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$SO(3)$-CONNECTIONS AND THE TOPOLOGY OF 4-MANIFOLDS

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1. Introduction

The early part of the 1980s has experienced a vast increase in our understanding of smooth 4-manifolds. This has been accomplished principally through the work of S. Donaldson, namely

Theorem 1.1 (*Donaldson* [3]). *Let M be a smooth closed oriented simply-connected 4-manifold with positive definite intersection form θ. Then θ is "standard", i.e. over the integers $\theta \cong (1) \oplus \cdots \oplus (1)$.*

Although this is a theorem about 4-dimensional topology, its proof is differential geometric and analytic. The main theme of Donaldson's work is to study the topology of the space of solutions of the self-dual Yang-Mills equations on an $SU(2)$-bundle over the Riemannian manifold M and relate it to the topology of M. Recently we attempted to apply these Yang-Mills techniques to problems we have been working on for several years; namely finding numerical invariants for homology 3-spheres bounding acyclic 4-manifolds and studying smooth pseudo-free circle actions on 5-manifolds. We were moderately successful in [4]. That work studied Yang-Mills equations invariant under a cyclic group action. However, we were unsuccessful in following the complete "Donaldson program" in this equivariant setting. In particular, we were unable to mimic the work of C. Taubes [10] in finding nicely parametrized solutions to the Yang-Mills equations and were forced to use ad hoc, less analytical, techniques. We then "unequivariantized" our proof only to realize that we had a "simpler" proof of a version of Donaldson's theorem under the weaker (more topologically useful) assumption that $H_1(M; \mathbf{Z})$ has no 2-torsion.

The goals of this paper are, then, to give a proof of a version of Donaldson's theorem which on the one hand is more accessible to topologists and on the

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other hand is more general in that it removes the restriction that $\pi_1(M) = 0$. We have achieved the first goal by substantially reducing the amount of analysis necessary to prove our result. As for the second goal, we have been only partially successful. Our proof shows that many nonstandard positive definite forms cannot be realized as intersection forms of closed 4-manifolds M with $H_1(M; \mathbf{Z})$ containing no 2-torsion; however the proof does not work for all such forms. Here is a sample theorem which follows from our techniques.

Theorem 1.2. *Let θ be any positive definite symmetric unimodular integral form. Then $E_8 \oplus \theta$ cannot be realized as the intersection form of any smooth closed 4-manifold M with $H_1(M; \mathbf{Z})$ containing no 2-torsion.*

Our result is stated in full generality in §2. Theorem 1.2 is more than enough, when combined with Freedman's work [6], to imply the existence of an exotic differentiable structure on $\mathbf{R}^4$ (see [7], [5]).

As in Donaldson's proof, the idea of our proof is to reduce to a simple cobordism argument, where the cobordism is obtained as a moduli space of connections. Our major innovation follows from the use of $SO(3)$ rather than $SU(2)$-connections. It turns out that this allows the possibility that the moduli space be compact (see 5.3 and 5.7) and accounts for the simplification of our technique. Also, a version of Theorem 1.2 for all nonstandard positive definite intersection pairings can be obtained from our techniques modulo a reasonable conjecture (see 5.7).

Since we believe that the techniques related to Donaldson's theorem and our Theorem 1.2 are important for topologists, we have written this paper in a somewhat expository manner. It is probably best read in conjunction with one or both of the excellent expositions of Donaldson's theorem due to Blaine Lawson [7] or Karen Uhlenbeck et al. [5]. In fact, we shall often refer to these papers for general background material.

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2. Self-dual $SO(3)$-connections on 4-manifolds

Let M be a smooth closed oriented 4-manifold. It is convenient to arbitrarily split

$$H_2(M; \mathbf{Z}) = \operatorname{Fr} H_2(M; \mathbf{Z}) \oplus \operatorname{Tor} H_2(M; \mathbf{Z})$$

into free and torsion parts. Our goal is to study the intersection pairing

$$\operatorname{Fr} H_2(M; \mathbf{Z}) \otimes \operatorname{Fr} H_2(M; \mathbf{Z}) \to \mathbf{Z}.$$

Basically, this is the same as studying the cup product

$$H^2(M;\mathbf{Z}) \otimes H^2(M;\mathbf{Z}) \overset{\cup}{\to} H^4(M;\mathbf{Z}) \cong \mathbf{Z}.$$

These groups are related to the geometry of M via deRham cohomology. The deRham complex is

$$0 \to \Omega^0 \overset{d}{\to} \Omega^1 \overset{d}{\to} \Omega^2 \overset{d}{\to} \Omega^3 \overset{d}{\to} \Omega^4 \to 0,$$

where Ω^k is the space of exterior k-forms on M and d is exterior differentiation. This complex has finite-dimensional cohomology groups $H^k_{\mathrm{DR}}(M)$ which are isomorphic to the real singular cohomology groups $H^k(M;\mathbf{R})$. There is a subgroup of $H^k_{\mathrm{DR}}(M)$ which is isomorphic to $\mathrm{Fr}\, H^k(M;\mathbf{Z})$; it consists of those classes represented by a k-form β such that $\int_\Gamma \beta \in \mathbf{Z}$ for every integral k-cycle Γ on M.

Fix a Riemannian metric on M. Then there is a canonical representative of each cohomology class in $H^k_{\mathrm{DR}}(M)$ obtained by minimizing the *energy functional* $\int_M |\beta|^2$ over β in a fixed cohomology class. Solutions to this minimization problem must satisfy the Euler-Lagrange equation $\delta\beta = 0$, where δ is the adjoint of d. Since also $d\beta = 0$, β is harmonic. The Hodge theorem states that in each cohomology class in $H_{\mathrm{DR}}(M)$ there is a unique harmonic representative. We let $\mathbf{H}^k$ denote the subspace of Ω^k consisting of harmonic forms. Thus we may identify $H^k_{\mathrm{DR}}(M) = \mathbf{H}^k$.

An important role is played by the relationship between $H^2(M;\mathbf{Z})$ and 2-plane bundles over M. There is 1-1 correspondence

$$H^2(M;\mathbf{Z}) \leftrightarrow \text{principal } SO(2)\text{-bundles over } M$$
$$e \leftrightarrow P_e,$$

where e is the Euler class $e(P_e)$. Equivalently we have the 1-1 correspondence

$$H^2(M;\mathbf{Z}) \leftrightarrow SO(2)\text{-vector bundles over } M$$
$$e \leftrightarrow L_e \equiv P_e \times_{SO(2)} \mathbf{R}^2.$$

Note that L_e comes equipped with a Riemannian metric induced from the standard inner product on $\mathbf{R}^2$.

This correspondence can be used to place a natural equivalence relation on $\mathrm{Fr}\, H_2(M;\mathbf{Z})$.

Definition 2.1. *For* $e_1, e_2 \in \mathrm{Fr}\, H_2(M;\mathbf{Z})$ *say that* $e_1 \sim e_2$ *if* $L_{e_1} \oplus \varepsilon^1 = L_{e_2} \oplus \varepsilon^1$, *where* ε^1 *is a trivial real line bundle.*

In other words $e_1 \sim e_2$ if and only if the corresponding $SO(2)$-bundles are equivalent after stabilizing to $SO(3)$.

