J. DIFFERENTIAL GEOMETRY
18 (1983) 791–809

UNIQUENESS, SYMMETRY, AND EMBEDDEDNESS OF MINIMAL SURFACES

RICHARD M. SCHOEN

In 1956, A. D. Alexandrov [1] proved that a closed embedded hypersurface of constant mean curvature in Euclidean space is a standard sphere. Besides the importance of this result in differential geometry, the method employed in its proof has been used on a variety of problems in partial differential equations and differential geometry, first by J. Serrin [13] and more recently by B. Gidas, W. M. Ni and L. Nirenberg [2]. In a surprising recent development, W. Y. Hsiang, Zhen-Huan Teng and Wen-ci Yu [4] have constructed non-spherical closed immersed hypersurfaces of constant mean curvature in $\mathbf{R}^4$. These examples show that the embeddedness hypothesis is essential in Alexandrov's theorem. In this paper we apply Alexandrov's method to minimal hypersurfaces. The main difficulty, of course, is that minimal surfaces are never closed, but either have boundary or are complete and noncompact. An interesting feature of our results is that the embeddedness is not required; in fact, a consequence of the method is that in certain cases immersed surfaces can be shown to be embedded. This can be partially attributed to the fact that minimal hypersurfaces do not have a distinguished side locally whereas surfaces of nonzero constant mean curvature do.

Also in 1956, M. Shiffman [14] posed the problem of understanding minimal surfaces in $\mathbf{R}^3$ whose boundary consists of a union of two Jordan curves Γ_1, Γ_2 lying in parallel planes. Shiffman proved the striking result that if M is an immersed minimal surface of genus zero with $\partial M = \Gamma_1 \cup \Gamma_2$ and if Γ_1, Γ_2 are convex curves (resp. circles), then M meets each intermediate plane transversally in a convex curve (resp. circle). In particular this shows that if Γ_1 and Γ_2 are circles situated so that the line joining their centers is perpendicular to the planes in which they lie, then M is a surface of rotation, hence a catenoid. In §1 of this paper we extend this result in various directions; for example, we remove the topological assumption on M in the above characterization of the catenoid, and extend the results to higher dimensions. We also show that if Γ is

Received January 20, 1983, and, in revised form, May 19, 1983. Centre for Mathematical Analysis Research Report No. R04, 1982.

any boundary consisting of convex curves (the result is actually much stronger) lying in a pair of parrallel planes which is invariant under reflection through some orthogonal plane which intersects each component of Γ, then every minimal surface spanning Γ is embedded and invariant under reflection through the same plane. There is a similar result in all dimensions. W. Meeks [7, p. 87] has conjectured that the genus zero hypothesis is unnecessary in Shiffman's theorem. In Corollary 4 we prove a special case of this conjecture. The precise context of the theorems (see Theorems 1 and 2) is minimal surfaces having boundary lying on the boundary of a cylinder having nonpositive mean curvature relative to the outward unit normal (convex in case of two dimensions in $\mathbf{R}^3$). An interesting case of the embeddedness conclusion is for a boundary Γ consisting of convex curves, one on each face of a cylinder over a convex polygon in the plane. Our result implies that if Γ is invariant under a reflection through a plane perpendicular to the axis of the cylinder, then every minimal surface spanning Γ is embedded and invariant. Thus one cannot have a pair of intersecting annular surfaces with boundaries on a pair of opposite faces of a cube or rectangular solid. We refer the reader to §1 for precise statements of results and a couple more examples.

In §2 of this paper we define a class of minimal hypersurfaces which are said to be *regular at infinity*. For two dimensional surfaces in $\mathbf{R}^3$, results of R. Osserman [10] show that this notion is equivalent to finite total curvature and embedded ends. In higher dimensions we show that a surface whose normal vectors behave reasonably well at infinity is, in fact, regular at infinity. (A similar result was obtained by Jorge and Meeks [5].) These hypersurfaces also have the property that they scale down homothetically to a limit which is a union of hyperplanes. In §3 we show that any complete minimal hypersurface which is regular at infinity and has two ends is a catenoid or a pair of planes. There is some similarity between our proof in the noncompact case and the methods of Gidas-Ni-Nirenberg [2] where they deal with solutions of certain elliptic equations satisfying a suitable regularity property at infinity. Our proof is complicated by the fact that there are two infinities which may behave, a priori, differently. This is a particular problem in the two-dimensional case where the ends may be unbounded. In a preliminary result, Lemma 2, we are able to relate the ends in a suitable way to enable us to apply the reflection method. A well-known general uniqueness question for minimal surfaces in $\mathbf{R}^3$ is the question of determining all embedded complete minimal surfaces of finite topological type. The only known examples are the plane, catenoid, and helicoid. Our method gives uniqueness of the plane and catenoid among complete embedded minimal surfaces of finite total curvature with at most two infinities. It does not seem to generalize to handle more than two ends. We remark that there are complete immersed surfaces of genus zero in $\mathbf{R}^3$ with finite total curvature and three simple ends so that the exact analogue of our theorem for more than two ends certainly fails.

Concerning our results for complete surfaces, there are a few previous papers on the subject which we would like to mention. In 1962, J. C. C. Nitsche [8] showed that the catenoid is the only complete minimal surface in $\mathbf{R}^3$ which intersects each plane parallel to a given plane transversally in a star-shaped Jordan curve. While the hypotheses of Nitsche do restrict the topological type of the surface, the allowable behavior at infinity is considerably more complicated than ours. In fact, Nitsche [9] has derived a local version of his theorem which perhaps could be used in our setting to weaken our regularity hypothesis. Secondly, if one assumes genus zero and regular at infinity with two ends, it is quite easy to see that the absolute total curvature must be 4π. That the surface is a catenoid then follows from a result of R. Osserman (see [10, p. 87]).

In the case of surfaces of genus zero, Jorge and Meeks [5] have shown that there are no embedded surfaces of finite total curvature with fewer than six ends besides the plane and the catenoid. Recently H. Rosenberg [12] has studied the question of C^1 rigidity for complete minimal surfaces in $\mathbf{R}^3$ and in flat three-dimensional manifolds. He has obtained rigidity results for a variety of surfaces including the catenoid.

1. Compact minimal surfaces

Throughout this section $B^{n-1} \subset \mathbf{R}^{n+1}$ will be a compact immersed C^2 boundary of dimension $n - 1$, and M^n will be a smooth immersed minimal hypersurface in $\mathbf{R}^{n+1}$ with $\partial M = B$; that is, M is smooth in the interior and C^2 at the boundary. We will distinguish the $(n+1)$st direction, so we identify $\mathbf{R}^n$ with the hyperplane $\{x_{n+1} = 0\}$ in $\mathbf{R}^{n+1}$. The coordinates of a point in $\mathbf{R}^{n+1}$ will be denoted (x, x_{n+1}) where $x \in \mathbf{R}^n$. Let Ω be a bounded domain in $\mathbf{R}^n$ with C^2 boundary. If ν denotes the outward pointing unit normal to $\partial\Omega$ in $\mathbf{R}^n$, the mean curvature function H of $\partial\Omega$ is given by

$$H(x) = \sum_{i=1}^{n-1} \left(\nabla_{e_i} e_i\right) \cdot \nu(x),$$

where $x \in \partial\Omega$, $\{e_1, \cdots, e_{n-1}\}$ is a local orthonormal basis tangent to $\partial\Omega$ at points near x, and ∇ denotes the directional derivative in $\mathbf{R}^n$. Note that the boundary of the unit ball in $\mathbf{R}^n$ has *negative* mean curvature under our sign convention.

