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MODULI SPACE OF STABLE CURVES FROM A HOMOTOPY VIEWPOINT

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Dedicated to Daniel Mostow on his 60th birthday

1. Introduction

(1.1) Let J: $R(S) \to \mathfrak{S}_g/\mathrm{SP}_g(\mathbb{Z})$ be the period mapping of the Riemann space $R(S)$ of a nonsingular curve S of genus g to the Siegel modular space $\mathfrak{S}_g/\mathrm{SP}_g(\mathbb{Z})$ of degree g. Both of these spaces $R(S)$ and $\mathfrak{S}_g/\mathrm{SP}_g(\mathbb{Z})$ can be compactified to projective varieties in a natural manner. The compactification $\hat{R}(S)$ of $R(S)$ is known as the *moduli space of stable curves* or the *augmented Riemann space*. As for the Siegel space, its compactification $\mathfrak{S}_g^*/\mathrm{SP}_g(\mathbb{Z})$ is called the *Satake space*, and in our previous paper [7] we studied the stable cohomology $H^*(\mathfrak{S}_g^*/\mathrm{SP}_g(\mathbb{Z}))$ of this space. From the work of Namikawa (see [17]) it is known that the classical period mapping can be extended to a map J: $\hat{R}(S) \to \mathfrak{S}_g^*/\mathrm{SP}_g(\mathbb{Z})$ of the compactifications. One of our original goals in writing this paper was to study the cohomological nature of this map. Our result follows (see also (7.1.4)).

Theorem. *The stable cohomology $H^*(\mathfrak{S}_g^*/\mathrm{SP}_g(\mathbb{Z}); \mathbb{Q})$ of the Satake space is a tensor product of two polynomial rings $\mathbb{Q}[x_i] \otimes \mathbb{Q}[y_j]$, degree $x_i = 4i + 2$, $0 \leqslant i < \infty$, degree $y_j = 4j + 2$, $0 < j < \infty$. The induced map on cohomology J^*: $H^*(\mathfrak{S}_g^*/\mathrm{SP}_g(\mathbb{Z}); \mathbb{Q}) \to H^*(\hat{R}(S); \mathbb{Q})$ kills the second polynomial ring $J^*(y_{4j+2}) = 0$.*

Recently E. Miller proved independently that J^* maps the first polynomial ring $\mathbb{Q}[x_i]$ injectively into $H^*(R(S); \mathbb{Q})$ in a stable range (see [16]). Thus his results, combined with the above theorem provide a complete answer for the induced mapping of J on stable cohomology.

(1.2) To establish our theorem, we have to overcome some of the technical difficulties which are typical in studying the homotopy nature of moduli

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spaces. One of our goals in this paper is to set up a convenient framework in which we can solve these problems. First, many of these moduli spaces are obtained by gluing Eilenberg-MacLane spaces $K(\pi, 1)$ together. Although each individual piece has a fixed homotopy type determined by the fundamental group, after gluing them together this rigid homotopy structure is lost because of the choice of base point involved. For these reasons, we formulate the construction in terms of a certain functor $\mathbf{Sim}_G X \to \mathbf{Top}$ from the simplices of a building with G-action to the category of topological spaces. We exploit carefully the notion of universal G-spaces so that at the end there is a rigid homotopy type. Our results are too technical to be summarized here (see (3.3.6)).

Moduli spaces are rarely smooth manifolds, but they can be triangulated into stratified polyhedra. Thus to control the topology near a stratum, it is convenient to adopt the theory of regular neighborhoods, familiar to piecewise linear topologists. This theory goes hand-in-hand with the categorical framework mentioned above, and seems to be ideal for our purposes.

(1.3) This paper is divided into eight sections. §1 is the Introduction. In §2, we give a brief summary of results required of Teichmüller theory. We adopt the point of view of L. Bers, B. Maskit, W. Abikoff (see [4], [13], [1]) and others because it is more accessible to topologists. In §3, we describe the categorical framework which is the backbone of this paper, and in the following section we present a theory of regular neighborhoods of stratified polyhedra. In §5, we begin to treat the augmented Riemann space $\hat{R}(S)$ using our theory, and this is completed in §6 by showing that there is a description of $\hat{R}(S)$ in terms of the category of stable curves. In §7, we recall the treatment of the Satake compactification in our previous paper [7]. The last section is devoted to the proof of our main theorem.

(1.4) We would like to acknowledge our debts to J. Harer and S. Wolpert for explaining to us various aspects of Teichmüller theory. To F. Raymond, we owe the treatment of regular neighborhoods in §4. Our original treatment of this topic was extremely cumbersome, and it is based upon a conversation with him that we arrived at the present formulation.

2. Moduli space of stable curves

In this section, we collect some of the basic facts and generalities concerning the moduli space of stable curves. Since these are well established facts, we will not provide proofs here but will give standard references where the proofs can be found.

(2.1) The word "moduli" comes from Riemann in his discussion of the space of Riemann surfaces. By a *Riemann surface*, we mean a connected, 1-dimensional complex manifold. Topologically, it is a two-dimensional differentiable, orientable manifold and up to homeomorphism it is completely determined by its genus [14]. However, a Riemann surface does not have a fixed Riemannian structure. Instead it has a conformal structure, for linear homomorphisms of $\mathbb{C}$ are conformal transformations, $z \to az$.

Modern development of this subject perhaps started from the work of A. Weil. Let G be a Lie group, and π a discrete subgroup of G. Then the *deformation space of* π *in* G, denoted by $\mathrm{Def}(\pi, G)$, is the space of all injective homomorphisms $\theta\colon \pi \to G$ such that the image $\theta(\pi)$ is closed. There is a natural topology on this space defined by considering this as a subspace in the function space G^{π}. Given an element θ in $\mathrm{Def}(\pi, G)$ we can deform θ by composing θ with the conjugation by an element α in G,

$$(\alpha\theta)(\gamma) = \alpha \cdot \theta(\gamma) \cdot \alpha^{-1}, \qquad \gamma \in \pi.$$

These are regarded as trivial deformations, and so it is natural to take the quotient space $\mathrm{Def}_0(\pi, G) = \mathrm{Def}(\pi, G)/G$ which is called the *reduced deformation space*.

(2.2) The above discussion is in a setting far more general than is necessary for us. Our situation belongs to the deformation of the *surface group* π,

$$\pi = \begin{cases} \text{group generated by } 2g \text{ generators } x_1, y_1, x_2, y_2 \cdots x_g, y_g \\ \text{and one relation } [x_1, y_1] \cdots [x_g, y_g] = 1, \end{cases}$$

into the *projective special linear group* $G = \mathrm{PSL}_2(\mathbb{R})$. Given an element θ in the deformation space $\mathrm{Def}(\pi, \mathrm{PSL}_2(\mathbb{R}))$, the quotient space of the upper half-plane $\mathfrak{S}_1 = \mathrm{SO}_2 \backslash \mathrm{PSL}_2(\mathbb{R})$ by the action of $\theta(\pi)$ is a surface $S_\theta = \mathfrak{S}_1/\theta(\pi)$ with fundamental group π. If two embeddings $\theta\colon \pi \to \mathrm{PSL}_2(\mathbb{R})$ and $\theta'\colon \pi \to \mathrm{PSL}_2(\mathbb{R})$ differ by the action of an element α in $\mathrm{PSL}_2(\mathbb{R})$, $\theta = \alpha \cdot \theta'$, then the surfaces S_θ and S'_θ are homeomorphic as complex manifolds. Hence every element in the reduced deformation space $\mathrm{Def}_0(\pi, G)$ gives rise to a Riemann surface.

Now a Riemann surface obtained in this manner has a natural Riemannian structure, namely the metric induced by the Poincare metric on the upper half-plane. With respect to this metric, the manifold S_θ has constant negative curvature, $K = -1$, and it is usually referred to as a hyperbolic manifold. From the classical uniformization theorem (see [2]), a necessary and sufficient condition for a Riemann surface to be hyperbolic is that it have negative Euler characteristic, $\chi(S) < 0$, or in other words its genus be bigger than 1, $g > 1$.

An element θ in the deformation space $\mathrm{Def}_0(\pi, \mathrm{PSL}_2(\mathbb{R}))$ contains more information than the Riemann surface S_θ itself. To see this, we choose a fixed embedding of π into $\mathrm{PSL}_2(\mathbb{R})$, $i\colon \pi \to \mathrm{PSL}_2(\mathbb{R})$, and let $S = S_i$ be the corresponding surface. By a topological theorem of Nielson, there is a homeomorphism $F\colon \mathfrak{S}_1 \to \mathfrak{S}_1$ which induces the action of θ, $\theta(\gamma) = F \circ i(\gamma) \circ F^{-1}$. Passing to the quotient spaces S and S_θ, we obtain a topological homeomorphism $\phi\colon S \to S_\theta$ between the two Riemann surfaces. Such homeomorphisms ϕ are not determined by the class $[\theta]$ in $\mathrm{Def}_0(\pi, \mathrm{PSL}_2(\mathbb{R}))$. First of all ϕ can be changed by isotopy, and second of all, there are trivial deformations mentioned before. Taking both of these into account, we consider the space of all triples (ϕ, S, S'), where S' is a Riemann surface and $\phi\colon S \to S'$ is a homeomorphism, not necessarily preserving the complex structures. Two such triples, (ϕ, S, S') and (ϕ'', S, S''), are said to be equivalent if and only if there exists a complex homeomorphism h such that

$$\begin{array}{ccc} S & \xrightarrow{\ \phi'\ } & S' \\ & \underset{\phi''}{\searrow} & \big\downarrow h \\ & & S'' \end{array}$$

is commutative up to homotopy. It is not difficult to verify that these equivalence classes of triples $[\phi, S, S']$ can be identified with the reduced deformation space $\mathrm{Def}_0(\pi, \mathrm{PSL}_2(\mathbb{R}))$ (see [8]). It is interesting to point out that the above formulation can be compared with the action of "the set of homotopy triangulations", or "the set of homotopy smoothings" developed in surgery theory of manifolds.

(2.3) Long before the modern theory of Weil and others, this last space was studied by Teichmüller, and so it is more appropriate to denote $\mathrm{Def}_0(\pi, \mathrm{PSL}_2(\mathbb{R}))$ by $T(S)$ and call this the *Teichmüller space*. Let $\Gamma(S)$ denote the *mapping class group*. By that we mean the orientation preserving automorphism group of π, $\mathrm{Aut}^+(\pi)$, modulo the subgroup $\mathrm{Inn}(\pi)$ of inner automorphisms, i.e., $\Gamma(S) = \mathrm{Aut}^+(\pi)/\mathrm{Inn}(\pi)$. Every element g in $\Gamma(S)$ determines, up to isotopy, a unique class of topological orientation-preserving homeomorphisms of S, i.e.

$$\Gamma(S) = \{g\colon S \to S, \text{ orientation-preserving homeomorphism}\}/\text{isotopy}.$$

There is an action on the Teichmüller space $T(S)$ defined by sending a triple $[\phi, S, S']$ to the triple $[\phi \circ g, S, S']$. In terms of the deformation space $\mathrm{Def}_0(\pi, \mathrm{PSL}_2(\mathbb{R}))$, this means changing an injection $\theta\colon \pi \to \mathrm{PSL}_2(\mathbb{R})$ to another injection $\theta \circ g$ by composing with the isomorphism g. It is not difficult to verify that this action on the Teichmüller space $T(S)$ is discontinuous (see [1, p. 80]).

The quotient space under this action is called the *Riemann space*, $R(S) = T(S)/\Gamma(S)$. In terms of equivalence classes $[\phi, S, S']$ in $T(S)$, this means to forget about the homeomorphism ϕ, and so elements in $R(S)$ coincide with the isomorphism classes of Riemann surfaces $[S']$ with the same genus as S. Riemann stated without proof that this space $R(S)$ may be holomorphically parametrized by $3g - 3$ complex parameters. To establish this was clearly the motivation behind the work of Fricke and Teichmüller.

(2.4) In the early 60's, W. L. Baily proved that $R(S)$ has the structure of a quasiprojective variety. In other words, it can be compactified into a projective variety with $R(S)$ as a Zariski open set. The starting point of Baily was the study of the Siegel modular space $\mathfrak{S}_g/\mathrm{SP}_g(\mathbb{Z})$ of degree g, and its compactification known as the Satake compactification $\mathfrak{S}_g^*/\mathrm{SP}_g(\mathbb{Z})$ (see [7] for additional information). Since Riemann's time, the space $R(S)$ has been studied by means of a period mapping J: $R(S) \to \mathfrak{S}_g/\mathrm{SP}_g(\mathbb{Z})$ which assigns a Riemann surface to a matrix (Riemann matrix) of period integrals. Baily not only proved that the Satake space is a projective variety, but that the Riemann space $R(S)$, embedded by J, can be compactified accordingly.

In the middle 60's, algebraic geometers began to formulate the modern definition of the word "moduli". Both the Siegel space $\mathfrak{S}_g/\mathrm{SP}_g(\mathbb{Z})$ and the Riemann space served as wonderful examples. As it turns out, the latter space coincides with the coarse moduli space M_g of algebraic curves of genus g over the complex field, $R(S) \cong M_g(\mathbb{C})$, $g =$ genus of S. In [12], Mumford and Deligne showed that M_g has a natural compactification as the course moduli space M_g^* of *stable* curves (see below). This compactification was quite different from that of Baily, and this was one of the topics in Mumford's address at the 1970 International Congress of Mathematicians.

(2.5) In the decade of the 70's, many papers appeared interpreting Mumford's compactification in the framework of the Teichmüller theory. This approach began with the work of L. Bers (see [4], [5]), and continued in a series of papers by B. Maskit and W. Abikoff (see [13], [1], [2]). There are also contributions by W. J. Harvey, A. Marden, and many others (see [8], [9], [12]).

The central idea of this approach relies on the concept of *Riemann surfaces with nodes*. By that we mean a compact, connected, complex analytic space S_0 such that every point P in S_0 has a local neighborhood isomorphic either to the disk $|z_1| < 1$ in $\mathbb{C}$ or to the cone $z_1 z_2 = 1$, $|z_1| < 1$, $|z_2| < 1$ in $\mathbb{C}^2$. The first represents the generic points in S_0 which are locally euclidean, and the second represents isolated singularities in S_0 which can be obtained topologically by collapsing a meridian circle in a cylinder to a point. The latter set of points are called the nodes of the surface, and since S_0 is compact, there are only a finite number of nodes $P_1, \cdots, P_k$.

The complement $S_0 - \{P_1, \cdots, P_k\}$ forms a nonsingular surface, and the components $\Sigma_1, \cdots, \Sigma_r$ are called *parts*. A surface with nodes is called *stable* (or *hyperbolic*) if every part Σ_i has the upper half-plane $\mathfrak{S}_1$ as its universal covering space. The last condition excludes the situation where a part of S_0 is the 2-sphere $P^1(\mathbf{C})$ with 0, 1, or 2 punctures, or where it is a torus with 0 or 1 puncture. Let g_i be the genus of the Riemann surface Σ_i, $1 \leqslant i \leqslant r$. Then the genus g of the stable surface S_0 with nodes is defined by the formula $g = (g_1 - 1) + \cdots + (g_r - 1) + k + 1$. Alternatively, each stable surface S_0 is homeomorphic to the quotient space of a nonsingular surface S of genus g after collapsing a system α of disjoint, simple, closed curves to node points, $S_0 \simeq S/\alpha$. Note that this system of curves has the property that no single curve bounds a disk and no two curves bound a cylinder. This is referred to as an *admissible system of curves in* S.

(2.6) From now on, we fix a nonsingular surface S of genus g. We define the *augmented Riemann space* $\hat{R}(S)$ as the set of all isomorphism classes of stable Riemann surfaces with nodes $[S_0]$ whose genus is the same as S. From the viewpoint of function spaces, L. Bers introduced in [4] a topology on this space $\hat{R}(S)$ which extends the topology on the subspace $R(S)$ and makes $\hat{R}(S)$ a complex analytic space.

For each stable surface S_0, we define $R(S_0)$ to be the subspace in $\hat{R}(S)$ consisting of all stable surfaces homeomorphic to S_0. This is referred to as the *Riemann space* $R(S_0)$ *associated* to S_0. Clearly $R(S_0) = R(S_0')$ if and only if S_0 and S_0' are homeomorphic, and we can write $\hat{R}(S)$ as the disjoint union $\hat{R}(S) = \coprod_{S_0} R(S_0)$, where S_0 runs through all representatives of distinct homomorphism types of stable surfaces of genus g.

With respect to Bers' topology, our original Riemann space $R(S)$ is contained in $\hat{R}(S)$ as an open dense set. Its boundary $\hat{R}(S) - R(S)$ consists of all Riemann spaces $\coprod_{S_0} R(S_0)$, where $S_0 \neq S$, and S can be deformed onto S_0. In the same manner, the closure $\hat{R}(S_0)$ of $R(S_0)$ can be regarded as the *augmented Riemann space* of S_0, for the boundary $\hat{R}(S_0) - R(S_0)$ is again the union of all Riemann spaces $\coprod_{S_0} R(S_0')$, where $S_0' \neq S_0$ and S_0 can be deformed onto S_0'.

From the viewpoint of algebraic geometry, the augmented Riemann space $\hat{R}(S)$ is a projective variety of dimension $3g - 3$, and each of the closures $\hat{R}(S_0)$ is a subvariety of codimension l, where l is the number of nodes in S_0 (see [4]). By an abuse of notation, we will also write $R(S_0)$ as $R(S/\alpha)$, where α is an admissible system of curves in S with $S_0 \simeq S/\alpha$. Then there is a filtration of subvarieties

$$\hat{R}(S) \supset \coprod_{|\alpha|=1} \hat{R}(S/\alpha) \supset \coprod_{|\alpha|=2} \hat{R}(S/\alpha) \supset \cdots \supset \coprod_{|\alpha|=3g-3} \hat{R}(S/\alpha),$$

where $|\alpha|$ denotes the number of curves in α. This filtration makes $\hat{R}(S)$ a stratified space in the sense of R. Thom and J. Mather (see [21] and [15]).

(2.7) To describe the topology on $\hat{R}(S)$ we can proceed as follows.

Let S_1 and S_2 be two Riemann surfaces with nodes and with the same genus g. A *deformation* $[F, S_1, S_2]$ of S_1 onto S_2 is a continuous map F: $S_1 \to S_2$ such that

(2.7.1) F is onto,

(2.7.2) F^{-1}(node) is either a node or a Jordan curve in S_1,

(2.7.3) if Σ is the disjoint union of all the parts in S_2, then the restriction of F onto the subspace $F^{-1}(\Sigma)$ is an orientation-preserving homeomorphism onto Σ.

Two deformations F: $S_1' \to S_2$ and G: $S_1'' \to S_2$ are said to be equivalent to each other if there exists a complex homeomorphism h: $S_1' \to S_1''$ such that the following diagram is commutative up to homotopy:

$$\begin{array}{ccc} S_1' & \xrightarrow{F} & S_2 \\ {\scriptstyle h}\downarrow & \nearrow_{G} & \\ S_1'' & & \end{array}$$

Given a stable Riemann surface S_0 with nodes and with genus g, the set of all equivalence classes of deformations $[F, S', S_0]$ onto S_0 is called the deformation space of S_0, and is denoted by $D(S_0)$.

There is a canonical mapping π: $D(S_0) \to \hat{R}(S)$ of the deformation space onto the Riemann space defined by taking a deformation $[F, S', S_0]$ onto the domain S'. Within a fixed genus g, there are only finitely many nonhomeomorphic stable Riemann surfaces $S_1, \cdots, S_d$ with nodes. For each type of surface $S_1, \cdots, S_d$, the above construction gives us a deformation space $D(S_i)$ and a projection π: $D(S_i) \to \hat{R}(S)$, $1 \leqslant i \leqslant d$, into $\hat{R}(S)$. It is easy to see that the union of all the images $\pi(D(S_i))$ contains every point in the augmented Riemann space $\hat{R}(S)$.

A *partition* of a Riemann surface S_0 with k nodes is a set of $3g - 3 - k$ disjoint geodesic Jordan curves on S_0, such that the curves and nodes separate the surface S_0 into disjoint regions each of which is topologically homeomorphic to a punctured 2-sphere $\mathbb{P}^1(\mathbb{C})$ with three holes. The Jordan curves and nodes are called the boundary elements of the partition, and the disjoint regions are called *pants*. An ordered partition is one with an ordering of pants together with an ordering on the three boundary elements of each region. Associated to an ordered partition, there are $3g - 3$ complex numbers $\zeta_1, \cdots, \zeta_{3g-3}$, known as the Fenchel-Nielsen coordinates, where the norm $|\zeta_i|$ is the length of the ith boundary element and, for $\zeta_i \neq 0$, the angle $\operatorname{Arg} \zeta_i$ is the twist parameter θ_i determined by the position of the endpoints of certain geodesic arcs joining and orthogonal to distinct boundary elements.

Let S_t be a *terminal* Riemann surface. By that we mean that it has $3g - 3$ nodes, or equivalently it is partitioned by nodes. We fix an order for this partition. For any deformation F: $S \to S_t$, there is an induced partition on S obtained by deforming every Jordan curve F^{-1}(node) into a geodesic, and by ordering the pants in S according to the ordering in S_t. The Fenchel-Neilson coordinates of S now give rise to a map $D(S_t) \to \mathbf{C}^{3g-3}$ of the deformation space $D(S_t)$ into the complex affine space of dimension $3g - g$. We call this the Fenchel-Nielson map. It follows from the well-known results of Fenchel-Nielson (see [1]) that the above mapping is a bijection, and so we can introduce a complex analytic structure on $D(S_t)$ by means of this mapping. If we write $R(S)$ as a union $\hat{R}(S) = \pi(D(S_{t_i})) \cup \cdots \cup \pi(D(S_{t_k}))$ (see also (2.6)), where S_{t_i} runs through a set of topologically distinct terminal surfaces S_{t_i}, then the topology on $\hat{R}(S)$ coincides with the weak topology induced by the deformation spaces $D(S_{t_i})$, $1 \leqslant i \leqslant k$.

(2.8) For the other deformation spaces $D(S_0)$, we choose a deformation h: $S_0 \to S_t$ of S_0 onto a terminal surface S_t. As mentioned before, this gives a partition on S_0 with a preferred ordering. For any deformation $\langle F, S', S_0 \rangle$ this ordered partition again gives rise to $3g - 3$ Fenchel-Nielson coordinates $\zeta_1, \cdots, \zeta_{3g-3}$ of S', and so an induced map h_*: $D(S_0) \to D(S_t) \cong \mathbf{C}^{3g-3}$. However, this is neither injective nor surjective. To every node P in S_t, there is a *distinguished subspace* $\langle P \rangle$ in $D(S_t)$ consisting of all deformations F: $S' \to S_t$ such that $F^{-1}(P)$ is a node in S'. Under the identification in (2.7.1) this distinguished set $\langle P \rangle$ is mapped onto a coordinate hyperplane $\zeta_i = 0$, for some i.

Let $\{P_1, \cdots, P_l\}$ be the set of nodes in S_t whose preimage $F^{-1}(P_i)$ in S_0 is a geodesic curve. Then the image $h_*(D(S_0))$ of the above mapping h_* in $D(S_t)$ coincides with the complement of the union of distinguished sets $\langle P_1 \rangle \cup \cdots \cup \langle P_l \rangle$, i.e. $h_*(D(S_0)) \cong D(S_t) - \langle P_1 \rangle \cup \cdots \cup \langle P_l \rangle$. Since each $\langle P_i \rangle$ is a hyperplane, we have an isomorphism of $h_*(D(S_0))$ with the product space $(S^1)^l \times \mathbf{R}^l \times \mathbf{C}^{3g-3-l}$. It turns out that there is in fact a canonical isomorphism of $D(S_0)$ with the universal covering space $\mathbf{C}^{3g-3}$ of $(S^1)^l \times \mathbf{R}^l \times \mathbf{C}^{3g-3-l}$, and the map h_* is the covering map. One way to achieve this isomorphism is to redefine each of the angle parameters $\theta_i = \operatorname{Arg}(\zeta_i)$ in the Fenchel-Neilson coordinates so that it is measured by a geodesic distance in the universal covering space of the open surface $S_0 - \{\text{nodes}\}$ (see [2, p. 93]).

(2.8.1) *The deformation space $D(S_0)$ of a Riemann surface with nodes is isomorphic to the complex affine space $\mathbf{C}^{3g-3}$ under an isomorphism which takes every distinguished subset $\langle P \rangle$ to a coordinate hyperplane.*

A deformation h: $S_1 \to S_2$ between two surfaces S_1, S_2 (not necessarily terminal) gives rise to a holomorphic mapping h_*: $D(S_1) \to D(S_2)$ defined by sending an element F: $S' \to S_1$ in $D(S_1)$ to an element $h \circ F$: $S' \to S_2$ in $D(S_2)$. It has the following properties.

