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CONTRACTIONS OF INVARIANT FINSLER FORMS ON THE CLASSICAL DOMAINS

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Abstract

The Schwarz-Pick inequality shows that every holomorphic map of the complex unit disk into itself contracts the Poincaré metric. We consider the analogous question for holomorphic maps and biholomorphically invariant Finsler forms, on any of the classical Cartan domains of rank > 1.

Theorem. *Suppose D is a classical Cartan symmetric domain of rank > 1. If f: $D \to D$ is a nonconstant holomorphic map that contracts all invariant Finsler forms, then f is a biholomorphism.*

The question of which maps contract all invariant Finsler forms contrasts with previous work giving a Schwarz-Pick inequality for the Bergman metric on bounded symmetric domains (see e.g., Kobayashi: *Hyperbolic manifolds and holomorphic mappings*) and also with work defining systems of metrics contracted by all holomorphic mappings (the "Schwarz-Pick" systems of pseudometrics of Harris).

1. Introduction

Let D be any one of the four classical Cartan domains in C^n. This paper answers the following question: which holomorphic maps f: $D \to D$ contract all biholomorphically invariant infinitesimal Finsler forms on D?

If D is the Poincare disk then of course the answer is familiar. The infinitesimal Poincare metric is (up to scalar multiples) the only invariant infinitesimal Finsler form on D and by the Schwarz-Pick inequality every holomorphic f: $D \to D$ is a contraction.

When the rank of D is greater than 1 the situation is radically different. We shall see that except for the biholomorphisms themselves, there are *no* nonconstant holomorphic contractions of every invariant Finsler form.

The question of which maps contract every invariant form seems to contrast with previous work on the Schwarz-Pick inequality. Koranyi [6] showed that if G is a bounded symmetric domain of rank k then every holomorphic f: $G \to G$

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satisfies $f^*ds \leqslant k^{1/2}ds$, where ds is the infinitesimal Bergman metric; generalizations of this result may be found in Kobayashi [5]. Harris [4] considered the question of assigning a pseudometric to each complex domain in such a way that holomorphic maps between domains are contractions and the metric assigned to the unit disk is the Poincare metric. The Caratheodory and Kobayashi pseudometrics are respectively the largest and smallest of these "Schwarz-Pick" systems; on any bounded symmetric domain the two coincide.

This paper was motivated by work of Ball and Helton [1] which characterized the set of "totally contractive" mappings of the $n \times n$ matrix ball (a type I classical domain); examples are the linear fractional mappings that commonly occur in electrical network design. "Totally contractive" mappings are contractions for many but not all invariant Finsler forms; the question then arises which maps do contract them all.

Figure 1 provides some insight into the answer in the case of the 2×2 matrix ball. The *indicatrix* of a Finsler form F is the set of matrices Y in the tangent space at 0 which satisfy $F(0, Y) < 1$. If F is biholomorphically invariant then $F(0, UYW) = F(0, Y)$ when U and W are unitary matrices; since there exist unitary U, W which reduce Y to diagonal form with real entries, the indicatrix of F is completely determined by the set Δ_F of real diagonal matrices contained in it. The figures show Δ_F as a subset of R^2; a matrix Y may be identified with the point (σ_1, σ_2) whose coordinates are the singular values of Y.

Suppose Λ is a linear map on the tangent space at 0 which contracts every Finsler form of the type represented in Figure 1 with increasingly sharp spikes; evidently Λ must preserve the ratio σ_2/σ_1 for each Y or map Y to 0. We will show that such a Λ must be a linear isometry or the zero map; thus the forms of Figure 1 force the conclusion of our main theorem.

In contrast, Figure 2 represents a Finsler form contracted by all 'totally contractive" maps, a large class. The requirement here is that the intersection of every horizontal and vertical line with Δ_F is an interval; this prevents sharp

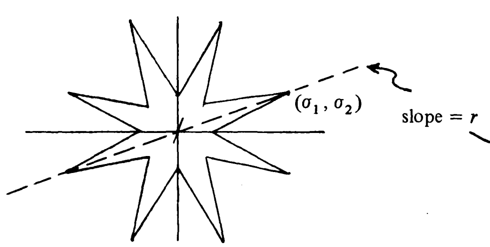

FIG. 1

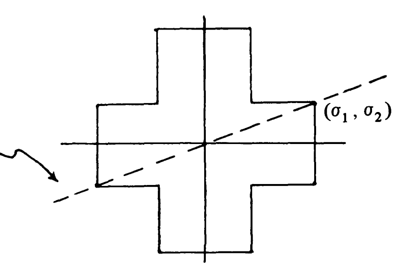

FIG. 2

spikes from forming. Clearly any convex indicatrix falls into this category, hence each Finsler *metric* is contracted by every totally contractive map. This and other results for the convex case are given in [2].