Consider principal $SO(3)$-bundles over a given oriented 4-manifold M. A simple obstruction theory argument shows that principal $SO(3)$-bundles over M-(4-ball) are classified by their second Stiefel-Whitney class w_2. Gluing in the bundle over the 4-ball corresponds to fixing trivializations over the boundary S^3 and prescribing a map $S^3 \to SO(3)$. This fixes the Pontryagin class p_1. In fact, a theorem of Dold and Whitney [2] states that principal $SO(3)$-bundles over 4-manifolds are classified by w_2 and p_1.

For a moment consider $SO(3)$-bundles over S^4. For such a bundle $w_2 = 0$; so it is double covered by a $\mathrm{Spin}(3) = SU(2)$-bundle. An $SU(2)$-bundle has $c_1 = 0$, so $p_1 = -2c_2$ and c_2 is the Euler class; so we see that any even integer can occur as $p_1[S^4]$ for a principal $SU(2)$-bundle over S^4. Our given $SO(3)$-bundle is double covered by an $SU(2)$-bundle; so it is not hard to see that for it, $p_1[S^4] = 4k$.

Now fix a principal $SO(3)$-bundle over M. Any other principal $SO(3)$-bundle over M with the same w_2 can be obtained by removing the part of the bundle over a 4-ball and regluing. In terms of the classifying map:

$$M \to M \vee S^4 \to BSO(3).$$

So the Pontryagin number is changed by $4k$. Thus we have:

Lemma 2.2. *Let E_1, E_2 be $SO(3)$-vector bundles over the oriented closed 4-manifold M. If $w_2(E_1) = w_2(E_2)$, then $p_1(E_1) \equiv p_1(E_2) \pmod 4$.*

We shall say that an $SO(3)$-vector bundle E over M *reduces* to an $SO(2)$-vector bundle L_e if $E = L_e \oplus \varepsilon^1$. In this case $p_1(E) = e^2$ and $w_2(E) \equiv e \pmod 2$. By the classification of Dold and Whitney [2], all other reductions correspond to $v \in H^2(M; \mathbf{Z})$ such that $v \equiv e \pmod 2$ and $v^2 = e^2$. Furthermore, returning to the equivalence relation on $\mathrm{Fr}\, H_2(M; \mathbf{Z})$:

Lemma 2.3. *If $e_1, e_2 \in \mathrm{Fr}\, H_2(M; \mathbf{Z})$, then $e_1 \sim e_2$ if and only if $e_1^2 = e_2^2$ and $e_1 \equiv e_2 \pmod 2$.*

Clearly $-e \sim e$, so we define

$$\mu(e) = \tfrac{1}{2}\#\left\{\, v \in \mathrm{Fr}\, H_2(M; \mathbf{Z}) | v \sim e \right\}.$$

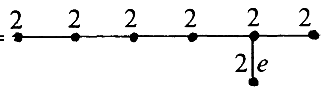
Also, define $e \in \mathrm{Fr}\, H_2(M; \mathbf{Z})$ to be *minimal* if

$$v \equiv e \pmod 2 \Rightarrow |e^2| \leqslant |v^2|.$$

These concepts make sense in any inner product space over $\mathbf{Z}$. Consider some examples.

Example 2.4. The standard form $(1) \oplus \cdots \oplus (1)$. Minimal vectors have the form $e = (\varepsilon_1, \cdots, \varepsilon_k)$, where $\varepsilon_i = 0$ or ± 1. Let n be the number of ± 1's in a particular minimal e. Then $\mu(e) = \frac{1}{2}2^n = 2^{n-1}$. Hence $\mu(e)$ is even unless e has just one ± 1, in which case $\mu(e) = 1$ and $e^2 = 1$.

Example 2.5.

$E_8 =$ 

Note $e^2 = 2$, e is minimal and $\mu(e) = 1$.

In fact we have the following characterization of the standard form in terms of these concepts.

Proposition 2.6. *A positive definite unimodular integral inner product space U is standard if and only if each minimal vector e with $e^2 > 1$ has $\mu(e)$ even.*

Proof. For each vector $v \in U$ such that $v^2 = 1$ we may form the orthogonal splitting $U = \langle v \rangle \oplus \langle v \rangle^{\perp}$. After inductively splitting off all such subspaces we either exhaust U, and thus we see that U is standard, or we are left with an orthogonal summand V of U in which each vector has length $v^2 \geqslant 2$. Let $e \in V$ be a vector with minimum length. If $v \sim e$, then $v \equiv e \pmod 2$ and $v^2 = e^2$; so there is a vector $w \in V$ such that $v = e + 2w$. Thus

$$e^2 = v^2 = (e + 2w)^2 = e^2 + 4e \cdot w + 4w^2;$$

i.e. $w \cdot (w + e) = 0$. So we have $e^2 = v^2 = ((e + w) + w)^2 = (e + w)^2 + w^2$. Since e has minimal length in V, unless $w = 0$ or $-e$, we have the contradiction $e^2 = (e + w)^2 + w^2 \geqslant 2e^2$. So $\mu(e) = 1$. The converse was proved in Example 2.4. q.e.d.

We thus pose the following

Conjecture 2.7. *Let M be a closed smooth oriented 4-manifold. Suppose that its intersection form is positive definite and has a minimal vector e with $e^2 > 1$. Then $\mu(e)$ is even.*

In other words the conjecture is that such a form is standard. Under the additional hypothesis that $\pi_1(M) = 0$ this is Donaldson's theorem. As we mentioned in §1, in this paper we prove a portion of this conjecture. More precisely we shall prove:

Theorem 2.8. *If U is a positive definite unimodular inner product space over $\mathbf{Z}$ and if there is a vector $u \in U$ such that $u^2 = 2$ or 3 and $\mu(u)$ is odd, then U is not the intersection form of any oriented 4-manifold M with $H_1(M; \mathbf{Z})$ containing no 2-torsion.*

If there is a minimal vector u with $u^2 = 4$ and $\mu(u)$ is odd, then U is not the intersection form of any oriented 4-manifold M with $H_1(M; \mathbf{Z})$ containing no 2-torsion and $H^2(\pi_1 M; \mathbf{Z}_2) = 0$.

We shall prove this theorem by studying the geometry of the bundle $E = L_e \oplus \varepsilon$ associated to a minimal vector $e \in \operatorname{Fr} H_2(M; \mathbf{Z})$. The geometric structure is provided by an *$SO(3)$-connection*, i.e. a linear map

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

satisfying $\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma$ and $d\langle\sigma_2, \sigma_2\rangle = \langle\nabla\sigma_1, \sigma_2\rangle + \langle\sigma_1, \nabla\sigma_2\rangle$, where $f\colon M \to R$, and $\langle\ ,\ \rangle$ is the Riemannian metric on $E = L_e \oplus \varepsilon$ which was described above.

It will be convenient for us to use the notation

$$\Omega^k(F) = \Gamma(\Lambda^k T^*M \otimes F)$$

for any bundle F over M. So, for example, $\nabla\colon \Omega^0(E) \to \Omega^1(E)$. A connection ∇ has a natural extension $d^{\nabla}\colon \Omega^k(E) \to \Omega^{k+1}(E)$ defined by $d^{\nabla}(\alpha \otimes \sigma) = d\alpha \otimes \sigma + (-1)^k\alpha \wedge \nabla\sigma$, where $\alpha \in \Omega^k$ and $\sigma \in \Omega^0(E)$.