(2.8.2) *If h is a homeomorphic deformation, then the induced map h_*: $D(S_1) \to D(S_2)$ is an isomorphism.*

(2.8.3) *If S_2 has l more nodes than S_1, then the induced map h_*: $D(S_1) \to D(S_2)$ is a universal covering of $D(S_1)$ onto $h_*(D(S_1))$, where $h_*(D(S_1))$ is the complement of l distinguished subsets $\langle P_i \rangle$, $1 \leqslant i \leqslant l$, with $h^{-1}(P_i)$ geodesic curves in S_1.*

(2.9) For a nonsingular surface S, the deformation space $D(S)$ can be identified with the Teichmüller space $T(S)$, and the mapping class group $\Gamma(S)$ operates on $D(S)$ with the Riemann space $R(S)$ as its orbit space.

The situation for other deformation spaces is more complicated. Given a Riemann surface S_0 with nodes, define the mapping class group $\Gamma(S_0)$ to be the group of isotopy classes of homeomorphisms f: $S_0 \to S_0$ of S_0 into itself. Note that every homeomorphism must keep the set of node points invariant but not necessarily pointwise fixed. Let h: $S \to S_0$ be a deformation of a nonsingular surface S onto S_0. Pulling back the node points P_i in S_0 to geodesic curves $h^{-1}(P_i)$ in S, we obtain an admissible system α of curves in S. Let $\Gamma(S, \alpha)$ be the subgroup of the mapping class group $\Gamma(S)$ consisting of homeomorphisms f_α: $S \to S$ which preserve the set α up to isotopy. Given a self-homeomorphism f: $S_0 \to S_0$ in $\Gamma(S)$, we can always lift this to a self-homeomorphism f_α: $S \to S$ in $\Gamma(S_0, \alpha)$ such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\ h\ } & S_0 \\ {\scriptstyle f_\alpha}\downarrow & & \downarrow{\scriptstyle f} \\ S & \xrightarrow{\ h\ } & S_0 \end{array}$$

is commutative up to isotopy. This is because we can always lift homeomorphisms locally in a conical neighborhood U of a node point P_i to homeomorphisms in a cylindrical neighborhood $f^{-1}(U)$ of the corresponding curve $f^{-1}(P_i)$, and outside these neighborhoods h is a homeomorphism h: $S_0 - \alpha \cong S - \{P_i\}$. However, such a lifting is not unique. They can differ by Dehn twists along the geodesic curves in α. In other words, we have a group extension

$$1 \to N \to \Gamma(S, \alpha) \to \Gamma(S_0) \to 1, \tag{2.9.1}$$

where N is the free abelian group generated by the above Dehn's twists. We can view this group extension in terms of the induced mapping h_*: $D(S) \to D(S_0)$ between the deformation spaces. There are natural actions of the mapping class groups $\Gamma(S, \alpha)$ and $\Gamma(S_0)$ on these spaces, and h_* is equivariant with respect to these actions. As in (2.8.3), the projection h_*: $D(S) \to h_*(D(S))$ of $D(S)$ onto its image $h_*(D(S))$ is a universal covering map with N as its covering transformation group. The projection map π: $h_*(D(S)) = D(S_0) - \langle P_1 \rangle \cup \cdots \cup \langle P_l \rangle \to R(S)$ from the complement of the distinguished hyperplanes $\langle P_i \rangle$ to the Riemann space $R(S)$ is a ramified covering map. This is because the projection π: $T(S) \to R(S)$ of the Teichmüller space $D(S) \cong T(S)$ to the Riemann space $R(S)$ is a ramified covering map. If we factor a ramified covering map π by the covering map h,

$$\text{(2.9.2)} \qquad \begin{array}{ccl} D(S) & \overset{h_*}{\searrow} & \\ \pi \downarrow & & h_*(D(S)) \cong D(S)/N \subset D(S_0), \\ R(S) & \swarrow & \end{array}$$

the result π: $h_*(D(S)) \to R(S)$ is again a ramified covering map. Note that the group $\Gamma(S_0) \cong \Gamma(S, \alpha)/N$ operates on $h_*(D(S))$ but $R(S)$ is not its orbit space. However, consider the intersection of all the distinguished hyperplanes $\bigcap_{i=1}^{l} \langle P_i \rangle$ in $D(S_0)$. Then this can be regarded as the *Teichmüller space of* S_0. For, this space consists of all deformations f: $S' \to S_0$ which are homeomorphisms and the group $\Gamma(S_0)$ operates on this space with the Riemann space $R(S_0)$ as its quotient space. In fact, a stronger version of this observation is known (see [5, p. 53]).

(2.9.3) *For every element x in $D(S_0)$, there exists a neighborhood U stable under the isotropy subgroup I_x of x in $\Gamma(S_0)$, such that two elements x_1, x_2 in U map onto the same element in $\hat{R}(S)$, $\pi(x_1) = \pi(x_2)$, if and only if $x_1 = x_2 \cdot \gamma$ for some γ in I_x.*

An immediate consequence of this is the following.

Lemma (2.9.4). *There exists a $\Gamma(S_0)$-equivariant, open, neighborhood $\tilde{W}$ of $\bigcap_{i=1}^{l} \langle P_i \rangle$ in $D(S_0)$ such that its orbit space $\tilde{W}/\Gamma(S_0)$ under the action of $\Gamma(S_0)$ is isomorphic to its projection W in $\hat{R}(S)$ which in turn is an open neighborhood of $R(S_0)$.*

Proof of (2.9.4). Consider the closure $\hat{R}(S_0)$ of $R(S_0)$ in $\hat{R}(S)$. From the above discussion, it is clear that it is compact and consists of a disjoint union of lower dimensional Riemann spaces $R(S_0/\alpha)$, $\hat{R}(S_0) = \coprod R(S_0/\alpha)$. Each of these Riemann spaces $R(S_0/\alpha)$ arises from a stable surface S_0/α which is homeomorphic to the quotient of S_0 after collapsing a system of admissible curves α in S_0 to node points, h_α: $S_0 \to S_0/\alpha$, h_α^{-1} (nodes) $= \alpha \cup$ nodes of S_0.

Note that S_0/α contains more node points than S_0. We proceed by induction based on the decreasing number of node points. In the case where S_0 is terminal, $\bigcap_{i=1}^{3g-3}\langle P_i\rangle$ is a point, so we are reduced to the situation of (2.9.3). As an induction hypothesis, we may assume that there exists a neighborhood U_α of $R(S_0/\alpha)$ in $\hat{R}(S)$ which can be lifted to a corresponding (S_0/α)-equivariant neighborhood $\tilde{U}_\alpha$ in $D(S_0/\alpha)$. As in the previous paragraph, the natural map $h_{\alpha *}\colon D(S_0) \to D(S_0/\alpha)$ is a covering map of $D(S_0)$ onto its image, and so the set $U_\alpha \cap R(S_0)$, open in $R(S_0)$, can be further lifted to a $\Gamma(S_0)$-equivariant open set $\tilde{\tilde{U}}_\alpha$ in $D(S_0)$, $\pi(\tilde{\tilde{U}}_\alpha \cap (\bigcap\langle P_i\rangle)) = U_\alpha \cap R(S_0)$.

From (2.9.3), given any point x in $R(S_0)$, there exists a neighborhood U_x in $\hat{R}(S)$ which can be lifted to an (S_0)-equivariant neighborhood U_x in $D(S_0)$ covering a $\Gamma(S_0)$-orbit $\tilde{x}\cdot\Gamma(S_0)$ in $\bigcap\langle P_i\rangle$. Since $\hat{R}(S_0)$ is compact, it can be covered by a finite number of these neighborhoods U_{x_i}, $i = 1,\cdots,k$, and U_{α_j}, $j = 1,\cdots,k'$. Set $\tilde{W} = \tilde{U}_{x_1} \cup \cdots \cup \tilde{U}_{x_k} \cup \tilde{\tilde{U}}_{\alpha_1} \cup \cdots \cup \tilde{\tilde{U}}_{\alpha_{k'}}$. Then $W = \pi(\tilde{W})$ satisfies all the requirements and so this proves (2.9.4).

(2.10) In conclusion, it is worthwhile to point out that the above procedure of compactifying the Riemann space $R(S)$ is analogous to the toroidal compactification of locally symmetric spaces due to D. Mumford and others (see [3]). First of all, the Teichmüller space $T(S)$ plays the role of the symmetric space $K\backslash G$. The subgroup $\Gamma(S;\alpha)$ plays the role of an integral parabolic subgroup in $\Gamma(S)$. It has a decomposition in (2.9.1) which is analogous to the Levi-decomposition with N as its unipotent radical and $\Gamma(S/\alpha)$ as its semi-simple component. The quotient space $T(S)/N$ has a torus structure given by the formula $T(S)/N \cong (\mathbf{C}^*)^l \times \mathbf{C}^{3g-3-l}$. Using this, we can follow the procedure of torodial compactification to arrive at the deformation space $D(S/\alpha)$ (see [2, p. 29] and [3]).

3. Simplicial G-spaces associated to G-functors

(3.1) Let X be a simplicial complex, and let G be a group operating on X simplicially. For every simplex σ in X, we denote by G_σ the isotropy subgroup in G which keeps σ invariant, $G_\sigma = \{g \in G \mid \sigma\cdot g = \sigma\}$. By taking the first derived subdivision, we may assume that the correspondence $\sigma \mapsto G_\sigma$ satisfies the following properties:

(3.1.1) *Functorial.* If σ is a face of σ', then $G_{\sigma'}$ is a subgroup of G_σ, $\sigma \subseteq \sigma' \Rightarrow G_{\sigma'} \subseteq G_\sigma$.

(3.1.2) *Equivariant.* For an element g in G, there is a canonical isomorphism of G_σ onto $G_{\sigma' g}$ defined by conjugation, $G_\sigma \to G_{\sigma\cdot g}$, $x \mapsto g^{-1}xg$. In other words, if we consider the category $\mathbf{Sim}_G\, X$ whose objects are simplices σ in X

and whose morphisms $\mathrm{Mor}(\sigma_1, \sigma_2)$ are elements $g \in G$ such that $\sigma_1 \cdot g \supseteq \sigma_2$, then there is a functor

$$\mathbf{G}\colon \mathbf{Sim}_G\, X \to \mathcal{G}\!\not{p}$$

into the category of groups which takes an object σ to $G_\sigma = \mathrm{Mor}(\sigma, \sigma)$ and a morphism $\sigma_1 \cdot g \supseteq \sigma_2$ to the composite homomorphism

$$G_{\sigma_1} \xrightarrow{\text{conj}} G_{\sigma_1 \cdot g} \xhookrightarrow{\text{inc}} G_{\sigma_2}.$$

Now suppose we have a functor

$$\mathbf{E}\colon \mathbf{Sim}_G\, X \to \mathbf{Top}, \qquad \sigma \to E_\sigma$$

from $\mathbf{Sim}_G\, X$ into the category of topological spaces (with the homotopy type of a CW-complex). Then we associate to $\mathbf{E}$ a simplicial space $\mathbf{E}_*$,

$$\mathbf{E}_k = \coprod_{\substack{k\text{-simplices} \\ \sigma \subset X}} E_\sigma, \qquad \mathbf{E}(\partial_i) = \coprod E(\sigma \supseteq \partial_i \sigma)\colon \mathbf{E}_k \to \mathbf{E}_{k-1},$$

and a topological space $|\mathbf{E}|$,

$$|\mathbf{E}| = \coprod_{\substack{\text{simplices} \\ \sigma \subset X}} |\sigma| \times E_\sigma / \sim ,$$

obtained by taking the disjoint union $\coprod |\sigma| \times E_\sigma$ and glueing the components $|\sigma| \times E_\sigma$ together according to the relation $(\partial_i x, y) \sim (x, \mathbf{E}(\partial_i) y)$. The space $|\mathbf{E}|$ will be referred to as the *geometric realization of the functor* $\mathbf{E}$. There is a natural G-action on this space given by the formula

$$(x, y) \cdot g = (x \cdot g, E(g)(y)), \qquad (x, y) \in |\sigma| \times E_\sigma,$$

where $E(g)\colon E_\sigma \to E_{\sigma \cdot g}$ is the map corresponding to $g \in \mathrm{Mor}(\sigma, \sigma \cdot g)$.

Here are some examples.

Example (3.1.3). The simplest example is the situation when E_σ is a point for all σ. For this trivial functor, the geometric realization is isomorphic as a G-space to the original simplicial complex X. Given any other functor $\mathbf{E}$: $\mathbf{Sim}_G\, X \to \mathbf{Top}$, there is a natural transformation of $\mathbf{E}$ to this trivial functor which induces a simplicial, equivariant map $\pi\colon |\mathbf{E}| \to X$.

Example (3.1.4). The other extreme is the situation for which E_σ is a *universal* G_σ-space. By that we mean E_σ is contractible, and the G_σ-action is free. An example of such a functor is obtained by considering a universal G-space E_G. Restricting to the subgroup G_σ, we have a universal G_σ-space. Take $\mathbf{E}$: $\mathbf{Sim}_G\, X \to \mathbf{Top}$ to be the functor which sends each σ to E_G and sends $(\sigma_1 \cdot g \supseteq \sigma_2)$ to the action of g, $g\colon E_G \to E_G$. In this case, $|\mathbf{E}| = X \times E_G$, where the G-action is the diagonal action.

Example (3.1.5). Recall in (2.7) we gave the definition of a partition of a nonsingular Riemann surface S. Now, such a partition consists of a system of (disjoint) Jordan curves $\alpha = \{\gamma_\lambda\}$ such that no single curve γ_λ bounds a disk nor any two curves γ_λ, γ'_λ bound a cylinder. As in (2.5), we call such a system of curves an *admissible system of curves*. Two such systems $\alpha = \{\gamma_\lambda\}$, $\alpha' = \{\gamma_\mu\}$ are said to be equivalent if there exists a homeomorphism $f: S \to S$ which is isotopic to the identity and brings α to α'. The collection of all the equivalence classes of admissible systems of curves forms a partially ordered set under inclusion. Define *the Tit's building* $\mathscr{T}(S)$ of S to be this partially ordered set, and use the notation $|\mathscr{T}(S)|$ to denote the associated simplicial complex.

In practice, it will be convenient to consider the empty set also as an admissible system of curves. This will enlarge $\mathscr{T}(S)$ by including a minimal element, and we will denote the resulting partially ordered set by $\mathscr{T}(S)^+$ and the corresponding simplicial complex by $|\mathscr{T}(S)^+|$.

Geometrically, a vertex in $|\mathscr{T}(S)^+|$ is an element in $\mathscr{T}(S)^+$, and so it is represented by a system α of admissible curves. A k-simplex consists of a filtration of $k+1$ such systems $\sigma = \{\alpha_0 \subset \cdots \subset \alpha_k\}$, and the boundary operation $\partial_i\sigma$, $0 < i \leqslant k$, is defined by deleting the ith-system α_i, $\partial_i\sigma = \{\alpha_0 \subset \cdots \subset \hat{\alpha}_i \subset \cdots \subset \alpha_k\}$.

The mapping class group $\Gamma(S)$ operates on $\mathscr{T}(S)^+$ in a natural manner. The correspondence $\sigma \to \Gamma_\sigma$ of a simplex $\sigma = (\alpha_0 \subset \alpha_1 \subset \cdots \subset \alpha_k)$ with its isotropy subgroup Γ_σ clearly satisfies (3.1.1) and (3.1.2), and so defines a functor Γ: $\mathbf{Sim}_\Gamma|\mathscr{T}(S)^+| \mapsto \mathscr{G}\!\mathit{p}$.

(3.2) More generally, we will be interested in functors $\mathbf{E}$: $\mathbf{Sim}_G X \to \mathbf{Top}$ such that E_σ is a universal space for some *quotient* of G_σ. For this we consider functors

$$\mathring{\mathbf{G}}: \mathbf{Sim}_G X \to \mathscr{G}\!\mathit{p}$$

together with a natural transformation η: $\mathbf{G} \to \mathring{\mathbf{G}}$ which is a surjection for each σ, η_σ: $G \twoheadrightarrow \mathring{G}_\sigma$. We call such a pair $(\mathring{\mathbf{G}}, \eta)$ a *reduced group functor* for $\mathbf{G}$. A *universal* $(\mathring{\mathbf{G}}, \eta)$*-functor is a functor*

$$\mathbf{E}: \mathbf{Sim}_G X \to \mathbf{Top}$$

such that for all σ, the action of G_σ on E_σ factors through $\mathring{G}_\sigma$ (via η_σ) and E_σ is a universal space for $\mathring{G}_\sigma$, that is, E_σ is contractible and the action of $\mathring{G}_\sigma$ is free. (In most cases, $\mathring{G}_\sigma$ will be a natural quotient of G_σ and η_σ will be the natural projection map. When this is clear, we omit η from the notation.)

Example (3.2.1). We define a reduced group functor $(\mathring{\Gamma}, \eta)$ for the functor Γ of Example (3.1.5). Given a simplex $\sigma = (\alpha_0 \subset \alpha_1 \subset \cdots \subset \alpha_k)$, we have a stable surface S/α_0 and a filtration of nodes α_0/α_0 and curves $\alpha_i/\alpha_0 - \alpha_0/\alpha_0$

upon this surface, denoted by σ/α_0. Let $\mathring{\Gamma}(\sigma) = \Gamma(S/\alpha_0, \sigma/\alpha_0)$ be the mapping class group of homeomorphisms of S/α_0 preserving this filtration σ/α_0. For $\sigma' = \partial_i\sigma = (\alpha_0 \subset \cdots \subset \hat{\alpha}_i \subset \cdots \subset \alpha_k)$, $1 \leqslant i \leqslant k$, there is a natural inclusion $\mathring{\Gamma}(\partial_i)$: $\Gamma(S/\alpha_0, \sigma/\alpha_0) \to \Gamma(S/\alpha_0, \sigma'/\alpha_0)$ of the corresponding mapping class groups. For $\sigma' = \partial_0(\sigma) = (\alpha_1 \subset \cdots \subset \alpha_k)$, there is a deformation $S/\alpha_0 \to S/\alpha_1$ obtained by collapsing the curves $\alpha_1/\alpha_0 - \alpha_0/\alpha_0$ to node points. Since this preserves the filtrations σ/α_0 and σ'/α_1, any element in $\Gamma(S/\alpha_0, \sigma/\alpha_0)$ gives rise to a corresponding element in $\Gamma(S/\alpha_1, \sigma'/\alpha_1)$ and so a map

(3.2.2) $$\mathring{\Gamma}(\partial_0)\colon \Gamma(S/\alpha_0, \sigma/\alpha_0) \to \Gamma(S/\alpha_1, \sigma'/\alpha_1).$$

These satisfy the simplicial identities and hence their composition gives rise to well-defined homomorphisms $\mathring{\Gamma}(\partial)$ for any inclusion of simplicies $\sigma_1 \supseteq \sigma_2$. In addition, for a group element $g \in \Gamma(S)$, the homeomorphism g: $S/\alpha_0 \to S/\alpha_0 \cdot g$ induces a homomorphism $\mathring{\Gamma}(g)$: $\Gamma(S/\alpha_0, \sigma/\alpha_0) \to \Gamma(S/\alpha_0 \cdot g, \sigma \cdot g/\alpha_0 \cdot g)$ which commutes with the boundary maps. Thus we have a functor $\mathring{\Gamma}$: $\mathbf{Sim}_\Gamma|\mathscr{T}(S)^+| \to \mathscr{G}\!\mathit{p}$ defined by $\sigma \mapsto \Gamma(S/\alpha_0, \sigma/\alpha_0)$, $(\sigma_1 \cdot g \supseteq \sigma_2) \mapsto \mathring{\Gamma}(\partial) \circ \mathring{\Gamma}(g)$.

Following a similar procedure we define a natural transformation from Γ to $\mathring{\Gamma}$. For any simplex $\sigma = (\alpha_0 \subset \alpha_1 \subset \cdots \subset \alpha_k)$, we let $\sigma^+ = (\varnothing \subseteq \alpha_0 \subset \cdots \subset \alpha_k)$ be the simplex in $|\mathscr{T}(S)^+|$ obtained by adding the empty set as the first system of curves. Since Γ_σ is the same as $\Gamma(S/\varnothing, \sigma^+/\varnothing)$ and $\sigma = \partial_0\sigma^+$, the homomorphism $\mathring{\Gamma}(\partial_0)$ of (3.2.2) gives us a map

$$\eta_\sigma = \mathring{\Gamma}(\partial_0)\colon \Gamma_\sigma \to \Gamma(S/\alpha_0, \sigma/\alpha_0).$$

This defines a natural transformation η: $\Gamma \to \mathring{\Gamma}$ and completes our example.

Proposition (3.2.3). *Let* $(\mathring{\mathbf{G}}, \eta)$ *and* $(\mathring{\mathbf{G}}', \eta')$ *be reduced group functors for* $\mathbf{G}$, *and let* $\mathbf{E}$, $\mathbf{E}'$ *be universal* $(\mathring{\mathbf{G}}, \eta)$, $(\mathring{\mathbf{G}}', \eta')$ *functors respectively. If* φ: $\mathring{\mathbf{G}} \to \mathring{\mathbf{G}}'$ *is a natural transformation such that* $\varphi \circ \eta = \eta'$, *then* $\exists$ *an equivariant map of G-spaces*

$$H_\varphi\colon |\mathbf{E}| \to |\mathbf{E}'|.$$

Moreover, if Φ: $\mathbf{E} \to \mathbf{E}'$ *is any natural transformation, and* $|\Phi|$: $|\mathbf{E}| \to |\mathbf{E}'|$ *is the induced map of G-spaces, then there is an equivariant homotopy* $H_\varphi \sim |\Phi|$ *between* H_φ *and* $|\Phi|$.

Remark (3.2.4). As will be seen in the proof, the map H_φ is "almost" simplicial in the sense that $\mathbf{E}$ can be replaced by a functor $\mathbf{E}_D$ such that $|\mathbf{E}_D|$ is naturally homeomorphic to $|\mathbf{E}|$ and H_φ: $|\mathbf{E}_D| \to |\mathbf{E}'|$ is the geometric realization of a natural transformation $\mathbf{E}_D \to \mathbf{E}'$. Moreover, the homeomorphism $|\mathbf{E}_D| \cong |\mathbf{E}|$, though not itself simplicial, is homotopic to a simplicial map.

Proof of (3.2.3). Let X^1 denote the first derived subdivision of X. For a simplex σ in X, let $D(\sigma)$ denote the dual complex of σ, namely the subcomplex of X^1 consisting of simplicies $(\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_k)$ with $\sigma \subset \sigma_0$,

$$D(\sigma) = \left\{ (\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_k) \in X^1 \middle| \sigma \subseteq \sigma_0 \right\}.$$

If $\sigma \cdot g \supseteq \sigma'$, then the action of g gives a map $D(\sigma) \xrightarrow{g} D(\sigma \cdot g) \subseteq D(\sigma')$, so these define a functor $\mathbf{D}$: $\mathbf{Sim}_G X \to \mathbf{Top}$. The associated G-space $|\mathbf{D}|$ is homeomorphic to the original space X (cf. [7, Proposition 1.4]).

We can carry out this "dual triangulation" for a functor $\mathbf{E}$: $\mathbf{Sim}_G X \to \mathbf{Top}$. Let π: $|\mathbf{E}| \to X$ be the natural projection defined in (3.1.3). Define the functor $\mathbf{E}_D$: $\mathbf{Sim}_G X \to \mathbf{Top}$ by sending an object σ to $\pi^{-1}|D(\sigma)|$ and a morphism $(\sigma \cdot g \supseteq \sigma')$ to

$$\pi^{-1}\left(|D(\sigma)|\right) \xrightarrow{\cdot g} \pi^{-1}\left(|D(\sigma \cdot g)|\right) \subseteq \pi^{-1}\left(|D(\sigma')|\right).$$

Then there exists an equivariant homeomorphism

$$d: |\mathbf{E}_D| \xrightarrow{\cong} |\mathbf{E}| \tag{3.2.5}$$

which covers the homeomorphism $|\mathbf{D}| \cong X$ mentioned above.

To prove (3.2.3), we construct a natural transformation $\mathbf{E}_D \to \mathbf{E}'$. First, for every simplex σ in X, there exists a G_σ-equivariant homotopy equivalence h_σ: $E_\sigma \to E'_\sigma$. This follows from equivariant obstruction theory since E'_σ is contractible and the G_σ-actions on both spaces factor through $\mathring{G}_\sigma$ which acts freely on E_σ (see [6]). Secondly, for $\sigma' \subseteq \sigma$, we consider the space $\mathrm{Map}_{G_\sigma}(E_\sigma, E'_{\sigma'})$ of G_σ-equivariant mappings from E_σ to $E'_{\sigma'}$. These mapping spaces have the following properties:

(3.2.6) If $\sigma_0 \subseteq \sigma_1 \subseteq \sigma_2 \subseteq \sigma_3$, then the induced maps $E'_{\sigma_1} \to E'_{\sigma_0}$, $E_{\sigma_3} \to E_{\sigma_2}$ give rise to

$$\mathrm{Map}_{G_{\sigma_2}}\left(E_{\sigma_2}, E'_{\sigma_1}\right) \to \mathrm{Map}_{G_{\sigma_3}}\left(E_{\sigma_3}, E'_{\sigma_1}\right).$$

(3.2.7) The space $\mathrm{Map}_{G_\sigma}(E_\sigma, E'_{\sigma'})$ is contractible.