We will need to introduce some notation. Let p: $\mathbf{R}^{n+1} \to \mathbf{R}^n$ denote the projection map $p(x, x_{n+1}) = x$, and for $t \in \mathbf{R}$ let Π_t denote the hyperplane $\{x_{n+1} = t\}$ so that $\mathbf{R}^n \approx \Pi_0$. If $\Sigma \subset \mathbf{R}^{n+1}$ is any subset and $t \in \mathbf{R}$, we let Σ_{t^+} denote the portion of Σ on and above Π_t, that is,

$$\Sigma_{t^+} = \{(x, x_{n+1}): (x, x_{n+1}) \in \Sigma, x_{n+1} \geqslant t\}.$$

Similarly Σ_{t^-} denotes the points of Σ on and below Π_t. We will let $\Sigma^*_{t^+}$ denote the reflection of Σ_{t^+} across Π_t, that is,

$$\Sigma^*_{t^+} = \{(x, 2t - x_{n+1}): (x, x_{n+1}) \in \Sigma_{t^+}\}.$$

For any set $S \subset \mathbf{R}^{n+1}$, we say that S is a *graph* if the projection of S into $\mathbf{R}^n$ is one-to-one. If S is the closure of a C^2 submanifold, then we say that S has *locally bounded slope* if the tangent plane $T_\rho S$, for any interior point $\rho \in S$, does not contain the unit vertical vector $v = (0, 1)$. Finally, if $A, B \subset \mathbf{R}^{n+1}$ are subsets, we say that $A \geqslant B$ provided for every $x \in \mathbf{R}^n$ for which $p^{-1}\{x\} \cap A \neq 0$ and $p^{-1}\{x\} \cap B \neq 0$ we have all points of $p^{-1}\{x\} \cap A$ lying above all points of $p^{-1}\{x\} \cap B$; that is, if $(x, x_{n+1}) \in p^{-1}\{x\} \cap A$ and $(x, y_{n+1}) \in p^{-1}\{x\} \cap B$, then $x_{n+1} \geqslant y_{n+1}$. We now state the main result of this section.

Theorem 1. *Suppose $B^{n-1} \subset \mathbf{R}^{n+1}$ is a compact immersed C^2 boundary, not necessarily connected. Suppose $\Omega \subset \mathbf{R}^n$ is a bounded C^2 domain whose boundary has nonpositive mean curvature at every point. Assume that B satisfies:* (i) $B \subset (\partial\Omega) \times \mathbf{R}$, (ii) *$B_{0^+}$ is a graph with locally bounded slope, and* (iii) $B^*_{0^+} \geqslant B_{0^-}$. *If M is any immersed minimal hypersurface with $\partial M = B$ and with all interior points of M contained in $\Omega \times \mathbf{R}$, then M satisfies:* (i) *M_{0^+} is a graph with locally bounded slope, and* (ii) $M^*_{0^+} \geqslant M_{0^-}$.

Remark 1. The hypothesis in Theorem 1 that all interior points of M lie in $\Omega \times \mathbf{R}$ is not serious because if any interior point ρ lies on $(\partial\Omega) \times \mathbf{R}$, one can apply the maximum principle (see Lemma 1) to assert that a neighborhood of ρ lies in $(\partial\Omega) \times \mathbf{R}$. Therefore the set of such ρ is open and closed and hence consists of certain connected components of M. After removing these, one can apply Theorem 1 to the remaining components. Notice that we do not require M to be either connected or embedded in Theorem 1.

Before discussing the proof of Theorem 1, we give a few consequences. We state the following known result to put Theorem 1 in context for the reader.

Corollary 1. *Suppose $\partial\Omega$ has nonpositive mean curvature, and B^{n-1} is a C^2 boundary contained in $(\partial\Omega) \times \mathbf{R}$ which is a graph with bounded slope. Then any smooth immersed minimal hypersurface M with $\partial M = B$ is the graph of a smooth function defined on $\overline{\Omega}$.*

The corollary follows from Theorem 1 by choosing coordinates so that $B \subset \{x_{n+1} > 0\}$ so that $B_{0^+} = B$ and observing that by Theorem 1 all of M must be a graph with bounded slope.

It is generally false that a minimal surface spanning a boundary inherits the symmetries of its boundary. One such example is discussed below. The following theorem gives sufficient conditions under which minimal surfaces do inherit symmetries. An interesting feature of the result is that it yields symmetry for certain boundaries which span a multitude of minimal surfaces.

Theorem 2. *Suppose the hypotheses of Theorem* 1 *are satisfied and in addition it is true that* $B^*_{0^+} = B_{0^-}$. *If* M *is a smooth immersed minimal surface spanning* B *such that every interior point of* M *lies in* $\Omega \times \mathbf{R}$, *then in addition to the conclusions of Theorem* 1, M *satisfies* $M^*_{0^+} = M_{0^-}$. *Moreover, if* B *is embedded, then* M *is embedded.*

We can derive Theorem 2 from Theorem 1 by first noting that Theorem 1 can be applied from above or below (i.e., changing x_{n+1} to $-x_{n+1}$) to assert that both $M^*_{0^+} \geqslant M_{0^-}$ and $M^*_{0^-} \leqslant M_{0^+}$, or equivalently $M^*_{0^+} \leqslant M_{0^-}$. Now if M_1 is any component of M, and B_1 is a boundary component of M_1 which is contained in Π_0, then $M_1 = M = \overline{\Omega}$ and hence the theorem holds. Otherwise, let $\rho \in B_1 \cap \{x_{n+1} > 0\}$ and let $\rho^* \in B$ be the reflection of ρ through Π_0. Now in a neighborhood of ρ, M is a smooth graph with bounded gradient, and this is likewise so in a neighborhood of ρ^*. The orderings for $M^*_{0^+}$, M_{0^-} taken together then imply that a neighborhood of ρ^* in M_1 coincides with a neighborhood of ρ^* in M^*_1. Since M_1 is connected, it follows that $M^*_1 \subset M$. But M_1 was any component of M, and so we must have $M^* = M$. Finally, suppose B is embedded. It follows that M is embedded in a neighborhood of B. Notice that the hypothesis that $\partial\Omega$ have nonpositive mean curvature implies that $\partial\Omega$ is *connected*. Let Σ be the set of points of self-intersection of M. Since both M_{0^+} and M_{0^-} are graphs, it follows that $\Sigma \subset \Pi_0$. Since M is embedded near B, we have Σ compactly contained in Ω. Since Σ consists locally of intersection points of distinct minimal surfaces, it follows that Σ is an $(n-1)$-dimensional real analytic variety. Thus there exists a domain $\Omega_1 \subset \Pi_0$ with $\partial\Omega_1 \subset \Sigma$ and hence $\partial\Omega_1 \cap \partial\Omega = \varnothing$. Since $\partial\Omega$ is connected, it follows that Ω_1 is compactly contained in Ω. Now go to a regular point x of $\partial\Omega_1$, and observe that there are two pieces of surface $D_1, D_2 \subset M$ such that $T_x D_1$ is not vertical and $D_2 = D^*_1$. Such D_1, D_2 must exist because the boundary point lemma implies (see Lemma 1) that no two pieces of M passing through x can be vertical since M_{t^+} is embedded for $t > 0$. Therefore it follows that $p(M_{0^+})$ contains a neighborhood of x. Let $y \in \Omega_1$ and $y_{n+1} > 0$ such that $(y, y_{n+1}) \in M_{0^+}$. For $t < y_{n+1}$, let G be the component of (y, y_{n+1}) in M_{t^+}. Thus G is a smooth graph, and we assert that $p(G) \subset \Omega_1$. This is so because M_{0^+} is a graph, and $\partial\Omega_1 \subset M_{0^+}$ so that $(\partial\Omega_1 \times \mathbf{R}) \cap M_{t^+} = \varnothing$ for any $t > 0$. Thus we must have $(\partial G) \cap B = \varnothing$ and hence $\partial G \subset \Pi_t$. This contradicts the fact that x_{n+1} cannot have an interior maximum in G. This shows that M is embedded and completes the proof of Theorem 2.