The *curvature* R^{∇} of an $SO(3)$-connection ∇ on E is a 2-form with values in $\mathfrak{g}_E$, where $\mathfrak{g}_E \subset \mathrm{Hom}(E, E)$ is the subbundle of those transformations which are skew-symmetric on each fiber. (In fact, for an $SO(3)$-vector bundle E, $\mathfrak{g}_E \sim E$ via the cross product "$\times$".) That is $R^{\nabla} \in \Omega^2(\mathfrak{g}_E)$. It is defined by

$$R^{\nabla}_{V,W} = \nabla_V\nabla_W - \nabla_W\nabla_V - \nabla_{[V,W]}.$$

Also we have $R^{\nabla} = d^{\nabla} \circ \nabla$.

Similarly we can study $SO(2)$-connections and their curvatures on the $SO(2)$-vector bundle L_e. It is easy to see that $\mathfrak{g}_{L_e}$ is just the trivial real line bundle over M; so if D is a connection on L_e, then $R^{\nabla} \in \Omega^2(\mathfrak{g}_{L_e}) = \Omega^2$. The real Euler class $e_{\mathbf{R}}(L_e) \in H^2_{\mathrm{DR}}(M)$ is represented by $R^{\nabla}/2\pi$. By the Hodge Theorem there is a unique harmonic 2-form θ in the deRham class of $e_{\mathbf{R}}(L_e)$. We shall be interested in those $SO(2)$-connections D on L_e such that $R^D/2\pi = \theta$. These connections provide a link between differential geometry and the group $\mathrm{Fr}\, H_2(M; \mathbf{Z})$.

Analogously we wish to consider those $SO(3)$-connections on E whose curvatures are harmonic. Now $\mathrm{Hom}(E, E)$ has the inner product $\langle A, B\rangle = \mathrm{tr}(A^t \cdot B)$. The *energy* or *Yang-Mills action* of ∇ is defined by

$$\mathcal{YM}(\nabla) = \tfrac{1}{2}\int_M \|R^{\nabla}\|^2.$$

A critical point of the Yang-Mills functional satisfies the Yang-Mills equations $\delta^{\nabla}R^{\nabla} = 0$. Here δ^{∇} is the adjoint of d^{∇} (we identify $\mathfrak{g}_E$ with E to get d^{∇}). The Bianchi identity $d^{\nabla}R^{\nabla} \equiv 0$ shows that any solution ∇ to the Yang-Mills equation has harmonic curvature, that is $(d^{\nabla}\delta^{\nabla} + \delta^{\nabla}d^{\nabla})R^{\nabla} = 0$.

To find such solutions we consider the $*$-operator $*\colon \Omega^2 \to \Omega^2$ on the 2-forms of M. For a local orthogonal basis $\theta^1, \cdots, \theta^4$, $*$ is defined by $*\,\theta^i \wedge \theta^j = \theta^k \wedge \theta^l$ where (i, j, k, l) is an even permutation of $(1, 2, 3, 4)$, or equivalently by $\phi \wedge *\,\psi = \langle\phi, \psi\rangle d\mathrm{vol}$ for any $\phi, \psi \in \Omega^2$. The $*$-operator has order 2 and its ± 1-eigenspaces define a splitting of Ω^2 into orthogonal subspaces $\Omega^2 = \Omega^2_+ \oplus \Omega^2_-$. Ω^2_+ (Ω^2_-) is called the space of *self-dual* (*anti self-dual*) 2-forms of M. Tensoring with $\mathfrak{g}_E$ we obtain the orthogonal splitting

$$\Omega^2(\mathfrak{g}_E) = \Omega^2_+(\mathfrak{g}_E) \oplus \Omega^2_-(\mathfrak{g}_E)$$

in which we get $R^\nabla = R^\nabla_+ + R^\nabla_-$ for any $SO(3)$-connection ∇ on E. Since this splitting is orthogonal,

$$\mathscr{YM}(\nabla) = \tfrac{1}{2}\int_M \|R^\nabla_+\|^2 + \|R^\nabla_-\|^2.$$

The Chern-Weil formula for the Pontryagin number of E is

$$p_1(E) = \frac{1}{4\pi^2}\int_M \operatorname{tr}(R^\nabla \wedge R^\nabla).$$

Using $R^\nabla = R^\nabla_+ + R^\nabla_-$ the formula may be rewritten as

$$p_1(E) = \frac{1}{4\pi^2}\int_M \|R^\nabla_+\| - \|R^\nabla_-\|^2.$$

Hence $\mathscr{YM}(\nabla) \geqslant 2\pi^2 p_1(E)$ with equality holding if and only if $\|R^\nabla_-\|^2 = 0$. Such ∇ are called *self-dual*. Evidently, ∇ is self-dual if and only if $R^\nabla_- = 0$, i.e., $*R^\nabla = R^\nabla$. Notice that if self-dual connections exist on a bundle E, they give an absolute minimum for $\mathscr{YM}$ on E; hence they satisfy the Yang-Mills equations. Furthermore self-dual connections form a bond between the topology and geometry of E via the equation $\mathscr{YM}(\nabla) = 2\pi^2 p_1(E)$ when ∇ is self-dual.

For an $SO(3)$-vector bundle E over M, let $\mathscr{C}$ denote the set of all $SO(3)$-connections, and let $\mathscr{A} \subset \mathscr{C}$ be the set of all self-dual connections. (In general we do not know that $\mathscr{A} \neq \varnothing$.) It is convenient to identify two connections on E which are related by a *gauge transformation*; i.e. a bundle map

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr] & & E \arrow[dl] \\ & M & \end{tikzcd}$$

such that on each fiber $g_x \in SO(E_x)$. Then g acts on forms by pullback and on connections by $g(\nabla) = g^{-1} \circ \nabla \circ g$. So $R^{g(\nabla)} = g^{-1} \circ R^\nabla \circ g$. Clearly $g(\nabla)$ is self-dual if and only if ∇ is self-dual. Hence we can form the moduli spaces $\mathscr{B} = \mathscr{C}/\mathscr{G}$ and $\mathfrak{M} = \mathscr{A}/\mathscr{G}$, where $\mathscr{G}$ is the group of gauge transformations of E.

3. Reducible connections

From here on we let M be a smooth oriented closed 4-manifold with $H_1(M; \mathbf{Z}_2) = 0$. If E is an $SO(3)$-vector bundle over M, then we define a *splitting* of E to be a real 2-plane bundle L over M such that $E \cong L \oplus \varepsilon^1$. Since

$H^1(M; \mathbf{Z}_2) = 0$, $w_1(L) = 0$; so L may be given an $SO(2)$-vector bundle structure. From the classification of $SO(3)$-vector bundles we see that up to orientation, splittings of E are in 1-1 correspondence with

$$\left\{ \pm e \in H^2(M; \mathbf{Z}) | e^2 = p_1(E) \text{ and } e \equiv w_2(E) \ (\mathrm{mod}\, 2) \right\}.$$

If E is a reducible $SO(3)$-vector bundle, say $E = L_e \oplus \varepsilon^1$, and if $\tau \in \operatorname{Tor} H^2(M; \mathbf{Z}) \cong \operatorname{Tor} H_1(M; \mathbf{Z}) \cong H_1(M; \mathbf{Z})$, then $(e + \tau)^2 = e^2$ and $e + \tau \equiv e \ (\mathrm{mod}\, 2)$, so also $E = L_{e+\tau} \oplus \varepsilon^1$. Hence up to orientation, the number of reductions of $E = L_e \oplus \varepsilon^1$ is just $\mu(e) \cdot |H_1(M; \mathbf{Z})|$ which (mod 2) is congruent to $\mu(e)$.