This last property follows once again from the equivariant obstruction theory mentioned above.

To construct the required natural transformation, we proceed by defining a map

$$f_\gamma: \Delta^k \to \mathrm{Map}_{G_{\sigma_k}}\left(E_{\sigma_k}, E'_{\sigma_0}\right)$$

for each $\gamma = (\sigma_0 \subset \cdots \subset \sigma_k)$ in X^1 such that (3.2.8) and (3.2.9) hold.

(3.2.8) If $\delta = (\sigma_{i_0} \subset \cdots \subset \sigma_{i_j}) \subset \gamma$, then

$$\begin{array}{ccc} \Delta^j & \xrightarrow{f_\delta} & \operatorname{Map}_{G_{\sigma_{i_j}}}\left(E_{\sigma_{i_j}}, E'_{\sigma_{i_0}}\right) \\ \Big\downarrow \text{incl.} & & \Big\downarrow \\ \Delta^k & \xrightarrow{f_\gamma} & \operatorname{Map}_{G_{\sigma_k}}\left(E_{\sigma_k}, E'_{\sigma_0}\right) \end{array}$$

commutes, where the right vertical map is given in (3.2.6).

(3.2.9) If $\gamma \cdot g = (\sigma_0 \cdot g \subset \cdots \subset \sigma_k \cdot g)$, then the following diagram commutes:

$$\begin{array}{ccc} \Delta^k & \xrightarrow{f_\gamma} & \operatorname{Map}_{G_{\sigma_k}}\left(E_{\sigma_k}, E'_{\sigma_0}\right) \\ \Big\downarrow \cdot g & & \Big\downarrow \cdot g \\ \Delta^k \cdot g & \xrightarrow{f_{\gamma \cdot g}} & \operatorname{Map}_{G_{\sigma_k \cdot g}}\left(E_{\sigma_k \cdot g}, E'_{\sigma_0 \cdot g}\right). \end{array}$$

This is achieved by induction on $\dim \gamma$. In dimension 0, we choose one vertex σ_0 for each G-orbit and define $f_\gamma = h_{\sigma_0}$. Then we extend this to all other vertices $\sigma_0 \cdot g$ in the same orbit by means of the equivariant condition in (3.2.9). Assume we have defined f_δ for all δ of dimension less than k. We choose among each G-orbit a representative $\gamma = (\sigma_0 \subset \cdots \subset \sigma_k)$. Then by (3.2.8), the maps f_δ, $\delta \subset \gamma$, fit together to get a map

$$f_{\partial\Delta}\colon \partial\Delta^k \to \operatorname{Map}_{G_{\sigma_k}}\left(E_{\sigma_k}, E'_{\sigma_0}\right).$$

Since the space $\operatorname{Map}_{G_{\sigma_k}}(E_{\sigma_k}, E'_{\sigma_0})$ is contractible, this extends to a map f_γ on all of Δ^k. By the definition of $f_\gamma | \partial\Delta^k$, this is compatible with the previously defined f_δ. For every other k-simplex $\gamma \cdot g$ in the same orbit, we define $f_{\gamma \cdot g}$ by means of the diagram (3.2.9).

We now define

$$h_\sigma\colon (E_D)_\sigma = \coprod_{\substack{\gamma = (\sigma_0 \cdots \sigma_k) \\ \sigma \subset \sigma_0}} \Delta^k \times E_{\sigma_k}/\sim \;\to E'_\sigma$$

by taking the composition

$$\Delta^k \times E_{\sigma_k} \xrightarrow{f_\gamma} E'_{\sigma_0} \xrightarrow{E'(\partial)} E'_\sigma.$$

From (3.2.8) and (3.2.9), it is not difficult to show that these maps commute with the boundary operations and also preserve the G-actions. Thus they define a natural transformation $\mathbf{h}\colon \mathbf{E}_D \to \mathbf{E}'$. On the level of simplicial spaces, this gives rise to a map

$$H_\varphi \colon |\mathbf{E}| \overset{d^{-1}}{\cong} |\mathbf{E}_D| \overset{|\mathbf{h}|}{\to} |\mathbf{E}'|,$$

which is equivariant with respect to the G-action.

To prove the second statement of the theorem, we make the following observation. The homeomorphism $d\colon |\mathbf{E}_D| \cong |\mathbf{E}|$ is not simplicial, that is, it does not arise as a natural transformation from $\mathbf{E}_D$ to $\mathbf{E}$. On the other hand, there is a natural transformation $\mathbf{r}\colon \mathbf{E}_D \to \mathbf{E}$ such that $|\mathbf{r}|$ is homotopic to d. To see this, we note that the deformation retraction of $|D(\sigma)|$ onto the barycenter b_σ of σ (b_σ corresponds to the vertex (σ) in $D(\sigma)$) is covered by a deformation retraction r_σ of $(\mathbf{E}_D)_\sigma = \pi^{-1}(|D(\sigma)|)$ onto $\mathbf{E}_\sigma = \pi^{-1}(b_\sigma)$,

$$r_\sigma \colon (\mathbf{E}_D)_\sigma = \coprod_{\substack{(\sigma_0 \subset \cdots \subset \sigma_k) \\ \sigma \subseteq \sigma_0}} |\Delta^k| \times E_{\sigma_k} \xrightarrow{E(\sigma_k \supseteq \sigma)} E_\sigma .$$

It is not difficult to check that these form a natural transformation $\mathbf{r}\colon \mathbf{E}_D \to \mathbf{E}$ whose geometric realization $|\mathbf{r}|\colon |\mathbf{E}_D| \to |\mathbf{E}|$ is homotopic to d.

Suppose now that $\phi\colon \mathbf{E} \to \mathbf{E}'$ is a natural transformation. Then ϕ induces a natural transformation $\phi_D\colon \mathbf{E}_D \to \mathbf{E}'_D$ in the obvious way such that the diagram

$$\begin{array}{ccc} \mathbf{E}_D & \xrightarrow{\phi_D} & \mathbf{E}'_D \\ {\scriptstyle \mathbf{r}}\downarrow & & \downarrow{\scriptstyle \mathbf{r}'} \\ \mathbf{E} & \xrightarrow{\phi} & \mathbf{E}' \end{array}$$

commutes. Since $H_\varphi = |\mathbf{h}| \circ d^{-1}$, it suffices to show that $\mathbf{h}$ and $\mathbf{r}' \circ \phi_D$ induce homotopic maps. The proof of this consists of constructing a natural transformation $\boldsymbol{\mu}\colon \mathbf{E}_D \times I \to \mathbf{E}'$ which restricts to $\mathbf{h}$ and $\mathbf{r}' \circ \Phi_D$ at $\mathbf{E}_D \times 0$ and $\mathbf{E}_D \times 1$ respectively. The construction is done inductively and the arguments are essentially the same as those used above in defining $\mathbf{h}$. We leave the details to the reader.

(3.3) In practice, there is a technical difficulty in applying the theory in (3.2). It arises when the group $\mathring{G}_\sigma$ operates on E_σ "almost" freely.

Suppose $\Gamma \subset G$ is a subgroup. Then the action of G on X restricts to an action of Γ on X and $\mathbf{Sim}_\Gamma X \subset \mathbf{Sim}_G X$. Let $(\mathring{\mathbf{G}}, \eta)$ be a reduced group functor for $\mathbf{G}$. Set

$$\mathring{\Gamma}_\sigma = \operatorname{image}\left(\Gamma \cap G_\sigma \subseteq G_\sigma \overset{\eta_\sigma}{\to} \mathring{G}_\sigma\right).$$

If $\sigma_1 \cdot g \supseteq \sigma_2$ is a morphism in $\mathbf{Sim}_\Gamma X$, then the corresponding morphism $\mathring{G}_{\sigma_1} \to \mathring{G}_{\sigma_2}$ takes $\mathring{\Gamma}_{\sigma_1}$ into $\mathring{\Gamma}_{\sigma_2}$, hence we obtain a functor

$$\mathring{\mathbf{\Gamma}}: \mathbf{Sim}_\Gamma X \to \mathcal{G}\!\mathit{p}.$$

Moreover, the natural transformation η: $\mathbf{G} \to \mathring{\mathbf{G}}$ restricts to a natural transformation

$$\eta^\Gamma: \mathbf{\Gamma} \to \mathring{\mathbf{\Gamma}}, \qquad \eta^\Gamma_\sigma: \Gamma_\sigma = \Gamma \cap G_\sigma \to \mathring{\Gamma}_\sigma$$

so $(\mathring{\mathbf{\Gamma}}, \eta^\Gamma)$ is a reduced group functor for Γ. Note that if Γ is a *normal* subgroup of g, then the definition of $\mathring{\mathbf{\Gamma}}$ makes sense on all of $\mathbf{Sim}_G X$ so we can (by abuse of notation) consider $\mathring{\mathbf{\Gamma}}$ to be a functor

$$\mathring{\mathbf{\Gamma}}: \mathbf{Sim}_G X \to \mathcal{G}\!\mathit{p}.$$

Now suppose $\mathbf{E}$: $\mathbf{Sim}_G X \to \mathbf{Top}$ is a universal $(\mathring{\mathbf{G}}, \eta)$-functor. Then clearly it is also a universal $(\mathring{\mathbf{\Gamma}}, \eta^\Gamma)$-functor. The converse, however, is not true. We say that $\mathbf{E}$ is an *almost universal* $(\mathring{\mathbf{G}}, \eta)$-functor if there exists a normal subgroup $\Gamma \subseteq G$ of finite index such that $\mathbf{E}$ is a universal $(\mathring{\mathbf{\Gamma}}, \eta^\Gamma)$-functor.

Proposition (3.3.1). *If* E: $\mathbf{Sim}_G X \to \mathbf{Top}$ *is an almost universal* $(\mathring{\mathbf{G}}, \eta)$-*functor such that each* E_σ *is a CW-complex, then* $\exists$ *a universal* $(\mathring{\mathbf{G}}, \eta)$-*functor* $\mathbf{E}^{\mathrm{un}}$ *and a natural transformation* μ: $\mathbf{E}^{\mathrm{un}} \to \mathbf{E}$ *such that the induced map on quotient spaces*

$$|\mu|/G: |\mathbf{E}^{\mathrm{un}}|/G \to |\mathbf{E}|/G$$

is a rational homology equivalence.

Proof. Let $\Gamma \subset G$ be a normal subgroup of finite index such that $\mathbf{E}$ is a universal $(\mathring{\mathbf{\Gamma}}, \eta^\Gamma)$-functor. Then we have a "quotient group" functor

$$\mathring{\mathbf{G}}/\mathring{\mathbf{\Gamma}}: \mathbf{Sim}_G X \to \mathcal{G}\!\mathit{p}, \qquad \sigma \mapsto \mathring{G}_\sigma/\mathring{\Gamma}_\sigma.$$

Composing this with the universal space functor $\mathcal{G}\!\mathit{p} \to \mathbf{Top}$ which takes a group G to a universal G-space, we obtain a functor

$$\mathbf{Y}: \mathbf{Sim}_G X \to \mathbf{Top}$$

which takes σ to a univesal $\mathring{G}_\sigma/\mathring{\Gamma}_\sigma$-space Y_σ. The product of this with $\mathbf{E}$,

$$\mathbf{E} \times \mathbf{Y}: \mathbf{Sim}_G X \to \mathbf{Top}, \qquad \sigma \mapsto E_\sigma \times Y_\sigma,$$

is easily seen to be a universal $(\mathring{\mathbf{G}}, \eta)$-functor. We set $\mathbf{E}^{\mathrm{un}} = \mathbf{E} \times \mathbf{Y}$. Next, consider the natural transformation μ: $\mathbf{E}^{\mathrm{un}} \to \mathbf{E}$ defined by projecting $E_\sigma \times Y_\sigma$ onto the first factor E_σ. This gives rise to a map of orbit spaces $\bar{\mu} = |\mu|/G$,

$$\bar{\mu}: |\mathbf{E}^{\mathrm{un}}|/G \to |\mathbf{E}|/G,$$

whose fiber $\bar{\mu}^{-1}(x)$ over a point x in the image of $|\sigma| \times E_\sigma$ in $|\mathbf{E}|/G$ is $\bar{\mu}^{-1}(x) = Y_\sigma/G_\sigma$. This last space is, by construction, a classifying space $B(\mathring{G}_\sigma/\mathring{\Gamma}_\sigma)$ for the finite group $\mathring{G}_\sigma/\mathring{\Gamma}_\sigma$. In particular, the rational cohomology

$H^*(\ ; \mathbb{Q})$ of every fiber is 0 for $* > 0$. It now follows from the Vietoris mapping theorem (see e.g. [19, Theorem 2.20]), that the restriction of $\bar{\mu}$ over any compact subset $A \subset |\mathbf{E}|/G$, $\bar{\mu}$: $\bar{\mu}^{-1}(A) \to A$, induces isomorphisms on $H^*(\ ; \mathbb{Q})$ and hence also on $H_*(\ ; \mathbb{Q})$. But

$$H_*(|\mathbf{E}|/G; \mathbb{Q}) = J \varinjlim_A H_*(A; \mathbb{Q}),$$

$$H_*(|\mathbf{E}| \times |\mathbf{Y}|/G; \mathbb{Q}) = \varinjlim_A H_*(\bar{\mu}^{-1}(A)),$$

where the limits are taken over all compact subsets $A \subset |\mathbf{E}|/G$. This proves the theorem.

4. Stratified polyhedra

(4.1) For the sake of completeness, we collect in this section some of the well-known facts concerning the homology of stratified polyhedra.

Let X be a finite, simplicial complex of dimension n. A *stratification* on X is a filtration of closed subcomplexes $X = X_n \supset X_{n-1} \supset \cdots \supset X_0$ such that:

(4.1.1) The subspace $X^i = X_i - X_{i-1}$, called the *i-dimensional stratum*, is either empty or of dimension i.

(4.1.2) If the intersection $\bar{X}^j \cap X^i \neq \varnothing$, then $i \leqslant j$ and $\bar{X}^j \cap X^i$ consists of connected components of X^i.

In particular, X is the disjoint union of its strata, $X = \coprod_{i \leqslant n} X^i$, and the boundary (frontier) of each stratum X^i is the union of the lower strata, $\partial X^i = \coprod_{j<i} X^j$.

Note that if, in addition to (4.1.1), we require X^i to be an i-dimensional manifold, then we arrive at the notion of stratified spaces as developed by J. Mather, R. Thom and others (see [15], [21]).

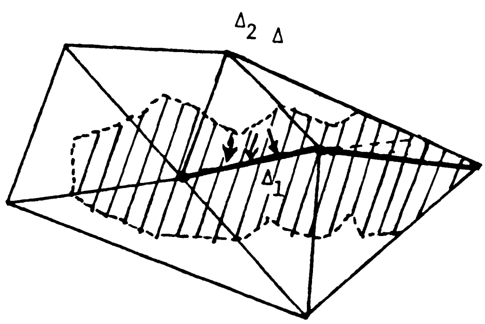
Let Y be a subcomplex in X. Recall the definition of a regular neighborhood $\bar{N}(Y)$ of Y in piecewise linear topology (see [11], [18]). First, form the second derived subdivision X'' of X. Then the regular neighborhood $\bar{N}(Y)$ is given by the *simplicial neighborhood* $\bar{N}(Y'', X'')$ of Y'' in X'', i.e.,

$$\begin{aligned} \bar{N}(Y) &= \bar{N}(Y'', X'') \\ &= \bigcup \{ \Delta \mid \Delta \text{ a closed simplex in } X'', \\ &\qquad\quad \Delta \text{ intersects } Y'' \text{ in a nonempty face} \} \\ &= \bigcup_{v \in Y''} \overline{\operatorname{Star}}(v, X'').^1 \end{aligned} \tag{4.1.3}$$

[1] The notation $\bar{N}(Y, X)$ stands for closed simplicial neighborhood, $\bar{N}(Y)$ for closed regular neighborhood, $N(Y)$ for open regular neighborhood, and $N(Y, X)$ for open simplicial neighborhood.

Abstractly, this is the smallest simplicial subcomplex in X'' which contains Y'' as a topological neighborhood. Note that in the common usage of the term regular neighborhood, it is not unique but depends on a triangulation of X. In our setting, this triangulation is fixed and so there is only one choice for the regular neighborhood.

For each i, let $\overline{N}(X_i)$ denote the regular neighborhood of X_i. The reason for taking the second derived is that a simplex Δ, in the first derived neighborhood $\overline{N}(X_i', X')$ (see [8]), can be written as a join $\Delta_1 * \Delta_2$, of two simplexes $\Delta_1 = \Delta \cap X_i'$, Δ_2 outside of X_i', $\Delta_2 \neq \varnothing$. From the structure of the join $\Delta_1 * \Delta_2$, there exists a canonical deformation retract of $\overline{N}(\Delta_1', \Delta')$ onto Δ' (see [10], [11]).



These deformations can be pieced together to get a deformation retract P_i: $\overline{N}(X_i) \to X_i$ of the entire regular neighborhood $\overline{N}(X_i) = \bigcup_\Delta \overline{N}(X_i'' \cap \Delta', \Delta')$ onto $X_i = \bigcup X_i \cap \Delta$. *This is referred to as the deformation retract associated to the regular neighborhood of* $\overline{N}(X_i)$.

Following the terminology of D. Stone (see, [20, p. 5]), a subspace Y in X is said to be a *subpolyspace* of X if both its closure $\overline{Y}$ and its boundary $\partial Y = \overline{Y} - Y$ are subpolyhedra in X. For instance, each stratum $X^i = X_i - X_{i-1}$ is a subpolyspace in X. Given such a subspace Y, consider the regular neighborhood $\overline{N}(\overline{Y})$ of its closure, the deformation retract P: $\overline{N}(\overline{Y}) \to \overline{Y}$, and the interior $N(\overline{Y})$ of $\overline{N}(\overline{Y})$. If we form the inverse image $P^{-1}(Y)$ in $N(\overline{Y})$ of the subspace Y in X, the result $N(Y) = N(\overline{Y}) \cap P^{-1}(Y)$ is an open neighborhood in X containing Y. This is referred to as the *open regular neighborhood* of Y, denoted by $N(Y)$. From another viewpoint, the open regular neighborhood $N(Y)$ can be defined as the smallest, subpolyspace in X'' which is an open topological neighborhood of Y. Or it can be written as the union of open stars $\operatorname{Star}(\nu, X'')$,

$$N(Y) = \bigcup_{\nu \in Y''} \operatorname{Star}(\nu, X''),$$

where ν runs through all vertices in Y''.

Let $N(X^i)$ denote the open regular neighborhood of X^i in X. From the above definition, there is a deformation retract P_i: $N(X^i) \to X^i$ of $N(X^i)$ onto X^i. Throughout this paper, we consider only subspaces which are subpolyspaces with respect to a fixed triangulation on X. In the following, we drop the term subpolyspace and refer to it simply as a subspace.

Suppose X is a stratified manifold in the sense that X_i is a submanifold in X_j, and suppose the triangulation is C^∞. Then, as is well known, the open regular neighborhood $N(X_i)$ has a structure of a normal bundle (a vector bundle), and the projection $N(X_i) \to X_i$ is homotopic to a locally trivial fibration (in the sense of Steenrod) (see [18]).

More generally, without the manifold structure, we still refer to the deformation retract X_i in $N(X_i)$ as the *zero section* of the open regular neighborhood.

(4.2) We index the connected components in the strata X^i, $0 \leqslant i \leqslant n$, by an index set I. Then there is a partial order relation defined on the elements of I:

(4.2.1) $\alpha < \beta$ if and only if $X_\beta \subseteq \overline{X}_\alpha$. Here X_α, X_β are the connected components corresponding to α and β respectively.

Since $N(X^i)$ has the same homotopy type as X^i, it breaks down into a disjoint union of connected components in one-to-one correspondence with components in X^i. Therefore, if we consider the connected components of $N(X^i)$, $0 \leqslant i \leqslant n$, we obtain an open covering $N(X_\alpha)$ of X indexed by I.

Proposition (4.2.2). *Let* $|I|$ *be the simplicial complex associated to the partially ordered set* $(I, <)$ *defined in* (4.2.1). *Then the nerve of the open covering* $\{N(X_\alpha) | \alpha \in I\}$ *is isomorphic to* $|I|$.

Proof. Two regular neighborhoods $N(X_\alpha)$ and $N(X_\beta)$ have nonempty intersection if and only if either X_α is in the closure of X_β ($\beta < \alpha$) or X_β is in the closure of X_α ($\alpha < \beta$). Hence the intersection $\bigcap_{\alpha \in I'} N(X_\alpha)$ of neighborhoods over $I' \subset I$ is nonempty if and only if I' can be linearly ordered. In other words, I' spans a simplex in $|I|$. This proves (4.2.2).

Proposition (4.2.3). *Given a simplex* $\sigma = (\alpha_0 < \alpha_1 < \cdots < \alpha_k)$ *in* $|I|$, *define* N_σ, M_σ *to be the following intersection*:

$$N_\sigma = \bigcap_{i=1}^{k} N(X_{\alpha_i}), \qquad M_\sigma = X_{\alpha_0} \cap N_\sigma.$$

Then N_σ *is the open regular neighborhood of* M_σ *in* X.

Proof. First, we reduce the proof to the case $k = 1$. Let Y be a subspace in X, $N(Y)$ its open regular neighborhood and U an open subspace of X with $U \cap Y \neq \varnothing$. It follows from the definition that the intersection $N(Y) \cap U$ is the smallest, open, polyspace containing $Y \cap U$, and this is its regular neighborhood. Suppose we have shown the case $k = 1$. Then the general case follows immediately by letting

$$U = \bigcap_{i=2}^{k} N(X_{\alpha_i}), \qquad N(Y) = N(X_{\alpha_0}) \cap N(X_{\alpha_1}), \qquad Y = X_{\alpha_0} \cap N(X_{\alpha_1}).$$

The proof for the case $k = 1$ can be reduced to the situation when X is a simplex. Let Δ be a simplex in the second derived subdivision X'' of X, and let $\Delta_i = X_{\alpha_i} \cap \Delta$. Then the intersection $N(X_{\alpha_i}) \cap \Delta$ of the regular neighborhood $N(X_{\alpha_i})$ with Δ is the same as the *open* simplicial neighborhood $N(\Delta_i, \Delta)$,

$$N(X_{\alpha_i}) \cap \Delta = N(X_{\alpha_i} \cap \Delta, \Delta) = N(\Delta_i, \Delta).$$

Applying the same principle to $N(X_{\alpha_0} \cap N(X_{\alpha_1}))$, we have

$$N\big(X_{\alpha_0} \cap N(X_{\alpha_1})\big) \cap \Delta = N(\Delta_0 \cap N(\Delta_1), \Delta).$$

Since $\bigcap_{i=0}^{1} N(X_{\alpha_i})$, $N(X_{\alpha_0} \cap N(X_{\alpha_1}))$ are unions of the above spaces as Δ runs through all simplices, it is enough to verify the formula

$$N(\Delta_0, \Delta) \cap N(\Delta_1, \Delta) = N(\Delta_0 \cap N(\Delta_1), \Delta)$$

for the polyspaces Δ_0, Δ_1 in Δ.

Let $\overline{\Delta}_i$ be the closure of Δ_i in Δ, $\Delta_1 \subset \overline{\Delta}_0$. We may assume that $\Delta_1 \neq \varnothing$ because otherwise $N(\Delta_1, \Delta) = \Delta_0 \cap N(\Delta_1) = \varnothing$ so there is nothing to prove. For $\overline{\Delta}_i \neq \varnothing$, the simplicial neighborhood $\overline{N}(\overline{\Delta}_i, \Delta)$ becomes the entire simplex $\overline{N}(\overline{\Delta}_i, \Delta) = \Delta$, and so from the definition the open simplicial neighborhood $N(\Delta_i, \Delta)$ is obtained from Δ by deleting faces which have nonempty intersection with $\overline{\Delta}_i - \Delta$. Hence $\bigcap_{i=0}^{1} N(\Delta_i, \Delta)$ can be obtained by deleting faces in Δ which intersect either $\overline{\Delta}_0 - \Delta_0$ or $\overline{\Delta}_1 - \Delta_1$. In the same manner, the open neighborhood $N(\Delta_1) \cap \Delta_0$ can be obtained from $\overline{\Delta}_0$ by subtracting faces which lie in $(\overline{\Delta}_0 - \Delta_0) \cap (\overline{\Delta}_1 - \Delta_1)$. To obtain $N(N(\Delta_1) \cap \Delta_0, \Delta)$ we have to subtract faces from Δ which intersect $\overline{\Delta}_1 - N(\Delta_1) \cap \Delta_0$. It follows from the description for $N(\Delta_1) \cap \Delta_0$ that these faces are precisely those which intersect $(\overline{\Delta}_0 - \Delta_0) \cap (\overline{\Delta}_1 - \Delta_1)$. Hence,

$$N(N(\Delta_1) \cap \Delta_0, \Delta) = \bigcap_{i=0}^{1} N(\Delta_i, \Delta),$$

and the proof of (4.2.3) is complete.