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2. Cartan domains and Finsler forms

For the first three classical Cartan domains we use the definitions given in [5, p. 34]. Let * denote conjugate transpose and let I_k denote the identity matrix of order k. Then

$$R_{\mathrm{I}}(m, n) \equiv \{m \times n \text{ matrices } Z\colon I_m - ZZ^* > 0\}.$$

We may assume that $m \geqslant n$. This domain has rank n.

$$R_{\mathrm{II}}(n) \equiv \{n \times n\, Z\colon Z = Z^T, I_n - ZZ^* > 0\}.$$

The rank is n.

$$R_{\mathrm{III}}(n) \equiv \{n \times n\, Z\colon Z = -Z^T, I_n - ZZ^* > 0\}.$$

The rank is $[n/2]$.

We define each *Cartan factor* $X_{\mathrm{I}}(m, n)$, $X_{\mathrm{II}}(n)$ and $X_{\mathrm{III}}(n)$ to be the tangent space at 0 of the corresponding Cartan domain; it is clear conversely that a Cartan domain is the open unit ball of its Cartan factor in the operator norm.

For domains of type IV it will be convenient to use a realization given in [3]. A factor of this type is a closed subspace X_{IV} of $X_{\mathrm{I}}(n, n)$ such that $Y \in X_{\mathrm{IV}}$ implies $Y^* \in X_{\mathrm{IV}}$ and $Y^2 = c_y I_n$ for some scalar c_y. There is an inner product on X_{IV} defined by $(Y, Z)I_n = \frac{1}{2}(YZ^* + Z^*Y)$.

A domain R_{IV} is then the open unit ball of a factor X_{IV} in the operator norm. The operator norm may be expressed in terms of the inner product by

$$\|Z\|^2 = (Z, Z) + \left((Z, Z)^2 - |(Z, Z^*)|^2\right)^{1/2},$$

which leads us to the standard definition of R_{IV} as given in [5]. The rank of any R_{IV} is 2 unless X_{IV} is one-dimensional.

Now let $T_Z(D)$ and $T(D)$ be respectively the tangent space at Z and the tangent bundle of the domain D. An infinitesimal Finsler form is a continuous nonnegative map $F\colon T(D) \to R$ satisfying $F(Z, cY) = |c|F(Z, Y)$ for every scalar c and $Y \in T_Z(D)$. We say F is biholomorphically invariant if for every biholomorphism $h\colon D \to D$ we have $F(h(Z), Dh(Z)\{Y\}) = F(Z, Y)$.

Let D be a Cartan domain and $Z \in D$. The Moebius transformation

$$t_{\Delta Z}(Y) = Z + (I - ZZ^*)^{1/2} Y (I + Z^*Y)^{-1} (I - Z^*Z)^{1/2} \tag{1}$$

is a biholomorphism of D such that $t_{\Delta Z}(0) = Z$, $t_{\Delta Z}^{-1} = t_{\Delta - Z}$ and the following derivative formulas hold:

$$\text{(2a)} \qquad Dt_{\Delta Z}(0)\{V\} = (I - ZZ^*)^{1/2} V (I - Z^*Z)^{1/2},$$

$$\text{(2b)} \qquad Dt_{\Delta - Z}(Z)\{V\} = (I - ZZ^*)^{-1/2} V (I - Z^*Z)^{-1/2}.$$

For the proof of these facts, see [3, Theorem 2].

