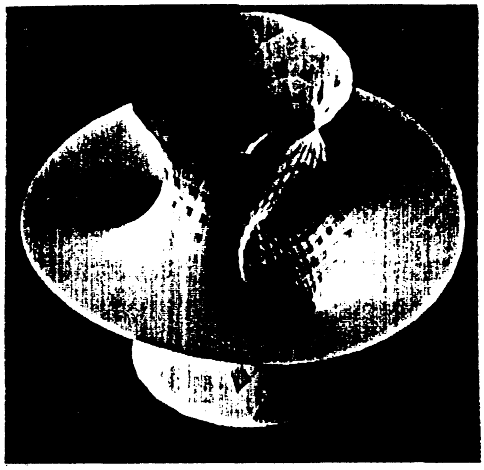


FIGURE 4

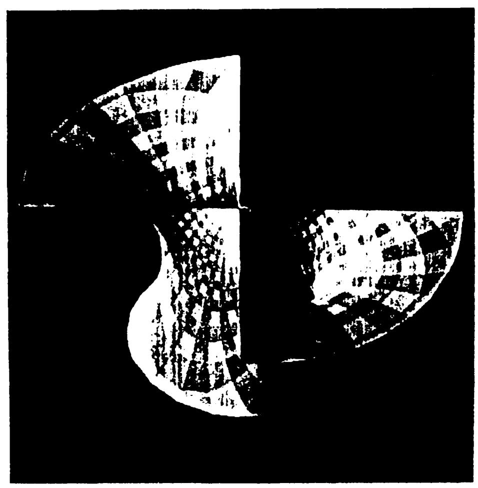
Remark 8. We note that in the proof of Proposition 3, the choice of the plane P is somewhat arbitrary. In fact we could use the same argument to show that $X(T)$ is a graph over the plane P_t, the plane orthogonal to $\vec{u}_t = (1, t, 0)$, for t, $0 < t < 1$. With slightly more work, it can be shown that $X(T)$ is a graph over P_0, the (x_2, x_3)-plane.

It is also the case that $X(R)$ is a graph over P_t, $0 \leqslant t < 1$.



$M \cap \{x \mid |x| < 3\}$

$M \cap \{x \mid |x| < 3, x_2 > 0\}$



$M \cap \{x \mid |x| < 3, x_3 < 0\}$

6. Computer graphics

All of the images are viewed from above the (x_1, x_2)-plane, looking toward the origin. The quadrilateral pattern on the surface is the image of a checkerboard pattern on the parameter domain D. Applying the symmetry B of (13) to the half of M in the second illustration produces the other half of the surface. Similarly R of (13) moves the half of the surface in the third illustration into the other half.

These images were created by James T. Hoffman. Assistance in the production of the photographs was provided by Richard Newton and the Digital Image Analysis Laboratory at the University of Massachusetts, Amherst.

References

[1] M. Abramovitz & I. Stegun, eds., *Handbook of mathematical functions*, Nat. Bureau Standards, Applied Math. Series, June, 1964, Chapter 18.

[2] D. Hoffman & W. H. Meeks, III, *Complete embedded minimal surfaces of finite total curvature*, Bull. Amer. Math. Soc. (N.S.), 1985, Vol. 12, 1, (January 1985) 134, 135.

[3] C. Costa, *Imersões minimas completas em $\mathbf{R}^3$ de gênero um e curvatura total finita*, Doctoral thesis, IMPA, Rio de Janeiro, Brasil, 1982 (Bol. Soc. Brasil. Mat., Vol. 15, No. 1 as "Example of a complete minimal immersion in $\mathbf{R}^3$ of genus one and three embedded ends"), to appear.

[4] D. Hoffman & R. Osserman, *The geometry of the generalized Gauss map*, Mem. Amer. Math. Soc., No. 236, 1980.

[5] L. Jorge and W. Meeks, III, *The topology of complete minimal surfaces of finite total Gaussian curvature*, Topology **22** (1983) 203–221.

[6] R. Osserman, *Global properties of complete minimal surfaces in E^3 and E^n*, Ann. of Math. (2) **80** (1964) 340–364.

[7] R. Schoen, *Uniqueness, symmetry, and embeddedness of minimal surfaces*, J. Differential Geometry **18** (1983) 791–809.

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