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THE DIRICHLET PROBLEM AT INFINITY FOR MANIFOLDS OF NEGATIVE CURVATURE

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This paper is concerned with the existence of bounded harmonic functions on simply connected manifolds N^n of negative curvature. It has been conjectured for some time with such manifolds admit a wealth of bounded harmonic functions provided the sectional curvature K_N satisfies $-a^2 \leqslant K_N \leqslant -b^2$, for some constants, a, $b > 0$, or even if $K_N \leqslant -b^2 < 0$; see [7], [18]. Justification for this comes from the fact that the model space $H^n(-1)$, the space form of curvature -1, admits many bounded harmonic functions; in fact, there is a Poisson integral representation 'at infinity' in $H^n(-1)$. (Similar results hold in more general noncompact symmetric spaces, cf. [12].) Furthermore, in case $n = 2$ the Ahlfors-Schwarz Lemma [1] shows that N^2 is conformally the unit disc provided $K_N \leqslant -b^2 < 0$, so that the model $H^2(-1)$ provides full information in this case.

It is natural to consider a Dirichlet problem at infinity for the Laplace-Beltrami operator Δ on N^n; there is a well-known compactification $\overline{N^n} = N^n \cup S^{n-1}(\infty)$ of N^n giving a homeomorphism of $(N^n, S^{n-1}(\infty))$ with the Euclidean pair (B^n, S^{n-1}). One can then state the

Asymptotic Dirichlet problem for Δ. Given a continuous function ρ on $S^{n-1}(\infty)$, find $f \in C^\infty(N^n) \cup C^0(\overline{N^n})$ satisfying

$$\Delta f = 0, \qquad f|_{S^{n-1}(\infty)} = \rho.$$

The main result of this paper is given by the following theorem (Theorem 3.2).

Theorem. *Let N^n be a complete simply connected Riemannian manifold with sectional curvature K_N satisfying $-a^2 \leqslant K_N \leqslant -b^2$, where $a^2 \geqslant b^2$ are arbitrary positive constants. Then the asymptotic Dirichlet problem for Δ is uniquely solvable, for any $\rho \in C^0(S^{n-1}(\infty))$.*

In particular, it follows that N^n has a large class of bounded harmonic functions. Using this one may show for instance that there are smooth proper

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harmonic maps F from N^n onto the Euclidean unit ball (or any other convex domain in $\mathbf{R}^n$), inducing a homeomorphism $S^{n-1}(\infty) \to S^{n-1} = \partial B^n$. For an open neighborhood of metrics sufficiently close to the hyperbolic metric, f will in fact be a diffeomorphism on N^n.

There have been a number of nonexistence results along the lines of the above theorem concerning generalizations of the Liouville theorem to Riemannian manifolds. Yau [17] proved that on any complete Riemannian manifold there are no globally defined harmonic functions in L^p for any $1 < p < \infty$. Greene and Wu [8] proved there are no bounded harmonic functions on manifolds $N^n \approx \mathbf{R}^n$ for which the exponential map from some point is a quasi-isometry. Further, Yau [16] proved that if N^n has nonnegative Ricci curvature, then there are no bounded harmonic functions on N^n. In the opposite direction, Choi [4] has recently obtained existence results for spherically symmetric metrics and also in dimension 2, in the case of negative curvature.

The proof of the theorem is based on the classical Perron method of solving the Dirichlet problem. Recall the success of the method hinges upon the existence of barrier functions, that is, subharmonic functions $B_x\colon N^n \to \mathbf{R}$ for $x \in S^{n-1}(\infty)$, such that $B_x \leqslant 0$ and $\lim_{y \to x} B_x(y) = 0$. Now manifolds N^n with $K_N \leqslant 0$, $\pi_1(N) = 0$ admit a wealth of convex, thus subharmonic functions. However, none of the familiar constructions of such functions give rise to barrier functions, since their behavior at infinity becomes too trivial; consider for instance Busemann functions or distance functions to complete geodesics. Thus our major contribution is the construction of global convex sets having nontrivial asymptotic behavior; from this we deduce the existence of barrier functions for Δ.

An outline of the contents of the paper is as follows. After presenting preliminary background material in §0, we discuss in §1 the Perron method, asymptotic maximum principle for harmonic functions and a characterization of the solvability of the Dirichlet problem in terms of convexity conditions at infinity (Theorem 1.4); the material for this section draws heavily on the work of Choi [4]. In §2 we construct a large family of global convex domains in N^n with controlled behavior at infinity; we refer to §2 for an outline of the construction. This is applied to give the solution to the Dirichlet problem in §3 (Theorem 3.2). We also show that the convex hull $C(S)$ of closed sets $S \subset S^{n-1}(\infty)$ is well behaved; in fact $C(S) \cap S^{n-1}(\infty) = S$, as is the case for hyperbolic space. This is useful in constructing barriers for complete minimal submanifolds in N^n and harmonic maps into N^n. In §4 we introduce harmonic measure on $S^{n-1}(\infty)$ and a Poisson integral representation of harmonic functions on $S^{n-1}(\infty)$; besides allowing one to construct larger classes of harmonic functions, one obtains in this fashion a satisfying correspondence between the space of bounded harmonic functions on N^n and $L^\infty(S^{n-1}(\infty), \mu)$; see Theorem 4.3. Finally, §5 closes with various extensions and remarks.

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We note that Sullivan [15] has recently obtained a proof of the asymptotic Dirichlet problem by quite different methods.

0. Preliminaries

Throughout this paper, N^n will denote a Cartan-Hadamard manifold, that is, a simply connected manifold of nonpositive curvature. The standard model for such spaces is the hyperbolic space form $H^n(-\lambda^2)$ of constant sectional curvature $-\lambda^2$. The *sphere at infinity* $S^{n-1}(\infty)$ of N^n is defined to be the set of asymptote classes of geodesic rays; two rays $\gamma_1, \gamma_2\colon [0,\infty) \to N^n$ define the same asymptote class if $\lim_{t\to\infty} \operatorname{dist}_N(\gamma_1(t), \gamma_2(t)) < \infty$. There is a natural topology on $\overline{N^n} = N^n \cup S^{n-1}(\infty)$, called the *cone topology*, given as follows: for any origin $\mathcal{O} \in N^n$, choose $v \in T_0N^n$ and let $C(v, \delta)$ be the cone around v of angle δ, i.e.,

$$C(v,\delta) = \left\{x \in N^n \cup S^{n-1}(\infty) \colon \measuredangle_{\mathcal{O}}\left(v, T_{\overline{\mathcal{O}x}}\right) < \delta\right\},$$

where $T_{\overline{\mathcal{O}x}}$ denotes the tangent vector to the geodesic ray $\overline{\mathcal{O}x}$ through $\mathcal{O}$ and x, and $\measuredangle_{\mathcal{O}}$ indicates angle in TN^n. Let $T(v, \delta, r)$ be the truncated cone of radius r, i.e., $T(v, \delta, r) = C(v, \delta)\backslash B_{\mathcal{O}}(r)$, $B_{\mathcal{O}}(r)$ the geodesic r-ball around $\mathcal{O}$. Eberlein and O'Neill [5] have shown that the family $T(v, \delta, r)$ for $v \in T_{\mathcal{O}}N$, $\delta > 0$, $r > 0$, together with the balls $B_q(r)$, $q \in N^n$, forms a local basis for the cone topology on $\overline{N^n}$; it turns out this topology is independent of the choice of $\mathcal{O}$. In this topology, $\overline{N^n}$ is homeomorphic to a closed ball $\overline{B}$ in $\mathbf{R}^n$, $S^{n-1}(\infty)$ being homeomorphic to the boundary sphere $S^{n-1} \subset \mathbf{R}^n$. In fact, if $\eta\colon [0,1] \to [0,\infty]$ is any homeomorphism, the map $E_\eta\colon D_1 \subset T_{\mathcal{O}}N^n \to N^n$ given by $E_\eta(v) = \exp \eta(|v|)\cdot v$ is a homeomorphism of the unit disc $D_1 \subset T_{\mathcal{O}}N^n$ onto N^n, inducing a homeomorphism of the sphere $S_1 = \partial D_1$ onto $S^{n-1}(\infty)$. We note that in general there is no natural (independent of 0) differentiable structure on $S^{n-1}(\infty)$. Especially in §4, we use the above homeomorphism to identify $S^{n-1}(1)$ with $S^{n-1}(\infty)$.