The *rank* k of a Cartan domain D is equal to the maximum number of distinct nonzero elements in the spectrum of Z^*Z as Z ranges over D; each Z has k *canonical values* $\sigma_1(Z) \geqslant \cdots \geqslant \sigma_k(Z) \geqslant 0$ belonging to the spectrum of $(Z^*Z)^{1/2}$. For domains of type I and II the canonical values and the singular values of Z coincide. For skew-symmetric matrices (domains of type III) the singular values occur in pairs, except for matrices of odd order, where the last singular value is 0. For $Z \in R_{\mathrm{IV}}$ there are at most two distinct eigenvalues of Z^*Z. Thus there are $[n/2]$ canonical values for $Z \in R_{\mathrm{III}}(n)$ and 2 canonical values for $Z \in R_{\mathrm{IV}}$.

3. Finsler form contractions

Theorem 1. *Let D be a Cartan domain of rank > 1. If f: $D \to D$ is a nonconstant holomorphic map that contracts all invariant Finsler forms then f is a biholomorphism.*

The proof uses the following three lemmas.

Lemma 1a. *If a linear map Λ: $T_0(D) \to T_0(D)$ does not increase any invariant Finsler form at 0, then for each $Y \in T_0(D)$ there is a constant $c \geqslant 0$ such that the canonical values of $\Lambda(Y)$ are c times those of Y.*

Proof. Define the following family of infinitesimal forms at 0, indexed by the parameters r, m, j with $0 \leqslant r \leqslant 1, j \in \{2,\cdots,k\}$ and $m \in \{1, 2, \cdots\}$:

$$F_{m,r,j}(0, Y) = \begin{cases} m\sigma_1(Y) - (m-1)\sigma_j(Y)/r & \text{if } \sigma_j/\sigma_1 \leqslant r, \sigma_1 \neq 0, \\ m\sigma_j(Y) - (mr-1)\sigma_1(Y) & \text{if } \sigma_j/\sigma_1 \geqslant r, \sigma_1 \neq 0, \\ 0 & \text{if } \sigma_1 = 0 \end{cases}$$

for $0 < r < 1$;

$$F_{m,0,j}(0, Y) = m\sigma_j(Y) + \sigma_1(Y); \quad F_{m,1,j}(0, Y) = m\sigma_1(Y) - (m-1)\sigma_j(Y).$$

It is easy to verify that each $F_{m,r,j}$ is continuous; Figure 1 shows Δ_F with $0 < r < 1$ for the case of a rank 2 domain.

Now suppose $\sigma_j(Y) = r'\sigma_1(Y)$ but $\sigma_j(\Lambda(Y)) \neq r'\sigma_1(\Lambda(Y))$ and $\sigma_1(\Lambda(Y)) \neq 0$. If $\sigma_j(\Lambda(Y)) = r'\sigma_1(\Lambda(Y)) + \delta$ for some $\delta > 0$, then

$$F_{m,r',j}(0, Y) = \sigma_1(Y), \quad F_{m,r',j}(0, \Lambda(Y)) = m\delta + \sigma_1(\Lambda(Y)),$$

so $F_{m,r',j}(0, \Lambda(Y)) > F_{m,r',j}(0, Y)$ as soon as $m > (\sigma_1(Y) - \sigma_1(\Lambda(Y)))/\delta$. A similar calculation gives the same result when $\sigma_j(\Lambda(Y)) < r'\sigma_1(\Lambda(Y))$.

If Λ contracts every invariant Finsler form at 0 then for each Y and each j it follows that $\sigma_j(\Lambda(Y))/\sigma_1(\Lambda(Y)) = \sigma_j(Y)/\sigma_1(Y)$ or $\Lambda(Y) = 0$. This proves the lemma.

Lemma 1b. *If* $\Lambda: T_0(D) \to T_0(D)$ *satisfies the conclusion of Lemma* 1a, *then* Λ *is a multiple of a linear isometry on* $T_0(D)$ *in the operator norm.*

Proof. We will show that if Λ is not the 0 map then there is a nonzero constant c such that $(1/c)\Lambda$ maps the set of extreme points of D into itself and additionally that Λ is injective. It will then follow by the Schwarz lemma that $(1/c)\Lambda$ is a linear isometry L.

Step 1. $(1/c)\Lambda$ maps extreme points to extreme points.

By a result of Kadison-Harris (see [3, Theorem 11]), the extreme points of D are the partial isometries of rank $n - 1$ when $D = R_{\mathrm{III}}(n)$ for odd n, and the rank n isometries in every other case. Thus for each domain the extreme points are the matrices of the corresponding Cartan factor having all canonical values equal to 1. Lemma 1a then tells us that Λ maps extreme points to multiples of extreme points.