If $E = L_e \oplus \varepsilon$ and $\nabla = D_e \oplus d$, where D_e is an $SO(2)$-connection on L_e and d is exterior differentiation on the trivial bundle, then ∇ is called a *reducible connection*. Let Γ^∇ denote $\{ g \in \mathcal{G} | g(\nabla) = \nabla \}$, the stabilizer of ∇ under the action of the gauge transformation group. We have:

Proposition 3.1. *For a nonflat $SO(3)$-vector bundle E over M the following are equivalent.*

(1) *∇ is a reducible connection on E.*

(2) $\Gamma^\nabla \cong S^1$.

(3) $\Gamma^\nabla \neq \{\mathrm{id}\}$.

(4) *There is a parallel section in $\mathfrak{g}_E$ (with respect to d^∇).*

Proof. If ∇ is reducible, then $\nabla = D_e \oplus d$ on $L_e \oplus \varepsilon^1$ for some L_e. Let $S^1 \subset \mathcal{G}$ be the subgroup which rotates the fibers of L_e (each fiber by the same amount) and acts trivially on the ε^1-summand. Then if $g \in S^1$ and $\sigma \in \Omega^0(E)$,

$$\nabla(g\sigma) = dg \otimes \sigma + g(\nabla\sigma) = g(\nabla\sigma),$$

i.e. $g(\nabla) = g^{-1} \nabla g = \nabla$. So $g \in \Gamma^\nabla$. Furthermore, if $g \in \Gamma^\nabla$, then g must fix ε^1 and rotate L_e; and as above, the equation $\nabla g = g \nabla$ shows that g must rotate each fiber of L_e by the same amount. Thus $\Gamma^\nabla = S^1$. So (1) $\Rightarrow$ (2). Clearly (2) $\Rightarrow$ (3).

To see that (3) $\Rightarrow$ (1), consider any $g \in \Gamma^\nabla$, $g \neq \mathrm{id}$. Over the point $x \in M$, $g_x | E_x$ has a real eigenvalue λ. However, g is parallel. (This means that g is parallel when $\mathcal{G}$ is viewed as the sections of the subbundle G_E of $\operatorname{Hom}(E, E)$ which are in $SO(E_x)$ on each fiber. The connection on $\operatorname{Hom}(E, E)$ induced from ∇ is $\nabla(L) \equiv [\nabla, L] = \nabla L - L \nabla$. So, since $g \in \Gamma^\nabla$, $[\nabla, g] = \nabla g - g \nabla = 0$ and g is parallel.) Thus parallel translation shows that the eigenvalue λ is constant throughout M. The corresponding eigenspace gives a real line in each fiber of E preserved by ∇. Its orthogonal complement yields a splitting $E \cong L \oplus \eta$, where L is an $O(2)$-bundle and η is an $O(1)$-bundle. However, since $H_1(M; \mathbf{Z}_2) = 0$, L must be an $SO(2)$-bundle, hence $E \cong L \oplus \varepsilon^1$. Since $g \in \Gamma^\nabla$, $\nabla g = g \nabla$ and ∇ preserves this splitting, so (3) $\Rightarrow$ (1).

We have seen that ∇ is reducible if and only if E has a parallel section; but $E \cong \mathfrak{g}_E$, so (1) $\Rightarrow$ (4). q.e.d.

Consider now an $SO(2)$-connection D on an $SO(2)$-vector bundle L. Locally, a connection on L has the form $d + w$ where $w \in \Omega^1$, and so locally, $R^D = dw$, and the deRham class $[R^D/2\pi]$ represents the real Euler class $e_{\mathbf{R}}(L)$.

Proposition 3.2 (*Compare* [7, II.8.14]). *The map $D \to R^D/2\pi$ induces a* 1-1 *correspondence between gauge equivalence classes of $SO(2)$-connections on L and closed* 2*-forms representing $e_{\mathbf{R}}(L)$.*

Proof. A gauge transformation of L is a section of $\mathrm{Hom}(L, L)$ which is in $SO(L_x)$ for each $x \in M$. Let exp: $R = so(2) \to SO(2)$ be the exponential map. Then a gauge transformation is just $\exp(s)$ for $s \in \Omega^0$.

Two connections D and D' are gauge equivalent if and only if $D' = \exp(-s) D \exp(s)$ for some $s \in \Omega^0$, i.e. if and only if

$$D' = \exp(-s)(\exp(s)\, ds + \exp(s) D) = D + ds.$$

Thus $R^{D'} = R^D + dds = R^D$. So $D \to R^D/2\pi$ depends only on the gauge equivalence class of D.

If $R^{D'} = R^D$, then $D' = D + w$ where $dw = 0$. Since we are assuming that $H_1(M; \mathbf{Z}_2) = 0$, also $H^1(M; \mathbf{R}) = 0$, thus w is exact; $w = ds$ for some $s \in \Omega^0$. So D and D' are gauge equivalent.

Finally, if Φ represents $e_{\mathbf{R}}(L)$ then fix any D and note $R^D/2\pi = \Phi + dw$ for some $w \in \Omega^1$. Set $D' = D - 2\pi w$; then $R^{D'}/2\pi = R^D/2\pi - dw = \Phi$. So $D \to R^D/2\pi$ is onto. q.e.d.

Now we move to self-dual connections. The action of the $*$-operator splits $\Omega^2 = \Omega^2_+ \oplus \Omega^2_-$ and preseves harmonic forms; so we have $\mathbf{H}^2 = \mathbf{H}^2_+ \oplus \mathbf{H}^2_-$. The cup product on $H^2_{\mathrm{DR}}(M)$ is given by $[\phi] \cdot [\psi] = \int_M \phi \wedge \psi$ for closed 2-forms ϕ and ψ. Thus, if $* \phi = \pm \phi$, then

$$[\phi]^2 = [\phi] \cdot [\phi] = \int_M \phi \wedge \phi = \pm \int_M \phi \wedge * \phi = \pm \langle \phi, \phi \rangle.$$

Consequently, the intersection form is $\pm$-definite on $\mathbf{H}^2_\pm$. If M has positive definite intersection form, then $\mathbf{H}^2 = H^2_+$ and every harmonic 2-form is self-dual.

Corollary 3.3. *Let M be a smooth oriented* 4*-manifold with $H^1_{\mathrm{DR}}(M) = 0$ and positive definite intersection form. Then each $SO(2)$-bundle L over M has a unique gauge equivalence class of self-dual connections.*

Corollary 3.4. *If $E \cong L_e \oplus \varepsilon^1$ is a reducible $SO(3)$-vector bundle on a smooth closed positive definite* 4*-manifold M with $H_1(M; \mathbf{Z}_2) = 0$, then there are exactly $\mu(e) \cdot |H_1(M; \mathbf{Z})|$ gauge equivalence classes of reducible self-dual $SO(3)$-connections on E.*

Proof. Recall that $m = \mu(E) \cdot |H_1(M; \mathbf{Z})|$ is just the number of reductions, up to orientations, of $E = L_e \oplus \varepsilon^1$.

Now a reducible connection ∇ gives a splitting $E \cong L \oplus \varepsilon^1$ and $\nabla = D \oplus d$. But $E \cong (-L) \oplus (-\varepsilon)^1$ and D is also a connection on $-L$, so that by (3.3) there are at most m gauge equivalence classes of reducible self-dual connections on E.

Any gauge equivalence of reducible $SO(3)$-connections must preserve parallel sections of E and hence induce a gauge equivalence of the corresponding $SO(2)$-connections. Thus there are exactly m gauge equivalence classes of reducible self-dual $SO(3)$-connections on E.