Proposition (4.2.4). *Let P_σ: $N_\sigma \to M_\sigma$ denote the deformation retract associated to the open regular neighborhood N_σ of M_σ in* (4.2.3). *Then there are commutative diagrams*:

(4.2.5) *For* $1 \leqslant i \leqslant k$, $\sigma = (\alpha_0 < \alpha_1 < \cdots < \alpha_k)$, $\partial_i\sigma = (\alpha_0 < \cdots \hat{\alpha}_i \cdots \alpha_k)$:

$$\begin{array}{ccc} N_{\alpha_0} \cap \cdots \cap N_{\alpha_k} & \xrightarrow{\text{incl.}} & N_{\alpha_0} \cap \cdots \hat{N}_{\alpha_i} \cdots \cap N_{\alpha_k} \\ \downarrow P_\sigma & & \downarrow P_{\partial_i \sigma} \\ X_{\alpha_0} \cap N_{\alpha_1} \cdots \cap N_{\alpha_k} & \xrightarrow{\text{incl.}} & X_{\alpha_0} \cap N_{\alpha_i} \cap \cdots \hat{N}_{\alpha_i} \cdots \cap N_{\alpha_k} \end{array}$$

(4.2.6) *For* $i = 0$, $\sigma = (\alpha_0 < \cdots < \alpha_k)$, $\partial_0\sigma = (\alpha_1 < \cdots < \alpha_k)$:

$$\begin{array}{ccc} N_{\alpha_0} \cap \cdots \cap N_{\alpha_k} & \xrightarrow{\text{incl.}} & N_{\alpha_1} \cap \cdots \cap N_{\alpha_k} \\ \downarrow{\scriptstyle P_\sigma} & & \downarrow{\scriptstyle P_{\partial_0\sigma}} \\ X_{\alpha_0} \cap N_{\alpha_1} \cdots N_{\alpha_k} & \xrightarrow{P_{\alpha_1}} & X_{\alpha_i} \cap N_{\alpha_2} \cdots N_{\alpha_k} \end{array}$$

where the bottom horizontal map in (4.2.6) *is obtained by the restriction* P_{α_1}: $X_{\alpha_0} \cap N_{\alpha_1} \to X_{\alpha_1}$ *to the subspace* $X_{\alpha_0} \cap N_{\alpha_1} \cdots N_{\alpha_k}$ (*see Figure* (4.2.10)).

Proof (*Sketch*). Let Y be a subspace in X, $N(Y)$ its open regular neighborhood, and U an open subspace in X. Then, as in (4.2.3), the intersection $N(Y) \cap U$ is the open regular neighborhood of $Y \cap U$. In addition, if P: $N(Y) \to Y$ is the deformation retract associated to $N(Y)$, then the restriction of P to $N(Y) \cap U$ is the deformation associated to $Y \cap U$. Diagram (4.2.5) follows by letting $Y = M_{\partial_i\sigma}$, $N(Y) = N_{\partial_i\sigma}$, $U = N_{\alpha_i}$, $U \cap Y = M_\sigma$, $U \cap N(Y) = N_\sigma$.

The above argument allows us to reduce the proof of (4.2.6) to the special case $k = 1$. To verify this case, let Δ be a simplex in the first derived subdivision X' of X, and let $\Delta_i = X'_{\alpha_i} \cap \Delta$. Our problem can be reduced to verifying that the diagram

$$\text{(4.2.7)} \qquad \begin{array}{ccc} N(\Delta_0, \Delta') \cap N(\Delta_1, \Delta') & \xhookrightarrow{\text{incl.}} & N(\Delta_1, \Delta') \\ \downarrow{\scriptstyle P_\sigma} & & \downarrow{\scriptstyle P_{\partial_0\sigma}} \\ \Delta_0 \cap N(\Delta_1, \Delta') & \xhookrightarrow{P_1} & \Delta_1 \end{array}$$

is commutative in the special case when $X = \Delta$ and $X_{\alpha_i} = \Delta_i$. Note that there is a corresponding diagram

$$\text{(4.2.8)} \qquad \begin{array}{ccc} \overline{N}(\overline{\Delta}_0, \Delta') \cap \overline{N}(\overline{\Delta}_1, \Delta') & \xhookrightarrow{\text{incl.}} & \overline{N}(\overline{\Delta}_1, \Delta') \\ \downarrow{\scriptstyle P_\sigma} & & \downarrow{\scriptstyle P_{\partial_0\sigma}} \\ \overline{\Delta}_0 \cap \overline{N}(\overline{\Delta}_1, \Delta') & \xhookrightarrow{P_1} & \overline{\Delta}_1 \end{array}$$

obtained by replacing Δ_i by its closure $\overline{\Delta}_i$ in Δ'. Maps in (4.2.7) are the restriction of the corresponding mappings in (4.2.8). Hence it is enough to prove (4.2.8) is commutative. From the definition, $\overline{\Delta}_1 \subset \overline{\Delta}_0$, and they are subsimplices in Δ'. The commutativity of (4.2.8) follows from a straightforward argument, and we will leave the details to the reader. This completes the proof of (4.2.4).

Remark (4.2.9). It is worthwhile to point out the following observation related to (4.2.6).

First, X_{α_1} is a subspace of $\overline{X}_{\alpha_0}$. The regular neighborhood of X_{α_1} is N_{α_1} and its intersection $N_{\alpha_1} \cap \overline{X}_{\alpha_0}$ gives the regular neighborhood of X_{α_1} in $\overline{X}_{\alpha_0}$. Deleting the zero section X_{α_1} from this regular neighborhood, we arrive at $X_{\alpha_0} \cap N_{\alpha_1}$ and the projection P_1: $X_{\alpha_0} \cap N_{\alpha_1} \to X_{\alpha_1}$.

In the same manner, the space $M_{\partial_0\sigma} = X_{\alpha_1} \cap N_{\alpha_2} \cdots N_{\alpha_k}$ in (4.2.6) is a subspace in the closure $\overline{X}_{\alpha_0} \cap \overline{N}_{\alpha_2} \cdots \overline{N}_{\alpha_k}$. The open regular neighborhood of $M_{\partial_0\sigma}$ in this closure is $\overline{X}_{\alpha_0} \cap N_{\alpha_1} \cap \cdots \cap N_{\alpha_k}$, and after deleting the zero section we arrive at $M_\sigma = X_{\alpha_0} \cap N_{\alpha_1} \cap \cdots \cap N_{\alpha_k}$, and the projection P_{α_1}: $M_\sigma \to M_{\partial_0\sigma}$ in (4.2.6).

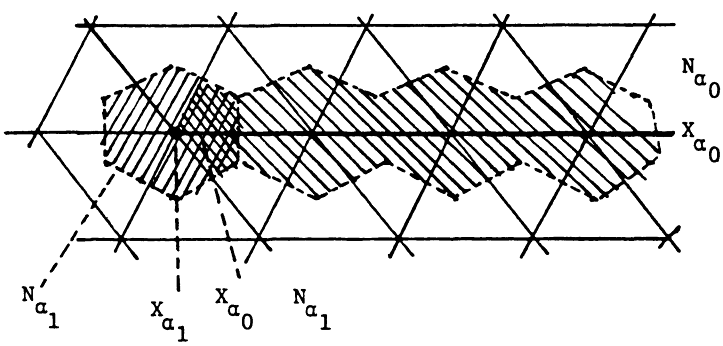


FIGURE (4.2.10). Open regular neighborhoods N_{α_0}, N_{α_1} of X_{α_0}, X_{α_1} and their intersections.

(4.3) The correspondence

(4.3.1) $$\sigma \mapsto N_\sigma, \qquad \partial_i \mapsto \text{incl},$$

gives rise to a functor of the category **Sim**$|I|$ of simplices in $|I|$ to the category **Top** of topological spaces, **N**: **Sim**$|I| \to$ **Top**. Since N_σ is open for every σ, $\bigcup N_\sigma = X$, it is well known that the geometric realization of this functor $|\mathbf{N}|$ has the same homotopy type as X.

There is another functor **M**: **Sim**$|I| \to$ **Top** defined by the correspondence:

(4.3.2) $$\sigma \mapsto M_\sigma;\ \partial_i \to \text{incl. for } 1 \leqslant i \leqslant k;\ \partial_0 \to P_{\alpha_1} \text{ for } i = 0.$$

From (4.2.4), there is a natural transformation

(4.3.3) $$\mathbf{P}: \mathbf{N} \to \mathbf{M}; \qquad \sigma \mapsto (P_\sigma: N_\sigma \to M_\sigma),$$

of these functors. It induces a mapping on the geometric realizations $|\mathbf{P}|$: $|\mathbf{N}| \to |\mathbf{M}|$. Since P_σ: $N_\sigma \to M_\sigma$ is a homotopy equivalence for every σ, the geometric realizations $|\mathbf{N}|$ and $|\mathbf{M}|$ have the same homotopy type.

Suppose we have two stratified polyhedra;

$$(4.3.4)\qquad \begin{aligned} X &= X_n \supset X_{n-1} \supset \cdots \supset X_0, \\ Z &= Z_{n'} \supset Z_{n'-1} \supset \cdots \supset Z_0, \end{aligned}$$

and suppose we have a simplicial map $g\colon X \to Z$ which preserves the stratifications $g(X_i) = Z_i$. Then, first of all, connected components X_α, $\alpha \in I$, are mapped to connected components $Z_{\alpha'}$, $\alpha' \in I'$. This induces an order preserving mapping of I into I', and so a simplicial map $h\colon |I| \to |I'|$ of the simplicial complex $|I|$ to $|I'|$. Secondly, since the open regular neighborhood is completely determined by the simplicial structure, there are induced mappings

$$g_\sigma\colon (N_\sigma, M_\sigma) \to (N'_{h(\sigma)}, M'_{h(\sigma)})$$

between the corresponding spaces defined in the previous section. They give rise to natural transformations between the corresponding functors $\mathbf{g}\colon \mathbf{N} \to \mathbf{N}'$, $\mathbf{g}\colon \mathbf{M} \to \mathbf{M}'$, and hence their geometric realizations $|\mathbf{g}|\colon |\mathbf{N}| \to |\mathbf{N}'|$, $|\mathbf{g}|\colon |\mathbf{M}| \to |\mathbf{M}'|$. It is easy to see that there is a commutative diagram

$$(4.3.5)\qquad \begin{array}{ccc} |\mathbf{N}| & \xrightarrow{|\mathbf{g}|} & |\mathbf{N}'| \\ \downarrow\sim & & \downarrow\sim \\ X & \xrightarrow{g} & Z \end{array}$$

where the vertical mappings are the homotopy equivalences mentioned before.

The above considerations can be applied to the situation when we have a group G operating simplicially on the stratified polyhedron X and preserving the strata

$$(4.3.6)\qquad G \times X \to X, \qquad g(X_i) = X_i, \qquad g \in G.$$

It follows that there is an induced action of G on the simplicial complex $|I|$,

$$(4.3.7)\qquad G \times |I| \to |I|$$

and, for every simplex σ, the isotropy subgroup G_σ operates on the spaces N_σ and M_σ. As a result, the functors $\mathbf{N}$, $\mathbf{M}$ are functors of $\mathbf{Sim}_G|I|$ to $\mathbf{Top}$ in the sense of §3.

5. A $K(\pi, 1)$-covering for $\hat{R}(S)$

(5.1) As mentioned in §2, the augmented Riemann space $\hat{R}(S)$ has the structure of a projective variety with the subspaces $\hat{R}(S)_i = \coprod_{|\alpha| \leqslant i} R(S/\alpha)$ as subvarieties. From the results of Hironaka (see [8]), this algebraic variety can be triangulated so that the subvarieties $\hat{R}(S)_i$ become subcomplexes. In this way, the space $\hat{R}(S)$ has the structure of a stratified polyhedron with the filtration

$$\hat{R}(S) = \hat{R}(S)_0 \supset \hat{R}(S)_1 \supset \cdots \supset \hat{R}(S)_{3g-3}.$$

The connected components $R(S/\alpha)$ in the strata are indexed by the vertices in the orbit space $|\mathscr{T}(S)^+|/\Gamma$. For each such vertex $\{\alpha\}$ in $|\mathscr{T}(S)^+|/\Gamma$, we define $U_{\{\alpha\}}$ to be the open regular neighborhood of $R(S/\alpha)$ in $\hat{R}(S)$ with respect to the above triangulation. This forms an open covering $\{U_{\{\alpha\}}\}$ of $\hat{R}(S)$, and from the results of §4, this covering has the following properties:

(5.1.1) The nerve of this covering $\{U_{\{\alpha\}}\}$ is isomorphic to $|\mathscr{T}(S)^+|/\Gamma$.

(5.1.2) For every simplex $\{\sigma\} = (\{\alpha_0\} < \{\alpha_1\} < \cdots < \{\alpha_k\})$ in $|\mathscr{T}(S)^+|/\Gamma$, let $U_{\{\sigma\}} = \bigcap_{i=0}^k U_{\{\alpha_i\}}$ and $V_{\{\sigma\}} = R(S/\alpha_0) \cap U_{\{\sigma\}}$. Then $U_{\{\sigma\}}$ is the open regular neighborhood of $V_{\{\sigma\}}$ in $\hat{R}(S)$.

(5.1.3) Let $P_{\{\sigma\}}\colon U_{\{\sigma\}} \to V_{\{\sigma\}}$ be the projection of the regular neighborhood of $U_{\{\sigma\}}$ onto its base $V_{\{\sigma\}}$. For $\{\sigma\} = (\{\alpha_0\} < \{\alpha_1\} < \cdots < \{\alpha_k\})$, $\partial_i\{\sigma\} = (\{\alpha_0\} < \cdots < \{\hat{\alpha}_i\} \cdots \{\alpha_k\})$, $1 \leqslant i \leqslant k$, $u(\partial_i) =$ incl. and $v(\partial_i) =$ incl., we have a commutative diagram:

$$\begin{array}{ccc}
U_{\{\alpha_0\}} \cap U_{\{\alpha_1\}} \cdots \cap U_{\{\alpha_k\}} & \xrightarrow{u(\partial_i)} & U_{\{\alpha_0\}} \cap \cdots \hat{U}_{\{\alpha_i\}} \cdots \cap U_{\{\alpha_k\}} \\
\Big\downarrow P_{\{\sigma\}} & & \Big\downarrow P_{\partial_i\{\sigma\}} \\
R(S/\alpha_0) \cap U_{\{\alpha_1\}} \cdots \cap U_{\{\alpha_k\}} & \xrightarrow{v(\partial_i)} & R(S/\alpha_0) \cap \cdots \hat{U}_{\{\alpha_i\}} \cdots \cap U_{\{\alpha_k\}}
\end{array}$$

(5.1.4) For $\{\sigma\} = (\{\alpha_0\} < \{\alpha_1\} < \cdots < \{\alpha_k\})$, $\partial_0\{\sigma\} = (\{\alpha_1\} < \{\alpha_2\} < \cdots < \{\alpha_k\})$, $u(\partial_0) =$ incl. and $\nu(\partial_0) = P_{\alpha_1}$, we have:

$$\begin{array}{ccc}
U_{\{\alpha_0\}} \cap U_{\{\alpha_1\}} \cdots \cap U_{\{\alpha_k\}} & \xrightarrow{u(\partial_0)} & U_{\{\alpha_1\}} \cap \cdots \cap U_{\{\alpha_k\}} \\
\Big\downarrow P_{\{\sigma\}} & & \Big\downarrow P_{\partial_0\{\sigma\}} \\
R(S/\alpha_0) \cap U_{\{\alpha_1\}} \cdots \cap U_{\{\alpha_k\}} & \xrightarrow{\nu(\partial_0)} & R(S/\alpha_1) \cap \cdots \cap U_{\{\alpha_k\}}
\end{array}$$

(5.1.5) The correspondence $\{\sigma\} \to V_{\{\sigma\}}$, $\partial_i \to v(\partial_i)$, defines a functor $\mathbf{V}$ from the category $\mathbf{Sim}|\mathscr{T}(S)^+|/\Gamma$ of simplices in $|\mathscr{T}(S)^+|/\Gamma$ to the category of topological spaces. The geometric realization of this functor has the same homotopy type as $\hat{R}(S)$.

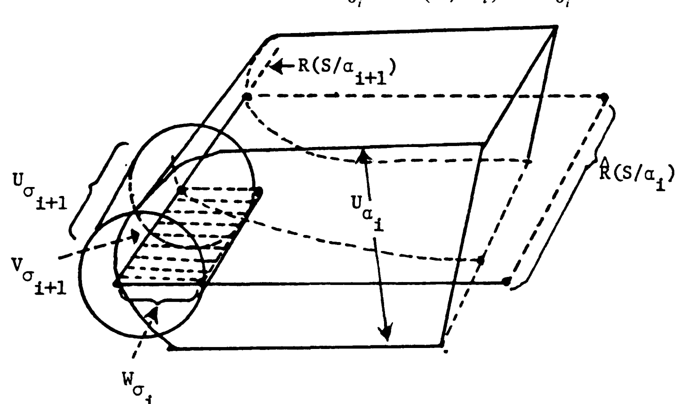
(5.2) We now lift the above functor $\mathbf{V}$ to an equivariant functor $\tilde{\mathbf{V}}$ defined over the category $\mathbf{Sim}_\Gamma|\mathscr{T}(S)^+|$.

First, when σ consists of a single system of curves α, we consider the deformation space $D(S/\alpha)$ and the projection map $h_\alpha\colon D(S/\alpha) \to \hat{R}(S)$ of $D(S/\alpha)$ into $\hat{R}(S)$. Define $\tilde{V}_\alpha$ to be the inverse image $h_\alpha^{-1}(V_{\{\alpha\}})$ in $D(S/\alpha)$ of the subspace $V_{\{\alpha\}} = R(S/\alpha)$ in $\hat{R}(S)$. From (2.9.3), it is clear that the group $\Gamma(S/\alpha)$ operates discontinuously on $\tilde{V}_\alpha$ with $V_{\{\alpha\}}$ as its orbit space.

To define $\tilde{V}_\sigma$ in general, we have to use the lifting procedure mentioned in Lemma (2.9.4). By subdividing our triangulation on $\hat{R}(S)$, we may assume that all the open regular neighborhoods U_σ are contained in the neighborhood W of $R(S/\alpha_0)$, mentioned in (2.9.4). Note that the group $\Gamma(S/\alpha)$ contains elements of finite order, and so the quotient map $\tilde{W} \to W$ in (2.9.4) is not a covering. The singular points of these actions lead to singular points of W. Since these are singular points of the projective variety $\hat{R}(S)$, we may assume that the filtration of the singular orbits gives a filtration of subcomplexes in our polyhedron. For an open regular neighborhood W, the spaces $\tilde{W}$ and $\tilde{V}$ are ramified coverings of W and V along branch sets which are subpolyspaces. In this situation the regular neighborhood map P: $W \to V$ can be lifted to a unique equivariant map $\tilde{P}$: $\tilde{W} \to \tilde{V}$ of $\tilde{W}$ onto $\tilde{V}$.

For a simplex $\sigma = (\alpha_0 \subset \alpha_1 \subset \cdots \subset \alpha_k)$ in $\mathbf{Sim}|\mathscr{T}(S)^+|$, set σ_i to be the subcomplex $\sigma_i = (\alpha_i \subset \alpha_{i+1} \subset \cdots \subset \alpha_k)$ in σ. Since $\sigma_{i+1} = \partial_0\sigma_i$, there is a sequence of maps $v(\partial_0)$: $V_{\sigma_i} \to V_{\sigma_{i+1}}$, $0 \leqslant i \leqslant k$. Recall that each V_{σ_i} can be obtained by the following procedure (see (4.2.9)). The space $V_{\sigma_{i+1}}$ is an open subspace in $R(S/\alpha_{i+1})$ and so lies in the closure $\hat{R}(S/\alpha_i)$ of $R(S/\alpha_i)$. Let W_{σ_i} be the regular neighborhood of $V_{\sigma_{i+1}}$ in this space $\hat{R}(S/\alpha_i)$. Then W_{σ_i} is the intersection of the open neighborhood $U_{\sigma_{i+1}}$ with $\hat{R}(S/\alpha_i)$, $W_{\sigma_i} = U_{\sigma_{i+1}} \cap \hat{R}(S/\alpha_i)$. The subspace V_{σ_i} can be obtained from W_{σ_i} by deleting the boundary elements in $W_{\sigma_i} \cap \partial\hat{R}(S/\alpha_i)$, i.e.,

$$
\begin{aligned}
W_{\sigma_i} \cap R(S/\alpha_i) &= U_{\sigma_{i+1}} \cap R(S/\alpha_i) \\
&= U_{\sigma_{i+1}} \cap \left[U_{\alpha_i} \cap R(S/\alpha_i)\right] \\
&= U_{\sigma_i} \cap R(S/\alpha_i) = V_{\sigma_i}.
\end{aligned}
\tag{5.2.1}
$$



Schematic picture of various subspaces

As for the map $v(\partial_0)$, it coincides with the restriction of the regular neighborhood map, $V_{\sigma_i} \xrightarrow{\text{incl.}} W_{\sigma_i} \xrightarrow{P} V_{\sigma_{i+1}}$.

To define $\tilde{V}_{\sigma_i}$, we proceed by downward induction on i. Suppose we have defined $\tilde{V}_{\sigma_{i+1}}$ as an open subset in $h^{-1}_{\alpha_{i+1}}(R(S/\alpha_{i+1})) = \tilde{V}_{\alpha_{i+1}}$. Since W_{σ_i} is an open neighborhood of $R(S/\alpha_{i+1})$ contained in $U_{\sigma_{i+1}}$, by (2.9.4) it can be lifted to a neighborhood $\tilde{W}_{\sigma_i}$ of $\tilde{V}_{\sigma_{i+1}}$ in $h^{-1}_{\alpha_{i+1}}(\hat{R}(S/\alpha_i))$. Now V_{σ_i} is an open subspace of W_{σ_i}, and so there is a corresponding neighborhood, $(V \times_W \tilde{W})_{\sigma_i} = h^{-1}_{\alpha_{i+1}}(V_{\sigma_i}) \cap \tilde{W}_{\sigma_i}$, in $\tilde{W}_{\sigma_i}$ covering V_{σ_i} (see diagram (5.2.2) below). This last space $(V \times_W \tilde{W})_{\sigma_i}$ can also be obtained from $\tilde{W}_{\sigma_i}$ by taking its intersection with $h^{-1}_{\alpha_{i+1}}(R(S/\alpha_i))$. Consider the natural projection $h^{\alpha_{i+1}}_{\alpha_i}$: $D(S/\alpha_i) \to D(S/\alpha_{i+1})$ of $D(S/\alpha_i)$ to $D(S/\alpha_{i+1})$. Since $h_{\alpha_{i+1}} \circ h^{\alpha_{i+1}}_{\alpha_i} = h_{\alpha_i}$, the image of $\tilde{V}_{\alpha_i} = h^{-1}_{\alpha_i}(R(S/\alpha_i))$ under this map $h^{\alpha_{i+1}}_{\alpha_i}$ is the same as $h^{-1}_{\alpha_{i+1}}(R(S/\alpha_i))$. In particular, this image contains $(V \times_W \tilde{W})_{\sigma_i}$ as an open subspace. Define $\tilde{V}_{\sigma_i}$ to be the inverse image $(h^{\alpha_{i+1}}_{\alpha_i})^{-1}(V \times_W \tilde{W})_{\sigma_i} \subset \tilde{V}_{\alpha_i}$ of the subspace $(V \times_W \tilde{W})_{\sigma_i} \subset \tilde{W}_{\sigma_i}$. This completes our inductive step, and so we can proceed by induction to get the space $\tilde{V}_\sigma = \tilde{V}_{\sigma_0}$.