If there exists a constant c such that $\|\Lambda(U)\| = c$ for each extreme point U, then by the maximum principle $c > 0$. We show in each case that c exists.

Case 1. D is any of the domains $R_{\mathrm{I}}(n, n)$, $R_{\mathrm{II}}(n)$, $R_{\mathrm{III}}(2n)$, R_{IV}.

For each of these domains the extreme points are the unitary matrices belonging to the corresponding Cartan factor.

Let U and V be two extreme points. If λ is an eigenvalue of V^*U then $U - \lambda V$ is singular. Since Λ by Lemma 1a is rank-preserving, $\Lambda(U) - \lambda\Lambda(V)$ is also singular; let $\vec{v}$ be a unit null vector. Then $\Lambda(U)\vec{v} = \lambda\Lambda(V)\vec{v}$; taking the norm of each side gives $\|\Lambda(U)\| = \|\Lambda(V)\|$ since $\Lambda(U)$ and $\Lambda(V)$ are each multiples of a unitary matrix and $|\lambda| = 1$.

Case 2. $D = R_{\mathrm{I}}(m, n)$ for $m > n$.

Here the extreme points are the rank n isometries.

We claim the following:

(i) If U and V are extreme points having a column in common then $\|\Lambda(U)\| = \|\Lambda(V)\|$.

Suppose the ith column of U and V is the vector $\vec{y}$. Then $U - V$ is singular, hence the same is true for $\Lambda(U) - \Lambda(V)$, and our claim follows from the same argument as used in Case 1.

(ii) If U and V are extreme points having the same range then $\|\Lambda(U)\| = \|\Lambda(V)\|$.

If U and V have the same range then V^*U is unitary on C^n. If λ is an eigenvalue of V^*U then $U-\lambda V$ is singular, since VV^* is the orthogonal projection onto range(V) = range(U); the claim then follows as before.

Now let U and V be any two extreme points. Let $\vec{u}$ be the first column of U, $\vec{v}$ the first column of V and find an n-dimensional subspace of C^m containing $\vec{u}$ and $\vec{v}$. Expand $\vec{u}$ to an orthonormal basis for this subspace and let U_1 be the isometry whose columns are these basis vectors with $\vec{u}$ in the first column. Likewise expand $\vec{v}$ to an orthonormal basis for this subspace and let V_1 be the isometry having this set of vectors as its columns with $\vec{v}$ as the first. Then we have

$$\begin{aligned}\|\Lambda(U)\| &= \|\Lambda(U_1)\| \quad \text{by (i)}\\ &= \|\Lambda(V_1)\| \quad \text{by (ii)}\\ &= \|\Lambda(V)\| \quad \text{by (i)}.\end{aligned}$$

Case 3. $D = R_{\mathrm{III}}(n)$ for odd n.

The extreme points are the partial isometries of rank $n-1$. Every matrix of $X_{\mathrm{III}}(n)$ is singular and its null space is odd-dimensional. In what follows $\mathscr{N}(Z)$ and $\mathscr{I}(Z)$ will denote respectively the null space and the initial space (i.e., $\mathscr{N}(Z)^\perp$) of Z; $\mathrm{n}(Z)$ denotes the dimension of the null space. We claim the following:

(i) If U and V are extreme points such that $\mathrm{n}(U-V) \geqslant 3$ then $\|\Lambda(U)\| = \|\Lambda(V)\|$.

If $n(u-v) \geqslant 3$, then by Lemma 1a we have also $\mathrm{n}(\Lambda(U)-\Lambda(V)) \geqslant 3$, hence there is a unit vector $\vec{v} \in \mathscr{I}(\Lambda(U)) \cap \mathscr{I}(\Lambda(V))$ such that $\Lambda(U)\vec{v} = \Lambda(V)\vec{v}$. The assertion then follows as before.

This assertion follows immediately from the fact that $\mathrm{n}(U-V) \geqslant 3$.

(ii) If $\mathscr{N}(U) = \mathscr{N}(V)$ then $\|\Lambda(U)\| = \|\Lambda(V)\|$.