We introduce some notation which will be used throughout the paper. Let $\overline{xy}$ denote the geodesic ray determined by x and y in $\overline{N^n}$. For a given $v \in T_{\odot}N^n$, $x_v \in S^{n-1}(\infty)$ will denote the asymptote class determined by the ray $\exp tv$, $t \geqslant 0$. Geodesic spheres of radius r will be denoted by $S(r)$ or $S_p(r)$ if p is the center of $S(r)$; similarly geodesic balls are denoted by $B(r)$ or $B_p(r)$. The notation above for cones and truncated cones will be kept throughout the paper. In addition, we adhere to the usual notation in Riemannian geometry and partial differential equations; our main references are [2] and [6] on these matters.

1. Dirichlet problem at infinity

In this section, we discuss the Perron method and barrier functions for the solution of the following Dirichlet problem. Much of the material in this section is contained in the work of Choi [4].

1.0. Dirichlet problem at infinity for Δ. Let N^n be a simply connected manifold of nonpositive curvature. Given a continuous function $\rho \in C^0(S^{n-1}(\infty))$, find $f \in C^\infty(N^n) \cup C^0(\overline{N^n})$ such that

$$f = 0 \quad \text{in } N^n, \qquad f|_{S^{n-1}(\infty)} = \rho.$$

We recall that the topology on $\overline{N^n}$ is given by the cone topology, and note that convergence in this topology is much stronger than radial convergence (convergence along rays). For example, the function $f(x, y) = x \cdot y$ is harmonic in the upper half plane (with hyperbolic metric); along all geodesic rays emanating from $i = (0, 1)$, f converges to 0 on $S^1(\infty)$. Nevertheless, f does not have continuous boundary values on $S^1(\infty)$ in the cone topology. The classical Phragmen-Lindelöf principle illustrates more precisely the difference between the two topologies.

Using the natural identifications $S_{\odot}^{n-1}(1) \approx S_{\odot}^{n-1}(t)$, for any t, given by the exponential map, it is easy to see that $f \in C^0(\overline{N^n})$ has asymptotic boundary values ρ if and only if the restrictions $f_t = f|_{S_{\odot}^{n-1}(t)}$, pulled back to functions on $S_{\odot}^{n-1}(1)$, converge to $\rho \in C^0(S_{\odot}^{n-1}(1))$.

The following maximum principle is a simple consequence of the definitions.

Proposition 1.1. (a) *Let f: $N^n \to \mathbf{R}$ be a subharmonic function such that*

$$\overline{\lim_{x \to x_\infty}} f(x) \leqslant 0, \quad \textit{for any } x_\infty \in S^{n-1}(\infty).$$

Then $f \leqslant 0$ on N^n.

(b) *If f is a subharmonic function on N^n, g is a superharmonic function on N^n, and*

$$\overline{\lim_{x \to x_\infty}}\, f(x) \leqslant \underline{\lim_{x \to x_\infty}}\, g(x), \quad \textit{for any } x_\infty \in S^{n-1}(\infty),$$

then $f \leqslant g$ on N^n.

Proof. We leave this to the reader, or see Choi [4].

Let S_ρ be the set of subfunctions on N^n relative to ρ: that is, given $\rho \in C^0(S^{n-1}(\infty))$, S_ρ is the set of C^0 subharmonic functions v: $N^n \to \mathbf{R}$ such that $\overline{\lim}_{x \to x_\infty} v(x) \leqslant \rho(x_\infty)$. Clearly, S_ρ is nonempty. Let $u(x) = \sup_{v \in S_\rho} v(x)$; it is well known that u is a globally defined harmonic function on N^n. The function u defined in this manner is a candidate for the solution of the Dirichlet problem; to show that u achieves the required boundary values, one needs to construct appropriate barrier functions.

Definition 1.2. Let $v \in T_0 N^n$ be a unit vector, and suppose $\delta > 0$. Then $\beta = \beta(v, \delta)$: $N^n \to \mathbf{R}$ is called a *barrier function* at v with angle δ if

(1) β is subharmonic,

(2) $\beta \leqslant 0$ and $\lim_{x \to x_v} \beta(x) = 0$,

(3) $\exists \mu > 0$ such that $\overline{\lim}_{x \to x_w} \beta(x) \leqslant -\mu$ for any $w \in T_0 N^n$ with $\sphericalangle_0(v, w) > \delta$.

This is a natural analogue of the classical barrier concept for domains in $\mathbf{R}^n$; see [11, 2.6.2] or [4, 2.6]. One then has

Theorem 1.3. *Suppose there exist barrier functions $\beta = \beta(v, \delta)$ with arbitrarily small angle δ at any $v \in T_0 N^n$. Then the Dirichlet problem* 1.0 *at infinity is uniquely solvable for any $\rho \in C^0(S^{n-1}(\infty))$.*

Proof. Let $u(x)$ be the Perron solution relative to $\rho \in C^0(S^{n-1}(\infty))$ defined above. We show that $\lim_{x \to x_\infty} u(x) = \rho(x_\infty)$ for any $x_\infty \in S^{n-1}(\infty)$, which implies the theorem. Fix any $v \in T_0 N^n$ and choose $\varepsilon > 0$. Let $\delta > 0$ be such that $|\rho(x_v) - \rho(x_w)| < \varepsilon$ if $\sphericalangle_0(v, w) \leqslant \delta$, and let $\beta = \beta(v, \delta)$ be a barrier function at v, angle δ. If μ is given by (3) above, choose k so that $\mu k \geqslant 2M$, where $M = \sup |\rho|$ on $S^{n-1}(\infty)$.

It is easy to see that $\psi_1(x) \equiv \rho(x_v) - \varepsilon + k\beta(\text{i}x)$ is a subfunction, and $\psi_2(x) \equiv \rho(x_v) + \varepsilon - k\beta(x)$ is a superfunction relative to ρ; since the proofs are almost identical, we verify this claim for ψ_2. It is clear that $\psi_2(x)$ is superharmonic on N^n. To show that $\underline{\lim}_{x \to x_w} \psi_2(x) \geqslant \rho(x_w)$ for any $w \in T_0 N^n$, suppose first that $\sphericalangle_0(x_v, x_w) \leqslant \delta$. Then $|\rho(x_v) - \rho(x_w)| \leqslant \varepsilon$ and since $\beta \leqslant 0$, $\psi_2(x) \geqslant \rho(x_w)$, whenever $x \in C(v, \delta)$. If $\sphericalangle_0(x_v, x_w) > \delta$, we have

$$\underline{\lim_{x \to x_w}}\, \psi_2(x) = \rho(x_v) + \varepsilon - k \cdot \overline{\lim_{x \to x_w}}\, \beta(x) \geqslant \rho(x_v) + \varepsilon + k\mu \geqslant \rho(x_w),$$

as required. From the definition of u as the Perron solution and by Proposition 1.1(b), we find $\psi_1(x) \leqslant u(x) \leqslant \psi_2(x)$, or $|u(x) - \rho(x_v)| \leqslant \varepsilon - k\beta(x)$. Since $\beta(x) \to 0$ as $x \to x_v$ and ε is arbitrary, it follows that $\lim_{x \to x_v} u(x) = \rho(x_v)$ as desired. This proves the solvability of the Dirichlet problem for arbitrary $\rho \in C^0(S^{n-1}(\infty))$. Uniqueness follows from the maximum principle (Proposition 1.1(a)) in the usual fashion. q.e.d.

It is well known that Cartan-Hadamard manifolds N^n possess a wealth of convex functions; typical examples are distance functions to a point or to a geodesic, or the mean square distance to a compact submanifold. In case $K_N \leqslant -c^2 < 0$, such convex functions may be reparametrized to give bounded subharmonic functions; see e.g. [4]. Thus one may hope to obtain the existence of barrier functions from suitable convex functions or sets, as is the case for bounded domains in $\mathbf{R}^n$.