As mentioned before, the regular neighborhood map P: $W_{\sigma_i} \to V_{\sigma_{i+1}}$ can be lifted to a map $\tilde{P}$: $\tilde{W}_{\sigma_i} \to \tilde{V}_{\sigma_{i+1}}$. Combining this with the projection $h^{\alpha_{i+1}}_{\alpha_i}$: $\tilde{V}_{\sigma_i} \to (V_W \times \tilde{W})_{\alpha_i}$, and the inclusion, incl:$(V \times_W \tilde{W})_{\sigma_i} \to \tilde{W}_{\sigma_i}$, we obtain a map $\tilde{P} \circ \text{incl} \circ h^{\alpha_{i+1}}_{\alpha_i}$ of $\tilde{V}_{\sigma_i}$ to $\tilde{V}_{\sigma_{i+1}}$:

(5.2.2)
$$\begin{array}{ccccccc}
\tilde{V}_{\sigma_i} & & \cdots\cdots & & \cdots\cdots & \dashrightarrow & D(S/\alpha_i) \\
\downarrow h^{\alpha_{i+1}}_{\alpha_i} & & & \searrow \tilde{P} \circ \text{incl} \circ h^{\alpha_{i+1}}_{\alpha_i} & & & \vdots\, h^{\alpha_{i+1}}_{\alpha_i} \\
(V \times_W \tilde{W})_{\sigma_i} & \xrightarrow{\text{incl.}} & \tilde{W}_{\sigma_i} & \xrightarrow{\tilde{P}} & \tilde{V}_{\sigma_{i+1}} & \dashrightarrow & D(S/\alpha_{i+1}) \\
\downarrow h_{\alpha_{i+1}} & & \downarrow h_{\alpha_{i+1}} & & \downarrow h_{\alpha_{i+1}} & & \vdots\, h_{\alpha_{i+1}} \\
V_{\sigma_i} & \xrightarrow{\text{incl.}} & W_{\sigma_i} & \xrightarrow{P} & V_{\sigma_{i+1}} & \dashrightarrow & \hat{R}(S)
\end{array}$$

Note the dotted arrows do not necessarily commute with the rest of the diagram.

We can now define the functor

$$\tilde{\mathbf{V}}\colon \mathbf{Sim}_\Gamma|\mathscr{T}(S)^+| \to \mathbf{Top}$$

which takes a simplex σ in $|\mathcal{T}(S)^+|$ to $\tilde{V}_\sigma$. For a k-simplex σ, we take the boundary maps $\sigma \supseteq \partial_i\sigma$, $1 \leqslant i \leqslant k$, to the inclusion $\tilde{V}_\sigma \subseteq \tilde{V}_{\partial_i\sigma}$ and the map $\sigma \supseteq \partial_0\sigma$ to the composite $\tilde{P} \circ \mathrm{incl} \circ h^{\alpha_1}_{\alpha_0}$, as in diagram (5.2.2). It is straightforward to check that these satisfy the necessary simplicial identities. For a group element $g \in \Gamma(S)$, there is a naturally induced map of deformation spaces $D(g)$: $D(S/\alpha_0) \to D(S/\alpha_0 \cdot g)$. Under this mapping, the subspace $\tilde{V}_\sigma$ in $D(S/\alpha_0)$ is sent to the corresponding subspace $\tilde{V}_{\sigma \cdot g}$ in $D(S/\alpha_0 \cdot g)$. It is again straightforward to verify that the diagram

$$\begin{array}{ccc} \tilde{V}_\sigma & \xrightarrow{D(g)} & \tilde{V}_{\sigma \cdot g} \\ {\scriptstyle \tilde{v}(\partial_i)} \downarrow & & \downarrow {\scriptstyle \tilde{v}(\partial_i)} \\ \tilde{V}_{\partial_i\sigma} & \xrightarrow{\cdot g} & \tilde{V}_{\partial_i\sigma \cdot g} \end{array}$$

commutes for any ∂_i. It follows that the correspondence $\sigma \mapsto \tilde{V}_\sigma$, $(\sigma \cdot g \supseteq \sigma') \mapsto \tilde{v}(\partial) \circ D(g)$ defines a functor $\tilde{\mathbf{V}}$ as above.

(5.3) Proposition (5.3.1). *The spaces $\tilde{V}_\sigma$ defined in* (5.2) *are contractible.*

Proof. Consider the sequence of spaces $\tilde{V}_{\sigma_i}$ and maps $\tilde{P} \circ \mathrm{incl} \circ h^{\alpha_{i+1}}_{\alpha_i}$: $\tilde{V}_{\sigma_i} \to \tilde{V}_{\sigma_{i+1}}$ used in defining $\tilde{V}_\sigma$. From (2.8), the space $\tilde{V}_{\sigma_k} = \tilde{V}_{\alpha_k}$ can be identified with the intersection $\bigcap_{\gamma \in \alpha_k} \langle P_\gamma \rangle$ of distinguished hyperplanes $\langle P_\gamma \rangle$, $\gamma \in \alpha_k$, in $D(S/\alpha_k)$, and so it is contractible. To prove (5.3.1), it is enough to show that, for every i, the map $v(\partial_0)$: $\tilde{V}_{\sigma_i} \to \tilde{V}_{\sigma_{i+1}}$ is a homotopy equivalence.

For this, we consider the spaces V_{σ_i}, W_{σ_i}, $V_{\sigma_{i+1}}$, and their coverings $(V \times_W \tilde{W})_{\sigma_i}$, $\tilde{W}_{\sigma_i}$, $\tilde{V}_{\sigma_{i+1}}$ as in (5.2.2). Write $\alpha_i = \{\gamma_1, \cdots, \gamma_{m_i}\}$, and $\alpha_{i+1} = \{\gamma_1, \cdots, \gamma_{m_i}, \cdots, \gamma_{m_{i+1}}\}$. Then, as in the previous paragraph, the subspace $h^{-1}_{\alpha_{i+1}}(R(S/\alpha_{i+1}))$ is the intersection $\bigcap_{j=1}^{m_{i+1}} \langle P_{\gamma_j} \rangle$ of distinguished hyperplanes in $D(S/\alpha_{i+1})$. Since $\tilde{V}_{\sigma_{i+1}}$ is an open subset in $h^{-1}_{\alpha_{i+1}}(R(S/\alpha_{i+1}))$, it has the structure of a complex manifold.

For each curve γ_j, $1 \leqslant j \leqslant m_i$, there is a complex codimension-one, subvariety $\hat{R}(S/\gamma_j)$ in $\hat{R}(S)$, and the intersection $\bigcap_{j=1}^{m_i} \hat{R}(S/\gamma_j)$ is the subvariety $\hat{R}(S/\alpha_i)$. Hence, as before, the inverse image $h^{-1}_{\alpha_{i+1}}(\hat{R}(S/\alpha_i))$ is the intersection $\bigcap_{j=1}^{m_i} \langle P_{\gamma_j} \rangle$ of the hyperplanes $\langle P_{\gamma_j} \rangle$, $1 \leqslant j \leqslant m_i$. From the definition, $\tilde{W}_{\sigma_i}$ is the open regular neighborhood of $\tilde{V}_{\sigma_{i+1}}$ in this last space $\bigcap_{j=1}^{m_i} \langle P_{\gamma_j} \rangle$. Since these are complex manifolds, $\tilde{W}_{\sigma_i}$ has the structure of a complex normal bundle, and the projection $\tilde{P}$: $\tilde{W}_{\sigma_i} \to \tilde{V}_{\sigma_{i+1}}$ is homotopic to a complex vector bundle (see (4.1)).

Each of the curves γ_j, $m_i < j \leqslant m_{i+1}$, gives a codimension-one, subvariety $\hat{R}(S/\alpha_0 \cup \gamma_j) = \hat{R}(S/\alpha_0) \cap \hat{R}(S/\gamma_j)$ in $\hat{R}(S/\alpha_0)$. From the definition, V_{σ_i} is obtained by removing the intersection $W_{\sigma_i} \cap \hat{R}(S/\alpha_0 \cup \gamma_j)$ from W_{σ_i}. Pulling this back to $D(S/\alpha_{i+1})$, this means that by subtracting the subspaces

$$h_{\alpha_{i+1}}^{-1}\big(\hat{R}(S/\alpha_i \cup \gamma_j)\big) = h_{\alpha_{i+1}}^{-1}\big(\hat{R}(S/\alpha_i)\big) \cap h_{\alpha_{i+1}}^{-1}\big(\hat{R}(S/\gamma_j)\big)$$
$$= \Big(\bigcap_{\gamma\in\alpha_i} \langle P_\gamma\rangle\Big) \cap \langle P_{\gamma_j}\rangle$$

from $\tilde{W}_\sigma$, we get the subspace $(V \times_W \tilde{W})_{\sigma_i}$. Let us describe this process locally at a point x in

$$h_{\alpha_{i+1}}^{-1}(\hat{R}(S/\alpha_{i+1})) = \bigcap_{j=1}^{m_{i+1}} \langle P_{\gamma_j}\rangle.$$

Let T_x be the tangent space of x in $h_{\alpha_{i+1}}^{-1}(\hat{R}(S/\alpha_i)) = \bigcap_{j=1}^{m_i} \langle P_{\gamma_j}\rangle$, and let N_x be the subspace in T_x normal to $\bigcap_{j=1}^{m_{i+1}} \langle P_{\gamma_j}\rangle$. Then N_x is a complex vector space of dimension $m_{i+1} - m_i - 1$. Each of the submanifolds $h_{\alpha_{i+1}}^{-1}(\hat{R}(S/\alpha_i \cup \gamma_j))$ gives rise to a codimension-one tangent space $N_{x,j}$ in N_x. If we remove these submanifolds $h_{\alpha_{i+1}}^{-1}(\hat{R}(S/\alpha_i \cup \gamma_j))$ from the normal bundle of $h_{\alpha_{i+1}}^{-1}(\hat{R}(S/\alpha_{i+1}))$, the effect on each of the fibers N_x amounts to subtracting $N_{x,j}$ from N_x. The resulting space $N_x - \bigcup N_{x,j}$ has the homotopy type of a torus

$$N_x - \bigcup N_{x,j} = \prod (\mathbf{C}^*)_j \sim \prod_{m_i < j \leqslant m_{i+1}} (S^1)_j,$$

where each circle factor $(S^1)_j$ represents the normal circle to $\langle P_{\gamma_j}\rangle$. Since $\tilde{V}_{\sigma_{i+1}}$ is an open submanifold in $h_{\alpha_{i+1}}^{-1}(R(S/\alpha_{i+1}))$, it inherits this structure on its normal bundle. In particular, the map $\tilde{P} \circ \text{incl}: (V \times_W \tilde{W})_{\sigma_i} \to \tilde{V}_{\sigma_{i+1}}$ is homotopic to a locally trivial fibration with fiber a torus as above.

From (2.9), the projection $h_{\alpha_i}^{\alpha_{i+1}}: D(S/\alpha_i) \to D(S/\alpha_{i+1})$ is a regular covering of $D(S/\alpha_i)$ onto its image, and its group of covering transformations is the free abelian group generated by the Dehn twists along curves γ_j, $m_i < j \leqslant m_{i+1}$. It is not difficult to prove that the Dehn twist along γ_j represents precisely the generator of the fundamental group of the above fiber circle $(S^1)_j$. In particular, the covering transformation group is the same as the fundamental group of the torus.

The homotopy fiber of the composition $\tilde{v}(\partial_0)$: $\tilde{V}_{\sigma_i} \to (V \times_W \tilde{W})_{\sigma_i} \to \tilde{V}_{\sigma_{i+1}}$ can be obtained by first forming the homotopy fiber of $\tilde{P} \circ \text{incl}$, and then forming the induced covering space over such a homotopy fiber. From the discussion in the previous paragraph, this is the universal covering space of the torus $\prod_j (S^1)_j$ and so is contractible. This proves $v(\partial_0)$ is a homotopy equivalence, and the proof of (5.3.1) follows immediately from induction.

(5.4) Proposition (5.4.1). *Let* $(\mathring{\Gamma}, \eta)$ *be as in example* (3.2.1). *Then the functor*

$$\tilde{\mathbf{V}}: \mathbf{Sim}_\Gamma|\mathscr{T}(S)^+| \to \mathbf{Top}$$

is an almost universal $(\mathring{\Gamma}, \eta)$*-functor, and the oribt space* $|\tilde{\mathbf{V}}|/\Gamma$ *of its geometric realization is the same as* $|\mathbf{V}|$.

Proof of (5.4.1). To prove the first statement of the proposition we must show that the action of Γ_σ on $\tilde{V}_\sigma$ factors through the natural transformation

$$\eta_\sigma: \Gamma_\sigma \twoheadrightarrow \mathring{\Gamma}_\sigma = \Gamma(S/\alpha_0, \sigma/\alpha_0)$$

and that this action is "almost universal". The action of Γ_σ on $\tilde{V}_\sigma$ is the restriction of the action of Γ_σ on $D(S/\alpha_0)$. From the definition of this action, it is clear that it factors through $\Gamma(S/\alpha_0, \sigma/\alpha_0)$. For "almost universal", we note that Γ contains a torsion-free subgroup Γ' of finite index such that the images $\mathring{\Gamma}'_\sigma = \eta_\sigma(\Gamma' \cap \Gamma_\sigma)$ are also torsion-free (see [22]). For $x \in D(S/\alpha_0)$, the isotropy subgroup of x in $\mathring{\Gamma}_\sigma$ is a torsion group, hence $\mathring{\Gamma}'_\sigma$ must act freely on $\tilde{V}_\sigma \supseteq D(S/\alpha_0)$. By Proposition (5.3.1), $\tilde{V}_\sigma$ is contractible, hence it is a universal $\mathring{\Gamma}'_\sigma$-space. This proves that $\tilde{\mathbf{V}}$ is an almost universal $(\mathring{\Gamma}, \eta)$-functor.

For the last statement in (5.4.1), we have to prove that the group $\Gamma(S/\alpha_0, \sigma/\alpha_0)$ operates properly discontinuously on $\tilde{V}_\sigma$ with $V_{\{\sigma\}}$ as its orbit space.

We proceed by induction as in (5.3.1). Let $\tilde{V}_\sigma \mapsto \tilde{V}_{\sigma_1} \to \cdots \to \tilde{V}_{\sigma_k}$ be the sequence of spaces V_{σ_i} defined in (5.3.2). In (2.9) we showed that the group $\Gamma(S/\alpha_k)$ operates properly discontinuously on the subspace $\tilde{V}_{\sigma_k}$ in $D(S/\alpha_k)$ and its orbit is the same as the image of $\tilde{V}_{\sigma_k}$ under the projection h_{α_k}, $h_{\alpha_k}(\tilde{V}_{\sigma_k}) = V_{\{\sigma_k\}}$.

Suppose we have shown that the group $\Gamma(S/\alpha_{i+1}, \sigma_{i+1}/\alpha_{i+1})$ operates properly discontinuously on $\tilde{V}_{\sigma_{i+1}}$ with $V_{\{\sigma_{i+1}\}}$ as its orbit space. From §2, there is a group extension

$$0 \to \mathbf{Z}\{\alpha_{i+1} - \alpha_i\} \to \Gamma(S/\alpha_i, \sigma_i/\alpha_i) \to \Gamma(S/\alpha_{i+1}, \sigma_{i+1}/\alpha_{i+1}) \to 0,$$

where the kernel $\mathbf{Z}\{\alpha_{i+1} - \alpha_i\}$ is the free abelian subgroup in $\Gamma(S/\alpha_i, \sigma_i/\alpha_i)$ generated by the Dehn twists along curves in $\alpha_{i+1} - \alpha_i$. As in (5.3), this free abelian group $\mathbf{Z}\{\alpha_{i+1} - \alpha_i\}$ operates properly discontinuously on $\tilde{V}_{\sigma_i}$, and its orbit space is $(V \times_W \tilde{W})_{\sigma_i}$.

The factor group $\Gamma(S/\alpha_{i+1}, \sigma_{i+1}/\alpha_{i+1})$ is a subgroup in $\Gamma(S/\alpha_{i+1})$, and so it operates on $(V \times_W \tilde{W})_{\sigma_i}$ properly discontinuously. From our inductive hypothesis, this group $\Gamma(S/\alpha_{i+1}, \sigma_{i+1}/\alpha_{i+1})$ is precisely the subgroup in $\Gamma(S/\alpha_{i+1})$ which keeps the subspace $\tilde{V}_{\sigma_{i+1}}$ and its regular neighborhood $\tilde{W}_{\sigma_i}$ in $D(S/\alpha_{i+1})$ invariant. Hence the orbit space of $\tilde{W}_{\sigma_i}$ under this action is the same as its image W_{σ_i} in $\hat{R}(S)$. Since $(V \times_W \tilde{W})_{\sigma_i}$ is a subspace in $\tilde{W}_{\sigma_i}$, its orbit space coincides with $V_{\{\sigma_i\}}$ in W_{σ_i}. Thus the group $\Gamma(S/\alpha_i, \sigma_i/\alpha_i)$ operates properly discontinuously on $\tilde{V}_{\sigma_i}$ with orbit space $V_{\{\sigma_i\}}$. This completes our induction, and so the proof of (5.4.1) is complete.

6. The category of stable curves

(6.1) Define the category of stable curves **SC** to be the category whose objects are stable Riemann surfaces S' and whose morphisms $S' \to S''$ are isotopy classes of deformations f: $S' \to S''$. For every $g \geq 2$, there is a full subcategory $\mathbf{SC}_g$ consisting of all stable Riemann surfaces of genus g. In order to have a deformation f: $S' \to S''$ between two stable Riemann surfaces, they must have the same genus and so belong to the same subcategory $\mathbf{SC}_g$. Hence the category **SC** can be decomposed into a disjoint union of subcategories, $\mathbf{SC} = \coprod_{g \geq 2} \mathbf{SC}_g$.

The object of this section is to establish the following:

Theorem (6.1.1). *Let S be a fixed nonsingular Riemann surface of genus g, and let $\mathbf{SC}_g$ be defined as above. Then there exists a rational homology equivalence of $|\mathbf{SC}_g|$ to the augmented Riemann space $\hat{R}(S)$.*

(6.2) Let S be fixed as in the statement of (6.1.1). By a marked Riemann surface, we mean a triple $(\varphi_\beta, S/\beta, R)$ consisting of a stable surface R, a vertex β in $|\mathcal{T}(S)^+|$, and an isotropy class of homeomorphisms φ_β: $S/\beta \to R$ of S/β to R. The collection of all marked Riemann surfaces $(\varphi_\beta, S/\beta, R)$ forms the objects of a category, denoted by $\widetilde{\mathbf{SC}}_g$. A morphism Φ: $(\varphi_\beta, S/\beta, R)$ $\to (\varphi'_{\beta'}, S/\beta', R')$ in this category is a diagram of deformations, commutative up to isotopy,

$$\text{(6.2.1)} \qquad \begin{array}{ccc} S/\beta & \xrightarrow{\varphi_\beta} & R \\ \downarrow{\scriptstyle p_\beta^{\beta'}} & & \downarrow{\scriptstyle f} \\ S/\beta' & \xrightarrow{\varphi_{\beta'}} & R' \end{array}$$

where $\beta' \supset \beta$, and $p_\beta^{\beta'}$ is the natural collapsing map. This is called the *category $\widetilde{\mathbf{SC}}_g$ of marked Riemann surfaces of genus g.*

There is an action of the mapping class group Γ on the category $\widetilde{\mathbf{SC}}_g$. Given an element in Γ, it can be represented by an isotopy class of homeomorphisms φ: $S \to S$. Given any marked Riemann surface $(\varphi_\beta, S/\beta, R)$, this element $[\varphi]$ in Γ sends $(\varphi_\beta, S/\beta, R)$ to the marked Riemann surface $(\varphi_\beta \circ \varphi^{-1}, S/\varphi(\beta), R)$,

$$S/\varphi(\beta) \xrightarrow{\varphi^{-1}} S/\beta \xrightarrow{\varphi_\beta} R.$$

Similarly, $[\varphi]$ sends a morphism Φ of the form in (6.2.1) to morphism $\Phi \cdot [\varphi]$ given by the square:

(6.2.2)
$$\begin{array}{ccccc} S/\varphi(\beta) & \xrightarrow{\varphi^{-1}} & S/\beta & \xrightarrow{\varphi_\beta} & R \\ \downarrow{\scriptstyle p^{\varphi(\beta')}_{\varphi(\beta)}} & & & & \downarrow{\scriptstyle f} \\ S/\varphi(\beta') & \xrightarrow{\varphi^{-1}} & S/\beta' & \xrightarrow{\varphi_{\beta'}} & R' \end{array}$$

This gives us an induced action of Γ on the classifying space $|\widetilde{\mathbf{SC}}_g|$. With respect to this action, we have

Proposition (6.2.3). *The orbit space* $|\widetilde{\mathbf{SC}}_g|/\Gamma$ *is isomorphic to* $|\mathbf{SC}_g|$.

Proof of (6.2.3). Let $\widetilde{\mathbf{SC}}_g/\Gamma$ denote the (quotient) category whose objects are Γ-orbits of objects in $\widetilde{\mathbf{SC}}_g$ and whose morphisms are Γ-orbits of morphisms. (It is an easy exercise to show that composition of such morphisms is well defined.) There is a natural isomorphisms $|\widetilde{\mathbf{SC}}_g|/\Gamma \cong |\widetilde{\mathbf{SC}}_g/\Gamma|$.

Let F: $\widetilde{\mathbf{SC}}_g \to \mathbf{SC}_g$ be the forgetful functor $F(\varphi_\beta, S/\beta, R) = R$ defined by forgetting the markings on a surface R. Clearly it takes a single value on each Γ-orbit, and so gives rise to a functor F/Γ: $\widetilde{\mathbf{SC}}_g/\Gamma \to \mathbf{SC}_g$. We claim the last functor F/Γ is an isomorphism.

First, given any stable surface R, it is isomorphic to S/β for some β. Hence F/Γ is a surjection on objects. For any two markings $(\varphi_\beta, S/\beta, R)$, $(\varphi_{\beta'}, S/\beta', R)$ on the same stable Riemann surface R, we have a composite homeomorphism $\varphi_{\beta'}^{-1} \circ \varphi_\beta$: $S/\beta \xrightarrow{\varphi_\beta} R \xrightarrow{\varphi_{\beta'}^{-1}} S/\beta'$ of S/β onto S/β'. Any such homeomorphism can be lifted to a homeomorphism φ: $S \to S$ such that $\varphi(\beta) = \beta'$, and the diagram

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & S \\ \downarrow{\scriptstyle \bmod \beta} & & \downarrow{\scriptstyle \bmod \beta'} \\ S/\beta & \xrightarrow{\varphi_{\beta'}^{-1} \circ \varphi_\beta} & S/\beta' \end{array}$$

commutes. Hence the two markings $(\varphi_\beta, S/\beta, R)$, $(\varphi_{\beta'}, S/\beta', R)$ lie on the same Γ-orbit, and so F/Γ is an injection on objects.

Suppose we are given a morphism f: $R \to R'$ in $\mathbf{SC}_g$. Let $(\varphi_\beta, S/\beta, R)$ be a marking of R. Then R' can be obtained from R by collapsing curves f^{-1}(nodes) $-$ nodes. Let $\beta' = (f \circ \varphi_\beta)^{-1}$(nodes). Then $\beta' \supset \beta$, and there is a composite homeomorphism $\varphi_{\beta'}$: $S/\beta' \xrightarrow{\sim} R/f^{-1}(\text{nodes}) - \text{nodes} \xrightarrow{f} R'$ such that the diagram

$$\begin{array}{ccc} S/\beta & \xrightarrow{\varphi_\beta} & R \\ \downarrow p_\beta^{\beta'} & & \downarrow f \\ S/\beta' & \xrightarrow{\varphi_{\beta'}} & R' \end{array}$$

commutes. This proves that F/Γ is surjective on morphisms.

Suppose we are given two morphisms Φ, Φ' in $\widetilde{\mathbf{SC}}_g$ which map onto the same morphism $f\colon R \to R'$ in $\mathbf{SC}_g$:

$$\Phi \equiv \begin{array}{ccc} S/\beta & \xrightarrow{\varphi_\beta} & R \\ \downarrow p_\beta^{\beta'} & & \downarrow f \\ S/\beta' & \xrightarrow{\varphi_{\beta'}} & R' \end{array} \qquad \Phi' \equiv \begin{array}{ccc} S/\gamma & \xrightarrow{\varphi_\gamma} & R \\ \downarrow p_\gamma^{\gamma'} & & \downarrow f \\ S/\gamma' & \xrightarrow{\varphi_{\gamma'}} & R' \end{array}$$

From the discussion in the previous paragraph, there exists a homeomorphism $\varphi\colon S \to S$ of S which brings β to γ and covers the homeomorphism $\varphi_\gamma^{-1} \circ \varphi_\beta$. It follows that φ brings β' to γ', and sends the commutative diagram Φ to Φ'. This proves the functor F/Γ is injective on morphism, and so completes the proof of (6.2.3).