If U and V have the same null space then because they are skew-symmetric they also have the same range. Thus $V^*U|_{\mathscr{I}(U)}$ is unitary. If λ is an eigenvalue of $V^*U|_{\mathscr{I}(U)}$ then $(U-\lambda V)|_{\mathscr{I}(U)}$ is singular since VV^* is the orthogonal projection onto range(V) = range(U). But $\mathscr{N}((U-\lambda V)|_{\mathscr{I}(U)})$ is orthogonal to $\mathscr{N}(U) = \mathscr{N}(V)$; since nullity is odd, we have $\mathrm{n}(U-\lambda V) \geqslant 3$ and the assertion follows from (i) and the fact that $|\lambda| = 1$.

Now let E be the partial isometry defined by $E_{i,i+1} = -E_{i+1,i} = 1$ for odd i, with all other entries equal to 0. Then $\mathscr{N}(E)$ is spanned by $\vec{e}_n = (0,\cdots,0,1)^T$. Let Y be another extreme point having the same first two columns as E and any skew-symmetric partial isometry Y' of rank $n-3$ in the remaining block of order $n-2$:

$$Y = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \cdot & \cdot & & Y' & \\ 0 & 0 & & & \end{bmatrix}, \quad W = \begin{bmatrix} & & & 0 & \cdot & \cdot & 0 \\ & W' & & 0 & \cdot & \cdot & 0 \\ 0 & & 0 & 1 & & & 0 \\ & & \cdot & & 1 & \cdot & 0 \\ 0 & & 0 & 0 & 0 & & 1 \end{bmatrix}.$$

Note that $\mathcal{N}(Y)$ is spanned by $\vec{y} = (0, 0, y_3, \cdots, y_n)^T$, where $\vec{y}' = (y_3, \cdots, y_n)^T$ spans $\mathcal{N}(Y')$. Let W be defined so that the upper left 3×3 block W' is unitary and the remaining nonzero entries are 1's along the diagonal.

Suppose V is an extreme point with unit null vector $\vec{v} = (v_1, \cdots, v_n)^T$. We will show that $\|\Lambda(V)\| = \|\Lambda(E)\|$. Pick Y' having null vector

$$\vec{v}' = \left(\left(\sum_1^3 |v_i|^2 \right)^{1/2}, v_4, \cdots, v_n \right)^T,$$

and W' satisfying

$$W'(v_1, v_2, v_3)^T = \left(0, 0, \left(\sum_1^3 |v_i|^2 \right)^{1/2} \right)^T.$$

Then $W\vec{v} = (0, 0, \vec{v}'^T)^T$ spans $\mathcal{N}(Y)$ so V and W^TYW have the same null space. Also $W\vec{e}_n = \vec{e}_n$ so W^TEW and E have the same null space. Then:

$$\begin{aligned} \|\Lambda(V)\| &= \|\Lambda(W^TYW)\| \quad \text{by (ii)} \\ &= \|\Lambda(W^TEw)\| \quad \text{by (i) since } \mathrm{n}(W^TYW - W^TEW) \\ &\qquad\qquad\qquad = \mathrm{n}(W^T(Y - E)W) \geqslant 3 \\ &= \|\Lambda(E)\| \quad \text{by (ii)}. \end{aligned}$$

Let $c = \|\Lambda(E)\|$.

Step 2. Λ is injective.

Suppose $\Lambda(Z) = 0$. Let U be an extreme point of D. Then for each nonzero scalar z, we have $\Lambda(Z + zU) = z\Lambda(U)$ is a nonzero multiple of an extreme point; by Lemma 1a the same must be true for $Z + zU$. If $Z \neq 0$ then letting $z \to 0$ shows that Z is also a nonzero multiple of an extreme point. But we have seen from Step 1 that Λ sends no extreme points to 0.

Thus Λ is injective so $(1/c)\Lambda \equiv L$ maps the extreme points of D onto themselves. Then L is a biholomorphism of D preserving the origin, so by the Schwarz lemma is a linear isometry. This proves Lemma 1b.