4. Local description of the moduli space

Once again let M be a smooth closed oriented Riemannian 4-manifold with $H_1(M; \mathbf{Z}_2) = 0$ and with a positive definite intersection form. Fix a reducible $SO(3)$-vector bundle $E \cong L_e \oplus \varepsilon^1$ over M. We want to give a local description of the moduli space $\mathfrak{M}$ of self-dual connections on E. If ∇ and ∇' are $SO(3)$-connections on E, consider the difference $A = \nabla' - \nabla$. For any smooth function f on M and $\sigma \in \Omega^0(E)$,

$$A(f\sigma) = \nabla'(f\sigma) - \nabla(f\sigma) = (df \otimes \sigma + f\nabla\sigma) - (df \otimes \sigma + f\nabla'\sigma).$$

So $A(f\sigma) = f(A\sigma)$. Furthermore if V is a vector field on M, A_V is linear and

$$\langle A_V\sigma_1, \sigma_2 \rangle + \langle \sigma_1, A_V\sigma_2 \rangle = d\langle \sigma_1, \sigma_2 \rangle_V - d\langle \sigma_1, \sigma_2 \rangle_V = 0.$$

Hence $A \in \Omega^1(\mathfrak{g}_E)$.

Conversely, if ∇ is an $SO(3)$-connection on E and $A \in \Omega^1(\mathfrak{g}_E)$, then $\nabla' = \nabla + A$ is an $SO(3)$-connection on E. Hence the space $\mathscr{C}$ of $SO(3)$-connections on E is an affine space. If $\nabla' = \nabla + A$, then $R^{\nabla'} = R^{\nabla} + d^{\nabla}A + [A, A]$, where $[A, A]_{V,W} = [A_V, A_W]$.

Now let ∇ be a fixed self-dual connection on E and $\nabla_t = \nabla + A_t$ be a 1-parameter family of self-dual connections through ∇. Then

$$R^{\nabla_t} = R^{\nabla} + d^{\nabla}A_t + [A_t, A_t],$$

$$R_-^{\nabla_t} = d_-^{\nabla}A_t + [A_t, A_t]_- = 0 \in \Omega^2(\mathfrak{g}_E)$$

by self-duality. Differentiating with respect to t and setting $t = 0$, we obtain $d_-^{\nabla}\dot{A} = 0$, where $\dot{A} = (d/dt)A_t|_{t=0}$. This is the linearized condition for self-duality. Thus the space of tangents to curves of self-dual connections at ∇_0 is $\ker d_-^{\nabla}$. To compute the space of tangents to the gauge equivalence class $[\nabla_0] \in \mathfrak{M}$ we must divide $\ker d_-^{\nabla}$ by the tangent space to the orbit $\mathscr{G}(\nabla)$.

The gauge transformations of E are sections of a bundle G_E of M with fiber $SO(3)$, and $\mathfrak{g}_E$ is its associated bundle of Lie algebras. So if we locally identify a gauge orbit with $\mathscr{G} = \Omega^0(G_E)$, then its tangent space at a point can be identified with $\Omega^0(\mathfrak{g}_E)$. Hence it is not surprising that $\operatorname{Im}\{d^\nabla\colon \Omega^0(\mathfrak{g}_E) \to \Omega^1(\mathfrak{g}_E)\}$ gives the space of tangents to $\mathscr{G}(\nabla)$ at ∇ (see [7, II.6.10] for a proof).

Thus the space of tangents to $\mathfrak{M}$ at $[\nabla]$ is the first cohomology H^1_∇ of the complex

$$0 \to \Omega^0(\mathfrak{g}_E) \xrightarrow{d^\nabla} \Omega^1(\mathfrak{g}_E) \xrightarrow{d^\nabla_-} \Omega^2_-(\mathfrak{g}_E) \to 0.$$

(Note that $d^\nabla_- \circ d^\nabla = R^\nabla_- = 0$.)

This complex is elliptic [1]. This has the following consequences. Let δ^∇ denote the formal adjoint to d^∇ and note that the adjoint of d^∇_- is the restriction of d^∇ to Ω^2_-. Then each of the Laplace operators $\Delta_0 = \delta^\nabla d^\nabla$, $\Delta_1 = \delta^\nabla d^\nabla_- + d^\nabla \delta^\nabla$ and $\Delta_2 = d^\nabla_- \delta^\nabla$ is elliptic and has finite dimensional kernel

$$\mathbf{H}^k_\nabla = \left\{ \phi \in \Omega^k(\mathfrak{g}_E) \middle| \Delta_k \phi = 0 \right\}.$$

Also, the Hodge theorem applies to identify $H^k_\nabla \cong \mathbf{H}^k_\nabla$.

Atiyah, Hitchin, and Singer [1] have computed the index of the above complex,

$$\begin{aligned} -\dim H^0_\nabla + \dim H^1_\nabla - \dim H^2_\nabla &= 2p_1(E) - \tfrac{3}{2}(\chi(M) - \sigma(M)) \\ &= 2e^2 - 3, \end{aligned}$$

since M is positive definite.

Note that H^0_∇ consists of the ∇-parallel sections of $\mathfrak{g}_E$. By Proposition 3.1 this has dimension 1 if ∇ is reducible and dimension 0 if ∇ is irreducible.

If ∇ is reducible and corresponds to the splitting $\nabla = D + d$ on $E = L \oplus \varepsilon^1 = \mathfrak{g}_E$, then Γ^∇ consists of the S^1 which rotates L and acts trivially on ε^1. So $\Omega^k(\mathfrak{g}_E) = \Omega^k(L) \oplus \Omega^k(\varepsilon^1)$ and Γ^∇ acts via the standard action of S^1 on $\Omega^k(L)$ and trivially on $\Omega^k(\varepsilon^1)$. The fixed point set of the action of Γ^∇ on $\Omega^k(\mathfrak{g}_E)$ is $\Omega^k(\varepsilon^1)$. For the harmonic spaces we have

$$\begin{aligned} \mathbf{H}^1_\nabla &= \mathbf{H}^1_D \oplus \mathbf{H}^1_d(\varepsilon) \cong \mathbf{H}^1_D \oplus \mathbf{H}^1 = \mathbf{H}^1_D, \\ \mathbf{H}^2_\nabla &= \mathbf{H}^2_D \oplus \mathbf{H}^2_d(\varepsilon) \cong \mathbf{H}^2_D \oplus \mathbf{H}^2_- = \mathbf{H}^2_D. \end{aligned}$$

So $\mathbf{H}^1_\nabla$ and $\mathbf{H}^2_\nabla$ have actions of S^1 with only the origin fixed. Thus they are even dimensional vector spaces.

We next need to study the ambient moduli space $\mathscr{B} = \mathscr{C}/\mathscr{G}$. At this point it is useful to complete our various spaces of connections with respect to appropriate Sobolev norms. For any Riemannian vector bundle ξ over M with a connection ∇, the Sobolev k-norm ($k \in \mathbf{Z}^+$) on $\Gamma(\xi)$ is

$$\|\sigma\|_k^2 = \int_M \left\{ \|\sigma\|^2 + \cdots + \left\| \underbrace{\nabla \cdots \nabla}_{k} \sigma \right\|^2 \right\}.$$

(Different choices of metrics and connections give equivalent norms.)