We explicitly mention a couple of corollaries relating to the case of boundaries lying in parallel hyperplanes.

Corollary 2. *Suppose $B = B_1 \cup B_2$ where each B_i is connected and lies in a hyperplane P_i. Assume that P_1 and P_2 are parallel and that B is invariant under reflection through a hyperplane Π which is orthogonal to P_1, P_2. Assume moreover that each piece of B_i bounded by Π is a graph over Π with locally bounded slope. Then every smooth immersed minimal surface bounding B is embedded and invariant under reflection through Π. If M is a connected minimal surface spanning B, then the part of M on either side of Π is a graph over Π with locally bounded slope.*

To prove Corollary 2 from Theorem 2 one simply chooses coordinates so that $\Pi = \{(x, 0)\}$ and observes that B lies on the boundary of a suitably chosen cylinder of nonpositive mean curvature. The following result also follows directly.

Corollary 3. *If $B = B_1 \cup B_2$ where B_1, B_2 are spheres in parallel planes with the line l joining their centres being orthogonal to these planes, then any immersed minimal surface M spanning B is a hypersurface of revolution with axis l. In particular, M is a catenoid or a pair of plane disks.*

Remark 2. Results of the type of Corollaries 1 and 2 were first proven by M. Shiffman [14]. For $n = 2$ Shiffman proved that if B_1, B_2 are convex curves (resp. circles), then any minimal *annulus* spanning B intersects every intermediate plane in a convex curve (resp. circle). Although we are not able to get as delicate information as Shiffman, our result has the advantages that the dimension is arbitrary and especially that we make no topological assumption on M. A few years ago, W. Meeks [7] conjectured that the topological assumption can be removed in Shiffman's theorem. We believe it likely that this is the case. We can prove a partial result in this direction.

Corollary 4. *Assume $n = 2$, and $B = B_1 \cup B_2$ is the union of two C^2 Jordan curves in parallel planes. Assume that there are two distinct planes Π_1, Π_2 orthogonal to the planes of the B_i such that B is invariant by reflection through both Π_1 and Π_2 and such that both Π_1 and Π_2 divide B into pieces which are graphs with locally bounded slope over the dividing plane. If M is any connected immersed minimal surface spanning B, then M is topologically an annulus and is an embedded surface meeting each parallel plane between the planes of the B_i transversally in smooth Jordan curves.*

Proof. Let P be any plane intermediate and parallel to the planes of B_1, B_2. Since M is connected, $M \cap P \neq \varnothing$. By Corollary 2, $\Gamma = M \cap P$ is invariant under reflection through distinct lines $l_i = P \cap \Pi_i$ for $i = 1, 2$. Moreover, each

l_i divides Γ into graphs over l_i of localy bounded slope. Let $\Omega_1,\cdots,\Omega_r$ be the bounded components of $P \sim \Gamma$. Since each line orthogonal to l_1 meets Π in at most two points, it follows that each Ω_i is simply connected. Let $\{x_0\} = l_1 \cap l_2$ and observe that if $x_0 \notin \Omega_i$ for a given i, then $\Omega_i \cap l_1$ lies in a component of $\Omega_i \sim l_2$, and it is not possible for Ω_1 to be symmetric in both l_1 and l_2. Therefore $r = 1$ and $P \sim \Gamma$ has a single bounded component. It follows that P intersects M transversally because otherwise Γ has a point from which at least four arcs emanate. In such a case, $P \sim \Gamma$ would have more than one bounded component. This proves Corollary 4.

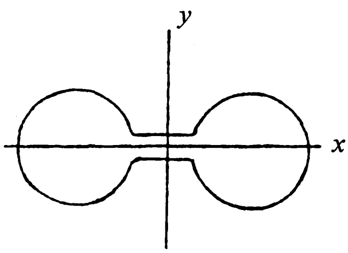

FIGURE 1

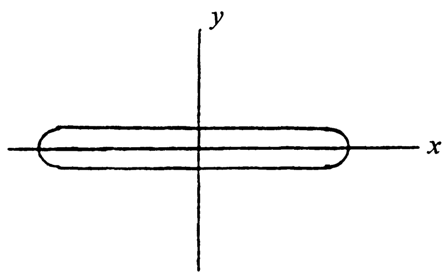

FIGURE 2

For the purpose of illustration we consider two examples. We consider the boundary Γ_1 in $\mathbf{R}^3$, which consists of two copies of Figure 1 in parallel planes. Similarly Γ_2 consists of two parallel copies of Figure 2. Observe that both Γ_1, Γ_2 are invariant under a pair of reflections in the vertical planes over the x and y axes in Figures 1, 2. We refer to these as the x and y reflections. Note that there are several minimal surfaces spanning Γ_1 which are not invariant under the y reflection. For example, we can connect the pair of plane disks spanning the left half of Γ_1 to the stable catenoid spanning the right half by a pair of thin bridges. On the other hand, Corollary 2 applies to show that any immersed minimal surface spanning Γ_1 is embedded and invariant under the x reflection. We can apply Corollary 4 to Γ_2 to assert that every connected minimal surface spanning Γ_2 is an embedded annulus. There will be at least two of these, one stable and one unstable, if we take the parallel planes close together.

We devote the remainder of this section to proving Theorem 1. The proof is a suitable version of the reflecion method of A. D. Alexandrov [1] and, as such, is based essentially of the Hopf maximum principle. We state a well-known lemma which summarizes the versions of the maximum principle which we require.

Lemma 1. *The following two assertions hold.*

(a) (*Boundary point lemma*) *Suppose M_1, M_2 are C^2 hypersurfaces with boundaries B_1, B_2. Suppose* 0 *is an interior point of both B_1 and B_2, and suppose the tangent planes of both M_1, M_2 and B_1, B_2 agree at* 0, *that is, suppose $T_0M_1 = T_0M_2$, $T_0B_1 = T_0B_2$. Assume that $T_0M_1 = \{x_{n+1} = 0\}$ so that both M_1, M_2 are given graphically near* 0. *Let H_1, H_2 be the mean curvature functions of M_1, M_2 computed with respect to the upward pointing normal. If $H_1 \leqslant 0$ and $H_2 \geqslant 0$ near* 0, *then it is not true that $M_1 \geqslant M_2$ in a neighborhood of* 0 *unless $M_1 = M_2$ in this neighborhood.*

(b) (*Interior maximum principle*) *Suppose* 0 *is an interior point of both M_1, M_2, and suppose $T_0M_1 = T_0M_2 = \{x_{n+1} = 0\}$. If $H_1 \leqslant 0$ and $H_2 \geqslant 0$ near* 0, *then it is not true that $M_1 \geqslant M_2$ near* 0 *unless $M_1 = M_2$ in a neighborhood of* 0.