Suppose f: $D \to D$ does not increase any of the Finsler forms defined in Lemma 1a, that is, for each $(Z, Y) \in T(D)$ and all r, m, j we have

$$F_{r,m,j}(f(Z), Df(Z)\{Y\}) \leqslant F_{r,m,j}(Z, Y).$$

By biholomorphic invariance of $F_{r,m,j}$, this is the same as

$$F_{r,m,j}\big(0, Dt_{\Delta-f(Z)}(f(Z))\circ Df(Z)\{Y\}\big) \leqslant F_{r,m,j}\big(0, Dt_{\Delta-Z}(Z)\{Y\}\big).$$

Defining a new function $f_Z = t_{\Delta-f(Z)}\circ f\circ t_{\Delta Z}$, we then rewrite the above inequality as

$$F_{r,m,j}\big(0, Df_Z(0)\{S\}\big) \leqslant F_{r,m,j}(0, S), \tag{3}$$

where $S = Dt_{\Delta-Z}(Z)\{Y\}$.

By Lemmas 1a and 1b it then follows that $Df_Z(0)$ is a multiple of a linear isometry.

Lemma 2. *Suppose $f\colon D\to D$ is a holomorphic map such that $f(0)=0$ and $Df(0)\neq 0$. If for each $Z\in D$ there exists a scalar a_Z and a linear isometry $L_{(Z)}$ such that $Df_Z(0) = a_Z L_{(Z)}$, then f is itself a linear isometry.*

Proof. Since $f(0)=0$ we have $f_0 = f$, so by hypothesis $Df(0) = a_0 L_{(0)}$ where $L_{(0)}$ is a linear isometry and $a_0\neq 0$. To conserve notation we will take $a\equiv a_0$ and $L\equiv L_{(0)}$.

Without loss of generality we may take a to be real. If we can show $a=1$ then it will follow from the Schwarz lemma that $f = Df(0) = L$ and our assertion will be proved.

We have the power series expansion $f = aL + \Sigma_2^\infty P_k$, where each $P_k = D^k f(0)/(k!)$ is a k-homogeneous polynomial. We first show that $P_2 = 0$, then apply this result to f_Z for small $\|Z\|$ to show that $a=1$.

Step 1. $P_2 = 0$.

To obtain a contradiction suppose otherwise. If z is a real scalar and $S\in D$, then we have the following expansions:

$$f(zS) = azL(S) + z^2P_2(S) + \cdots, \tag{4}$$

$$Df(zS)\{Y\} = aL(Y) + zDP_2(S)\{Y\} + \cdots. \tag{5}$$

From (2a), (2b), the definition of f_{zS} and the chain rule, we have

$$\begin{aligned} Df_{zS}(0)\{Y\} &= (I - f(zS)f(zS)^*)^{-1/2}\\ &\quad\times\left[Df(zS)\{(I-SS^*)^{1/2}Y(I-S^*S)^{1/2}\}\right]\\ &\quad\times(I - f(zS)^*f(zS))^{-1/2}. \end{aligned} \tag{6}$$

Now substitute the expansions (4) and (5) in (6); collect like powers of z on the right side and set the left side equal to $a_{zS}L_{(zS)}$ by hypothesis. The result is

$$a_{zS}L_{(zS)}(Y) = aL(Y) + zDP_2(S)\{Y\} + O(z^2). \tag{7}$$

Case 1. $D = R_{\text{III}}(n)$ for n odd or $R_{\text{I}}(m,n)$ for $m>n$, $n\geqslant 2$.

Formula (7) implies that $\operatorname{rank}(DP_2(S)\{Y\}) \leqslant \operatorname{rank}(Y)$, since for small enough $|z|$ the rank of the right side of (7) is at least $\operatorname{rank}(DP_2(S)\{Y\})$ and $L_{(zS)}$ is rank-preserving. Now recall that $P_2(Z) = A_2(Z, Z)$ where A_2 is a symmetric bilinear map. Thus $DP_2(S)\{Y\} = 2A_2(S, Y) = 2A_2(Y, S) = DP_2(Y)\{S\}$ implies also that $\operatorname{rank}(DP_2(S)\{Y\}) \leqslant \operatorname{rank}(S)$.