Theorem 1.4 (*Choi*). *Suppose for any distinct $x, y \in S^{n-1}(\infty)$ there exist disjoint open neighborhoods V_x, V_y of x, y in $\overline{N^n}$ such that $V_x \cap N^n$ and $V_y \cap N^n$ are strictly convex. Then if $K_N \leqslant -1$, the Dirichlet problem at infinity for Δ is uniquely solvable for any $\rho \in C^0(S^{n-1}(\infty))$.*

Proof. By Theorem 1.3 we need to construct a barrier $\beta = \beta(v, \delta)$ for $v \in T_0N^n$ and any small $\delta > 0$. Given $x_v \in S^{n-1}(\infty)$, let $S(v, \delta) = \{x_w \in S^{n-1}(\infty)\colon \sphericalangle_0(x_v, x_w) = \delta\}$. Since $S(v, \delta)$ is compact, we may cover it by a finite number of convex open sets $\{V_{w_i}\}_{i=1}^m$ such that

$$V_{w_i} \cap V_{x_v} = \varnothing, \quad \text{for all } i,$$

where V_{x_v} is an open neighborhood of x_v in $\overline{N^n}$; this much follows from the hypothesis of the theorem. Let $\Omega = N^n - \bigcup_1^m V_{w_i}$, and let $s_i\colon \Omega \to \mathbf{R}^+$ be the distance function from V_{w_i}. By an approximation theorem of Greene and Wu [9, Proposition 2.2], we may assume each ∂V_{w_i}, and thus each s_i is a smooth function. Using the second variational formula one may show that

$$\operatorname{Hess}(s_i) \geqslant \tanh\left(\frac{s_i}{2}\right) \cdot H_0,$$

where $H_0 = ds^2 - ds_i \otimes ds_i$, ds^2 is the metric on N^n; [4]. From this it follows easily that

$$\Delta \tanh \frac{s_i}{2} \geqslant 0,$$

on Ω. Now let $\bar{\beta} = \sum_{i=1}^m \tanh(s_i/2) - m$; clearly $\bar{\beta}$ is subharmonic, nonpositive and $\lim_{x \to x_\infty} \bar{\beta}(x) = 0$ for any $x_\infty \in V_{x_v} \cap S^{n-1}(\infty)$. Choose $R > 0$ so large that Ω disconnects as the disjoint union $\Omega_1 \cup \Omega_2$ outside $B_0(R)$. Let Ω_1 be the set of $q \in \Omega$ such that $\mathrm{dist}_N(\mathcal{O}, q) > R$ and $\sphericalangle_{\mathcal{O}}(\overline{\mathcal{O}q}, v) < \delta$. Let $\overline{\Omega} = \{x \in \Omega_1 : \overline{\beta}(x) \geq c\}$, where $-m < c < 0$ is a fixed constant, sufficiently close to 0. Define β on N^n by

$$\beta(x) = \begin{cases} \overline{\beta}(x), & x \in \overline{\Omega}, \\ c, & x \in (\overline{\Omega})^c. \end{cases}$$

Then β is a barrier at v, of angle δ ($\mu = -c$). Since v and δ are arbitrary, the result follows. q.e.d.

It appears that none of the standard constructions of convex functions on manifolds N^n of negative curvature give rise to convex sets satisfying the conditions of Theorem 1.4. In fact, there are no examples of convex sets C in general N^n such that $C \cap S^{n-1}(\infty) \neq S^{n-1}(\infty)$ is a set with nonempty interior in the cone topology. Thus our aim in the next section is to construct convex sets with nontrivial behavior at infinity.

2. Construction of convex sets

In this section we will construct unbounded convex domains in manifolds of negative curvature, having prescribed asymptotic behavior. Recall that horoballs H_x in N^n are strictly convex sets intersecting $S^{n-1}(\infty)$ at a single point x; the construction is based on the idea that locally one may produce larger convex sets containing H_x (due to the strict convexity): this is done in Step I below. In Step II we use an iteration procedure to construct global convex domains $\mathcal{C}$. In Step III by using comparison with negative space forms we show convergence and are able to control the behavior of $\mathcal{C}$ at infinity.

Throughout this section N^n will denote a simply connected Cartan-Hadamard manifold of sectional curvature K_N satisfying $-a^2 \leq K_N \leq -b^2$; using a homothety of the metric, we may assume $b = 1$.

Step I. An important feature of the spaces N^n is the convexity of large geodesic spheres. In fact, if II_R denotes the second fundamental form (with respect to inward normal) of a geodesic R-sphere in N^n, then standard arguments involving Jacobi fields show that

$$I \leq \coth R \leq \mathrm{II}_R \leq a \coth aR \cdot I, \tag{2.1}$$

where I denotes the identity matrix.

In order to find local convex expansions of spheres we use the following lemma.

Lemma 2.1. *Given any $p \in N^n$, there is an $f_p \in C^\infty(N^n, \mathbf{R})$ such that $0 \leqslant f_p \leqslant 1, f_p(p) = 0, f \equiv 1$ outside $B_p(1)$ and*

$$(2.2) \qquad |df_p| < C_1, \qquad |D^2_{ij} f_p| < C_2,$$

where C_1, C_2 are constants depending only on a^2.

Proof. Let ρ be the distance function from p, and consider functions $f_p = \phi(\rho)$ where ϕ: $\mathbf{R}^+ \to \mathbf{R}$ satisfies $\phi(0) = 0$, $\phi(t) = 1$, for $t \geqslant 1$. Then $df_p = \phi' \cdot d\rho$, so that $|df_p| = |\phi'|$. A simple computation shows that $D^2\phi(\rho) = \phi'' d\rho \otimes d\rho + \phi' D^2\rho$; thus

$$|D^2_{ij}\phi(\rho)| \leqslant |\phi''| + |\phi'|\, a \coth a\rho.$$

Choosing ϕ appropriately, e.g., as a fixed approximation to the characteristic function $\chi_{[1/2,\infty]}$, one easily obtains the estimates (2.2). q.e.d.

Now given a fixed origin $\mathcal{O}$, consider the functions $\rho_{\mathcal{O}} + \varepsilon f_p$ where $\rho_{\mathcal{O}}$ is the distance function from $\mathcal{O}$; these may be considered as local perturbations of $\rho_{\mathcal{O}}$, provided $\rho_{\mathcal{O}}(p) \geqslant 10$ say.

Proposition 2.2. *There is an $\varepsilon > 0$ depending only on a^2 such that the sublevel sets of $\rho_{\mathcal{O}} + \varepsilon f_p$ are totally convex subsets of N^n.*

Proof. Note that the level sets of $\rho_{\mathcal{O}} + \varepsilon f_p$ are smooth submanifolds of N^n; since N^n has negative curvature, it is sufficient to show they have positive definite second fundamental form $\mathrm{II}_{\varepsilon f}$. For a given tangent vector X we have

$$\mathrm{II}_{\varepsilon f}(X, X) = \langle \nabla_X N, X \rangle = \frac{1}{|d(\rho_{\mathcal{O}} + \varepsilon f_p)|} \cdot D^2(\rho_{\mathcal{O}} + \varepsilon f_p)(X, X).$$

We have $D^2(\rho_{\mathcal{O}}) \geqslant I$ by (2.1) and $D^2(\varepsilon f_p) \leqslant \varepsilon C_2$ by (2.2); choosing ε sufficiently small then gives the result.

Remark 2.3. It is clear that Proposition 2.2 remains valid when $f_p = \phi \circ \rho$ is replaced by $\phi_R \circ \rho$ where $\phi_R(t) = \phi(R \cdot t)$; then ε depends on R as well as a^2. In order to simplify the computations in Step III, we will choose R to satisfy the following requirement. Let $S_{\mathcal{O}}(p)$ be the sphere around $\mathcal{O}$ containing p, and $S_{\mathcal{O}}(\varepsilon)$ the concentric spheres with radius $\operatorname{dist}(\mathcal{O}, p) + \varepsilon$. Then the level set of $\rho_{\mathcal{O}} + \varepsilon f_p$ through p should equal $S_{\mathcal{O}}(\varepsilon)$ outside the intrinsic ball of radius 1 in $S_{\mathcal{O}}(\varepsilon)$ centered at $\overline{op} \cap S_{\mathcal{O}}(\varepsilon)$ (instead of the extrinsic ball of radius 1 centered at p). It is not difficult to see that for $\operatorname{dist}(\mathcal{O}, p) \geqslant C$, e.g., $\operatorname{dist}(\mathcal{O}, p) \geqslant 10$, such an R may be found independent of $\mathcal{O}$, p.

Step II. We now construct global convex domains, using iteration of the local result contained in Proposition 2.2. Thus let $S = S_{\mathcal{O}}(R)$ be a fixed geodesic R-sphere and let $S(t) = S_{\mathcal{O}}(R + t)$. Choose $p \in S$ and define C_1 to be the sublevel set of $\rho_{\mathcal{O}} + \varepsilon f_p$ with $p \in \partial C_1$; here and in what follows, ε and f_p are defined by Proposition 2.2 and Remark 2.3. One may view C_1 as the ball

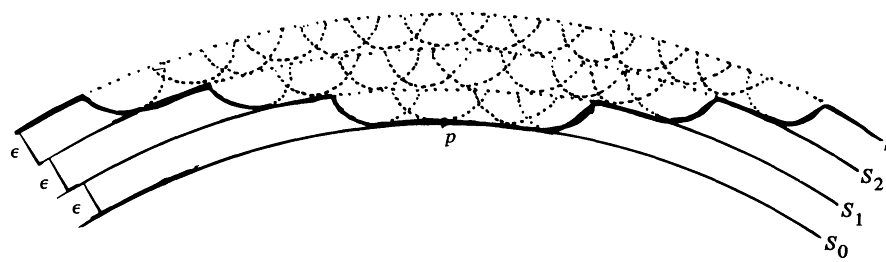
$B(\varepsilon) = B_{\mathcal{O}}(R + \varepsilon)$ with a 'scallop' cut out around p; C_1 is a totally convex domain.