Proof. Observe that if M_1, M_2 are the graphs of f, g respectively, then the hypotheses $H_1 \leqslant 0$, $H_2 \geqslant 0$ imply

$$\sum_{i,j=1}^{n} \left(\delta_{ij} - \frac{f_i f_j}{1 + |\nabla f|^2} \right) f_{x_i x_j} \leqslant 0,$$

$$\sum_{i,j=1}^{n} \left(\delta_{ij} - \frac{g_i g_j}{1 + |\nabla g|^2} \right) g_{x_i x_j} \geqslant 0.$$

Setting $u = f - g$, one then observes that u satisfies

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} \leqslant 0,$$

where

$$a_{ij} = \delta_{ij} - \frac{f_i f_j}{1 + |\nabla f|^2},$$

$$b_i = - \sum_{j,k=1}^{n} \left(\int_0^1 \frac{\partial \beta_{jk}}{\partial p_i} (\nabla f + t(\nabla g - \nabla f))\, dt \right) g_{x_j x_k},$$

$$\beta_{jk}(p_1, \cdots, p_n) = \frac{p_j p_k}{1 + |p|^2}.$$

Since f and g are C^2, one can apply the usual maximum principle; see [3.3.2] to prove Lemma 1.

Proof of Theorem 1. First observe that the hypotheses on B imply that for $t \geqslant 0$ we have $B^*_{t^+} \geqslant B_{t^-}$. We let $\bar{t} = \max\{x_{n+1}\colon (x, x_{n+1}) \in B$ for some $x \in \mathbf{R}^n\}$. Note that if $\bar{t} \leqslant 0$, then M_{0^+} either has no interior points or is a region in Π_0 by Lemma 1. In any case, the conclusion of Theorem 1 is true. Thus we may assume $\bar{t} > 0$. Define T to be the set of $t \in [0, \bar{t}]$ such that M_{t^+} is a graph with locally bounded slope and $M_{t^+}^* \geqslant M_{t^-}$. It is clear that $\bar{t} \in T$. The proof of Theorem 1 will be finished if we can show that $0 \in T$. This will be accomplished by showing that T is an open and closed subset of $[0, \bar{t}]$. Notice that if $t_1 \in T$ and $t_2 \in (t_1, \bar{t}]$, then it follows that $t_2 \in T$. Thus T is an interval.

We first show that T is closed. Assume $(t, \bar{t}] \subset T$ for some $t \in [0, \bar{t})$, and we must show $t \in T$. To show that M_{t^+} is a graph we observe that if both (x, x_{n+1}) and (x, y_{n+1}) belong to M_{t^+} with $x_{n+1} > y_{n+1}$, then we must have $y_{n+1} = t$ since $s \in T$ for $s > t$. By the hypothesis on B we must have $x \in \Omega$ so that (x, t) is an interior point of M. But recall that the slope of M at (x, x_{n+1}) is finite so that a neighborhood of (x, x_{n+1}) can be represented as a graph over a neighborhood of (x, t) in Π_t. This implies that for $(y, t) \in \Pi_t$ sufficiently close to (x, t), $p^{-1}\{y\} \cap M_{s^+}$ contains a point near (x, x_{n+1}) for s slightly larger than t. Since $s \in T$, this is the unique point of $p^{-1}\{y\} \cap M_s$, and hence it follows that a neighborhood of (x, t) in M lies below Π_t. Thus by Lemma 1 and a continuation argument, a component of M is contained in Π_t, and hence a component of B is equal to $((\partial\Omega) \times \mathbf{R}) \cap \Pi_t$. This contradicts the hypothesis on B since we are assuming $\bar{t} > 0$. Therefore M_{t^+} is a graph, and the fact that it has locally bounded slope follows immediately. The fact that $M_{t^+}^* \geqslant M_{t^-}$ follows because the contrary would mean that there are points $(x, x_{n+1}) \in M_{t^+}$, $(x, y_{n+1}) \in M_{t^-}$ with $2t - x_{n+1} < y_{n+1}$. It follows that $x_{n+1} > t$ and hence for $s > t$ sufficiently close to t we contradict $M_{s^+}^* \geqslant M_{s^-}$. This completes the proof that T is closed.

To prove that T is an open subset of $[0, \bar{t}]$, we let $t > 0$ with $t \in T$ and show that a neighborhood of t is contained in T. To carry out this plan we first show that every point $p = (x, t) \in M \cap \Pi_t$ has the property that $v \notin T_pM$ where $v = (0, 1)$ is the unit vertical vector. To see this, first observe that for $p \in B \cap \Pi_t$, if $v \in T_pM$, then we must have $T_pM = T_p((\partial\Omega) \times \mathbf{R})$ since $v \notin T_pB$, and an application of Lemma 1 shows that a neighborhood of p in M is contained in $(\partial\Omega \times \mathbf{R})$ contrary to assumption. Therefore $v \notin T_pM$ for $p \in B \cap \Pi_t$. If $p \in M \cap \Pi_t$ is an interior point of M, we restrict attention to a simple embedded disk D in M containing p. (There might be several if p is a point of self-intersection.) Since $t \in T$, we have $D_{t^+}^* \geqslant D_{t^-}$, but if T_pD contains v, then the half disks $D_{t^+}^*$, D_{t^-} meet tangentially along a smooth boundary at p, and hence by Lemma 1 we have $D_{t^+}^* = D_{t^-}$ in a neighborhood of p. This would imply that the component of M containing D, called M_1, is invariant by reflection through Π_t, and this clearly contradicts the assumption on B. Therefore we must have $v \notin T_pD$ for any smooth embedded piece D of M containing p. We now show that there are no points of self-intersection of M lying in Π_t. Suppose on the contrary that D, $\hat{D}$ are smooth embedded disks in M which both contain p. We then have $D_{t^+}^* \geqslant \hat{D}_{t^-}$ and $\hat{D}_{t^+}^* \geqslant D_{t^-}$ which is the same as saying $\hat{D}_{t^-}^* \geqslant D_{t^+}$, $\hat{D}_{t^+}^* \geqslant D_{t^-}$ which implies $\hat{D}^* \geqslant D$. Since $p \in \hat{D}^* \cap D$ we can thus apply Lemma 1 to conclude $\hat{D}^* = D$. Thus if M_1 is the union of the components of M containing $D \cup \hat{D}$, then we must have $M_1^* = M_1$ again contradicting the assumption on B. We have thus shown that every point of $M \cap \Pi_t$ is a point of embedding of M and a point of finite slope. Therefore we can find a sufficiently small positive number ε_0 such that for $\varepsilon \in (0, \varepsilon_0]$, the set $U_\varepsilon = M \cap \{|x_{n+1} - t| < \varepsilon\}$ is a graph with bounded slope over a subset of $\mathbf{R}^n$. If we take $s \in (0, \bar{t}]$ with $|s - t| < \varepsilon_0/2$ and denote by ρ_s reflection in Π_s, then we see that $\rho_s(U_{\varepsilon_0/2}) \subset U_{\varepsilon_0}$ and so that $\rho_s(M_{s^+} \cap U_{\varepsilon_0/2}) \geqslant M_{s^-}$. On the other hand, $M_{s^+} \sim U_{\varepsilon_0/2}$ is a compact set with the property that its image under ρ_t is disjoint from M_{t^-}. Therefore by continuity if s is sufficiently close to t, we will have $\rho_s(M_{s^+} \sim U_{\varepsilon_0/2}) \geqslant M_{s^-}$. Thus we have $M_{s^+}^* \geqslant M_{s^-}$ for s near t, as well as the fact that M_{s^+} is a graph with locally bounded slope. This completes the proof that T is open, and finishes the proof of Theorem 1.

2. Preliminaries on complete minimal surfaces

In the next section we will extend some of our uniqueness results to the case of complete minimal hypersurfaces. These extensions will apply only to hypersurfaces which are well behaved at infinity. We now define the precise class we will consider.