Now suppose that S has rank 1 (if $S \in R_{\mathrm{I}}(m, n)$) or rank 2 (if $S \in R_{\mathrm{III}}(n)$) and Y is an extreme point. Then $L(Y)$ and $L_{(zS)}(y)$ are also extreme points. Divide both sides of (7) by a_{zS}, then multiply each side by its conjugate transpose. The resulting equation has the projection $L_{(zS)}(Y)^*L_{(zS)}(Y)$ on the left side. Subtract the square of each side from this equation to obtain

$$
\begin{aligned}
0 &= a'^2(1 - a'^2)L(Y)^*L(Y) \\
&\quad + z(a'/a_{zS})\big[(1 - a'^2)\big(L(Y)^*DP_2(S)\{Y\} + DP_2(S)\{Y\}^*L(Y)\big) \\
&\quad - a'^2\big(L(Y)^*L(Y)DP_2(S)\{Y\}^*L(Y) \\
&\quad + L(Y)^*DP_2(S)\{Y\}L(Y)^*L(Y)\big)\big] + O(z^2),
\end{aligned} \tag{8}
$$

where $a' = a/a_{zS}$. If the second summand in the linear term is nonzero, then (8) gives $(1 - a'^2) = O(z)$. But we can pick a vector $\vec{v}$ such that $DP_2(S)\{Y\}^*L(Y)\vec{v} = 0$, since $DP_2(S)\{Y\}$ has rank 1 or 2 while $L(Y)$ is full rank (n or $n - 1$). Then multiplying (8) on the left by $\vec{v}^*$ and on the right by $\vec{v}$ gives

$$0 = a'(1 - a'^2)\vec{v}^*L(Y)^*L(Y)\vec{v} + O(z^2),$$

in other words $(1 - a'^2) = O(z^2)$. From this contradiction we conclude that the summand in question is 0; the same argument holds when we replace S by iS in (8). But then we have that $L(Y)^*L(Y)DP_2(S)\{Y\}^*L(Y)$ is both hermitian and skew-hermitian, hence equal to 0. Because $L(Y)$ has full rank, this means $DP_2(S)\{Y\} = 0$.

So we have shown that $A_2(S, Y) = 0$ whenever S has rank 1 (in $R_{\mathrm{I}}(m, n)$) or rank 2 (in $R_{\mathrm{III}}(n)$) and Y is an extreme point. But the $m \times n$ rank 1 matrices contain a basis for $X_{\mathrm{I}}(m, n)$ and the skew-symmetric rank 2 matrices contain a basis for $X_{\mathrm{III}}(n)$. In addition, each $Z \in D$ may be expressed as a convex combination of extreme points. It then follows by linearity that $A_2(S, Z) = 0$ for arbitrary $S, Z \in D$, hence $P_2 = 0$.

Case 2. $D = R_{\mathrm{I}}(n, n), R_{\mathrm{II}}(n), R_{\mathrm{III}}(2n)$ or R_{IV}.

In this case it will suffice to show that A_2 vanishes on pairs of unitary matrices. Let Y be unitary. Then $L(Y)$ and $L_{(zS)}(Y)$ are unitary; multiplying each side of (7) by its conjugate transpose gives

$$a_{zS}^2 I = a^2 I + za\big[L(Y)^*DP_2(S)\{Y\} + DP_2(S)\{Y\}^*L(Y)\big] + O(z^2),$$

which holds for all z sufficiently close to 0. This implies that the linear term on the right side is a multiple of I. Replacing S by iS in the above and taking the two equations together then gives that $L(Y)^*DP_2(S)\{Y\}$ is a multiple of I, in other words $DP_2(S)\{Y\} = c_s L(Y)$.

Now suppose V is another unitary matrix not a multiple of Y. Then we have $c_v L(V) = DP_2(Y)\{V\} = DP_2(V)\{Y\} = c_y L(Y)$ which implies that $c_v = c_y = 0$. Thus $A_2(V, Y) = 0$. If V is a multiple of Y, take a sequence of unitary matrices $\{V_k\}$ such that $V_k \to Y$ and each V_k is not a multiple of Y. This gives $A_2(Y, Y) = 0$. Since each $Z \in D$ may be expressed as a convex combination of unitary matrices, linearity implies $P_2 = 0$.

We have now shown that $f = aL + P_3 + \cdots$.

Step 2. $a = 1$.

From the definition of f_Z it is easy to see that f_Z contracts all invariant Finsler forms on D if f does. Also $f_Z(0) = 0$ and for Z close to 0 we have $Df_Z(0) \neq 0$ since by hypothesis $Df(0) \neq 0$. Thus the hypotheses of Lemma 2 apply to f_{zS} as well as to f for small $|z|$, so by Step 1 we have $D^2 f_{zS}(0) = 0$.