One may proceed to cut out successively larger 'scallops' from successively larger spheres as follows. Let T_1 be the 'seam' of C_1: $T_1 = \partial(C_1 \cap S(\varepsilon))$. Let U_{T_1} be the collection of points on $S_1 = S(\varepsilon)$ whose outward intrinsic distance to T_1 is $\leqslant 1$; note this includes all points of S_1 in the interior of T_1 (the component containing $\overline{op}$). For any $q \in U_{T_1}$, define the convex set $C_2(q)$ to be the sublevel set of $\rho_0 + \varepsilon f_q$ such that $q \in \partial C_2(q)$, and set

$$C_2' = \bigcap_{q \in U_{T_1}} C_2(q).$$

It is clear that C_2' is totally convex. Let $C_2 = C_2' \backslash (B(\varepsilon) \backslash C_1)$.

Note that $C_1 \subset B_1 = B(\varepsilon)$, and $B(\varepsilon) \backslash C_1$ is the 'scallop' bored out of $B(\varepsilon)$. It is easy to see that C_2 is also totally convex. In fact let $x, y \in \partial C_2$ and suppose not both x and y are in ∂C_1. Then $\gamma = \overline{xy}$ is obviously contained in C_2'; suppose γ entered the scallop $B(\varepsilon) \backslash C_1$. By convexity of C_1, at least one end z of $\gamma \cap B(\varepsilon)$ does not lie in ∂C_1. However, then one of the two geodesic arcs zx or zy must intersect the complement of $C_2(q)$ for some q, contradicting the convexity of $C_2(q)$.



It is now clear how we proceed inductively; the 'seam' T_i of C_i, $T_i = \partial(C_i \cap S(i\varepsilon))$, serves to construct C_{i+1}' as

$$C_{i+1}' = \bigcap_{q \in U_{T_i}} C_{i+1}(q),$$

where $C_{i+1}(q)$, U_{T_i} are defined as before. Then C_{i+1} is given by $C_{i+1} = C_{i+1}' - (B_i - C_i)$. We thus have a nested sequence of totally convex sets

$$C_1 \subset C_2 \subset \cdots \subset C_k \subset C_{k+1} \subset \ldots.$$

Let $\mathcal{C} = \bigcup_{i=1}^{\infty} C_i$; this is the desired global totally convex set. Note that we construct such $\mathcal{C}$ for any $p \in S$ and any S (of radius $\geqslant 10$).

Step III. Having constructed global convex domains $\mathcal{C}$ in N^n, we need to show that $\mathcal{C}$ has 'nontrivial' asymptotic behavior. In particular, we need to control the size of $\mathcal{C} \cap S^{n-1}(\infty)$. The study of $\mathcal{C} \cap S^{n-1}(\infty)$ uses comparison of $\mathcal{C}$ with models in the spaces $H^n(-1)$ and $H^n(-a^2)$. It turns out it is actually sufficient to compare $\mathcal{C}$ with models in $H^2(-1)$ and $H^2(-a^2)$. Thus choose an origin o in $H^2(-1)$ and $H^2(-a^2)$ together with R-spheres $S(R)$ centered at o, and let $p \in S(R)$. We construct the models $\overline{\mathcal{C}} \subset H^2(-1)$ exactly according to the prescription given for $\mathcal{C}$ in Steps I and II above (using the same or any equivalent f). Similarly, construct models $\underline{\mathcal{C}} \subset H^2(-a^2)$, but replacing the intrinsic distances 1 by $\frac{1}{2}$ everywhere.

We measure the asymptotic behavior of $\mathcal{C}$ (and $\overline{\mathcal{C}}$, $\underline{\mathcal{C}}$) by means of the angle at o from the ray $\overline{op}$; in other words, for any $x \in \mathcal{C} \subset S^{n-1}(\infty)$, consider $\sphericalangle_{\mathcal{O}}(\overline{op}, \overline{ox})$. Let

$$\alpha_{\mathcal{C}} = \sup\{\sphericalangle_{\mathcal{O}}(\overline{op}, \overline{ox}) \colon x \in \mathcal{C} \cap S^{n-1}(\infty)\},$$
$$\beta_{\mathcal{C}} = \inf\{\sphericalangle_{\mathcal{O}}(\overline{op}, \overline{ox}) \colon x \in \mathcal{C} \cap S^{n-1}(\infty)\}.$$

Similarly we define $\alpha_{\overline{\mathcal{C}}}$, $\beta_{\overline{\mathcal{C}}}$ and $\alpha_{\underline{\mathcal{C}}}$, $\beta_{\underline{\mathcal{C}}}$; it is easy to see that in fact $\alpha_{\overline{\mathcal{C}}} = \beta_{\overline{\mathcal{C}}}$ and $\alpha_{\underline{\mathcal{C}}} = \beta_{\underline{\mathcal{C}}}$.

Lemma 2.4. *In the above notation, we have*

$$\alpha_{\mathcal{C}} \leqslant \alpha_{\overline{\mathcal{C}}}, \qquad \beta_{\underline{\mathcal{C}}} \leqslant \beta_{\mathcal{C}}. \tag{2.5}$$

Proof. (i) $\alpha_{\mathcal{C}} \leqslant \alpha_{\overline{\mathcal{C}}}$. Referring to Step II in the construction of $\mathcal{C}$, let $x_N \in T_N$ be a sequence such that $x_N \to x_\infty$ and $\sphericalangle_o(\overline{ox_\infty}, \overline{op}) = \alpha_{\mathcal{C}}$. For each fixed N, let $x_N^i \in T_i$ be chosen inductively so that $x_N^N = x_N$ and $\operatorname{dist}_{S(i)}(x_N^i, (x_N^{i+1})') \leqslant 2$ where $(x_N^{i+1})'$ is the normal geodesic projection of x_N^{i+1} onto $S(i) = S(R + i\varepsilon)$. Consider the geodesic hinge $x_N^i o x_N^{i+1}$; set $\alpha_N^i = \sphericalangle_o(\overline{ox_N^i}, \overline{ox_N^{i+1}})$ and note that $l(\overline{ox^i}) = R + \varepsilon$ for any N. Now consider analogous hinges in $\overline{\mathcal{C}} \subset H^2(-1)$. In this case, each T_i consists of two points $\{\chi_i, \xi_i\}$; we let $\chi_i \in T_i$ be chosen so that $\{T_{\overline{o\chi_i}}, T_{\overline{op}}\}$ is an oriented basis for $T_o(H^2(-1))$. Let $\bar{\alpha}^i = \sphericalangle_o(\overline{op}, \overline{o\chi_i})$; we have $l(\overline{o\chi_i}) = R + i\varepsilon$ and $\operatorname{dist}_{S(i)}(\chi_i, \chi'_{i+1}) = 2$. It now follows from the Rauch comparison theorem (see [2, 1.28, 1.30]), that

$$\alpha_N^i \leqslant \overline{\alpha^i}, \quad \text{for all } i, \text{ any } N.$$

Thus we have shown

$$\alpha_{\mathcal{C}} \leqslant \lim_{N \to \infty} \sum_{i=1}^{N} \alpha_i^N \leqslant \sum_{i=1}^{\infty} \overline{\alpha^i} = \alpha_{\overline{\mathcal{C}}},$$

where the last equality follows from the fact that $H^2(-1)$ is 2-dimensional.