Definition. A complete minimal immersion $M^n \subset \mathbf{R}^{n+1}$ is said to be *regular at infinity* if there is a compact subset $K \subset M$ such that $M \sim K$ consists of r components $M_1, \cdots, M_r$ such that each M_i is the graph of a function u_i with bounded slope over the exterior of a bounded region in some hyperplane Π_i. Moreover, if $x_1, \cdots, x_n$ are coordinates in Π_i, we require the u_i have the following asymptotic behaviour for $|x|$ large and $n = 2$:

$$u_i(x) = a \log|x| + b + \frac{c_1 x_1}{|x|^2} + \frac{c_2 x_2}{|x|^2} + O\left(|x|^{-2}\right).$$

While for $n \geqslant 3$ we require

$$u_i(x) = b + a|x|^{2-n} + \sum_{j=1}^{n} c_j x_j |x|^{-n} + O\left(|x|^{-n}\right)$$

for constants a, b, c_j depending on i. The expression $O(|x|^{-n})$ in the above equations is used to indicate a term which is bounded in absolute value by a constant times $|x|^{-n}$ for $|x|$ large. We refer to the M_i as the *ends* of M.

Notice that our definition of regular at infinity requires that each M_i be embedded but it does not prohibit two different M_i's from intersecting. We first analyze the case $n = 2$ in the following.

Proposition 1. *A complete minimal immersion $M^2 \subset \mathbf{R}^3$ is regular at infinity if and only if M has finite total curvature and each end of M is embedded.*

Proof. That regular at infinity implies finite total curvature and embedded ends follows from the fact that $|K| = O(|x|^{-4})$ on each M_i which can be seen directly from the given asymptotic expansion of u_i (we assume the expression can be differentiated).

To prove the converse we need a few facts about finite total curvature surfaces which are due to R. Osserman [10, Chapter 9]. First we need the fact that each infinity is conformally a punctured disk, the Gauss map extends to infinity, and the surface M_i is given by

$$x_j(u, v) = \operatorname{Re} \int^{(u,v)} \phi_j(w)\, dw, \qquad j = 1, 2, 3,$$

where $w = u + \sqrt{-1}\, v \in D \sim \{(0,0)\}$, and ϕ_j are holomorphic in $D \backslash \{(0,0)\}$ with at most poles at $(0,0)$. The ϕ_j satisfy $\Sigma_{j=1}^{3} \phi_j^2 = 0$. Having chosen coordinates in $\mathbf{R}^3$ so that the limiting normal vector at infinity on M_i is $(0,0,1)$, we deduce that ϕ_3 has a milder pole than ϕ_1 and ϕ_2. The embeddedness of M_i implies that both ϕ_1, ϕ_2 have poles of order 2, and hence ϕ_3 is either regular or has a pole of order 1. One checks from the relation on ϕ_1, ϕ_2, ϕ_3 and the condition that the x_j be single valued that ϕ_1, ϕ_2 have no w^{-1} term in their power series. Thus we have

$$\phi_1(w) = \alpha w^{-2} + O(1), \qquad \phi_2(w) = \beta w^{-2} + O(1),$$
$$\phi_3(w) = \gamma w^{-1} + \tau + O(|w|), \qquad \alpha^2 + \beta^2 = 0, \qquad \gamma \in \mathbf{R}.$$

By changing coordinates in the $x_1 x_2$-plane we can assume α is real and $\beta = \sqrt{-1}$. Upon integration we conclude

$$x_1(u, v) = -\alpha \frac{u}{|w|^2} + O(|w|), \qquad x_2 = -\alpha \frac{v}{|w|^2} + O(|w|),$$
$$x_3 = \gamma \log|w| + \tau_1 u - \tau_2 v + O\left(|w|^2\right),$$

where $\tau = \tau_1 + \sqrt{-1}\, \tau_2$. From these expressions we observe

$$u = -\alpha^{-1} \frac{x_1}{|x|^2} + O\left(|x|^{-3}\right), \qquad v = -\alpha^{-1} \frac{x_2}{|x|^2} + O\left(|x|^{-3}\right).$$

Thus we have $|w| = |\alpha|^{-1} |x|^{-1} + O(|x|^{-3})$, where we are using $x = (x_1, x_2)$. Substituting this information into the expression for x_3 we get

$$x_3 = a \log|x| + b + \frac{c_1 x_1}{|x|^2} + \frac{c_2 x_2}{|x|^2} + O\left(|x|^{-2}\right)$$

for suitable constants a, b, c_1, c_2. This shows that M is regular at infinity and completes the proof of Proposition 1.

The general principle concerning minimal immersions is that they should either be very pathological at infinity or be regular at infinity. A very strong characterization of this type has been proven by R. Osserman [11].

Proposition 2 (*Osserman*). *A complete minimal surface M in $\mathbf{R}^3$ either has finite total curvature or the normals to M assume all values on the sphere infinitely often with the exception of at most a set of logarithmic capacity zero.*

For $n \geqslant 3$, it is not possible for such a strong result to hold. We can prove a weaker version of Proposition 2 for $n > 2$, which states that the asymptotic expansions required of a hypersurface to be regular at infinity follow from the condition that each end be a graph of bounded slope.

Proposition 3. *Assume $n \geqslant 3$, and $M^n \subset \mathbf{R}^{n+1}$ is a minimal immersion with the property that $M \sim K$, for some compact K, is a union of $M_1, \cdots, M_r$ where each M_i is a graph of bounded slope over the exterior of a bounded region in a hyperplane P_i. Then M is regular at infinity.*

Proof. We work with a given M_i and show that the asymptotic expansion is valid over some plane Π_i which may differ from P_i. The first step is to show that the tangent plane to M_i has a limit at infinity. Suppose $x_{n+1} = v(x)$, where $x = (x_1, \cdots, x_n) \in P_i$ is the graphical representation of M_i defined on $P_i \sim \Omega$ for some bounded open set $\Omega \subset P_i$. We will show that for each $k = 1, \cdots, n$ the function $\partial v / \partial x_k$ has a limit at infinity. We set $w(x) = \partial v / \partial x_k$ and recall that w satisfies the equation

$$Lw = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial w}{\partial x_j}\right) = 0,$$

$$a_{ij} = \left(1 + |\nabla v|^2\right)^{-1/2}\left(\delta_{ij} - \frac{v_i v_j}{1 + |\nabla v|^2}\right).$$

Since $|\nabla v|$ is assumed to be bounded, we see that L is uniformly elliptic on $P_i \sim \Omega$. For convenience we extend both L and w smoothly inside Ω so that L remains elliptic and

$$Lw = f, \qquad f = 0 \quad \text{outside } \Omega.$$

We now recall a result of Littman, Stampachia, and Weinberger [6, Theorem 7.1] which asserts the existence of a Green's function $G(x, y)$ for L on P_i satisfying

$$K^{-1}|x - y|^{2-n} \leqslant G(x, y) \leqslant K|x - y|^{2-n}$$

for a suitable constant K and all $x, y \in P_i$. Since f has compact support, we can define