For example the space $\Omega^p(E) = \Gamma(\Lambda^p T^*M \otimes E)$ may be completed in the Sobolev k-norm to obtain the Hilbert space $\Omega_k^p(E)$. The point is that after completion we have important analytical tools at our disposal, for example, the implicit and inverse function theorems, the open mapping theorem and regularity theorems for solutions for elliptic partial differential equations. If we fix a base connection $\nabla_0 \in \mathscr{C}$ we define the Sobolev space of connections $\mathscr{C}_k$ to be $\mathscr{C}_k = \{\nabla_0 + A | A \in \Omega_k^1(\mathfrak{g}_E)\}$. Because of the affine structure, this does not depend on the base connection.

Since $\mathscr{G} = \Omega^0(G_E)$, the gauge transformation group of $\mathscr{C}_k$ is defined to be $\mathscr{G}_{k+1} = \Omega_{k+1}^0(G_E)$. It is known that $\mathscr{G}_4$ is a Hilbert Lie group with Lie algebra $\Omega_4^0(\mathfrak{g}_E)$, and that the action of $\mathscr{G}$ on $\mathscr{C}$ extends to a smooth action of $\mathscr{G}_4$ on $\mathscr{C}_3$. Furthermore if we restrict the action of $\mathscr{G}_4$ to the irreducible connections $\mathscr{C}_3^* = \mathscr{C}_3 - \mathscr{C}_{3\,\mathrm{red}}$, then the orbit map $\mathscr{C}_3^* \to \mathscr{B}_3^*$ is a principal bundle projection and $\mathscr{B}_3^*$ is a smooth Hausdorff Hilbert manifold. From now on we drop the Sobolev subscripts and use $\mathscr{C}$ to mean $\mathscr{C}_3$ and $\mathscr{G}$ to mean $\mathscr{G}_4$.

We also have the following "slice theorem". For a proof see [7, §II.10] or [1].

Proposition 4.1. *The action of $\mathscr{G}$ on $\mathscr{C}$ has slices of the form*

$$\mathscr{O}_{\nabla,\varepsilon} = \left\{ \nabla + A | \delta^\nabla A = 0 \text{ and } \|A\|_3 < \varepsilon \right\}, \qquad \varepsilon \text{ small.}$$

This means

(1) *At an irreducible connection ∇, $\mathscr{O}_{\nabla,\varepsilon}$ forms a local coordinate chart for $\mathscr{B}$.*

(2) *At a reducible connection ∇, $\mathscr{B}$ is locally homeomorphic to $\mathscr{O}_{\nabla,\varepsilon}/\Gamma^\nabla$ and this homeomorphism is a diffeomorphism away from ∇.*

Using the slice theorem we can now follow Atiyah, Hitchin and Singer in describing $\mathfrak{M}$ in the neighborhood of a given self-dual connection. Fix a self-dual connection ∇ on E and suppose $\nabla' = \nabla + A \in \mathscr{O}_{\nabla,\varepsilon}$. Then $R^{\nabla'} = R^\nabla + dA + [A, A]$. So $R_-^{\nabla'} = d_-^\nabla A + [A, A]_-$. Thus we want to solve the system of equations

$$\delta^\nabla A = 0, \qquad d_-^\nabla A + [A, A]_- = 0$$

for $\|A\|_3 < \varepsilon$. The group $\mathbf{H}_\nabla^1 = \{A | d_-^\nabla A = 0, \delta^\nabla A = 0\}$ represents the solution to the linearized equations and we have seen that $\mathbf{H}_\nabla^2$ may be viewed as the formal tangent space of $\mathfrak{M}$ at $[\nabla]$. One can now apply a technique known as the Kuranishi method to the current situation. This essentially amounts to a smooth change of variables in $\Omega_3^1(\mathfrak{g}_E)$ and $\Omega_2^2(\mathfrak{g}_E)$ (see [7, IV.2.1.] for details).

One obtains

Proposition 4.2 (*Atiyah, Hitchin and Singer*). *Let ∇ be a self-dual $SO(3)$-connection on E. Then there is a neighborhood $\mathcal{O}$ of $0 \in \mathbf{H}^1_\nabla$ and a smooth map $\Phi\colon \mathcal{O} \to \mathbf{H}^2_\nabla$ such that $\Phi(0) = 0$ and which is S^1-equivariant if ∇ is reducible and such that*

(1) $\mathfrak{M} \cap \mathcal{O} \cong \Phi^{-1}(0)$ *if ∇ is irreducible.*

(2) $\mathfrak{M} \cap (\mathcal{O}/S^1) \cong \Phi^{-1}(0)/S^1$ *if ∇ is reducible.*

So here is a local description of $\mathfrak{M}$ in a special case. If ∇ is irreducible (so $\mathbf{H}^0_\nabla = 0$) and if $\mathbf{H}^2_\nabla = 0$, then near ∇, $\mathfrak{M}$ is diffeomorphic to $\mathbf{H}^1_\nabla$, i.e. it is a manifold of dimension $2e^2 - 3$. If ∇ is reducible (so $\dim \mathbf{H}^1_\nabla = 1$) and if $\mathbf{H}^2_\nabla = 0$, then near ∇, $\mathfrak{M}$ is the quotient of a $2e^2 - 2$ dimensional real vector space by a linear action of S^1 which fixes only the origin, so it is a cone on $\mathbf{CP}^{e^2-2}$.

In general we do not have $\mathbf{H}^2_\nabla = 0$. (Although Uhlenbeck has proved that for a generic metric on M this cohomology group vanishes [5].) However we shall be able to perturb the system of equations to turn the moduli space into a $2e^2 - 3$ dimensional manifold with point singularities of the type described above.

5. The Compactness Theorem

We now return to our original conjecture (2.7):

Conjecture. *Let M be a smooth closed oriented 4-manifold. Suppose that its intersection form is positive definite and has a minimal vector e with $e^2 > 1$. Then $\mu(e)$ is even.*

So suppose we have such an M and an $e \in \operatorname{Fr} H_2(M; \mathbf{Z})$. Form the $SO(2)$-vector bundle L_e over M and stabilize to get the $SO(3)$-vector bundle $E = L_e \oplus \varepsilon^1$. We wish to study the moduli space of self-dual connections on E. Two theorems of Karen Uhlenbeck are fundamental.

Proposition 5.1 (*The Bubble Theorem* [11]). *Let $\{\nabla_i\}$ be any sequence of self-dual connections on E. Then one of the following holds:*

(1) *There is a subsequence $\{\nabla_{i'}\}$ and gauge equivalent connections $\{\tilde{\nabla}_{i'}\}$ such that $\tilde{\nabla}_{i'} \to \nabla_\infty$, a self-dual connection on E, in the C^∞-topology (so $[\nabla_{i'}] \to [\nabla_\infty]$ in $\mathfrak{M}$).*

(2) *There are a finite number of points $x_1, \cdots, x_k$ in M and a subsequence $\{\nabla_{i'}\}$ and gauge equivalent connections $\{\tilde{\nabla}_{i'}\}$ such that $\tilde{\nabla}_{i'} \to \nabla_\infty$, a self-dual connection on $E|M_0$ ($M_0 = M - \{x_1, \cdots, x_k\}$) in the C^∞-topology.*

Proposition 5.2 (*Removability of Singularities* [12]). *Let ∇ be a self-dual $SO(3)$-connection on a bundle E_0 defined over $M_0 = M - \{x_1, \cdots, x_k\}$. Suppose $\mathcal{YM}(\nabla) < \infty$. Then (E_0, ∇) extend smoothly over M.*

An important consequence is that the moduli space $\mathfrak{M}$ is compact when $p_1(E)$ is small enough.