(6.3) There is another forgetful functor $F_0\colon \widetilde{\mathbf{SC}}_g \to \mathscr{T}(S)^+$ of the category $\widetilde{\mathbf{SC}}_g$ to the partially ordered set $\mathscr{T}(S)^+$, defined by assigning to a marked Riemann surface $(\varphi_\beta, S/\beta, R)$ the system of curves β (forgetting the surface R), and assigning to a morphism $\Phi\colon (\varphi_\beta, S/\beta, R) \to (\varphi_{\beta'}, S/\beta', R)$ the inclusion relation $\beta \subseteq \beta'$. This functor induces a Γ-equivariant map

$$|F_0|\colon |\widetilde{\mathbf{SC}}_g| \to |\mathscr{T}(S)^+| \tag{6.3.1}$$

on the classifying spaces. For a simplex $\sigma = (\beta_0 \subset \cdots \subset \beta_k)$ in $|\mathscr{T}(S)^+|$, we consider the fiber category $\widetilde{\mathbf{SC}}_\sigma$ whose objects are

$$(\varphi_{\beta_0}, S/\beta_0, R_0) \xrightarrow{\Phi_0} (\varphi_{\beta_1}, S/\beta_1, R_1) \to \cdots \to (\varphi_{\beta_k}, S/\beta_k, R_k) \tag{6.3.2}$$

and whose morphisms are commutative diagrams:

$$\begin{array}{ccccccc} (\varphi_{\beta_0}, S/\beta_0, R_0) & \xrightarrow{\Phi_0} & (\varphi_{\beta_1}, S/\beta_1, R_1) & \to & \cdots & \to & (\varphi_{\beta_k}, S/\beta_k, R_k) \\ \downarrow \Psi_0 & & \downarrow \Psi_1 & & & & \downarrow \Psi_k \\ (\varphi'_{\beta_0}, S/\beta_0, R'_0) & \xrightarrow{\Phi'_0} & (\varphi'_{\beta_1}, S/\beta_1, R'_1) & \to & \cdots & \to & (\varphi'_{\beta_k}, S/\beta_k, R'_k) \end{array} \tag{6.3.3}$$

It is easy to see from the definition that the classifying space $|\widetilde{\mathbf{SC}}_\sigma|$ of this category is homotopic to the inverse image $(F_0)^{-1}(\sigma)$ in $|\widetilde{\mathbf{SC}}_g|$ of the simplex σ in $|\mathscr{T}(S)^+|$. In fact, $|\widetilde{\mathbf{SC}}_\sigma|$ is precisely the inverse image of the barycenter b_σ of σ. It follows that the functor $\mathbf{SC}_g^F\colon \mathbf{Sim}_\Gamma|\mathscr{T}(S)^+| \to \mathbf{Top}$ defined by $\sigma \mapsto |\widetilde{\mathbf{SC}}_\sigma|$,

$\partial_i \mapsto |\text{omit } i\text{th term}|$ has geometric realization homeomorphic to that of the category $\widetilde{\mathbf{SC}}_g$, $|\widetilde{\mathbf{SC}}_g^F| \cong |\widetilde{\mathbf{SC}}_g|$.

The isotropy subgroup Γ_σ of σ operates on $\widetilde{\mathbf{SC}}_\sigma$, but this action is not effective because the Dehn twists along curves in β_0 have no effect. Instead, it gives rise to an action of $\Gamma(S/\beta_0, \sigma/\alpha_0)$ on the category $\widetilde{\mathbf{SC}}_\sigma$ and so an induced action on the classifying space $|\widetilde{\mathbf{SC}}_\sigma|$.

Proposition (6.3.4). *For every* $\sigma = (\beta_0 \subset \cdots \subset \beta_k)$ *in* $|\mathscr{T}(S)^+|$, *the space* $|\widetilde{\mathbf{SC}}_\sigma|$ *is a universal* $\Gamma(S/\alpha_0, \sigma/\alpha_0)$*-space.*

Proof of (6.3.4). Suppose we are given two objects

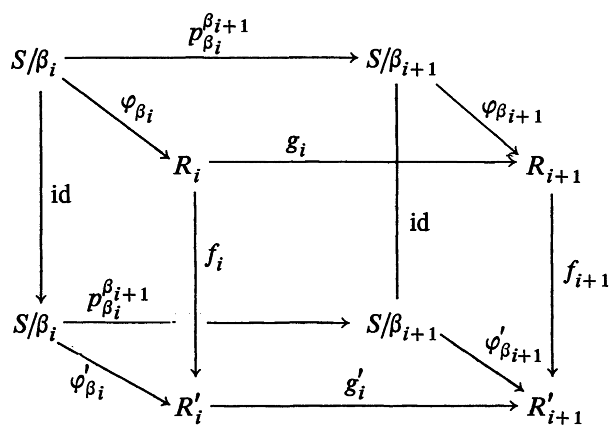
$$(\varphi_{\beta_0}, S/\beta_0, R_0) \overset{\Phi_0}{\to} (\varphi_{\beta_1}, S/\beta_1, R_1) \to \cdots \to (\varphi_{\beta_k}, S/\beta_k, R_k),$$

$$(\varphi'_{\beta_0}, S/\beta_0, R'_0) \overset{\Phi'_0}{\to} (\varphi_{\beta_1}, S/\beta_1, R'_1) \to \cdots \to (\varphi_{\beta_k}, S/\beta_k, R'_k)$$

in $\widetilde{\mathbf{SC}}_\sigma$. Recall that a morphism Ψ_i: $(\varphi_{\beta_i}, S/\beta_i, R_i) \to (\varphi'_{\beta_i}, S/\beta_i, R'_i)$ connecting the two objects $(\varphi_{\beta_i}, S/\beta_i, R_i)$, $(\varphi'_{\beta_i}, S/\beta_i, R'_i)$ in $\mathbf{SC}_g$ is a commutative diagram:

$$\begin{array}{ccc} S/\beta_i & \xrightarrow{\varphi_{\beta_i}} & R_i \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f_i} \\ S/\beta_i & \xrightarrow{\varphi'_{\beta_i}} & R'_i \end{array}$$

Such a morphism always exists and is unique because there is only one choice for f_i, $f_i = \varphi'_{\beta_i} \circ \varphi_{\beta_i}^{-1}$. This shows that the vertical maps in (6.3.3) always exist and are unique. To verify the commutative relation $\Psi_{i+1} \circ \Phi_i = \Phi'_i \circ \Psi_i$, we consider the cubic diagram of maps:



$$\begin{array}{ccc} \cdot & \xrightarrow{\Phi_i} & \cdot \\ \downarrow{\scriptstyle \Psi_i} & & \downarrow{\scriptstyle \Psi_{i+1}} \\ \cdot & \xrightarrow{\Phi_{i+1}} & \cdot \end{array}$$

where the maps g_i, g_i' are given by the definition of Φ_i, Φ_i'. Note that all the squares except the front face are commutative by definition, and φ_{β_i}, $\varphi_{\beta_{i+1}}$, φ'_{β_i}, $\varphi'_{\beta_{i+1}}$ are isomorphisms. It follows that the front face is commutative, the entire cube is commutative, and $\Psi_{i+1} \circ \Phi_i = \Phi_{i+1} \circ \Psi_{i+1}$. This proves that between any two objects in $\widetilde{\mathbf{SC}}_\sigma$, there exists a unique isomorphism and so $|\widetilde{\mathbf{SC}}_\sigma|$ is contractible.

The group $\Gamma(S/\alpha_0, \sigma/\alpha_0)$ operates freely on objects in $\widetilde{\mathbf{SC}}_\sigma$ because it operates freely on the set of marked Riemann surfaces $(\varphi_\beta, S/\beta, R)$. Since morphisms are uniquely determined by objects, it also operates freely on morphisms. This completes the proof of (6.3.4).

(6.4) *Proof of Theorem* (6.1.1). The proof is an immediate consequence of (6.3.4). For, in the previous section, we proved that the augmented Riemann space has the same homotopy type as $|\mathbf{V}|$ which, in turn, is isomorphic to the orbit space $|\tilde{\mathbf{V}}|/\Gamma$. For each σ, $|\tilde{V}_\sigma|$ is contractible, the action of $\Gamma(S/\alpha_0, \sigma/\alpha_0)$ is properly discontinuous but not free. Since Γ contains a torsion free subgroup of finite index, this fits into the framework of almost universal. As a result, $\hat{R}(S)$ has the rational homology of the orbit space $|\mathbf{E}|/\Gamma$ of a universal $\mathring{\Gamma}$-functor $\sigma \mapsto E_\sigma$. On the other hand, from (6.3.4), the functor $\mathbf{SC}_g^F$: $\sigma \mapsto |\widetilde{\mathbf{SC}}_\sigma|$ is precisely one of these functors, its geometric realization is precisely $|\widetilde{\mathbf{SC}}_g|$, and the orbit space of $|\widetilde{\mathbf{SC}}_g|$ is the same as $|\mathbf{SC}_g|$, $|\widetilde{\mathbf{SC}}_g|/\Gamma = |\mathbf{SC}_g|$. The proof of (6.1.1) follows.

7. Satake compactification

(7.1) Let $\mathfrak{S}_g$ denote the Siegel upper half-space of degree g,

$$\mathfrak{S}_g = \left\{ Z \in M_g(\mathbf{C}) \middle| Z = Z^t, \operatorname{Im} Z > 0 \right\},$$

and $\mathrm{SP}_g(\mathbf{Z})$ the integral symplectic group

$$\mathrm{SP}_g(\mathbf{Z}) = \left\{ M \in \mathrm{GL}_{2g}(\mathbf{Z}) \middle| M \cdot \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \cdot M^t = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \right\}.$$

Then the group $\mathrm{SP}_g(\mathbf{Z})$ operates on the space $\mathfrak{S}_g$ by the formula

$$Z \cdot M = (ZC + D)^{-1} \cdot (ZA + B), \qquad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

The orbit space of $\mathfrak{S}_g$ under this action is called the *Siegel modular space* $\mathfrak{S}_g/\mathrm{SP}_g(\mathbf{Z})$. Classically there is an embedding of the Riemann space $R(S)$ onto the Siegel modular space

$$J\colon R(S) \to \mathfrak{S}_g/\mathrm{SP}_g(\mathbf{Z}) \tag{7.1.1}$$

known as the period mapping. This is defined as follows. Let (ϕ, S, S') be an element in the Teichmüller space $T(S)$. Fix a symplectic basis α_i, β_i in

$H^1(S;\ \mathbb{Z})$, i.e., $\langle \alpha_i \cup \alpha_j; [S] \rangle = \langle \beta_i \cup \beta_j; [S] \rangle = 0$, $\langle \alpha_i \cup \beta_j; [S] \rangle = \delta_{ij}$. The homotopy equivalence ϕ induces a corresponding symplectic basis $\alpha_i' = \phi_*(\alpha_i)$, $\beta_i' = \phi_*(\beta_i)$ in $H^1(S';\ \mathbb{Z})$. Choose g linearly independent holomorphic 1-forms $\omega_1, \cdots, \omega_g$ in $H^{0,1}(S';\ \mathbb{C})$. We consider the period matrix

$$\left(\int_{\alpha_j'} \omega_i \right)^{-1} \cdot \left(\int_{\beta_j'} \omega_i \right)$$

whose entries are defined by period integrals. It is a classical result of Riemann that this matrix represents an element in $\mathfrak{S}_g$. The matrix is independent of the choice of $\omega_1, \cdots, \omega_g$, and so gives rise to a map $\tilde{J}$: $T(S) \to \mathfrak{S}_g$. There is a natural homomorphism of the mapping class group Γ onto the symplectic group $\mathrm{SP}_g(\mathbb{Z})$ defined by sending a homeomorphism φ: $S \to S$ to the induced homomorphism φ_*: $H_1(S;\ \mathbb{Z}) \to H_1(S;\ \mathbb{Z})$ preserving the intersection pairing λ,

$$\Gamma(S) \to \mathrm{Aut}(H_1(S;\ \mathbb{Z}); \lambda) \simeq \mathrm{SP}_g(\mathbb{Z}), \qquad \varphi \to \varphi_*.$$

It is not difficult to show that the above mapping J: $T(S) \to \mathfrak{S}_g$ is equivariant with respect to the action of $\Gamma(S)$ on $T(S)$, the action of $\mathrm{SP}_g(\mathbb{Z})$ on $\mathfrak{S}_g$, and the above homomorphism of $\Gamma(S)$ onto $\mathrm{SP}_g(\mathbb{Z})$. The map induced by J on the orbit spaces is the period mapping

$$J\colon R(S) \to \mathfrak{S}_g/\mathrm{SP}_g(\mathbb{Z}).$$

Let $\mathfrak{S}_g^*$ be the Satake compactification of $\mathfrak{S}_g$, and let $\mathfrak{S}_g^*/\mathrm{SP}_g(\mathbb{Z})$ be the orbit space of $\mathfrak{S}_g^*$ under the action of $\mathrm{SP}_g(\mathbb{Z})$ (see [7]). Then it is a result of Namikawa (see [17]) that the above map J can be extended to a morphism

(7.1.2) $$J\colon \hat{R}(S) \to \mathfrak{S}_g^*/\mathrm{SP}_g(\mathbb{Z})$$

of the corresponding compactifications. There is an explicit description of this extended period mapping. Given a Riemann surface S' with nodes, we normalize the surface by replacing each node point by two nonsingular points and replacing each neighborhood $Z_1Z_2 = 0$ of a node point by two disjoint affine neighborhoods. The result is a nonsingular surface $N(S')$, called the normalization of S', and the period matrix $J(N(S'))$ is well defined on the normalization $N(S')$. The period map is extended to $\hat{R}(S)$ by assigning to S' the period matrix $J(N(S'))$ of its normalization, i.e., $J(S') = J(N(S'))$.

In [7], it was proven that the cohomology $H^i(\mathfrak{S}_g^*/\mathrm{SP}_g(\mathbb{Z});\ \mathbb{Q})$ in degree lower than g is isomorphic to the subspace of corresponding degree in the polynomial algebra $\mathbb{Q}[x_2, x_6, \cdots, x_{4j+2}, \cdots] \otimes \mathbb{Q}[y_6, y_{10}, \cdots, y_{4j+2}, \cdots]$, where $\deg x_{4j+2} = \deg y_{4j+2} = 4j + 2$. The extended period mapping J induces a map on the rational cohomology

(7.1.3) $$J^*\colon H^*(\mathfrak{S}_g^*/SP_g(\mathbb{Z}); \mathbb{Q}) \to H^*(\hat{R}(S); \mathbb{Q})$$

and so gives us cohomology classes $J^*(x_{4j+2})$ and $J^*(y_{4j+2})$ in $H^*(\hat{R}(S); \mathbf{Q})$.

The object of the rest of this paper is devoted to the proof of the main theorem:

Theorem (7.1.4). *Let* y_{4j+2} *be the stable cohomology class in* $H^{4j+2}(\mathfrak{S}_g^*/\mathrm{SP}_g(\mathbf{Z}); \mathbf{Q})$, $4j \leqslant g-3$, *defined as above. Then under the induced map* J^* *it becomes zero in* $H^*(\hat{R}(S); \mathbf{Q})$, *i.e.*, $J^*(y_{4j+2}) = 0$.

(7.2) Let us recall the method of computing the stable cohomology of the Satake space

$$H^*\Big(\varinjlim \mathfrak{S}_g^*/\mathrm{SP}_g(\mathbf{Z}); \mathbf{Q}\Big) \cong \mathbf{Q}[x_{4j+2}] \otimes \mathbf{Q}[y_{4j+2}]$$

in the paper [7]. There are two categories **W** and **Esp** obtained from algebraic K-theory. The first, **W**, referred to as Giffen's category, is obtained by considering objects which are pairs (P, λ): P a finitely generated free abelian group, λ a nonsingular, skew-symmetric, bilinear pairing $\lambda\colon P \times P \to \mathbf{Z}$. A morphism $(P, \lambda) \to (P', \lambda')$ is also a pair (L, φ) consisting of a direct summand L in P, $L \subset L^{\perp}$, and an isometry $\varphi\colon (L^{\perp}/L, \lambda_L) \to (P', \lambda')$. The category **W** is filtered by full subcategories $\mathbf{W}_g$ which consist of objects (P, λ) whose rank is less than or equal to $2g$, rank $P \leqslant 2g$,

(7.2.1) $$\mathbf{W}_0 \subset \mathbf{W}_1 \subset \cdots \subset \mathbf{W}_g \subset \cdots.$$

The classifying space $|\mathbf{W}_g|$ of the subcategory $\mathbf{W}_g$ has the same rational cohomology as the Satake space, $H^*(|\mathbf{W}_g|; \mathbf{Q}) \cong H^*(\mathfrak{S}_g^*/\mathrm{SP}_g(\mathbf{Z}); \mathbf{Q})$ (see [7]). Therefore the problem of computing the rational cohomology of $\mathfrak{S}_g^*/\mathrm{SP}_g(\mathbf{Z})$ is reduced to one for the category $\mathbf{W}_g$.

For this, the second category **Esp** was introduced. The objects in **Esp** are again pairs (P, λ) except λ is no longer required to be nonsingular. Let $P^{\perp}$ be the null space in P, $P^{\perp} = \{x \in P \mid \lambda(x, y) = 0\ \forall y \in P\}$. Then there is an exact sequence $0 \to P^{\perp} \to P \to P/P^{\perp} \to 0$. The requirement is that $P/P^{\perp}$ is free abelian and the induced pairing $\lambda\colon P/P^{\perp} \times P/P^{\perp} \to \mathbf{Z}$ is nonsingular. A morphism $(P, \lambda) \to (P', \lambda')$ is an injection $f\colon P \to P'$, preserving the pairing and $f(P^{\perp}) = f(P)^{\perp}$ in P'. In terms of group extensions, a morphism connecting $0 \to P^{\perp} \to P \to P/P^{\perp} \to 0$ and $0 \to P'^{\perp} \to P' \to P'/P'^{\perp} \to 0$ is equivalent to a 3-by-3 commutative diagram:

(7.2.2)
$$\begin{array}{ccccccccc}
0 & \to & P^{\perp} & \to & P & \to & P/P^{\perp} & \to & 0\\
 & & \uparrow & & \| & & \uparrow & & \\
0 & \to & P'^{\perp} & \xrightarrow{f^{-1}} & P & \to & f(P)/P'^{\perp} & \to & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & P'^{\perp} & \to & P' & \to & P'/P'^{\perp} & \to & 0
\end{array}$$

Note that $f(P^{\perp})/P'^{\perp}$ is an isotropic subspace in $P'/P'^{\perp}$, its perpendicular subspace is $f(P)/P'^{\perp}$, and the quotient of these two spaces is isomorphic to $P/P^{\perp}$,

$$\text{(7.2.3)} \qquad \varphi\colon \frac{f(P)/P'^{\perp}}{f(P^{\perp})/P'^{\perp}} \cong \frac{f(P)}{f(P^{\perp})} \cong P/P^{\perp}.$$

In particular, the right vertical arrows represent a morphism in **W**. The above discussion allows us to define a functor

$$\text{(7.2.4)} \qquad \theta\colon \mathbf{Esp} \to \mathbf{W}$$

of the extension category **Esp** to **W** by sending (P, λ) to the nonsingular, skew-symmetric pairing $(P/P^{\perp}, \lambda)$ and sending a morphism $f\colon P \rightarrowtail P'$ to the morphism $(f(P^{\perp})/P'^{\perp}, \varphi)$ in **W**.

It was shown in [7] that the functor in (7.2.4) gives rise to a fibration of classifying spaces $|\theta_s|\colon |S^{-1}\mathbf{Esp}| \to |\mathbf{W}|$, where $S^{-1}\mathbf{Esp}$ is the stabilization of **Esp** with respect to a monoid S (see [7]). The rational cohomology of $|S^{-1}\mathbf{Esp}|$ can be computed, and is the same as the rational cohomology of the infinite symplectic group,

$$\text{(7.2.5)} \quad H^*\left(|S^{-1}\mathbf{Esp}|; \mathbb{Q}\right) \cong H^*(\mathrm{BSP}(\mathbb{Z}); \mathbb{Q}) \cong Q\left[\bar{x}_{4j+2} | j = 0, \cdots, \infty\right].$$

The rational cohomology of the homotopy fiber, $\mathrm{fib}(|\theta_s|)$ is the same as that of the general linear group $H^*(\mathrm{BGL}; \mathbb{Q}) \cong \Lambda[\bar{y}_{4j+1} | j = 1, \cdots, \infty]$. The computation of $H^*(|\mathbf{W}|; \mathbb{Q})$ follows from an argument in Hopf-algebras, with the classes x_{4j+2}, y_{4j+2} arising from $\bar{x}_{4j+2}$, $\bar{y}_{4j+1}$ respectively. Since $|\theta|$ factors through its stabilization $|\theta_s|$, it is a consequence of this approach that the induced map $|\theta|^*\colon H^*(|\mathbf{W}|; \mathbb{Q}) \to H^*(|\mathbf{Esp}|; \mathbb{Q})$ takes the generators y_{4j+2} to zero.

(7.3) Given a Riemann surface S' with nodes, consider the pairing

$$\text{(7.3.1)} \qquad \begin{gathered} \lambda\colon H^1(S'; \mathbb{Z}) \times H^1(S'; \mathbb{Z}) \to \mathbb{Z}, \\ \lambda(x, y) = \langle (x \cup y; [S']) \rangle \end{gathered}$$

defined by the evaluation of the cup product $x \cup y$ against the fundamental class $[S']$. This pairing is nonsingular if the surface S' is nonsingular.

Proposition (7.3.2). *There exists a covariant functor* **H**: **SC** $\to$ **Esp** *of the category* **SC** *of stable curves to the extension category* **Esp** *which sends a Riemann surface S' with nodes to the skew-symmetric pairing $(H^1(S'; \mathbb{Z}), \lambda)$ defined in* (7.3.1).

Proof. Given a Riemann surface S' with nodes, we consider its normalization $N(S')$ as in (7.1). Note that S' can be recovered from its normalization $N(S')$ by identifying pairs of points in $N(S')$ to the node points. Hence there is a natural surjection $\varphi\colon N(S') \to S'$ of $N(S')$ onto S'.

(7.3.3) $$0 \to \operatorname{Ker}\varphi' \to H^1(S'; \mathbb{Z}) \xrightarrow{\varphi'} H^1(N(S'); \mathbb{Z}) \to 0.$$

It is easy to see that φ induces an injection φ_*: $H_1(N(S'); \mathbb{Z}) \to H_1(S'; \mathbb{Z})$ on first homology groups and that this injection splits. Hence it induces a surjection on first cohomology, and so the right-hand side of (7.3.3) is exact.

Next we observe that the pairing λ: $H^1(N(S'); \mathbb{Z}) \times H^1(N(S'); \mathbb{Z}) \to \mathbb{Z}$ is nonsingular. Since φ' preserves the pairings on $H^1(S'; \mathbb{Z})$ and $H^1(N(S'); \mathbb{Z})$, the subspace $\operatorname{Ker}\varphi'$ represents the null space in $H^1(S'; \mathbb{Z})$, i.e. $\operatorname{Ker}\varphi' = \{H^1(S'; \mathbb{Z})\}^{\perp}$. It follows that the pair $(H^1(S'; \mathbb{Z}), \lambda)$ represents an object in the category **Esp**.

Suppose we are given a deformation f: $S' \to S''$ of S' onto S''. Then there exists a system of admissible curves $\alpha = f^{-1}$(nodes)-nodes such that f factors through a homeomorphism f': $S'/\alpha \to S''$ of S'/α onto S'':

$$\begin{array}{ccc} S' & \xrightarrow{\quad f \quad} & S'' \\ & \searrow \qquad \nearrow_{f'} & \\ & S'/\alpha & \end{array}$$

The above system of admissible curves α in S' can be lifted to an admissible system of curves $\beta = \varphi^{-1}(\alpha)$ in the normalization $N(S')$ by means of the normalization map φ: $N(S') \to S'$. If we collapse the curves β to points, the result is a surface $N(S')/\beta$ with nodes and the normalization of this surface $N(N(S')/\beta)$ is homeomorphic to $N(S'')$. It follows from the definition $N(S'') \cong N(N(S')/\beta)$ that there is a commutative diagram:

$$\begin{array}{ccccccccc} 0 \to & \operatorname{Ker}\varphi & \to & H^1(S'; \mathbb{Z}) & \xrightarrow{\varphi} & H^1(N(S'); \mathbb{Z}) & \to 0 \\ & \| & & \uparrow \bmod \alpha & & \uparrow \bmod \beta & \\ 0 \to & \operatorname{Ker}\varphi & \to & H^1(S'/\alpha\mathbb{Z}) & \xrightarrow{\varphi} & H^1(N(S')/\beta; \mathbb{Z}) & \to 0 \\ & \downarrow & & \downarrow \cong f' & & \downarrow & \\ 0 \to & \operatorname{Ker}\varphi' & \to & H^1(S''; \mathbb{Z}) & \xrightarrow{\varphi'} & H^1(N(S''); \mathbb{Z}) & \to 0 \\ & & & & & \| & \\ & & & & & H^1(N(N(S')/\beta); \mathbb{Z}) & \end{array}$$

In other words, we have a morphism in the category **Esp**. It is easy to show that this defines a covariant functor.