$$w_1(x) = \int_{\mathbf{R}^n} G(x, y) f(y)\, dy,$$

and we have $Lw_1 = f$ and $w_1 = O(|x|^{2-n})$. Therefore $w - w_1$ is a solution of $Lu = 0$, which is bounded and hence constant by the DeGiorgi-Nash theory. Thus we have shown $(\partial v/\partial x_k)(x) = \alpha_k + O(|x|^{2-n})$ for $k = 1,\cdots,n$ and suitable constants α_k. Hence the upward normal vector to M_i has the limit $(1 + |\alpha|^2)^{-1/2}(-\alpha, 1)$ at infinity, and we let Π_i be the hyperplane orthogonal to this limit. It follows that after removing a compact subset from M_i if necessary we can express M_i as the graph of a function u defined in the exterior of a bounded open subset $\mathcal{O}$ of Π_i. Moreover, we have $\lim_{|x|\to\infty} |\nabla u| = 0$ since Π_i is the limiting tangent plane to M_i at infinity. We choose coordinates $(x_1,\cdots,x_n) \in \Pi_i$ and show that u has the desired expansion. We extend u smoothly inside $\mathcal{O}$ and argue as above that $w = \partial u/\partial x_k$ satisfies

$$Lw = f, \qquad f = 0 \quad \text{outside } \mathcal{O}$$

for L a uniformly elliptic operator on Π_i. Let $w_1(x)$ be the function considered above, and observe that both w and w_1 satisfy the same equation and have limit zero at infinity. Therefore $w \equiv w_1$ and we have shown $|\nabla u|(x) = O(|x|^{2-n})$. This implies, by integration along rays, that u grows at most logarithmically (in fact, $|u| = O(1)$ for $n > 3$). Thus we can repeat the above argument using the fact that u satisfies a uniformly elliptic divergence form equation to show $u(x) = b + O(|x|^{2-n})$ for some constant b. It is now a simple matter to apply elliptic theory to assert $|\nabla u| = O(|x|^{1-n})$, $|\nabla\nabla u| = O(|x|^{-n})$. We can now derive the expansion by writing the minimal surface equation

$$\Delta u = f_1, \qquad f_1 = \sum_{j,k}\left[\frac{u_{x^j}u_{x^k}}{1 + |\nabla u|^2}\right]u_{x^jx^k} + f_2,$$
$$f_2 \equiv 0 \quad \text{outside } \mathcal{O}.$$

We have the bound $f_1 = O(|x|^{2-3n})$ for $|x|$ large, and hence we can show

$$u(x) = b - (n-2)^{-1}\omega_n^{-1}\int_{\mathbf{R}^n} |x-y|^{2-n} f_1(y)\,dy.$$

Now observe that for $|y| \leq \frac{1}{2}|x|$ we have

$$|x-y|^{2-n} = |x|^{2-n} - (n-2)|x|^{-n}x\cdot y + O\left(|x|^{-n}|y|^2\right).$$

Using this together with the decay rate on f_1 one can then obtain

$$u(x) = b + a|x|^{2-n} + \sum_{j=1}^{n} c_j x_j |x|^{-n} + O(|x|^{-n}),$$

$$a = -(n-2)^{-1}\omega_n^{-1}\int_{\mathbf{R}^n} f_1(y)\,dy,$$

$$c_j = \omega_n^{-1}\int_{\mathbf{R}^n} y_j f_1(y)\,dy.$$

This completes the proof of Proposition 3.

Remark 3. So far as the author knows, the only known nonplanar example of a complete minimal hypersurface in $\mathbf{R}^{n+1}$, for $n > 2$, which is regular at infinity, is the rotationally symmetric higher dimensional catenoid. In the next section we show that it is the only such hypersurface with *two* ends. We certainly believe that there are many with more than two ends. For $n = 2$, of course, there is hope to construct such examples by complex analytic methods. While there are a number of examples of finite total curvature surfaces known, some being regular at infinity, it is not known whether an embedded example exists besides the catenoid and the plane.

3. A uniqueness theorem for complete minimal surfaces

In this section we will apply the reflection method of §1 to complete immersions which are regular at infinity. Our results apply to immersions with two ends. The proof is most delicate in the case $n = 2$, so we prove the following preliminary lemma for that case.

Lemma 2. *Let $M^2 \subset \mathbf{R}^3$ be a complete minimal immersion which is regular at infinity. If M has two ends, then either both of the ends are bounded (i.e., $a = 0$ in the expansions), or the ends are parallel. In case the ends are parallel, we can expand both in the same coordinate system, and if $a^{(1)}$ and $a^{(2)}$ denote the coefficients of $\log|x|$ on the ends, we have $a^{(1)} + a^{(2)} = 0$, and neither $a^{(1)}$ nor $a^{(2)}$ is zero.*

Proof. Assume the ends are given by

$$x_3 = u_1(x) = a_1 \log|x| + O(1), \quad y_3 = u_2(y) = a_2 \log|y| + O(1),$$

where $X = AY$, $X = {}^t(x_1, x_2, x_3)$, $Y = {}^t(y_1, y_2, y_3)$, and $A = (\alpha_{ij})$ is an orthogonal matrix. Let $\beta = (\beta_1, \beta_2, \beta_3)$ be any vectors, and consider the function

$$h = \beta X = \sum_{i=1}^{3} \beta_i x_i.$$

Since M is minimal, h is a harmonic function on M. For any R large let M_R be the compact part of M bounded by the vertical cylinders of radius R over the limiting planes at the two infinities. Thus we have $\partial M_R = C_R^1 \cup C_R^2$ where $C_R^1 = \{(x, u_1(x)): |x| = R\}$, $C_R^2 = \{(y, u_2(y)): |y| = R\}$. An easy calculation shows that the normal vector n_i to C_R^i is given by

$$n_1 = R^{-1}(x_1, x_2, a_1) + O(R^{-2}), \quad n_2 = R^{-1}(y_1, y_2, a_2) + O(R^{-2}).$$

Since h is harmonic, we have $\sum_{i=1}^2 \int_{C_R^i} (\partial h/\partial n_i)\, ds = 0$ for any R. Since $h = \beta X = \beta A Y$, we let $R \to \infty$ to obtain

$$\beta_3 a_1 + \sum_{i=1}^{3} \beta_i \alpha_{i3} a_2 = 0,$$

for any $\beta \in \mathbf{R}^3$. First assume $\beta_3 = 0$, and allow β_1, β_2 to be arbitrary to conclude $\alpha_{13} a_2 = 0 = \alpha_{23} a_2$. If $a_2 = 0$, then from above we also have $a_1 = 0$, and both ends are bounded. The other possibility is $\alpha_{13} = 0 = \alpha_{23}$. Since A is an orthogonal matrix, we would then have $\alpha_{31} = 0 = \alpha_{32}$ and $\alpha_{33} = \pm 1$. Hence we have $x_3 = \pm y_3$, and we have shown that the ends are parallel. We can thus take $y_3 = x_3$ and $X = Y$, and the linear equation above becomes $a_1 + a_2 = 0$. Since M is conformally a surface with punctures, x_3 cannot be bounded, and hence neither a_1 nor a_2 can be zero. This completes the proof of Lemma 2.