(ii) $\beta_{\underline{\mathcal{C}}} \leqslant \beta_{\mathcal{C}}$. Again we choose $x_\infty \in \mathcal{C} \cap S^{n+1}(\infty)$ realizing $\beta_{\mathcal{C}}$. Let $P \subset T_0 N^n$ be the 2-plane spanned by the vectors $\{T_{\overline{op}}, T_{\overline{ox_\infty}}\}$, and let $\mathcal{P} = \exp_0 P$. Choose $x_i \in T_i \cap \mathcal{P}$ so that $\operatorname{dist}_{S(i)}(x_i, (x_{i+1})') \geqslant 1$ where $(x_{x+})'$ is as in (i); it is clear that such choices of x_i always exist and $x_i \to x_\infty$. We let $\beta_i = \sphericalangle_o(\overline{ox_i}, \overline{ox_{i+1}})$

and recall $l(\overline{ox_i}) = R + i\varepsilon$. Now compare this data with the model $\underline{\mathcal{C}} \subset H^2(-a^2)$; choose $\underline{\chi}_i \subset \underline{T}_i \subset \underline{\mathcal{C}}$ as in (i) and set $\underline{\beta}_i = \sphericalangle_o(\overline{op}, \overline{o\chi_i})$ in $H^2(-a^2)$. Since $l(\overline{o\chi_i}) = R + i\varepsilon$ and $\mathrm{dist}_{S_i}(\chi_i, (\chi_{i+1})') = \bar{1}$ (recall 1 is replaced by $\frac{1}{2}$ in $\underline{\mathcal{C}}$), it follows by Rauch comparison as above that

$$\underline{\beta}_i \leqslant \beta_i, \quad \text{for all } i.$$

Since all angles β_i are measured in P, we have $\beta_{\mathcal{C}} = \sum_{i=1}^{\infty} \beta_i$, which gives

$$\beta_{\underline{\mathcal{C}}} = \sum_{i=1}^{\infty} \underline{\beta}_i \leqslant \sum_{i=1}^{\infty} \beta_i = \beta_{\mathcal{C}}. \quad \text{q.e.d.}$$

We are now ready to obtain estimates for $\alpha_{\bar{\mathcal{C}}}$ and $\beta_{\underline{\mathcal{C}}}$ in terms of ε, a, R. For the calculations we use the Poincaré model for $H^2(-\lambda^2)$; recall the hyperbolic metric on the Euclidean ball $B^2(1)$ is given by

$$ds_\lambda^2 = \frac{4}{\lambda^2(1 - r^2)^2} ds_E^2,$$

where ds_E^2 is the Euclidean metric. One easily computes that at Euclidean distance $r \in (0, 1)$, the hyperbolic distance d_λ is given by

$$d_\lambda = \frac{1}{\lambda} \log \frac{1 + r}{1 - r}.$$

Also, the intrinsic hyperbolic distance S_λ on $S(R)$ is $S_\lambda = 2S_E/[\lambda(1 - r^2)]$, for S_E the intrinsic Euclidean distance.

Now given $T_i \subset \underline{\mathcal{C}}$ or $\bar{\mathcal{C}}$ in $H^2(-\lambda^2)$ for $\lambda^2 = 1, a^2$, we measure the position of T_i with respect to o and the ray $\overline{op}$ by means of 'polar coordinates': $T_i = (l_i, \theta_i)$, where l_i is the radius of the Euclidean sphere with $T_i \in S(l_i)$, and θ_i is the hyperbolic angle at o between $\overline{op}$ and $\overline{oT_i}$. We have $T_0 = (l_0, 0)$ where $l_0 = (e^{\lambda R} - 1)/(e^{\lambda R} + 1)$, since $T_0 = p$ is on the geodesic R-sphere centered at o. By construction, l_k is given by the formula

$$k\varepsilon = \frac{1}{\lambda} \int_{l_0}^{l_k} \frac{2}{1 - t^2} dt.$$

Writing $l_k = l + \mu_k$, one finds that

$$\mu_k = \frac{(1 + l_0)[e^{k\lambda\varepsilon} - 1]}{1 + e^{k\lambda\varepsilon}((1 + l_0)/(1 - l_0))}.$$

To compute θ_k, recall that

$$S_\lambda = \frac{2}{\lambda}\left(\frac{1}{1 - r^2}\right) \cdot S_E;$$

by construction we require $S_\lambda = \text{dist}_{S(i)}(\chi_i, (\chi_{i+1})') = 2\sigma$, where $\sigma = 1$ in $H^2(-1)$ and $\sigma = \frac{1}{2}$ in $H^2(-a^2)$. This gives $\theta_1 = S_E/l_0 = \lambda\sigma(1 - l_0^2)/l_0$ and generally

$$\theta_k = \frac{\lambda\sigma(1 - l_{k-1}^2)}{l_{k-1}}.$$

For the coordinates of T_k, we then have

$$T_k = \left(l_0 + \mu_k, \lambda\sigma \sum_{i=0}^{k-1} \frac{1 - l_i^2}{l_i} \right).$$

Set $\Omega_\lambda(R, \varepsilon) = \lim_{k\to\infty} \theta_k$. Then $\mu_k \to 1 - l_0$ as $k \to \infty$ and a lengthy but straightforward computation shows that

$$\text{(2.6)} \quad \Omega_\lambda(R, \varepsilon) = 4(1 - l_0^2)\lambda\sigma \cdot \sum_{i=0}^{\infty} \frac{1}{(1 + l_0)^2 e^{i\lambda\varepsilon} - (1 - l_0)^2 e^{-i\lambda\varepsilon}}$$

$$= 4\lambda\sigma \sum_{i=0}^{\infty} \frac{1}{e^{2\lambda\varepsilon} e^{i\lambda\varepsilon} - e^{-i\lambda\varepsilon}}.$$

It is clear that this series converges uniformly on compact sets in both variables $R > 0$ and $\varepsilon > 0$; thus Ω_λ is a continuous function of R and ε. In particular, setting $\lambda = 1$ and $\lambda = a$, and substituting in the value for σ above, we obtain bounds on $\alpha_{\bar{\mathcal{C}}}$ and $\beta_{\underline{\mathcal{C}}}$:

$$\text{(2.7)} \quad \alpha_{\bar{\mathcal{C}}} = 4 \sum_{i=0}^{\infty} \frac{1}{e^{2R} e^{i\varepsilon} - e^{-i\varepsilon}},$$

$$\beta_{\underline{\mathcal{C}}} = 2a \sum_{i=0}^{\infty} \frac{1}{e^{2aR} e^{ia\varepsilon} - e^{-ia\varepsilon}}.$$

Of course, we are only interested in the case when $\beta_{\underline{\mathcal{C}}} \leqslant \alpha_{\bar{\mathcal{C}}}$; this occurs for example for R satisfying $(a - 1)R \geqslant \ln(a/2)$; so $R \geqslant 1$ is sufficient.

Recall that ε is determined solely in terms of the constant a from Proposition 2.2. The estimates (2.7) in conjunction with Lemma 2.4 provide estimates for the behavior of $\mathcal{C}$ at infinity. In the next section, these will be used to discuss the convexity of $S^{n-1}(\infty)$ and the solution of the Dirichlet problem.

3. Convexity of $S^{n-1}(\infty)$: Solution of the Dirichlet problem

We use the results of §2 to discuss the convexity of N^n at infinity. First, we prove the existence of arbitrarily 'small' convex neighborhoods for $x \in S^{n-1}(\infty)$ in the cone topology; this leads to the solution of the Dirichlet problem at infinity for Δ. We also prove an important property of the convex hull of closed sets $S \subset S^{n-1}(\infty)$, namely, $\mathcal{C}(S) \cap S^{n-1}(\infty) = S$; this property is well known in hyperbolic space $H^n(-1)$.

We continue to assume that N^n satisfies $-a^2 \leqslant K_N \leqslant -1$.

Theorem 3.1. *Given any $v \in T_0(N^n)$ and $\delta > 0$, there are convex domains $K_{v,\delta}(l)$ in N^n satisfying*

(i) $K_{v,\delta}(l) \subset T(v, \delta, l)$, for $l \geqslant \bar{l}$, where $\bar{l}$ depends continuously on δ and a,

(ii) $C(v, \delta') \cap S^{n-1}(\infty) \subset K_{v,\delta}(l) \cap S^{n-1}(\infty)$, where $\delta' > 0$ depends continuously on a and l.

Proof. Let $x_v \in S^{n-1}(\infty)$ be the asymptote class determined by v, and $\overline{ox_v}$ the ray from o to x, and let $o_l = S_o(l) \cap \overline{ox_v}$ for $l > 0$. Consider the spheres $S_{o_l}(R)$ of radius $R < l$ around o_l and set $p_R = S_{o_l}(R) \cap \overline{ox_v}$. Define $\mathcal{C}(p_R, o_l)$ to be the convex domain constructed in §2 determined by the center o_l and point p_R; we will show that for appropriate choices of R and l, $K_{v,\delta}(l) \equiv \mathcal{C}(p_R, o_l)$ satisfies (i) and (ii).