(7.4) Consider the composite functor $\mathbf{J} = \boldsymbol{\theta} \circ \mathbf{H}$,

(7.4.1) $$\mathbf{J}: \mathbf{SC} \xrightarrow{\mathbf{H}} \mathbf{Esp} \xrightarrow{\boldsymbol{\theta}} \mathbf{W},$$

which sends a surface with nodes S' to the skew-symmetric pairing $(H^1(N(S'); \mathbb{Z}), \lambda)$. This functor is referred to in the following as the period functor. Clearly it preserves the filtrations on **SC** and **W**:

$$\text{(7.4.2)} \qquad \mathbf{J}\colon \mathbf{SC}_g \to \mathbf{W}_g.$$

Since **J** factors through **Esp**, the induced maps $|\mathbf{J}|^*\colon H^*(|\mathbf{W}_g|; \mathbb{Q}) \to H^*(|\mathbf{SC}_g|; \mathbb{Q})$ on cohomology map the classes y_{4j+2} to zero. As a result, to prove Theorem (7.1.4), it is enough to prove the following:

Proposition (7.4.3). *Let* $J\colon \hat{R}(S) \to \mathfrak{S}_g^*/\mathrm{SP}_g(\mathbb{Z})$ *be the extended period mapping, and let* $|\mathbf{J}|\colon |\mathbf{SC}_g| \to |\mathbf{W}_g|$ *be the map on the classifying spaces induced by the functor* **J** *in* (7.4.2). *Then there exists a commutative diagram*

$$\begin{array}{ccc} H^*(\mathfrak{S}_g^*/\mathrm{SP}_g(\mathbb{Z}); \mathbb{Q}) & \xrightarrow{J^*} & H^*(\hat{R}(S); \mathbb{Q}) \\ \uparrow \cong & & \uparrow \cong \\ H^*(|\mathbf{W}_g|; \mathbb{Q}) & \xrightarrow{\mathbf{J}^*} & H^*(|\mathbf{SC}_g|; \mathbb{Q}) \end{array}$$

where the horizontal maps are induced by J *and* $|\mathbf{J}|$, *and the vertical maps are isomorphisms.*

8. A $K(\pi, 1)$-covering for $\mathfrak{S}_g^*/\mathrm{SP}_g(\mathbb{Z})$

(8.1) For the proof of (7.4.3), we have to analyze the projective variety $\mathfrak{S}_g^*/\mathrm{SP}_g(\mathbb{Z})$ in the same manner as the Riemann space $\hat{R}(S)$.

Let $\mathcal{T}_g^+$ be the partially ordered set of isotropic subspaces L in the standard skew-symmetric pairing $(\mathbb{Z}^{2g}, \lambda)$. Let $|\mathcal{T}_g^+|$ denote the simplicial complex associated to this partially ordered set $\mathcal{T}_g^+$. The symplectic group $\mathrm{SP}_g(\mathbb{Z})$ operates on $|\mathcal{T}_g^+|$, and the isotropy group P_σ for a simplex σ satisfies the conditions in (3.1). Denote by **SP**: $\mathbf{Sim}_{\mathrm{SP}}|\mathcal{T}_g^+| \to \mathcal{G}\!\mathit{p}$ the covariant functor given by these isotropy subgroups $\sigma \mapsto P_\sigma$.

There is a reduced group functor associated to **SP**. Given a simplex $\sigma = (L_0 \subset L_1 \subset \cdots \subset L_k)$, we denote by $P(L_0^\perp/L_0, \sigma/L_0)$ the subgroup in the automorphism group $\mathrm{Aut}(L_0^\perp/L_0, \lambda)$ of the pairing $(L_0^\perp/L_0, \lambda)$ which keeps the flag of isotropy subspaces $0 \subset L_1/L_0 \subset \cdots \subset L_k/L_0$ in $L_0^\perp/L_0$ invariant. Let $\mathring{\mathbf{SP}}(\sigma) = P(L_0^\perp/L_0, \sigma/L_0)$. For a boundary map ∂_i, $1 \leqslant i \leqslant k$, there is a natural inclusion $\mathring{\mathbf{SP}}(\partial_i)\colon P(L_0^\perp/L_0, \sigma/L_0) \to P(L_0^\perp/L_0, \partial_i\sigma/L_0)$. For the boundary ∂_0, the filtration $L_0 \subset L_1 \subset L_1^\perp \subset L_0^\perp$ gives rise to a homomorphism $\mathring{\mathbf{SP}}(\partial_0)\colon P(L_0^\perp/L_0, \sigma/L_0) \to P(L_1^\perp/L_1, \partial_0\sigma/L_1)$ defined by sending an automorphism M in $P(L_0^\perp/L_0, \sigma/L_0)$ to the induced automorphism on

$L_1^\perp/L_1$. These satisfy the simplicial identities, hence, for any inclusion $\sigma_1 \supseteq \sigma_2$, there is a well-defined homomorphism $\mathring{\mathbf{SP}}(\partial)$: $\mathring{\mathbf{SP}}(\sigma_1) \to \mathring{\mathbf{SP}}(\sigma_2)$.

For a group element $g \in \mathrm{SP}_g(\mathbb{Z})$, conjugation by g gives rise to homomorphisms $P(L_0^\perp/L_0, \sigma/L_0) \to P(L_0 \cdot g^\perp/L_0 \cdot g, \sigma \cdot g/L_0 \cdot g)$ which commute with the boundary maps. Composing the group action and the boundary map we obtain homomorphisms $\mathring{\mathbf{SP}}(\sigma_1 \cdot g \supseteq \sigma_2)$: $\mathring{\mathbf{SP}}(\sigma_1) \to \mathring{\mathbf{SP}}(\sigma_2)$. This defines our functor $\mathring{\mathbf{SP}}$: $\mathrm{Sim}_{\mathrm{SP}}|\mathcal{T}_g^+| \to \mathcal{G}p$.

To see that there is a natural transformation η: $\mathbf{SP} \to \mathring{\mathbf{SP}}$, we set $\sigma^+ = (0 \subseteq L_0 \subset \cdots \subset L_k)$, $\sigma = \partial_0\sigma^+$. Then we can identify P_σ with $P(0^\perp/0, \sigma^+/0)$, and the homomorphism $\mathring{\mathbf{SP}}(\partial_0)$ defined above gives a surjection $\eta_\sigma = \mathring{\mathbf{SP}}(\partial_0)$: $P_\sigma \twoheadrightarrow P(L_0^\perp/L_0, \sigma/L_0)$. This defines a natural transformation η which makes $(\mathring{\mathbf{SP}}, \eta)$ a reduced $\mathbf{SP}$-functor. (In the following, η will be understood and dropped from the notation.)

As in (7.1), we fix an isometry between the pairing $(H^1(S; \mathbb{Z}), \lambda)$ and the standard skew-symmetric pairing $(\mathbb{Z}^{2g}, \lambda)$ on $\mathbb{Z}^{2g}$. Given a system α of admissible curves, there is a corresponding isotropic subspace L_α defined by the image in $H^1(S; \mathbb{Z})$ of the null space of $H^1(S/\alpha; \mathbb{Z})$, i.e., $L_\alpha = \mathrm{Im}(H^1(S/\alpha; \mathbb{Z})^\perp \to H^1(S; \mathbb{Z}))$. This correspondence $\alpha \mapsto L_\alpha$ defines an order preserving map $\mathcal{T}(S)^+ \to \mathcal{T}_g^+$, and so a functor $\tilde{\mathbf{I}}$: $\mathbf{Sim}_\Gamma|\mathcal{T}(S)^+| \to \mathbf{Sim}_{\mathrm{SP}}|\mathcal{T}_g^+|$, $\sigma \mapsto \tilde{I}(\sigma)$ between the categories of simplices.

We compare the reduced group functors $\mathring{\Gamma}$: $\mathbf{Sim}_\Gamma|\mathcal{T}(S)^+| \to \mathcal{G}p$ and $\mathring{\mathbf{SP}}$: $\mathbf{Sim}_{\mathrm{SP}}|\mathcal{T}_g^+| \to \mathcal{G}p$ via the functor $\tilde{\mathbf{I}}$. For every simplex $\sigma = (\alpha_0 \subset \alpha_1 \subset \cdots \subset \alpha_k)$ in $|\mathcal{T}(S)^+|$, there is a natural homomorphism $\Gamma(S/\alpha_0, \sigma/\alpha_0) \to P(L_0^\perp/L_0, \sigma/L_0)$ defined by sending a homeomorphism φ in $\Gamma(S/\alpha_0, \sigma/\alpha_0)$ to the induced automorphism φ^* on the cohomology $H^1(N(S/\alpha_0); \mathbb{Z}) \cong L_0^\perp/L_0$ of $N(S/\alpha_0)$ (see (7.3.3)). This defines a natural transformation μ: $\mathring{\Gamma} \to \mathring{\mathbf{SP}} \circ \tilde{\mathbf{I}}$ of functors from $\mathbf{Sim}_\Gamma|\mathcal{T}(S)^+|$ to $\mathcal{G}p$.

(8.2) The Satake compactification $\mathfrak{S}_g^*/\mathrm{SP}_g(\mathbb{Z})$ can be analyzed in terms of the Borel-Serre compactification $\overline{\mathfrak{S}}_g/\mathrm{SP}_g(\mathbb{Z})$ by the following push-out diagrams:

(8.2.1)
$$\begin{array}{ccc} \partial\overline{\mathfrak{S}}_g & \xrightarrow{\text{incl.}} & \overline{\mathfrak{S}}_g \\ \downarrow \tilde{f}_g & & \downarrow \tilde{f}_g \\ \partial\mathfrak{S}_g^* & \xrightarrow{\text{incl.}} & \mathfrak{S}_g^* \end{array} \qquad \begin{array}{ccc} \partial\overline{\mathfrak{S}}_g/\mathrm{SP}_g(\mathbb{Z}) & \xrightarrow{\text{incl.}} & \overline{\mathfrak{S}}_g/\mathrm{SP}_g(Z) \\ \downarrow f_g & & \downarrow f_g \\ \partial\mathfrak{S}_g^*/\mathrm{SP}_g(\mathbb{Z}) & \xrightarrow{\text{incl.}} & \mathfrak{S}_g^*/\mathrm{SP}_g(\mathbb{Z}) \end{array}$$

(see [7]).[2] The Borel-Serre space $\overline{\mathfrak{S}}_g$ is a disjoint union of cells $e(P_\sigma)$ indexed

[2]We take the topology on $\mathfrak{S}_g^*$ to be the quotient topology induced by the surjection $\tilde{f}_g$. This induces the usual topology on $\mathfrak{S}_g^*/\mathrm{SP}_g(\mathbb{Z})$.

by the simplices in $|\mathcal{T}_g^+|$, $\overline{\mathfrak{S}}_g = \coprod_\sigma e(P_\sigma)$. Similarly, the Satake compactification is a disjoint union of cells $e(\mathrm{Aut}(L^\perp/L, \lambda))$ indexed by the vertices L in $|\mathcal{T}_g^+|$, $\mathfrak{S}_g^* = \coprod_L e(\mathrm{Aut}(L^\perp/L, \lambda))$. For every simplex $\sigma = (L_0 \subset L_1 \subset \cdots \subset L_k)$ the map $\tilde{f}_g$ sends the cell $e(P_\sigma)$ to the subspace $e(\mathrm{Aut}(L_k^\perp/L_k, \lambda))$ in $\mathfrak{S}_g^*$. Passing to the orbit space, we have corresponding decompositions on the Borel-Serre compactification $\overline{\mathfrak{S}}_g/\mathrm{SP}_g(\mathbf{Z}) = \coprod_{\{\sigma\}} e'(P_{\{\sigma\}})$, and on the Satake compactification $\mathfrak{S}_g^*/\mathrm{SP}_g(\mathbf{Z}) = \coprod_{\{L\}} e'(\mathrm{Aut}(L^\perp/L, \lambda))$, and a map f_g between them. Since these are semianalytic sets, they can be triangulated into stratified polyhedra so that f_g is simplicial.

Applying the theory of open regular neighborhoods in §4 to the stratified space $\mathfrak{S}_g^*/\mathrm{SP}_g(\mathbf{Z})$, we obtain two functors $\mathbf{X}$: $\sigma \mapsto X_{\{\sigma\}}$, $\mathbf{Y}$: $\{\sigma\} \mapsto Y_{\{\sigma\}}$ of $\mathbf{Sim}|\mathcal{T}_g^+|/\mathrm{SP}_g(\mathbf{Z})$ to the category **Top** of topological spaces. The first consists of an open covering $X_{\{\sigma\}} = \bigcap X_{\{L_i\}}$, where $X_{\{L_i\}}$ is the open regular neighborhood of $e'(\mathrm{Aut}(L_i^\perp/L_i, \lambda))$ and the second $Y_{\{\sigma\}} = X_{\{\sigma\}} \cap e'(\mathrm{Aut}(L_0^\perp/L_0, \lambda))$ is a deformation retract of the first. The geometric realizations $|\mathbf{X}|$, $|\mathbf{Y}|$ of these functors share the same homotopy type as $\mathfrak{S}_g^*/\mathrm{SP}_g(\mathbf{Z})$.

The projection $\overline{\mathfrak{S}}_g \to \overline{\mathfrak{S}}_g/\mathrm{SP}_g(\mathbf{Z})$ is a ramified covering space with finite isotropy subgroups. Without loss of generality, we may assume that the branch sets are subcomplexes and so the triangulation on $\overline{\mathfrak{S}}_g/\mathrm{SP}_g(\mathbf{Z})$ can be lifted to one on $\overline{\mathfrak{S}}_g$. Since the isotropy subgroups are finite, the result is a locally finite simplicial structure on $\overline{\mathfrak{S}}_g$. The same procedure can be applied to $\mathfrak{S}_g^*/\mathrm{SP}_g(\mathbf{Z})$ to get a "triangulation"[3] on $\mathfrak{S}_g^*$. The map $\tilde{f}_g$ is simplicial with respect to these two triangulations.

For a simplex $\sigma = (L_0 \subset L_1 \subset \cdots \subset L_k)$ in $|\mathcal{T}_g^+|$, we define $\tilde{Y}_\sigma$ to be the inverse image in $e(\mathrm{Aut}(L_0^\perp/L_0, \lambda))$ of the subspace Y_σ in $e'(\mathrm{Aut}(L_0^\perp/L_0, \lambda))$. It is not difficult to show that the boundary map $\mathbf{Y}(\partial_i)$: $Y_\sigma \to Y_{\partial_i\sigma}$ can be lifted to a corresponding boundary map $\tilde{\mathbf{Y}}(\partial_i)$: $\tilde{Y}_\sigma \to \tilde{Y}_{\partial_i\sigma}$. In this way, we obtain a functor $\tilde{\mathbf{Y}}$: $\mathbf{Sim}_{\mathrm{SP}}|\mathcal{T}_g^+| \to$ **Top** which is equivariant with respect to the reduced group functor $\mathring{\mathbf{SP}}$, and $|\tilde{\mathbf{Y}}|/\mathrm{SP}_g(\mathbf{Z}) \cong |\mathbf{Y}|$.

Proposition (8.2.2). *The space $\tilde{Y}_\sigma$ is contractible.*

It is well known that any torsion-free subgroup of $\mathrm{Aut}(L_0^\perp/L_0, \lambda)$ acts freely on $e(\mathrm{Aut}(L_0^\perp/L_0, \lambda))$. As an immediate consequence of (8.2.2), therefore, we know that the functor $\tilde{\mathbf{Y}}$ is almost universal and its quotient space $|\tilde{\mathbf{Y}}|/\mathrm{SP}_g(\mathbf{Z})$ has the same rational homology as the quotient space of a universal $\mathring{\mathbf{SP}}$-functor. This is, of course, not new in view of the results already obtained in [7].

[3] The induced triangulation on $\mathfrak{S}_g^*$ is not locally finite. However the notion of regular neighborhood is well defined, as is the deformation retraction of a regular neighborhood onto its zero section.

Proof of (8.2.2). Given a simplex $\sigma = (L_0 \subset L_1 \subset \cdots \subset L_k)$ in $|\mathscr{T}_g^+|$, we have a filtration of subcomplexes $\sigma_1 \supset \cdots \supset \sigma_k$, where $\sigma_i = (L_i \subset L_{i+1} \subset \cdots \subset L_k)$. From the functorial properties, there is a sequence of mappings $\tilde{\mathbf{Y}}(\partial_0)$: $\tilde{Y}_{\sigma_i} \to \tilde{Y}_{\sigma_{i+1}} \sigma_{i+1} = \partial_0\sigma_i$ which covers a corresponding map $\mathbf{Y}(\partial_0)$: $Y_{\sigma_i} \to Y_{\sigma_{i+1}}$. To prove $\tilde{Y}_\sigma$ contractible, it is enough to show that $\tilde{\mathbf{Y}}(\partial_0)$: $\tilde{Y}_{\sigma_i} \to \tilde{Y}_{\sigma_{i+1}}$ is a homotopy equivalence. For once this is achieved, we have a sequence of homotopy equivalences

$$\tilde{Y}(P_\sigma) \to \tilde{Y}(P_{\sigma_1}) \to \cdots \to \tilde{Y}(P_{\sigma_k}),$$

and the last space $\tilde{Y}(P_{\sigma_k}) = e(\mathrm{Aut}(L_k^\perp/L_k, \lambda))$ is contractible.

Recall the map $\mathbf{Y}(\partial_0)$ can be obtained as follows. First, let $g_i = g - \operatorname{rank} L_i$, then the space $Y_{\sigma_{i+1}} = X_{\sigma_{i+1}} \cap \mathfrak{S}_{g_{i+1}}/\mathrm{SP}_{g_{i+1}}(\mathbf{Z})$ is a subspace in $\mathfrak{S}^*_{g_i}/\mathrm{SP}_{g_i}(\mathbf{Z})$, and it has the open subspace $X_{\sigma_{i+1}} \cap \mathfrak{S}^*_{g_i}/\mathrm{SP}_{g_i}(\mathbf{Z})$ as its regular neighborhood in $\mathfrak{S}^*_{g_i}/\mathrm{SP}_{g_i}(\mathbf{Z})$. After deleting the boundary elements $\partial\mathfrak{S}^*_{g_i}/\mathrm{SP}_{g_i}(\mathbf{Z})$ from this regular neighborhood, this becomes

$$Y_{\sigma_i} = \mathfrak{S}_{g_i}/\mathrm{SP}_{g_i}(\mathbf{Z}) \cap X_{\sigma_i} = \mathfrak{S}_{g_i}/\mathrm{SP}_{g_i}(\mathbf{Z}) \cap X_{\sigma_{i+1}}$$

and the projection map of the regular neighborhood gives our map $\mathbf{Y}(\partial_0)$: $Y_{\sigma_i} \to Y_{\sigma_{i+1}}$. The same holds for the covering spaces $\tilde{Y}_{\sigma_i}$, $\tilde{Y}_{\sigma_{i+1}}$: the space $\tilde{Y}_{\sigma_i}$ is obtained by deleting the boundary elements from the regular neighborhood of $\tilde{Y}_{\sigma_{i+1}}$ in the closure $e(\mathrm{Aut}(L_i^\perp/L_i, \lambda))^* \cong \mathfrak{S}^*_{g_i}$ of $e(\mathrm{Aut}(L_i^\perp/L_i, \lambda))$.

Without loss of generality therefore, we may assume that $g = g_i$, $L_i = 0$, $\sigma_i = (0 \subset L_{i+1} \subset \cdots \subset L_k)$, and the problem reduces to considering the regular neighborhood of $\tilde{Y}_{\sigma_{i+1}}$ in $\mathfrak{S}^*_g$. Note that $\tilde{Y}_{\sigma_{i+1}}$ is an open subspace in $e(\mathrm{Aut}(L_{i+1}^\perp/L_{i+1}, \lambda))$. Let $\overline{Y}_{\sigma_{i+1}} = \tilde{f}_g^{-1}(\tilde{Y}_{\sigma_{i+1}})$ be the pullback of the subspace $\tilde{Y}_{\sigma_{i+1}}$ in the Borel-Serre space $\overline{\mathfrak{S}}_g$ by means of $\tilde{f}_g$. Let $N(\overline{Y}_{\sigma_{i+1}})$ be the regular neighborhood of $\overline{Y}_{\sigma_{i+1}}$ in $\overline{\mathfrak{S}}_g$. Note that the image of $N(\overline{Y}_{\sigma_{i+1}})$ under $\tilde{f}_g$ is the regular neighborhood of $\tilde{Y}_{\sigma_{i+1}}$ in $\mathfrak{S}^*_g$. In particular, since $\tilde{f}_g$ is the identity map on the interior $\mathfrak{S}_g$ of $\overline{\mathfrak{S}}_g$, after deleting the boundary element from $N(\overline{Y}_{\sigma_{i+1}})$, we recover the subspace $\tilde{Y}_{\sigma_i}$ in $\mathfrak{S}_g$. It follows that the map $\tilde{Y}(\partial_0)$: $\tilde{Y}_{\sigma_i} \to \tilde{Y}_{\sigma_{i+1}}$ factors through $\overline{Y}_{\sigma_{i+1}}$,

$$\tilde{Y}_{\sigma_i} \xrightarrow{P} \overline{Y}_{\sigma_{i+1}} \xrightarrow{\tilde{f}_g} \tilde{Y}_{\sigma_{i+1}},$$

where the first map P: $\tilde{Y}_{\sigma_i} \to \overline{Y}_{\sigma_{i+1}}$ coincides with the projection of the regular neighborhood $N(\overline{Y}_{\sigma_{i+1}}) \to \overline{Y}_{\sigma_{i+1}}$.

Note that $\overline{Y}_{\sigma_{i+1}}$ is an open subspace in $\overline{e(P_{L_{i+1}})}$, and so it is a manifold with corners. It is not difficult to see that the normal bundle of such a submanifold with corners in $\overline{\mathfrak{S}}_g$ exists and has the structure of a locally trivial fibration with fiber a product of half lines $\prod \mathbf{R}^+$. Since the regular neighborhood is homotopic to such a fibration, it follows that the map P: $\tilde{Y}_{\sigma_i} \to \overline{Y}_{\sigma_{i+1}}$ is a homotopy equivalence.

As for the second map $\tilde{f}_g$: $\overline{Y}_{\sigma_{i+1}} \to \tilde{Y}_{\sigma_{i+1}}$, we observe that a manifold corners $\overline{Y}_{\sigma_{i+1}}$ has the same homotopy type as its interior $\overline{Y}_{\sigma_{i+1}} \cap e(P_{L_{i+1}})$, and the restriction of $\tilde{f}_g$ to this interior also maps onto $\tilde{Y}_{\sigma_{i+1}}$. It follows from the work of Zucker (see [Z]) that the projection $\tilde{f}_g$: $e(P_{L_{i+1}}) \to e(\mathrm{Aut}(L_{i+1}^{\perp}/L_i, \lambda))$ is a locally trivial fibration with contractible fiber and hence so is its restriction $\tilde{f}_g$: $e(P_{L_{i+1}}) \cap \overline{Y}_{\sigma_{i+1}} \to \tilde{Y}_{\sigma_{i+1}}$. This proves (8.2.2).

(8.3) We now reinterpret the period map J: $\hat{R}(S) \to \mathfrak{S}_g^*/\mathrm{SP}_g(\mathbf{Z})$ in terms of the functors **V** and **Y**. Since the period map is a morphism in the category of projective varieties, we may assume that it is simplicial with respect to our triangulations on these spaces. It follows that J maps the subspaces U_σ, V_σ of $\hat{R}(S)$ into the subspaces $X_{I(\sigma)}$, $Y_{I(\sigma)}$ of $\mathfrak{S}_g^*/\mathrm{SP}_g(\mathbf{Z})$, where **I**: $\mathbf{Sim}|\mathscr{T}(S)^+|/\Gamma \to \mathbf{Sim}|\mathscr{T}_g^+|/\mathrm{SP}$ is the quotient of the functor $\tilde{\mathbf{I}}$ described in (8.1). The first of these maps, J: $U_\sigma \to X_{I(\sigma)}$, commutes with the boundary maps (which in this case are just inclusions) and thus defines a natural transformation η_J: $\mathbf{U} \to \mathbf{X}$. Clearly, the diagram

$$\begin{array}{ccc} \hat{R}(S) & \xrightarrow{J} & \mathfrak{S}_g^*/\mathrm{SP}_g(\mathbf{Z}) \\ \uparrow \simeq & & \uparrow \simeq \\ |\mathbf{U}| & \xrightarrow{|\eta_J|} & |\mathbf{X}| \end{array}$$

commutes. The maps J: $V_\sigma \to Y_{I(\sigma)}$, on the other hand, do not commute with boundary maps as the following example shows.