Theorem 3. *The only complete minimal immersions $M^n \subset \mathbf{R}^{n+1}$, which are regular at infinity and have two ends, are the catenoids and pairs of planes.*

Proof. Let Π_1, Π_2 denote the limiting tangent planes at the two infinities. We distinguish two cases depending on whether Π_1 and Π_2 are parallel or transverse to one another. The first case we consider is:

Case 1. Π_1 is not parallel to Π_2. In this case we can apply Lemma 2 to assert that both of the ends are bounded. We thus have expansions for the ends

$$x_{n+1} = b_1 + O\left(|x|^{-1}\right), \qquad y_{n+1} = b_2 + O\left(|y|^{-1}\right),$$

in suitable coordinate systems. By a change of coordinates we can assume $b_1 = 0 = b_2$ and $x_i = y_i$ for $i = 1, \cdots, n-1$. Thus if $e_1, \cdots, e_{n+1}$ are the orthonormal basis vectors corresponding to the x coordinate system, we have the representations for Π_1 and Π_2 given by

$$\Pi_1 = \{X \cdot e_{n+1} = 0\}, \qquad \Pi_2 = \{X \cdot (\alpha e_n + \beta e_{n+1}) = 0\},$$

where $\alpha^2 + \beta^2 = 1$, $\alpha \neq 0$. Now define a vector v to be

$$v = (\alpha e_n + \beta e_{n+1} + e_{n+1}) / \left(\alpha^2 + (\beta + 1)^2\right)^{1/2},$$

and let $P_t = \{X \cdot v = t\}$ for $t \in \mathbf{R}$. Observe that $\Pi_1 \cup \Pi_2$ is invariant by reflection through P_0 and that $(\Pi_1 \cup \Pi_2)_{0^+} = \{X \in \Pi_1 \cup \Pi_2 \colon X \cdot v \geqslant 0\}$ is a graph over P_0. Since M is pointwise close to $\Pi_1 \cup \Pi_2$ at infinity, we can show that M is invariant under reflection through P_0 and that M_{0^+} is a graph over P_0. To see this, let $z_1, \cdots, z_n, z_{n+1}$ be a Euclidean coordinate system corresponding to P_0, i.e., $z_{n+1} = X \cdot v$. For R large, let C_R be the cylinder $C_R = \{(z, z_{n+1}) \colon |z| = R\}$ perpendicular to P_0. Let $M^R = M \cap \{(z, z_{n+1}) \colon |z| \leqslant R\}$ so that $\partial M^R \subset C_R$. Now for any $t > 0$ we can choose R sufficiently large so that the hypotheses of Theorem 1 are satisfied by M^R relative to the plane P_t. Thus we can apply Theorem 1 to assert that $M_{t^+}^{R*} \geqslant M_{t^-}^R$ and that $M_{t^+}^R$ is a graph over P_t. Letting t tend to zero we conclude $M_{0^+}^* \geqslant M_{0^-}$. Applying a similar argument from below we conclude that $M^* = M$ where * denotes reflection through P_0.

To finish the proof observe that the above argument could have been applied with the vector

$$w = (\alpha e_n + \beta e_{n+1} - e_{n+1}) / \left(\alpha^2 + (\beta - 1)^2\right)^{1/2}$$

replacing v, and the planes $Q_t = \{X \cdot w = t\}$ in place of P_t. We therefore can assert that M is also invariant under reflection through Q_0 and that M is embedded outside of Q_0. It follows that the self-intersection set S of M coincides with the $(n-1)$-plane $P_0 \cap Q_0$. We now see that $M \sim S$ is disjoint from both P_0 and Q_0 because for example if $(z, 0) \in S \subset M$ and also $(z, z_{n+1}) \in M$ for some $z_{n+1} > 0$, then we have contradicted the fact that $M \cap \{z_{n+1} \geqslant 0\}$ is a graph. (Note that P_0 and Q_0 are *orthogonal* planes.) Therefore $M \sim S$ consists of at least four components lying in the four "quadrants" of $\mathbf{R}^{n+1} \sim (P_0 \cup Q_0)$. The graphical property of $M \sim S$ allows us to conclude that $M \sim S$ consists of exactly four components which are arranged so that the union of components in opposite quadrants join together to form embedded minimal surfaces M_1, M_2 each having only one end asymptotic to Π_1, Π_2 respectively. The maximum principle now implies $M_i = \Pi_i$ and hence $M = \Pi_1 \cup \Pi_2$, a pair of planes. This completes the analysis of Case 1.

Case 2. Π_1 and Π_2 are parallel planes. We assume that M is connected, for otherwise the maximum principle implies that $M = \Pi_1 \cup \Pi_2$. We also assume that Π_1 and Π_2 are parallel to $\{x_{n+1} = 0\}$. We will show that M is a hypersurface of revolution and hence a catenoid. We first handle the case $n = 2$. We then have by Lemma 2

$$u_i(x) = a^{(i)} \log|x| + b^{(i)} = O\left(|x|^{-1}\right),$$

where $a^{(1)} + a^{(2)} = 0$, and neither $a^{(i)}$ is zero. If $u_1 < u_2$, we write $a = a^{(2)} > 0$ so that $a^{(1)} = -a$. By translation of the x_3-coordinate we may assume $b^{(1)} + b^{(2)} = 0$, so if we let $b = b^{(2)}$, the expansions become

$$u_2(x) = a\log|x| + b + O\left(|x|^{-1}\right),$$

$$u_1(x) - a\log|x| - b + O\left(|x|^{-1}\right).$$

We now observe that if $t > 0$, then for $|x|$ sufficiently large we have $2t - u_2(x) > u_1(x)$ and hence choosing $V = \{|x| < R\} \times \mathbf{R}$, $B = M \cap \partial V$ we have that for R large $B^*_{t^+} \geqslant B_{t^-}$, and B_{t^+} is a graph with bounded slope. Thus by Theorem 1 we have $(M \cap V)^*_{t^+} \geqslant (M \cap V)_{t^-}$. Since R is arbitrarily large, it follows that for any $t > 0$ we have $M^*_{t^+} \geqslant M_{t^-}$. Hence it follows that $M^*_{0^+} \geqslant M_{0^-}$. If we had applied the same argument from below (i.e., change x_{n+1} to $-x_{n+1}$) we would have $M^*_{0^-} \leqslant M_{0^+}$ which is equivalent to $M^*_{0^+} \leqslant M_{0^-}$. This immediately implies that $u_1 = -u_2$ in an open set, and hence by a continuation argument we get $M^* = M$. For $n > 2$, the above argument simplifies. We again translate x_{n+1} so that $b^{(2)} = b = -b^{(1)}$, and thus we have $u_2(x) = b + O(|x|^{2-n})$, $u_1(x) = -b + O(|x|^{2-n})$. A similar argument then shows $M^* = M$ where $*$ denotes reflection in $\{x_{n+1} = 0\}$.

We now show that M is rotationally symmetric. The argument is slightly different for $n = 2$ and for $n > 2$, so we first do the case $n = 2$. We have shown that $u_1 = -u_2$ while

$$u_2(x) = a\log|x| + b + \frac{c_1 x_1}{|x|^2} + \frac{c_2 x_2}{|x|^2} + O\left(|x|^{-2}\right),$$

for constants $a > 0$, b, c_1, c_2. We must locate the axis of symmetry, so we observe that if we set $x_1 = y_1 + \alpha_1$, $x_2 = y_2 + \alpha_2$, then the expansion for u_2 in terms of $y = (y_1, y_2)$ becomes

$$u_2(y) = a\log|y| + b + \frac{\tilde{c}_1 y_1}{|y|^2} + \frac{\tilde{c}_2 y_2}{|y|^2} + O\left(|y|^{-2}\right),$$

where $\tilde{c}_i = c_i + a\alpha_i$ for $i = 1, 2$. Thus if we choose $\alpha_i = -a^{-1}c_i$, and relabel y again as x, we may assume $u_2(x) = a\log|x| + b + O(|x|^{-2})$, $u_1(x) = -u_2(x)$. We now show that in these coordinates the x_3-axis is an axis of symmetry for M. It suffices to show that M is invariant under reflection in every plane $\{\beta_1 x_1 + \beta_2 x_2 = 0\}$. Since the expansions of u_1, u_2 are invariant under a rotation of x_1-, x_2-coordinates, it is sufficient to show that M is invariant under reflection in the plane $\{x_1 = 0\}$. This we now do by choosing a large number Λ and writing

$$M \cap \left\{|x_3| = \Lambda\right\} = B^1 \cup B^2,$$

where $B^1 = M_1 \cap \{x_3 = -\Lambda\}$, and $B^2 = M_2 \cap \{x_3 = \Lambda\}$. For $t \in \mathbf{R}$ we let $\Pi_t = \{x_1 = t\}$, and we wish to show that M is invariant under reflection through Π_0. Let $t > 0$ be given and let S_{t^+}, S_{t^-} be as in §1 with the x_1 and x_3 coordinates interchanged. We now show that for Λ sufficiently large (depending on t) we have $B^i_{t^+}$ is a graph with bounded slope over Π_0 and that $B^{i*}_{t^+} \geqslant B^i_{t^-}$ for $i = 1, 2$. We do the analysis only for B^2 as a similar argument works for B^1. First observe that from the expansion for u_2 we have

$$\frac{\partial u_2}{\partial x_1} = \frac{a x_1}{|x|^2} + O\left(|x|^{-3}\right).$$