(i) Given any $\delta > 0$, we claim there is an $\bar{l}$ such that $l \geqslant \bar{l}$ implies $\mathcal{C}(p_R, o_l) \subset T(v, \delta, l)$ for any $R_0(a) \leqslant R \leqslant R_1(a)$, where R_0, R_1 are fixed constants depending only on a. To show this, recall that $\beta_{\mathcal{C}} = \inf\{\sphericalangle_{o_l}(\overline{o_l p_R}, \overline{o_l x})\colon x \in \mathcal{C} \cap S^{n-1}(\infty)\}$, and $\beta_{\mathcal{C}} \geqslant \underline{\beta}_{\mathcal{C}}$ by Lemma 2.4. Consider the geodesic triangle $oo_l x_\infty$ where $x_\infty \in S^{n-1}(\infty)$ realizes $\beta_{\mathcal{C}}$, and let $\Omega = \sphericalangle_o(\overline{oo_l}, \overline{ox_\infty})$. Let $oo_l\chi_\infty$ be a similar triangle in $H^2(-1)$ where $\operatorname{dist}(o, o_l) = l$, $\chi_\infty \in S^1(\infty)$ and $\sphericalangle_{o_l}(\overline{o_l o}, \overline{o_l\chi_\infty}) = \underline{\beta}_{\mathcal{C}}$. Setting $\bar{\Omega} = \sphericalangle_o(\overline{oo_l}, \overline{o\chi_\infty})$ in $H^2(-1)$, the Rauch comparison theorem applied to the two triangles implies that

$$\Omega \leqslant \bar{\Omega}.$$

Now $\underline{\beta}_{\mathcal{C}}$ depends on a and R according to (2.7); in particular, there are constants $R_0(a)$ and $R_1(a)$ so that $R_0(a) \leqslant R \leqslant R_1(a)$ implies that $\pi/2 \leqslant \underline{\beta}_{\mathcal{C}} \leqslant 3\pi/4$, independent of l. It follows by elementary hyperbolic geometry that $\bar{\Omega}$ can be made arbitrarily small by choosing l sufficiently large; thus there exists $\bar{l}$ so that $\Omega \leqslant \bar{\Omega} < \delta$ for $l \geqslant \bar{l}$.

We have proved that $\mathcal{C}(p_R, o_l) \cap S^{n-1}(\infty) \subset C_o(v, \delta) \cap S^{n-1}(\infty)$ for $l \geqslant \bar{l}$, $R_0(a) \leqslant R \leqslant R_1(a)$. By examining the construction of $\mathcal{C}(p_R, o_l)$ in §2, this is easily seen to imply that

$$\mathcal{C}(p_R, o_l) \subset T(v, \delta, l - R_1(a)), \qquad l \geqslant \bar{l}.$$

(ii) To see that $K_{v,\delta}(l)$ intersects $S^{n-1}(\infty)$ in a set of nonempty interior, choose $y_\infty \in S^{n-1}(\infty)$ realizing $\alpha_{\mathcal{C}} = \sup\{\sphericalangle_{o_l}(\overline{o_l p_R}, \overline{o_l x})\colon x \in \mathcal{C} \cap S^{n-1}(\infty)\}$, where $\mathcal{C} = \mathcal{C}(p_R, o_l)$ as in (i). Consider the geodesic triangle $oo_l y_\infty$ and let $\omega = \sphericalangle_o(\overline{oo_l}, \overline{oy_\infty})$. We form the comparison triangle $oo_l\xi_\infty$ in $H^2(-a^2)$ where $\operatorname{dist}(o, o_l) = l$ and $\sphericalangle_{o_l}(\overline{o_l o}, \overline{o_l\xi_\infty}) = \bar{\alpha}_{\mathcal{C}}$ in $H^2(-a^2)$. Let $\underline{\omega} = \sphericalangle_o(\overline{oo_l}, \overline{o\xi_\infty})$; since

$\alpha_{\mathcal{C}} \leqslant \alpha_{\bar{\mathcal{C}}}$ by Lemma 2.4, Rauch comparison applied to the pair of triangles gives

$$\underline{\omega} \leqslant \omega.$$

This shows that $C_o(v, \underline{\omega}) \cap S^{n-1}(\infty) \subset \mathcal{C}(p_R, o_l) \equiv K_{v,\delta}(l)$. It is clear that $\delta' \equiv \underline{\omega} > 0$ depends continuously on a, l and R. Since R is bounded by $R_0(a) \leqslant R \leqslant R_1(a)$, δ' depends only on a and l. q.e.d.

As a consequence of Theorem 3.1, we deduce our main theorem on the solvability of the Dirichlet problem.

Theorem 3.2 (*Dirichlet problem at infinity*). *Let N^n be a complete simply connected Riemannian manifold of sectional curvature K_N satisfying $-a^2 \leqslant K_N \leqslant -1$, where $a^2 \geqslant 1$ is an arbitrary constant. Then the Dirichlet problem at infinity for Δ,* (1.0), *is uniquely solvable for any $\rho \in C^0(S^{n-1}(\infty))$.*

Proof. By Theorem 1.4 we need to prove that for pairs $x, y \in S^{n-1}(\infty)$, $x \neq y$, there exist disjoint open sets V_x, V_y in $\overline{N^n}$ relative to the cone topology so that $V_x \cap N$ and $V_y \cap N$ are convex. Let $v, w \in T_0N^n$ be chosen so that x, y are the asymptote classes of the rays determined by v, w. Choose $\delta > 0$ so that $C_0(v, \delta) \cap C_0(w, \delta) = \varnothing$. By Theorem 3.1 we may choose convex domains K_x and K_y so that $K_x \subset C_0(v, \delta)$, $K_y \subset C_0(w, \delta)$ and $K_x \cap S^{n-1}(\infty)$, $K_y \cap S^{n-1}(\infty)$ contain the intersection of δ'-cones around v, w with $S^{n-1}(\infty)$. In particular, the domains K_x, K_y satisfy the above conditions. q.e.d.

Given a set $S \subset \overline{N^n}$, we define the *convex hull* $\mathcal{C}(S)$ of S to be the smallest geodesically closed set in $\overline{N^n}$ containing S. The manifold N^n is said to satisfy the *Visibility property* if given any distinct $x, y \in S^{n-1}(\infty)$, there is a complete geodesic γ in N^n asymptotic to x and y; see [5]. As a simple special case, N^n is Visibility if $K_N \leqslant -b^2 < 0$. The Visibility manifolds are the natural class of manifolds in which to study the convexity of $S^{n-1}(\infty)$. A characteristic property of the convexity of the model space $H^n(-1)$ at infinity is the fact $\mathcal{C}(S) \cap S^{n-1}(\infty) = S$ for any closed set $S \subset S^{n-1}(\infty)$. We generalize this to other Riemannian manifolds as follows.

Theorem 3.3. *Let N^n be a complete simply connected manifold satisfying the conditions of Theorem* 3.2. *If S is a closed set in $S^{n-1}(\infty)$, then*

$$\mathcal{C}(S) \cap S^{n-1}(\infty) = S. \tag{3.3}$$

Proof. We note that it is sufficient to prove the existence of 'large' convex sets $\mathcal{C}_{v,\delta}$ for any $\delta > 0$, $v \in T_0N^n$ such that

$$S^{n-1}(\infty)/\big(C(v,\delta) \cap S^{n-1}(\infty)\big) \subset \mathcal{C}_{v,\delta} \cap S^{n-1}(\infty),$$

but $x_v \notin \mathcal{C}_{v,\delta} \cap S^{n-1}(\infty)$. For given such, let $x \in S^{n-1}(\infty)\backslash S$; then there exists δ and v with $x = x_v \in C_0(v, \delta) \cap S^{n-1}(\infty)$ satisfying $C_0(v, \delta) \cap S^{n-1}(\infty) \subset S^{n-1}(\infty)\backslash S$. Choose $\mathcal{C}_{v,\delta}$ as above; it follows that $x \notin \mathcal{C}_{v,\delta}$ yet $S \subset \mathcal{C}_{v,\delta}$. Since $\mathcal{C}(S) \subset \mathcal{C}_{v,\delta}$, we have $x \notin \mathcal{C}(S)$ as required.

The existence of $\mathcal{C}_{v,\delta}$ follows from the results of §2; we choose $\mathcal{O}$ and $p \in S_{\mathcal{O}}(R)$ so that v is the unit tangent vector to $\overline{\mathcal{O}p}$. Choose R so large that $\alpha_{\mathcal{C}} < \delta$; this is possible for any $\delta > 0$ by combining Lemma 2.4 with the estimate (2.7). On the other hand, we have the estimate $\beta_{\mathcal{C}} \geqslant \beta_{\mathcal{C}} \geqslant \delta'(a, \delta, R) > 0$ again by (2.7) and Lemma 2.4. Thus $x_v \notin \mathcal{C}_{v,\delta}$, and so $\mathcal{C}_{v,\delta}$ satisfies the required properties.

Remark. The property (3.3) is useful in providing barriers for systems of partial differential equations satisfying certain maximum principles; in particular, it can be applied to the study of complete minimal submanifolds in N^n and harmonic maps of complete manifolds into N^n. Gromov [10, 3.2] has also called attention to property (3.3), partly in regard to generalizing the theory of Kleinian groups.