Example (8.3.1). Let $\sigma = \{\phi \subset \alpha\}$, where α is a system of null homologous curves. Then α represents the zero subspace in $H_1(S;\, \mathbf{Z}) = \mathbf{Z}^{2g}$, so $I(\sigma) = \{0\}$. By definition, therefore $Y_{I(\sigma)} = Y_{I(\partial_0\sigma)} = \mathfrak{S}_g/\mathrm{SP}_g(\mathbf{Z})$. Consider the diagram

$$\begin{array}{ccc} V_\sigma & \hookrightarrow & Y_{I(\sigma)} \\ v(\partial_0)\downarrow & & \downarrow y \circ I(\partial_0) \\ V_{\partial_0} & \hookrightarrow & Y_{I(\partial_0\sigma)}\,. \end{array}$$

On the one hand, $y \circ I(\partial_0)$ is the identity map and on the other hand $v(\partial_0)$ is a nontrivial deformation. Hence the diagram does not commute.

To avoid this problem we use the theory of universal spaces developed in §3. From (5.4.1) and (8.2.3) we know that **V** and **Y** are quotients of functors $\tilde{\mathbf{V}}$ and $\tilde{\mathbf{Y}}$ which are almost universal with respect to $\mathring{\Gamma}$ and $\mathring{\mathbf{SP}}$ respectively. By Proposition (3.3.1), these can be replaced by universal functors $\tilde{\mathbf{V}}^{\mathrm{un}}$, $\tilde{\mathbf{Y}}^{\mathrm{un}}$ whose quotients, $\mathbf{V}^{\mathrm{un}} = \tilde{\mathbf{V}}^{\mathrm{un}}/\Gamma$, $\mathbf{Y}^{\mathrm{un}} = \tilde{\mathbf{Y}}^{\mathrm{un}}/\mathrm{SP}$, have the same rational cohomology as the original functors **V**, **Y**. (The construction of the universal functors depends on choices of torsion free subgroups Γ' in Γ and Γ'_{SP} in $\mathrm{SP}_g(\mathbb{Z})$. These can be chosen so that Γ' is the inverse image of Γ'_{SP} under the natural map $\Gamma \twoheadrightarrow \mathrm{SP}_g(\mathbb{Z})$ described in (7.1). Here we assume such a choice.)

Lemma (8.3.2). *There is a map* $f_J\colon |\mathbf{V}^{\mathrm{un}}| \to |\mathbf{Y}^{\mathrm{un}}|$ *such that the diagram*

$$\begin{array}{ccc}
|\mathbf{U}| & \xrightarrow{|\eta_J|} & |\mathbf{X}| \\
\downarrow \simeq & & \downarrow \simeq \\
|\mathbf{V}| & & |\mathbf{Y}| \\
\uparrow & & \uparrow \\
|\mathbf{V}^{\mathrm{un}}| & \xrightarrow{f_J} & |\mathbf{Y}^{\mathrm{un}}|
\end{array}$$

induces a commutative diagram on cohomology groups.

Proof of (8.3.2). Let $\tilde{\mathbf{I}}\colon \mathbf{Sim}_\Gamma|\mathscr{T}(S)^+| \to \mathbf{Sim}_{\mathrm{SP}}|T_g^+|$ be as in (8.1) and consider the composite functor $\mathring{\mathbf{SP}}\circ\tilde{\mathbf{I}}\colon \mathbf{Sim}_\Gamma|\mathscr{T}(S)^+| \to \mathscr{G}p$. Then the natural map $\Gamma \twoheadrightarrow \mathrm{SP}_g(\mathbb{Z})$ described in (7.1) gives rise to a natural transformation $\Gamma \to \mathring{\Gamma} \to \mathring{\mathbf{SP}}\circ\tilde{\mathbf{I}}$ which is a surjection for each $\sigma \in |\mathscr{T}(S)^+|$, $\Gamma_\sigma \twoheadrightarrow \mathring{\Gamma}_\sigma \twoheadrightarrow \mathring{\mathrm{SP}}_{\tilde{I}(\sigma)}$. This makes $\mathring{\mathbf{SP}}\circ\tilde{\mathbf{I}}$ a reduced Γ-functor. Now the functors $\tilde{\mathbf{V}}^{\mathrm{un}}$, $\tilde{\mathbf{Y}}^{\mathrm{un}}\circ\tilde{\mathbf{I}}\colon \mathbf{Sim}_\Gamma|\mathscr{T}(S)^+| \to \mathbf{Top}$ are almost universal with respect to $\mathring{\Gamma}$ and $\mathring{\mathbf{SP}}\circ\tilde{\mathbf{I}}$ respectively. We can therefore apply Proposition (3.2.3) to get a Γ-equivariant map $|\tilde{\mathbf{V}}^{\mathrm{un}}| \to |\tilde{\mathbf{Y}}^{\mathrm{un}}\circ\tilde{\mathbf{I}}|$. Replacing $\tilde{\mathbf{V}}^{\mathrm{un}}$ by its "dualization" $\tilde{\mathbf{V}}_D^{\mathrm{un}}$ if necessary (see Remark (3.2.4)), we may assume that this map is simplicial, that is, that it arises from a natural transformation $\tilde{\mathbf{J}}^{\mathrm{un}}\colon \tilde{\mathbf{V}}_D^{\mathrm{un}} \to \tilde{\mathbf{Y}}^{\mathrm{un}}\circ\tilde{\mathbf{I}}$. Let $\mathbf{J}^{\mathrm{un}}\colon \mathbf{V}_D^{\mathrm{un}} \to \mathbf{Y}^{\mathrm{un}}\circ\mathbf{I}$ denote the induced transformation of quotient functors.

Note that there is a canonical map

$$\begin{array}{ccc}
|\mathbf{Y}^{\mathrm{un}}\circ\mathbf{I}| & \longrightarrow & |\mathbf{Y}^{\mathrm{un}}| \\
\| & & \| \\
\coprod_{\sigma\in|\mathscr{T}(S)^+|/\Gamma} |\sigma| \times Y^{\mathrm{un}}_{I(\sigma)}/\sim & \longrightarrow & \coprod_{\tau\in|\mathscr{T}_g^+|/\mathrm{SP}} |\tau| \times Y^{\mathrm{un}}_\tau/\sim
\end{array}$$

defined by identifying $|\sigma| \times Y^{\mathrm{un}}_{I(\sigma)}$ with $|I(\sigma)| \times Y^{\mathrm{un}}_{I(\sigma)}$. Similarly for $|\mathbf{X}\circ\mathbf{I}|$ and $|\mathbf{Y}\circ\mathbf{I}|$. Consider the diagram:

$$\begin{array}{ccccc}
|\eta_J|\colon |\mathbf{U}| & \longrightarrow & |\mathbf{X}\circ\mathbf{I}| & \longrightarrow & |\mathbf{X}| \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
|\mathbf{V}| & & |\mathbf{Y}\circ\mathbf{I}| & \longrightarrow & |\mathbf{Y}| \\
\uparrow & & \uparrow & & \uparrow \\
f_J\colon |\mathbf{V}_D^{\mathrm{un}}| & \xrightarrow{|\mathbf{J}^{\mathrm{un}}|} & |\mathbf{Y}^{\mathrm{un}}\circ\mathbf{I}| & \longrightarrow & |\mathbf{Y}^{\mathrm{un}}|
\end{array}$$

Clearly the two squares on the right commute, so to prove the lemma, it suffices to verify that the left square induces a commutative diagram on cohomology. But since all of the maps in the diagram arise from natural transformations, the induced maps on cohomology are completely determined by their component parts:

$$\begin{array}{ccc} U_\sigma & \xrightarrow{J} & X_{I(\sigma)} \\ \downarrow\simeq & & \downarrow\simeq \\ V_\sigma & & Y_{I(\sigma)} \\ \uparrow & & \uparrow \\ (V_D^{\mathrm{un}})_\sigma & \xrightarrow{J^{\mathrm{un}}} & Y_{I(\sigma)}^{\mathrm{un}} \end{array}$$

Recall from the discussion preceding the lemma that the period map J maps V_σ into $Y_{I(\sigma)}$ (but does not respect boundary maps). The maps $U_\sigma \to V_\sigma$ and $X_{I(\sigma)} \to Y_{I(\sigma)}$ are deformation retractions hence are homotopy inverses of the inclusions $V_\sigma \hookrightarrow U_\sigma$, $Y_{I(\sigma)} \hookrightarrow X_{I(\sigma)}$. It follows that the diagrams

$$\begin{array}{ccc} U_\sigma & \xrightarrow{J} & X_{I(\sigma)} \\ \downarrow\simeq & & \downarrow\simeq \\ V_\sigma & \xrightarrow{J} & Y_{I(\sigma)} \end{array}$$

are homotopy commutative. Thus it remains only to show that

$$\begin{array}{ccc} V_\sigma & \xrightarrow{J} & Y_{I(\sigma)} \\ \uparrow & & \uparrow \\ (V_D^{\mathrm{un}})_\sigma & \xrightarrow{J^{\mathrm{un}}} & Y_{I(\sigma)}^{\mathrm{un}} \end{array} \tag{8.3.3}$$

is homotopy commutative. For this it suffices to show that $J\colon V_\sigma \to Y_{I(\sigma)}$ can be lifted to a map $\tilde{J}\colon \tilde{V}_{\tilde{\sigma}} \to \tilde{Y}_{\tilde{I}(\tilde{\sigma})}$, for in this case, the diagram (8.3.3) lifts to a diagram

$$\begin{array}{ccc} \tilde{V}_{\tilde{\sigma}} & \xrightarrow{\tilde{J}} & \tilde{Y}_{\tilde{I}(\tilde{\sigma})} \\ \uparrow & & \uparrow \\ (\tilde{V}_D^{\mathrm{un}})_{\tilde{\sigma}} & \xrightarrow{\tilde{J}^{\mathrm{un}}} & \tilde{Y}_{\tilde{I}(\tilde{\sigma})} \end{array} \tag{8.3.4}$$

of $\mathring{\Gamma}_{\tilde{\sigma}}$-equivariant maps. Since $\mathring{\Gamma}_{\tilde{\sigma}}$ acts freely on $(\tilde{V}_D^{\mathrm{un}})_{\tilde{\sigma}}$, and $\tilde{Y}_{\tilde{I}(\tilde{\sigma})}$ is contractible, there is a *unique* such map $(\tilde{V}_D^{\mathrm{un}})_{\tilde{\sigma}} \to \tilde{Y}_{\tilde{I}(\tilde{\sigma})}$, up to $\mathring{\Gamma}_{\tilde{\sigma}}$-equivariant homotopy. It follows that for *any* lifting $\tilde{J}$ of J, (8.3.4) commutes up to $\mathring{\Gamma}_{\tilde{\sigma}}$-equivariant homotopy and hence (8.3.3) commutes up to homotopy.

It remains to construct $\tilde{J}$: $\tilde{V}_{\tilde{\sigma}} \to \tilde{Y}_{\tilde{I}(\tilde{\sigma})}$. Consider first the case of a 0-simplex $\tilde{\sigma} = (\alpha_0)$, $\tilde{I}(\tilde{\sigma}) = (L_0)$. In this case, $\tilde{V}_{(\alpha_0)} = h_{\alpha_0}^{-1}(R(S/\alpha_0))$, where h_{α_0}: $D(S/\alpha_0) \to \hat{R}(S)$ is as in (5.2). Let $N(S/\alpha_0)$ denote the normalization of S/α_0. Then there is a natural identification of $\tilde{V}_{(\alpha_0)}$ with $D(N(S/\alpha_0))$ (or equivalently, with the Teichmüller space $T(N(S/\alpha_0)))$ given by sending a homeomorphism $S' \to S/\alpha_0$ to the induced homeomorphism $N(S') \to N(S/\alpha_0)$ of the normalizations. Since $N(S/\alpha_0)$ is nonsingular, the period map J, restricted to $R(S/\alpha_0)$, lifts to a map $\tilde{J}_{\alpha_0}$:

$$\begin{array}{ccccc} \tilde{V}_{(\alpha_0)} = D(N(S/\alpha_0)) & \xrightarrow{\tilde{J}_{\alpha_0}} & e\big(\mathrm{Aut}\big(L_0^{\perp}/L_0, \lambda\big)\big) & \subset & \mathfrak{S}_g^* \\ \downarrow h_{\alpha_0} & & \downarrow \pi & & \downarrow \pi \\ R(S/\alpha_0) & \xrightarrow{J} & e'\big(\mathrm{Aut}\big(L_0^{\perp}/L_0, \lambda\big)\big) & \subset & \mathfrak{S}_g^*/\mathrm{SP}_g(\mathbb{Z}) \end{array}$$

(see (7.1)). In particular, since $\tilde{Y}_{\tilde{I}(\alpha_0)} = e(\mathrm{Aut}(L_0^{\perp}/L_0, \lambda))$, we have $\tilde{J}_{\alpha_0}(\tilde{V}_{(\alpha_0)}) \subset \tilde{Y}_{\tilde{I}(\alpha_0)}$.

In general, for any $\tilde{\sigma} = (\alpha_0 < \alpha_1 < \cdots < \alpha_k)$ in $|\mathscr{T}(S)^+|$, $\tilde{V}_{\tilde{\sigma}}$ is contained in $\tilde{V}_{(\alpha_0)}$, so $\pi \circ \tilde{J}_{\alpha_0}(\tilde{V}_{\tilde{\sigma}}) = J \circ h_{\alpha_0}(\tilde{V}_{\tilde{\sigma}}) = J(V_\sigma) \subseteq Y_{I(\sigma)}$, hence

$$\tilde{J}_{\alpha_0}(\tilde{V}_{\tilde{\sigma}}) \subseteq \pi^{-1}\big(Y_{I(\sigma)}\big) \cap e\big(\mathrm{Aut}\big(L_0^{\perp}/L_0, \lambda\big)\big) = \tilde{Y}_{\tilde{I}(\tilde{\sigma})}.$$

Thus $\tilde{J}_{\alpha_0}$: $\tilde{V}_{\tilde{\sigma}} \to \tilde{Y}_{\tilde{I}(\tilde{\sigma})}$ is the desired lift of J: $V_\sigma \to Y_{I(\sigma)}$. This completes the proof of the lemma.

(8.4) Let $\mathbf{J}$: $\mathbf{SC}_g \to \mathbf{W}_g$ be the functor of (7.4.1). To complete the proof of Proposition (7.4.3), and hence also of Theorem (7.1.4), it remains to verify the following lemma.

Lemma (8.4.1). *There exist homotopy equivalences* $|\mathbf{SC}_g| \xrightarrow{\simeq} |\mathbf{V}^{\mathrm{un}}|$, $|\mathbf{W}_g| \xrightarrow{\simeq} |\mathbf{Y}^{\mathrm{un}}|$ *such that the diagram*

$$\begin{array}{ccc} |\mathbf{V}^{\mathrm{un}}| & \xrightarrow{f_J} & |\mathbf{Y}^{\mathrm{un}}| \\ \uparrow \simeq & & \uparrow \simeq \\ |\mathbf{SC}_g| & \xrightarrow{|\mathbf{J}|} & |\mathbf{W}_g| \end{array}$$

commutes up to homotopy.

Proof of (8.4.1). The proof is another application of the theory of universal functors. Recall from the proof of (8.3.2) that f_J is a composite

$$|\mathbf{V}^{\mathrm{un}}| = |\mathbf{V}_D^{\mathrm{un}}| \xrightarrow{|\mathbf{J}^{\mathrm{un}}|} |\mathbf{Y}^{\mathrm{un}} \circ \mathbf{I}| \to |\mathbf{Y}^{\mathrm{un}}|.$$

By definition, $\mathbf{V}_D^{\text{un}}$, $\mathbf{Y}^{\text{un}} \circ \mathbf{I}$ are quotients of functors $\tilde{\mathbf{V}}_D^{\text{un}}$, $\tilde{\mathbf{Y}}^{\text{un}} \circ \tilde{\mathbf{I}}$, universal with respect to $\mathring{\Gamma}$ and $\mathring{\mathbf{SP}} \circ \tilde{\mathbf{I}}$ respectively. The natural transformation $\mathbf{J}^{\text{un}}$ is the quotient of the unique (up to Γ-equivariant homotopy) natural transformation $\tilde{\mathbf{J}}^{\text{un}}$: $\tilde{\mathbf{V}}_D^{\text{un}} \to \tilde{\mathbf{Y}}^{\text{un}} \circ \tilde{\mathbf{I}}$. We need to interpret the map $|\mathbf{J}|$: $|\mathbf{SC}_g| \to |\mathbf{W}_g|$ similarly as the quotient of a map of universal functors. In (6.3) it is shown that there is a universal $\mathring{\Gamma}$-functor $\widetilde{\mathbf{SC}}_g^F$: $\mathbf{Sim}_\Gamma|\mathscr{T}(S)^+| \to \mathbf{Top}$ such that $|\widetilde{\mathbf{SC}}_g^F|/\Gamma$ is canonically homeomorphic to $|\mathbf{SC}_g|$. The same procedure can be applied to the category $\mathbf{W}_g$ to obtain a universal $\mathring{\mathbf{SP}}$-functor $\tilde{\mathbf{W}}_g^F$: $\mathbf{Sim}_{\text{SP}}|\mathscr{T}_g^+| \to \mathbf{Top}$ such that $|\tilde{\mathbf{W}}_g^F|/\text{SP}_g(\mathbb{Z})$ is canonically homeomorphic to $|\mathbf{W}_g|$. For this, we first lift $\mathbf{W}_g$ to a category $\tilde{\mathbf{W}}_g$ with $\text{SP}_g(\mathbb{Z})$-action whose objects are quadruples (φ, L, P, λ) such that L is a vertex in $|\mathscr{T}_g^+|$, (P, λ) is an object in $\mathbf{W}_g$, and φ: $(L^\perp/L, \lambda_L) \to (P, \lambda)$ is an isometry. A morphism ϕ: $(\varphi, L, P, \lambda) \to (\varphi', L', P', \lambda')$ in $\tilde{\mathbf{W}}_g$ is defined by an inclusion relation $L \subset L'$ together with a morphism (L'', ψ): $(P, \lambda) \to (P', \lambda')$ in $\mathbf{W}_g$ such that the diagram

$$\begin{array}{ccc} (L^\perp/L, \lambda_L) & \xrightarrow{(0,\varphi)} & (P, \lambda) \\ \big\downarrow{\scriptstyle (L'/L, \text{id})} & & \big\downarrow{\scriptstyle (L'', \psi)} \\ (L'^\perp/L', \lambda_{L'}) & \xrightarrow{(0,\varphi')} & (P', \lambda') \end{array}$$

is commutative in $\mathbf{W}_g$. The action of $M \in \text{SP}_g(\mathbb{Z})$ on $\tilde{W}_g$ is given by $M \cdot (\varphi, L, P, \lambda) = (\psi_M \circ \varphi, L \cdot M, P, \lambda)$, where ψ_M: $((L \cdot M)^\perp/(L \cdot M), \lambda_{L \cdot M}) \to (L^\perp/L, \lambda_L)$ is the isometry induced by M^{-1}.

Now, arguing as in (6.2) and (6.3), one has forgetful functors F: $\tilde{\mathbf{W}}_g \to \mathbf{W}_g$ and F_0: $\tilde{\mathbf{W}}_g \to \mathscr{T}_g^+$. The first induces a homeomorphism $|\tilde{\mathbf{W}}_g|/\Gamma \cong |\mathbf{W}_g|$. The second is used to define "fiber categories" $\tilde{\mathbf{W}}_\sigma$ whose objects are sequences of morphisms in $\tilde{\mathbf{W}}_g$ lying over $\sigma \in |\mathscr{T}_g^+|$ and whose morphisms are commutative diagrams. The resulting spaces $|\tilde{\mathbf{W}}_\sigma|$ are universal spaces for $\mathring{\text{SP}}_\sigma = P(L_0^\perp/L_0, \sigma/L_0)$ and thus define a universal $\mathring{\mathbf{SP}}$-functor, $\tilde{\mathbf{W}}_g^F$: $\mathbf{Sim}_{\text{SP}}|\mathscr{T}_g^+| \to \mathbf{Top}$, such that $|\tilde{\mathbf{W}}_g^F| \cong |\tilde{\mathbf{W}}_g|$ and hence $|\tilde{\mathbf{W}}_g^F|/\text{SP}_g(\mathbb{Z}) \cong |\mathbf{W}_g|$. The details are exactly as in (6.2), (6.3) and we leave their verification to the reader. Composing with $\tilde{\mathbf{I}}$: $\mathbf{Sim}_\Gamma|\mathscr{T}(S)^+| \to \mathbf{Sim}_{\text{SP}}|\mathscr{T}_g^+|$, we obtain a universal $\mathring{\mathbf{SP}} \circ \tilde{\mathbf{I}}$-functor, $\tilde{\mathbf{W}}_g^F \circ \tilde{\mathbf{I}}$: $\mathbf{Sim}_\Gamma|\mathscr{T}(S)^+| \to \mathbf{Top}$, and a canonical map $|\tilde{\mathbf{W}}_g^F \circ \tilde{\mathbf{I}}| \to |\tilde{\mathbf{W}}_g^F|$.

Consider the functor $\mathbf{J}$: $\mathbf{SC}_g \to \mathbf{W}_g$, $R \mapsto (H^1(N(R); \mathbb{Z}), \lambda)$, of (7.4). This lifts to an equivariant functor $\tilde{\mathbf{J}}$: $\widetilde{\mathbf{SC}}_g \to \tilde{\mathbf{W}}_g$, $(\varphi, S/\alpha, R) \mapsto (N(\varphi)^*, L_\alpha, H^1(N(R); \mathbb{Z}), \lambda)$, where $L_\alpha = \text{Im}(H^1(S/\alpha; \mathbb{Z})^\perp \to H^1(S; \mathbb{Z}))$ and $N(\varphi)^*$: $L_\alpha^\perp/L_\alpha \cong H^1(N(S/\alpha); \mathbb{Z}) \to H^1(N(R); \mathbb{Z})$ is the isometry induced by the normalization of φ, $N(\varphi)$: $N(S/\alpha) \to N(R)$. Recalling (8.1), the correspondence $\alpha \mapsto L_\alpha$ induces the map $\tilde{I}$: $|\mathscr{T}(S)^+| \to |\mathscr{T}_g^+|$, so there is a commutative diagram:

$$\begin{array}{ccc} |\widetilde{\mathbf{SC}}_g| & \xrightarrow{|\tilde{\mathbf{J}}|} & |\tilde{\mathbf{W}}_g| \\ \downarrow & & \downarrow \\ |\mathscr{T}(S)^+| & \xrightarrow{\tilde{I}} & |\mathscr{T}_g^+| \end{array}$$

It follows that $\tilde{\mathbf{J}}$ induces maps of fiber categories $\tilde{\mathbf{J}}_\sigma \colon \widetilde{\mathbf{SC}}_\sigma \to \tilde{\mathbf{W}}_{\tilde{I}(\sigma)}$ and hence a natural transformation of universal functors $\tilde{\mathbf{J}}^F \colon \widetilde{\mathbf{SC}}_g^F \to \tilde{\mathbf{W}}_g^F \circ \tilde{\mathbf{I}}$. Comparing this with the natural transformation $\tilde{\mathbf{J}}^{\mathrm{un}} \colon \tilde{\mathbf{V}}_D^{\mathrm{un}} \to \tilde{\mathbf{Y}}^{\mathrm{un}} \circ \tilde{\mathbf{I}}$, the rigidity of universal functors (Proposition (3.2.3)) implies that there exists homotopy equivalences $|\tilde{\mathbf{V}}_D^{\mathrm{un}}| \xrightarrow{\simeq} |\widetilde{\mathbf{SC}}_g^F|$ and $|\tilde{\mathbf{Y}}^{\mathrm{un}} \circ \tilde{\mathbf{I}}| \xrightarrow{\simeq} |\tilde{\mathbf{W}}_g^F \circ \tilde{\mathbf{I}}|$ such that the diagram

$$\begin{array}{ccc} |\tilde{\mathbf{V}}_D^{\mathrm{un}}| & \xrightarrow{|\tilde{\mathbf{J}}^{\mathrm{un}}|} & |\tilde{\mathbf{Y}}^{\mathrm{un}} \circ \tilde{\mathbf{I}}| \\ \downarrow \simeq & & \downarrow \simeq \\ |\widetilde{SC}_g^F| & \xrightarrow{|\tilde{\mathbf{J}}^F|} & |\tilde{\mathbf{W}}_g^F| \end{array}$$

is commutative up to Γ-equivariant homotopy. Taking quotients by the action of Γ, Lemma (8.4.1) follows immediately.

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