This implies that for $x_1 \geqslant t$ and $|x|$ sufficiently large (depending on t) we have $\partial u_2/\partial x_1 > 0$. Since for Λ large every point of B^2 has $|x|$ large, we see immediately that $B^2_{t^+}$ is a graph over Π_0 for Λ large. The fact that $B^2_{t^+}$ has bounded slope over Π_0, follows from the fact that its normal vector η in the plane $\{x_3 = \Lambda\}$ is

$$\eta = \left(\frac{a x_1}{|x|^2} + O\left(|x|^{-3}\right), \frac{a x_2}{|x|^2} + O\left(|x|^{-3}\right), 0\right),$$

and its first coordinate is nonzero for $x_1 \geqslant t$ and Λ large. To see that $B^{2*}_{t^+} \geqslant B^2_{t^-}$ for Λ large, observe that on B^2 we have

$$\log|x| + O\left(|x|^{-2}\right) = a^{-1}(\Lambda - b)$$

and hence $|x| e^{O(|x|^{-2})} = R$ for a large R. Since $e^{O(|x|^{-2})} = 1 + O(|x|^{-2})$, it follows that $|x| = R + O(|x|^{-1})$, and hence if $|x|$ is large, B^2 is close in distance to a plane circle. But note that if C is the circle of radius R centred at the origin in $\{x_3 = \Lambda\}$, we have

$$\operatorname{dist}\left(C^*_{t^+}, C_{t^-/2}\right) \geqslant \varepsilon(t),$$

where $\varepsilon(t) > 0$ is a number depending only on t. It follows that if Λ is sufficiently large, we will have $B^{2*}_{t^+} \geqslant B^2_{t^+/2}$. Since we have already shown that for Λ large $B_{t^+/2}$ is a graph over Π_0, it follows that $B^{2*}_{t^+} \geqslant B^2_{t^-} \cap \{x_1 > t/2\}$. Therefore we have shown $B^{2*}_{t^+} \geqslant B^2_{t^-}$. Likewise we can assert $B^{1*}_{t^+} \geqslant B^1_{t^-}$, and $B^1_{t^+}$ is a graph with bounded slope. Hence Theorem 1 can be applied to assert $(M \cap \{|x_3| \leqslant \Lambda\})^*_{t^+} \geqslant (M \cap \{|x_3| \leqslant \Lambda\})_{t^-}$ for Λ large. Thus for any $t > 0$ we have $M^*_{t^+} \geqslant M_{t^-}$, and hence we can assert $M^*_{0^+} \geqslant M_{0^-}$. Repeating the argument with x_1 replaced by $-x_1$ we can likewise assert $M^*_{0^-} \leqslant M_{0^+}$ which can be combined with the first to assert $M^* = M$ where * denotes reflection through Π_0. This completes the proof of Theorem 3 for the case $n = 2$.

The argument for $n > 2$ is quite similar, and we give its outline. First translate the x-coordinates so that the expansion for u_2 becomes

$$u_2(x) = b + a|x|^{2-n} + O\left(|x|^{-n}\right),$$

and recall that $u_2 = -u_1$, $b > 0$. (Observe that we must have $a \neq 0$ as we can see, for example, by applying Stokes Theorem on $M \cap \{\lambda \leqslant x_3 \leqslant \mu\}$ to $\Delta x_3 = 0$ for $\lambda < \mu < b$ and letting $\mu \uparrow b$ with λ fixed and close to b.) One then writes for $\Lambda < b$ and near to b

$$M \cap \left\{|x_{n+1}| = \Lambda\right\} = B^1 \cup B^2,$$

where $B^i = M_i \cap \{x_{n+1} = (-1)^i\Lambda\}$ for $i = 1, 2$. Again we must assert reflection symmetry in all planes of the form $\Sigma_{j=1}^n \beta_j x_j = 0$ and by rotation symmetry of the expansion we need to consider only the plane $x_1 = 0$. Let $\Pi_t = \{x_1 = t\}$ and we show that for Λ sufficiently near b and any $t > 0$ we have $B_{t^+}^{i*} \geqslant B_{t^-}^i$, and $B_{t^+}^i$ is a graph with bounded slope. The argument for this is similar to that given so we omit it. We then apply Theorem 1 to assert $M_{t^+}^* \geqslant M_{t^-}$ for all $t > 0$ and hence $M_{0^+}^* \geqslant M_{0^-}$. Repeating the arguments from the opposite direction we then get $M^* = M$ as desired.

References

[1] A. D. Alexandrov, *Uniqueness theorems for surfaces in the large*. I, (Russian) Vestnik Leningrad Univ. Math. **11** (1956) 5–17.

[2] B. Gidas, W. M. Ni & L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979) 209–243.

[3] D. Gilbarg & N. Trudinger, *Elliptic partial differential equations of second order*, Springer, Berlin, 1977.

[4] W. Y. Hsiang, Zhen-huan Teng & Wen-ci Yu, *New examples of constant mean curvature immersions of* $(2k - 1)$*-sphere into euclidean* $2k$*-space*, Preprint.

[5] L. Jorge & W. Meeks III, *The topology of complete minimal surfaces of finite total Gaussian curvature*, Topology **22** (1983) 203–221.

[6] W. Littman, G. Stampacchia & H. Weinberger, *Regular points for elliptic equations with discontinuous coefficients*, Ann. Scuola Norm. Sup. Pisa, (III) **17** (1963) 45–79.

[7] W. Meeks III, *Lectures on Plateau's problem*, Inst. Mat. Pura Apl., Rio de Janeiro, 1978.

[8] J. C. C. Nitsche, *A characterization of the catenoid*, J. Math. Mech. **11** (1962) 293–301.

[9] ______, *Über die Ausdehnung gewisser zweifach zusammenhängender Minimalfächen*, Math. Ann. **149** (1963) 144–149.

[10] R. Osserman, *A survey of minimal surfaces*, Math. Studies 25, Van Nostrand, New York, 1969.

[11] ______, *Global properties of minimal surfaces in* E^3 *and* E^n, Ann. of Math. (2) **80** (1964) 340–364.

[12] H. Rosenberg, *Deformations of complete minimal surfaces*, Preprint.

[13] J. Serrin, *A symmetry problem in potential theory*, Arch. Rational Mech. Anal. **43** (1971) 304–318.

[14] M. Shiffman, *On surfaces of stationary area bounded by two circles, or convex curves, in parallel planes*, Ann. of Math. **63** (1956) 77–90.

UNIVERSITY OF CALIFORNIA, BERKELEY