We will use this fact to prove our result by calculating the linear term of the series expansion in z for $D^2 f_{zS}(0)\{Y, Y\}$ and setting it equal to 0 (we continue taking z to be real).

Applying the chain rule twice to f_{zS} we have

$$\begin{aligned} &D^2 f_{zS}(0)\{Y, Y\} \\ &= D^2 t_{\Delta - f(zS)}(f(zS))\{Df(zS)\circ Dt_{\Delta_{zS}}(0)\{Y\}, Df(zS)\circ Dt_{\Delta_{zS}}(0)\{Y\}\} \\ &+ Dt_{\Delta - f(zS)}(f(zS))\{D^2 f(zS)\{Dt_{\Delta_{zS}}(0)\{Y\}, Dt_{\Delta_{zS}}(0)\{Y\}\} \\ &\qquad + Df(zS)\{D^2 t_{\Delta_{zS}}(0)\{Y, Y\}\}\}. \end{aligned} \tag{9}$$

We will also need the following formula which holds for arbitrary points V, W, X:

$$\begin{aligned} &D^2 g(X)\{V, W\} \\ &= \lim_{t\to 0} [g(X + tV + tW) - g(X + tV) - g(X + tW) + g(X)]/t^2. \end{aligned} \tag{10}$$

It will only be necessary to calculate the constant and linear terms for each summand on the right side of (9). We start with the second summand. Applying (1) and (10) to $t_{\Delta_{zS}}$ and expanding the result as a series in z shows that $D^2 t_{\Delta_{zS}}(0)\{Y, Y\}$ has linear term equal to $-2zYS^*Y$. Then by (5) we have

$$Df(zS)\{D^2 t_{\Delta_{zS}}(0)\{Y, Y\}\} = -2azL(YS^*Y) + O(z^2). \tag{11}$$

Next, by (2a) we have $Dt_{\Delta_{zS}}(0)\{Y\} = Y$ to order 1 in z, so

$$(12) \qquad D^2f(zS)\{Dt_{\Delta_{zS}}(0)\{Y\}, Dt_{\Delta_{zS}}(0)\{Y\}\} = zD^2P_3(S)\{Y, Y\}$$

to order 1 in z. Finally, $Dt_{\Delta_{-f(zS)}}(f(zS))$ acting on the sum of the expressions in (11) and (12) contributes no new terms of order 1 or lower in z, as can be seen from (2b) and (4), so the second summand on the right of (9) is

$$(13) \qquad z\left[D^2P_3(S)\{Y, Y\} - 2aL(YS^*Y)\right] + O(z^2).$$

Now we calculate the first summand on the right side of (9). Using (2a), (5) and the fact that $P_2 = 0$, we see that to order 1 in z we have $Df(zS) \circ Dt_{\Delta_{zS}}(0)\{Y\} = aL(Y)$. Then in the same way as we did for $D^2t_{\Delta_{zS}}(0)\{Y, Y\}$, using (10) (and the fact that $L(Y)L(S)^*L(Y) = L(YS^*Y)$; see [3]) we determine that to order 1 in z, the first summand on the right of (9) is

$$(14) \qquad 2a^3zL(YS^*Y).$$

Combining (13) and (14) yields

$$D^2f_{zS}(0)\{Y, Y\} = z\left[D^2P_3(S)\{Y, Y\} + 2(a^3 - a)L(YS^*Y)\right] + O(z^2).$$

Since we must have $D^2f_{zS}(0)\{Y, Y\} = 0$ for z close to 0, we set the coefficient of the linear term equal to 0 and obtain

$$D^2P_3(S)\{Y, Y\} = 2(a - a^3)L(YS^*Y).$$

The right side of this equality is conjugate linear in S, while the left side is linear in S. This is possible only if $a = 1$, so the lemma is proved.

Proof of Theorem 1. Let $f\colon D \to D$ be a holomorphic map contracting all invariant infinitesimal Finsler forms. If $Df_Z(0) = 0$ for each $Z \in D$ then by the definition of f_Z we have $Df(Z) = 0$ for each Z, so f is constant. Suppose f is not constant. Then for some Z we have $Df_Z(0) \neq 0$; by applying Lemma 2 to f_Z, it follows that $f_Z = L$ is a linear isometry. Then $f = t_{\Delta f(Z)} \circ L \circ t_{\Delta -Z}$ is a biholomorphism.

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