4. Harmonic measure and general boundary values

In this section we generalize our results on solvability of the Dirichlet problem to more general boundary values; we begin by introducing a Poisson integral representation for globally defined harmonic functions via the harmonic measure on $S^{n-1}(\infty)$. Theorem 4.3 then gives a satisfying relation between the class of bounded harmonic functions on N^n and the class of L^∞ functions on $S^{n-1}(\infty)$.

Given $f \in C^0(S^{n-1}(\infty))$, let $P[f]$ denote the unique harmonic extension of f into N^n; thus $P[f]$ is the harmonic function on N^n with asymptotic boundary values f on $S^{n-1}(\infty)$. For each $x \in N^n$, define a linear functional L_x on $C^0(S^{n-1}(\infty))$ by

$$L_x(f) = P[f](x). \tag{4.1}$$

Note that since $|P[f](x)| \leqslant \max_{S^{n-1}(\infty)} |f|$ by the maximum principle (Proposition 1.1(a)), L_x is a bounded linear functional of norm 1. Again the maximum principle shows that $f \geqslant 0 \Rightarrow L_x(f) \geqslant 0$ so that L_x is a positive functional. Applying the Riesz representation theorem gives the existence of a regular positive Borel measure μ_x on $S^{n-1}(\infty)$ such that

$$P[f](x) = \int_{S^{n-1}(\infty)} f \, d\mu_x,$$

for any $x \in N^n$, $f \in C^0(S^{n-1}(\infty))$. We note that the above remarks show that $S^{n-1}(\infty)$ is the Šilov boundary of N^n in the sense of harmonic analysis. The measure μ_x is called the *harmonic measure* of $S^{n-1}(\infty)$ at x; clearly $\mu_x(S^{n-1}(\infty)) = 1$ for all x.

The following result gives a means of constructing harmonic functions with more general behavior at infinity.

Theorem 4.1. *Let $f \in S^{n-1}(\infty) \to \mathbf{R}$ be μ_x-integrable for some $x \in N^n$. Then f is μ_x-integrable for all $x \in N^n$, and the function $P[f]$ given by*

$$P[f](x) = \int_{S^{n-1}(\infty)} f \cdot d\mu_x \tag{4.2}$$

is a smooth harmonic function on N^n.

The function $P[f]$ defined by (4.2) is called the *harmonic extension* of f. In particular, if $E \subset S^{n-1}(\infty)$ is a Borel set, then $\mu_x(E)$ is harmonic in x.

Proof. The proof is a straightforward adaptation of the same result for bounded regular domains in $\mathbf{R}^n$; see [11, §3.6]. We sketch the argument for f upper-semicontinuous and bounded. Choose a sequence of continuous functions $\{f_n\}$ decreasing montonically to f as $n \to \infty$. The corresponding sequence of harmonic extensions $P[f_n]$ given by (4.2) decrease to a limit $u(x)$, and u is harmonic in N^n; this follows easily from the Harnack-type convergence of harmonic functions in N^n. It is not difficult to see that u is independent of $\{f_n\}$. By monotone convergence, $\int_{S^{n-1}(\infty)} f_n \, d\mu_x \to \int_{S^{n-1}(\infty)} f \, d\mu_x$ for any x, so that $u = P[f]$ is harmonic. To prove (4.2) for Borel measurable f, one uses the fact that f is the monotone limit of upper-and lower-semicontinuous functions; see [11] for details. q.e.d.

We now show that the harmonic extension $P[f]$ of f has the correct boundary values, at least in certain cases.

Given a continuous function u: $N^n \to \mathbf{R}$, let u_t denote the restriction to $S^{n-1}(t)$, and u^t the pullback of u_t to $S^{n-1}(1)$ via the exponential map. If v: $S^{n-1}(\infty) \to \mathbf{R}$, we also view v: $S^{n-1}(1) \to \mathbf{R}$ via the homeomorphism η: $S^{n-1}(1) \to S^{n-1}(\infty)$ (see §0). Finally, if λ denotes the measure induced by the volume form on $S^{n-1}(1)$, we view λ as a measure on $S^{n-1}(t)$ for $t \in [1, \infty]$ via the above homeomorphisms.

Theorem 4.2. *Let $f \in L^p(S^{n-1}(\infty), \mu_x) \cap L^r(S^{n-1}(\infty), \lambda)$, $1 < r < \infty$, and let $P[f]$ denote the harmonic extension of f. Then $\|f - P[f]^t\|_r \to 0$ as $t \to \infty$ in $L^r(S^{n-1}(1), \lambda)$. In particular, $P[f]_t \to f$ almost everywhere in the cone topology on N^n.*

Proof. Since $1 < r < \infty$, C^0 is dense in L^r. Let $f_n \in C^0(S^{n-1}(\infty))$ be chosen so that $f_n \to f$ in $L^r(S^{n-1}(\infty), \lambda) \cap L^p(S^{n-1}(\infty), \mu)$, and set $u_n = P[f_n]$. By the solution to the Dirichlet problem, Theorem 3.2, $(u_n)_t$ converges to f_n uniformly in the cone topology so that $u_n^t \to f_n$ uniformly on $S^{n-1}(1)$. Let $u = P[f]$. Then $u_n \to u$ uniformly in N^n since

$$\begin{aligned} |u(x) - u_n(x)| &\leq \int_{S^{n-1}(\infty)} |f - f_n| d\mu_x \\ &\leq \left(\int_{S^{n-1}(\infty)} |f - f_n|^p \, d\mu_x \right)^{1/p} = \|f - f_n\|_{p,\mu}. \end{aligned}$$

We have

$$\|f - u^t\|_{r,\lambda} \leq \|f - f_n\|_{r,\lambda} + \|f_n - u_n^t\|_{r,\lambda} + \|u_n^t - u^t\|_{r,\lambda};$$

the above remarks show that each term is arbitrarily small for t, n sufficiently large. q.e.d.

We are interested in the converse of the above theorems, i.e., given harmonic functions u on N^n, when does u have boundary values in L^p? For the case of bounded harmonic functions, we have the following theorem.

Theorem 4.3. *Let β denote the Banach space of bounded harmonic functions on N^n under the sup norm. Then the linear mapping*

$$P\colon L^\infty(S^{n-1}(\infty), \mu) \to \beta, \qquad P(f) = P[f]$$

is a norm-nonincreasing isomorphism onto β. Further $\|P[f]^t - f\|_p \to 0$ as $t \to \infty$ for any $1 < p < \infty$ provided $f \in L^\infty(S^{n-1}(\infty), \lambda)$.

Proof. It is clear that P is linear and 1-1. P is norm-nonincreasing since

$$|P[f](x)| = \int_{S^{n-1}(\infty)} f\, d\mu_x \leq \|f\|_\infty \cdot \int_{S^{n-1}(\infty)} d\mu_x = \|f\|_\infty.$$

Thus the major task lies in showing P is surjective; the proof of this is a variation of the proof by Ullrich in the case of the Bergmann ball; see [13, §4.3].

There is a natural action of $SO(n)$ on $S^{n-1}(t)$ induced by the linear action of $SO(n)$ on the Euclidean t-spheres in TN^n; of course, the action on $S^{n-1}(t)$ is not by isometries. Let dg denote Haar measure on $SO(n)$ and choose a continuous nonnegative function h: $SO(n) \to \mathbf{R}$ such that $\int_{SO(n)} h\, dg = 1$. Given $F(z)$: $N^n \to \mathbf{R}$ harmonic, define

$$G(z) = \int_{SO(n)} F(gz) \cdot h(g)\, dg. \tag{4.3}$$

We claim there is a fixed constant k, depending only on the geometry of N such that

$$\int_{SO(n)} \left|F^t(g\xi)\right|^p dg \leq k \cdot \int_{S^{n-1}(1)} \left|F^t(x)\right|^p d\lambda, \tag{4.4}$$

for any $\xi \in S^{n-1}(1)$. To see this we have

$$\int_{S^{n-1}(1)}\left[\int_{SO(n)}|F^t(g\eta)|^p\,dg\right]d\lambda=\int_{SO(n)}\left[\int_{S^{n-1}(1)}|F^t(g\eta)|^p\,d\lambda\right]dg$$
$$=\int_{SO(n)}\left[\int_{S^{n-1}(1)}|F^t(\eta)|^p\,d((g^{-1})^*\lambda)\right]dg$$
$$\leqslant k\int_{SO(n)}\left[\int_{S^{n-1}(1)}|F(\eta)|^p\,d\lambda\right]dg$$
$$=k\cdot\int_{S^{n-1}(1)}|F(\eta)|^p\,d\lambda,$$

where $k=\sup_g\{\sup[h_g(x)\colon x\in S^{n-1}(1)]\}$, $h_g=$ Radon-Nikodym derivative of $(g^{-1})^*\lambda$ with respect to λ. On the other hand, since $SO(n)$ acts transitively on $S^{n-1}(t)$,