Theorem 5.3 (*Compactness Theorem*). *Let E be an $SO(3)$-vector bundle over an oriented 4-manifold M and suppose that $0 \leqslant p_1(E) \leqslant 3$. Then $\mathfrak{M}$ is compact. This also holds if $p_1(E) = 4$ if we also assume that $w_2(E) \neq 0$ and $H^2(\pi_1(M); \mathbf{Z}_2) = 0$.*

Proof. Consider a sequence $\{[\nabla_i]\}$ in $\mathfrak{M}$. If $\{[\nabla_i]\}$ has no convergent subsequence, then the Bubble Theorem implies that there is some subsequence $\{\nabla_{i'}\}$ and gauge equivalent connections $\{\tilde{\nabla}_{i'}\}$ such that $\tilde{\nabla}_{i'} \to \nabla_\infty$, a self-dual connection on $E|M_0$, where $M_0 = M - \{x_1, \cdots, x_k\}$ for some finite number of points of M. Since each $\tilde{\nabla}_{i'}$ is a self-dual connection on E we have

$$\mathscr{YM}(\tilde{\nabla}_{i'}) = 2\pi^2 p_1(E).$$

So from Fatou's Lemma

$$\begin{aligned} 0 \leqslant \mathscr{YM}(\nabla_\infty) = \tfrac{1}{2}\int_{M_0} \|R^{\nabla_\infty}\|^2 &\leqslant \tfrac{1}{2}\int_{M_0} \|R^{\tilde{\nabla}_{i'}}\|^2 \\ &= \mathscr{YM}(\tilde{\nabla}_{i'}) = 2\pi^2 p_i(E) < \infty. \end{aligned}$$

Thus the Removability of Singularities Theorem applies; so ∇_∞ extends to a self-dual connection on a bundle E_∞ over all of M. Sedlacek [9] shows that $w_2(E_\infty) = w_2(E)$; thus $p_1(E_\infty) \equiv p_1(E) \pmod 4$ (see Lemma 2.2). However since $0 \leqslant \frac{1}{2}\mathscr{YM}(\nabla_\infty)/\pi^2 = p_1(E_\infty)$ we have $0 \leqslant p_1(E_\infty) \leqslant p_1(E)$. If $0 \leqslant p_1(E) \leqslant 3$ this implies that $p_1(E_\infty) = p_1(E)$. So by the classification of $SO(3)$-bundles, $E_\infty \cong E$ and $\{[\nabla_i]\}$ actually has a convergent subsequence in $\mathfrak{M}$.

In case $p_1(E) = 4$, $w_2(E) \neq 0$ and $H^2(\pi_1(M); \mathbf{Z}_2) = 0$, we might have $p_1(E_\infty) = 0$. However this would imply that the bundle E_∞ was flat and so had finite structure group G. So the classifying space $BG = K(G, 1)$. Thus the classifying map for the vector bundle E_∞ must factor through $K(\pi_1(M), 1)$. But the second Stiefel-Whitney class $w_2(E_\infty)$ was pulled back through $H^2(\pi_1(M); \mathbf{Z}_2) = 0$; i.e. $w_2(E_\infty) = 0$, a contradiction. Hence $E_\infty \cong E$ in this case as well. q.e.d.

We now want to deal with the possibility that there are self-dual connections ∇ on E such that $\mathbf{H}^2_\nabla \neq 0$. Assume that we are in a situation where the moduli space $\mathfrak{M}$ is compact. Since $R^{g(\nabla)} = g^{-1} \circ R^\nabla \circ g$ for $g \in \mathscr{G}$, $\nabla \to R^\nabla_-$ does not induce a map $\mathscr{B} \to \Omega^2_-(\mathfrak{g}_E)$. However, recall that away from the reducible connections we have a principal G-bundle $\mathscr{C}^* \to \mathscr{B}^*$. If we form the associated bundle with fiber $\Omega^2_-(\mathfrak{g}_E)$

$$\mathscr{C}^* \times_{\mathscr{G}} \Omega^2_-(\mathfrak{g}_E) \to \mathscr{B}^*,$$

then $\nabla \to R_-^{\nabla}$ defines a section of this bundle. More generally let

$$\mathscr{F}^2 = \mathscr{C} \times_{\mathscr{G}} \Omega^2_{2-}(\mathfrak{g}_E).$$

We have an obvious projection $\mathscr{F}^2 \to \mathscr{B}$ and section $R_-\colon \nabla \to R_-^{\nabla}$. The moduli space $\mathfrak{M}$ is the zero set of this section. The idea is to perturb R_- until the image of the section cuts transversely across the zero set of $\mathscr{F}^2$. This can be carried out precisely because $\mathfrak{M}$ is assumed compact.

One first compactly perturbs the section R_- near the reducible connections $\nabla^1, \cdots, \nabla^{\mu}$ so that the perturbed section R'_- is equal to R_- outside small neighborhoods of the ∇^i's and so that the new zero set is a cone on the complex projective space $\mathbf{CP}^{e^2-2}$ in a neighborhood of each ∇^i. Now, using the compactness of $\mathfrak{M}$ continue to perturb R'_- to be transverse to the zero set of $\mathscr{F}^2$ (see [7, IV.4] for details).

In order to show that our new zero set is still compact we need the following lemma of Donaldson.

Lemma 5.4 [3, p. 293]. *If $R_- + \sigma$ is a compact perturbation of R_- over $\mathscr{O}_{\nabla,\varepsilon}$, then for any $r \geqslant 0$ and any closed set $\overline{N} \subset \mathscr{O}_{\nabla,\varepsilon}$, the set*

$$\left\{ \nabla + A \in \overline{N} \,\middle|\, \|(R_- + \sigma)(\nabla + A)\|_3 = 0 \right\}$$

is compact (in the L^2_3-topology on $\mathscr{O}_{\nabla,\varepsilon}$).

This combined with the above perturbation argument yields

Theorem 5.5. *Let M be a smooth closed oriented 4-manifold with positive definite intersection form and $H^1(M; \mathbf{Z}_2) = 0$. If $\mathfrak{M}$ is compact there is a compact perturbation $\Psi = R_- + \sigma$ of the self-duality equations on $\mathscr{C}$ so that the new moduli space $\mathfrak{M}' = \{[\nabla] \in \mathscr{B} | \Psi(\nabla) = 0\}$ is a compact smooth manifold of dimensional $2p_1(E) - 3 = 2e^2 - 3$ with $\mu(e) \cdot |H_1(M; \mathbf{Z})|$ singular points such that each has a neighborhood which is a cone on the complex projective space $\mathbf{CP}^{e^2-2}$.*

We are now in a position to prove a theorem which has a direct bearing on our conjecture.

Theorem 5.6. *Let M be a smooth closed oriented 4-manifold with positive definite intersection form and $H_1(M; \mathbf{Z}_2) = 0$. Suppose $e \in H^2(M; \mathbf{Z})$ such that $e^2 > 1$. Let $E = L_e \oplus \varepsilon^1$ and suppose that $\mathfrak{M}$, the moduli space of self-dual connections on E, is compact. Then $\mu(e)$ is even.*

Proof. If $e^2 > 1$, after perturbation the moduli space becomes a manifold $\mathfrak{M}'$ of dimension $2e^2 - 3$ with a finite number,

$$\mu(e) \cdot |H_1(M; \mathbf{Z})| \equiv \mu(e) \pmod 2,$$

of point singularities whose neighborhoods are cones on $\mathbf{CP}^{e^2-2}$. Removing the interiors of these cones, we obtain a smooth $2e^2 - 3$ dimensional manifold W

whose boundary is a disjoint union of copies of $\mathbf{CP}^{e^2-2}$. Mod 2, there are $\mu(e)$ boundary components of W. If e^2 is even this immediately implies that $\mu(e)$ is even, since an odd number of $\mathbf{CP}^{2k}$'s cannot bound a smooth manifold. ($\mathbf{CP}^{2k}$ has odd Euler characteristic.)

If e^2 is odd, consider $\mathscr{G}_0$, the subgroup of $\mathscr{G}$ consisting of those g which are the identity on the fiber of E over a basepoint x of M. The normal subgroup $\mathscr{G}_0$ acts freely on $\mathscr{C}^*$ and $\mathscr{G}/\mathscr{G}_0 \cong SO(E_x) \cong SO(3)$. The fibration π: $\mathscr{C}^* \to \mathscr{B}^*$ factors into π_0: $\mathscr{C}^* \to \mathscr{C}^*/\mathscr{G}_0$ a principal $\mathscr{G}_0$-bundle projection and π_1: $\mathscr{C}^*/\mathscr{G}_0 \to \mathscr{B}^*$ a principal $SO(3)$-bundle projection.

Consider a reducible self-dual connection ∇ on E. By (4.1) there is a slice $\mathscr{O}_{\nabla,\varepsilon}$ and by (4.2) and (5.5) in $\mathscr{O}_{\nabla,\varepsilon}$ there is a real vector space $\mathscr{O}$ of dimension $2e^2 - 2$ on which $\Gamma^\nabla \cong S^1$ acts in the standard manner, and such that $\mathscr{O}/S^1$ is a neighborhood of $[\nabla]$ in $\mathfrak{M}'$. Let S denote a sphere in $\mathscr{O}$ which is invariant under the $S^1 = \Gamma^\nabla$-action. If $g \in \mathscr{G}$ moves a connection in S to another connection in S, then (since $\mathscr{O}_{\nabla,\varepsilon}$ is a slice) $g \in \Gamma^\nabla$. So $g(S) = S$.

Recall that Γ^∇ acts on $E \cong L \oplus \varepsilon^1$ (where $\nabla = D \oplus d$) by action as $SO(2)$ on L and trivially on ε^1. Thus $\Gamma^\nabla \cap \mathscr{G}_0 = \{\mathrm{id}\}$. So each $g \in \mathscr{G}_0$ moves S off itself. Thus the projection π_0: $\mathscr{C} \to \mathscr{C}/\mathscr{G}_0$ is 1-1 on S (and in fact $\mathscr{O}_{\nabla,\varepsilon}$). In $\mathscr{C}/\mathscr{G}_0$ a $g \in SO(3)$ takes a connection in $\pi_0(S)$ to another connection in $\pi_0(S)$ if and only if $g \in \Gamma^\nabla$ (as above). Let $\nabla \in S$ and consider the $SO(3)$-orbit $SO(3)(\pi_0(\nabla))$. The intersection $SO(3)(\pi_0(\nabla)) \cap \pi_0(S) = \Gamma^\nabla(\pi_0(\nabla))$ is a circle. Hence the $SO(3)$-bundle

$$\pi_1^{-1}(\pi(S)) \to \pi(S) \cong \mathbf{CP}^{e^2-2}$$

reduces to an S^1-bundle

$$S = \pi_0(S) \to \pi(S) \cong \mathbf{CP}^{e^2-2}.$$

This is just the Hopf bundle. So this $SO(3)$-bundle over $\pi(S)$ has

$$w_2 = w_2(\text{Hopf-bundle}) \neq 0 \in H^2(\pi(S); \mathbf{Z}_2) = H^2\left(\mathbf{CP}^{e^2-2}; \mathbf{Z}_2\right).$$

Hence $w_2^{e^2-2} \neq 0 \in H^{2e^2-4}(\mathbf{CP}^{e^2-2}; \mathbf{Z}_2)$. This means that an odd number of these $SO(3)$-bundles cannot bound a principal $SO(3)$-bundle over a smooth $(2e^2 - 3)$-manifold. Of course this is just what occurs in $\pi_1^{-1}(W) \to W$. So $\mu(e)$ must be even. q.e.d.

For a smooth closed 4-manifold M we can surger out the free part of $H_1(M; \mathbf{Z})$ and the intersection form is unaffected. From Theorem 5.6 and the Compactness Theorem we obtain Theorem 2.8, which we restate for the convenience of the reader.

Theorem 2.8. *If U is a positive definite unimodular inner product space over $\mathbf{Z}$ and if there is a vector $u \in U$ such that $u^2 = 2$ or 3 and $\mu(u)$ is odd, then U is not the intersection form of any oriented 4-manifold M, with $H_1(M; \mathbf{Z})$ containing no 2-torsion.*

If there is a minimal vector u with $u^2 = 4$ and $\mu(u)$ is odd then U is not the intersection form of any oriented 4-manifold M with $H_1(M; \mathbf{Z})$ containing no 2-torsion and $H^2(\pi_1(M); \mathbf{Z}_2) = 0$.

For example this implies that $E_8 \oplus$ (any positive definite form) cannot be the intersection form of such an M^4 (see Example 2.5). Also since each even form of rank < 48 has a minimal u with $u^2 \leqslant 4$ [8, p. 29], the theorem applies to these forms as well.

We close with a conjecture whose truth would give compactness of $\mathfrak{M}$ for a general minimal e.

Conjecture 5.7. *Let M be a smooth oriented closed Riemannian 4-manifold with positive definite intersection form. Let $e \in \operatorname{Fr} H_2(M; \mathbf{Z})$ be minimal. If E' is an $SO(3)$-vector bundle such that $w_2(E') \equiv e \pmod 2$ and $p_1(E') < e^2$, then E' supports no self-dual connections.*

References

[1] M. F. Atiyah, N. J. Hitchin & I. M Singer, *Self-duality in 4-dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A **362** (1978) 425–461.

[2] A. Dold & H. Whitney, *Classification of oriented sphere bundles over a 4-complex*, Ann. of Math. (2) **69** (1959) 667–677.

[3] S. Donaldson, *An application of gauge theory to four dimensional topology*, J. Differential Geometry **18** (1983) 279–315.

[4] R. Fintushel & R. Stern, *Pseudofree orbifolds*, preprint.

[5] D. Freed & K. K. Uhlenbeck, *Instantons and 4-manifolds*, Math. Sci. Res. Inst. Series, Vol. 1, Springer, Berlin, 1984.

[6] M. Freedman, *The topology of four dimensional manifolds*, J. Differential Geometry **17** (1982) 357–454.

[7] H. B. Lawson, Jr., *The theory of gauge fields in four dimension*, Lecture Notes for NSF-CBMS conference, University of California at Santa Barbara, 1983.

[8] J. Milnor & D. Husemoller, *Symmetric bilinear forms*, Springer, New York, 1973.

[9] S. Sedlacek, *A direct method for minimizing the Yang-Mills functional*, Comm. Math. Phys. **86** (1982) 515–528.

[10] C. H. Taubes, *Self-dual Yang-Mills connections on non-self-dual 4-manifolds*, J. Differential Geometry **17** (1982) 139–170.

[11] K. K. Uhlenbeck, *Connections with L^p bounds on curvature*, Comm. Math. Phys. **83** (1982) 31–42.

[12] ______, *Removable singularities in Yang-Mills fields*, Comm. Math. Phys. **83** (1982) 11–30.

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