$$\int_{S^{n-1}(1)}\left[\int_{SO(n)}|F^t(t\eta)|^p\,dg\right]d\lambda=\operatorname{vol}S^{n-1}(1)\cdot\int_{SO(n)}|F^t(g\xi)|^p\,dg,$$

for any $\xi\in S^{n-1}(1)$; this proves (4.4). By the Hölder inequality applied to (4.3), using (4.4) we have

$$|G(z)|\leqslant k\cdot M_p\|h\|_q, \tag{4.5}$$

where p, q are conjugate indices, and $M_p=\sup_t\|F^t\|_p$; in particular,

$$|G(z)|\leqslant k\cdot M_\infty. \tag{4.6}$$

These estimates show that $\{G^t\}$, $t\in(1,\infty)$, is an equicontinuous family on $S^{n-1}(1)$. In fact, given $\varepsilon>0$, choose a neighborhood U of $I\subset SO(n)$ such that

$$|h(g)-h(gg_0^{-1})|<\varepsilon$$

for $g_0\in U$, $g\in SO(n)$. Choose $\delta>0$ so that if $\xi,\eta\in S^{n-1}(1)$ and $\operatorname{dist}(\xi,\eta)<\delta$, then $\xi=g_0\eta$ for $g_0\in U$. Since $G(g_0z)=\int_{SO(n)}F(gz)h(gg_0^{-1})\,dg$, one finds

$$|G^t(\xi)-G^t(\eta)|\leqslant\int_{SO(n)}|F^t(g\eta)|\cdot|h(gg_0^{-1})-h(g)|dg\leqslant M_1\varepsilon,$$

provided $\operatorname{dist}(\xi,\eta)<\delta$. By (4.6), $\{G^t\}$ is uniformly bounded, so it follows that there is a sequence $\{t_i\}\to\infty$ such that $\{G^{t_i}\}$ converges uniformly to $g\in C^0(S^{n-1}(\infty))$.

Now replace h by h_j in the definition of G, and let $\operatorname{supp}\{h_j\}$ shrink to $I\in SO(n)$. Then G_j converges pointwise to F. Consider the sequence $\{g_j\}$; by (4.6) and the uniform convergence of G_j^t to g_j, $\|g_j\|_{p,\mu}<k\cdot M_\infty$ for all

$1 < p \leqslant \infty$. Thus there is an $f \in L^{\infty}(S^{n-1}(\infty), \mu)$ such that $\{g_j\}$ has a subsequence, call it $\{g_j\}$, converging to f in the weak $*$-topology of L^{∞}. The function

$$P[f](x) = \int_{S^{n-1}(\infty)} f \, d\mu_x$$

is well defined and harmonic in N^n; we will show that $F = P[f]$. To do this we first prove that

$$(4.7) \quad F^t - P[f]^t \underset{t \to \infty}{\to} 0 \text{ weakly in } L^{\infty}(S^{n-1}(\infty), \lambda)^* \cap L^{\infty}(S^{n-1}(\infty), \mu)^*.$$

Clearly,

$$(4.8) \quad F^t - P[f]^t = \left(F^t - G_j^t\right) + \left(G_j^t - P[g_j]^t\right) + \left(P[g_j]^t - P[f]^t\right),$$

and we will analyze each of these terms separately. Let σ denote either of the measures λ or μ.

(i) $(F^t - G_j^t)$: We compute, for $l \in L^1(S^{n-1}(1), \sigma)$,

$$\begin{aligned}
&\int_{S^{n-1}(1)} \left(F^t(z) - G_j^t(z)\right) l(z)\, d\sigma \\
&\quad = \int_{S^{n-1}(1)} \left[\int_{SO(n)} (F^t(z) - F^t(gz)) \cdot l \cdot h_j \, dg\right] d\sigma \\
&\quad = \int_{SO(n)} \left[\int_{S^{n-1}(1)} (F^t(z) - F^t(gz)) \cdot l \, d\sigma\right] h_j \, dg.
\end{aligned}$$

Note that since F is bounded on N, there are a sequence $\{t_i\} \to \infty$ and $\phi \in L^{\infty}(S^{n-1}(1), \sigma)$ such that $\{F^{t_i}\} \to \phi$ in the weak $*$-topology on L^{∞}. Thus

$$\begin{aligned}
&\lim_{t_i \to \infty} \int_{S^{n-1}(1)} \left(F^{t_i}(z) - G_j^{t_i}(z)\right) l(z)\, d\sigma \\
&\quad = \int_{SO(n)} \left[\int_{S^{n-1}(1)} (\phi(z) - \phi(gz)) l \, d\sigma\right] \cdot h_j \, dg.
\end{aligned}$$

This shows that, given any $\varepsilon > 0$ and $l \in L^1(\sigma)$, there is a J such that for $j \geqslant J$, the last term is bounded, in absolute value, by ε. The same statement holds for any sequence $\{t_j\} \to \infty$.

In the next case, we can estimate in the strong L^p-norm.

$$\begin{aligned}
\text{(ii)}\quad \lim_{t \to \infty} \left\|G_j^t - P[g_j]^t\right\|_{p,\sigma}^p &= \lim_{t \to \infty} \int_{S^{n-1}(1)} \left|G_j^t(z) - P[g_j]^t(z)\right|^p d\sigma \\
&\leqslant \lim_{t \to \infty} \int_{S^{n-1}(1)} \left|G_j^t(z) - g_j(z)\right|^p d\sigma \\
&\quad + \lim_{t \to \infty} \int_{S^{n-1}(\infty)} \left|P[g_j]^t(z) - g_j(z)\right|^p d\sigma.
\end{aligned}$$

Since $G_j^t \to g_j$ and $P[g_j]^t \to g_j$ uniformly on $S^{n-1}(1)$, the last two terms vanish.

(iii) To estimate $\lim_{t\to\infty}(P[g_j]^t \to P[f]^t)$ in the weak *-topology, we just note that $P[g_j]^t \to g_j$ uniformly for fixed j, and $P[f]^t \to f$ in L^p-norm, $1 < p < \infty$; further $g_j \to f$ weakly in $L^\infty(S^{n-1}(1), \sigma)$.

Combining the estimates in (i), (ii) and (iii) with (4.8) gives the desired (4.7). Finally, since $F - P[f]$ is harmonic, bounded and $(F - P[f])^t \to 0$ weakly as $t \to \infty$ we claim that $F = P[f]$. Thus let $\psi = F - P[f]$ and let $P_t^x\, d\lambda$ denote harmonic measure on $S^{n-1}(t)$ at x. Then we have

$$\begin{aligned}\psi(x) &= \int_{S^{n-1}(1)} \psi^t(y) P_t^x(y)\, d\lambda \\ &= \int_{S^{n-1}(1)} \psi^t(y)\cdot d\mu^x(y) + \int_{S^{n-1}(1)} \psi^t(y)\cdot\left[P_t^x(y)\, d\lambda - d\mu^x(y)\right].\end{aligned}$$

By assumption the first term converges to 0 as $t \to \infty$, and by general principles the second term also converges to o since $\|\psi\|_\infty$ is finite.

This shows $F = P[f]$ as desired. Finally, the fact that $\|P[f]^t - f\|_p \to 0$ as $t \to \infty$ follows immediately from Theorem 4.2.

5. Concluding remarks

1. It is not difficult to see that Theorem 3.2 remains valid provided the conditions on the curvature hold outside some compact set B of N^n; in other words $a^2 \leqslant K_N \leqslant -1$ on $N^n - B$, where B is any compact set in N^n, and $N^n - B$ is diffeomorphic to $\mathbf{R}^n - \bar{B}$, $\bar{B}$ being a closed ball. This is in line with the philosophy of Greene-Wu that much of the function theory on Cartan-Hadamard manifolds should be determined by asymptotic conditions only.

2. Let $0 \in N^n$ and let $\partial/\partial x_i$ be global normal coordinates for N^n around 0. Setting $g_{ij} = \langle \partial/\partial x_i, \partial/\partial x_j \rangle$, the Laplace-Beltrami operator takes the form

$$\Delta u = \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x_i}\left(\sqrt{g}\, g^{ij} \frac{\partial u}{\partial x_j}\right),$$

where $g = \det(g_{ij})$, and $g^{ij} = (g_{ij})^{-1}$. Then Theorem 3.2 may be interpreted as showing that this operator admits many bounded solutions in $\mathbf{R}^n$, provided $a^{ij} = \sqrt{g}\, g^{ij}$ satisfies certain curvature conditions—certain bounds on 2nd and 3rd derivatives of a^{ij}.

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