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MICROLOCAL HYPO-ANALYTICITY AND EXTENSION OF CR FUNCTIONS

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INTRODUCTION

Since the publication of H. Lewy's paper [12] the question of local holomorphic extension of CR functions defined on submanifolds of $\mathbf{C}^m$ has been studied by a number of mathematicians (to name a few: Andreotti and Hill [1], Hill and Taiani [9], Hunt and Wells [11], Wells [17]).

In Baouendi & Treves [2] (and in [3]; for a systematic and updated description see Treves [16]) it has been shown that such a question is actually relevant for all solutions of a system of equations

$$(*) \qquad L_j h = 0, \quad j = 1, \cdots, n,$$

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where $(L_1,\cdots,L_n)$ is a *locally integrable* system of C^∞ complex vector fields in some manifold Ω. If we assume that the dimension of Ω is equal to $m+n$, the latter means that in the neighborhood of each point there are m C^∞ solutions $Z^1,\cdots,Z^m$ of $(*)$ such that $dZ^1,\cdots,dZ^m$ are linearly independent. One of the basic facts established in [2] is that any continuous solution of $(*)$ is of the form $h = \tilde{h} \circ Z$ near the point under consideration, where $\tilde{h}$ is a function on the range of $Z = (Z^1,\cdots,Z^m)$. Even when that range is not a submanifold of $\mathbf{C}^m$, the push-forward $\tilde{h}$ can be regarded as a CR function—in the sense that it is (locally) the limit of a sequence of polynomials in $z^1,\cdots,z^m$, a fact also established in [2]. For distributions similar properties can be deduced from an adapted representation formula (see [16, Chap. II, Theorem 3.1]).

If one intends to regard the holomorphic extendability of the push-forward $\tilde{h}$ as a property of the solution $\tilde{h} \circ Z$ of $(*)$ one has no choice but to take the "pushing" maps $Z = (Z^1,\cdots,Z^m)$ as the basic objects—rather than the system of vector fields $L_1,\cdots,L_n$. Holomorphic substitutions of the Z^j's are of course permitted—but not substitution by arbitrary sets of independent C^∞ solutions of $(*)$. This viewpoint leads to structures which are very similar to the (real) analytic ones. They are analytic structures when the Z^j's are all real and $n = 0$. Otherwise they are somewhat "poorer", and we call them *hypo-analytic*. A solution h of $(*)$ is hypo-analytic at a point ω of its domain of definition if it is the pull-back, via an admissible map Z, of a holomorphic function in a neighborhood of $Z(\omega)$ in $\mathbf{C}^m$. This concept is introduced and rapidly studied in Chapter I of the present article (among other things, hypo-analytic functions admit an infinite "Taylor expansion" of a special kind, and cannot vanish to infinite order at a point unless they vanish identically in a full neighborhood of that point). But the main point of Chapter I is the formulation and the proof (in §4) of a necessary and sufficient condition of hypo-analyticity based on the Fourier transform in the version of Bros-Iagolnitzer [7]. Certain modifications are forced upon us by the nature of the "first integrals" Z^j—which are neither real nor analytic (in general). An additional integration introduced by Sjöstrand [15] helps us out of the technical difficulties (basically, those of dealing with "complex phase functions" whose imaginary parts must be controlled).

A serious drawback of the necessary and sufficient condition of hypo-analyticity established in Chapter I is that it relies on one and the same set of Z^j's in a full neighborhood of the central point—regardless of the value of ξ, the variable on the Fourier transform side. In order to break away from such a limitation one *microlocalizes*. This is the purpose of Chapter II and the main reason for the present article. We adopt (and adapt) the definition of Sato [13] and show, afterwards, that it is equivalent to the one derived from the

Fourier-Bros-Iagolnitzer transform.[1] At first this is only done in the case in which $n = 0$ (when there are no vector fields!), and then applied to the traces of solutions of $(*)$ on the submanifolds which we call *maximally real*: these are the m-dimensional C^∞ submanifolds of Ω on which the pull-backs of the differentials $dZ^1, \cdots, dZ^m$ remain independent. A fortunate occurrence is that the definition of the *hypo-analytic wave-front set* of a solution h arrived at in this manner does not depend on the maximally real submanifold on which the trace of h is taken (Chapter II, Theorem 4.1).

Under favorable circumstances (Chapter II, Theorem 2.3) circumscribing the hypo-analytic wave-front set yields the holomorphic extension of the push-forward functions $\tilde{h}$ to certain *conoids*. Therein lies the main difference between our approach, linking the extension to conoids to the exponential decay of the Fourier-Bros-Iagolnitzer transform, and that of most other papers dealing with holomorphic extension of CR functions, which use the technique of families of analytic discs.

The last two sections of Chapter II describe an application of the machinery. The application is to the first cases one encounters, when the available information is provided by the Levi form or, if one prefers, by the first brackets $[L_j, \bar{L}_k]$ $(1 \leqslant j, k \leqslant n)$. It is shown that if the Levi form, associated with a given characteristic covector, has at least one negative eigenvalue, then the covector in question does not belong to the hypo-analytic wave-front set of any solution. This is the microlocal version of the classical result of H. Lewy about hypersurfaces in $\mathbf{C}^2$. Our proof requires that the first integrals Z^j be very precisely chosen so as to reflect the existence of a negative eigenvalue of the Levi form at the characteristic covector under consideration, thus justifying microlocalization. It should be mentioned that, through all this, the "CR set" in complex space (i.e., the image of Ω under the map Z) is not necessarily a manifold and its codimension is arbitrary. Under the additional assumption that the "CR set" is a manifold, Boggess and Polking [5] have recently obtained a local version of the same result, using analytic discs techniques together with the approximation theorem of [2]. The last result of the present paper is that if the Levi form at a characteristic covector is positive definite, the covector belongs to the hypo-analytic wave-front set of at least one solution.

We believe that the microlocal approach proposed here will be helpful in untangling more sophisticated situations, some of those where higher commutation brackets determine the outcome. This is definitely suggested by the

[1] See Bony [6] and Sjöstrand [15]. We have not been able to adapt the concept of *analytic wave front set* as defined in Hörmander [10].

results in the "tube" situation in Baouendi and Treves [4], and by those in a forthcoming paper by Chang [8].

I. HYPO-ANALYTIC STRUCTURES

1. Hypo-analytic structures. Definitions

Throughout this chapter Ω will denote an N-dimensional manifold of class C^q, countable at infinity. We suppose $N \geqslant 1$ and $2 \leqslant q \leqslant +\infty$. The integer q measures the regularity of the structures which we shall be dealing with.

Definition 1.1. By a hypo-analytic structure on Ω we mean the data of an open covering $\{U_j\}_{j\in J}$ of Ω and, for each $j \in J$, of m complex-valued C^q functions $Z_j^1,\cdots,Z_j^m$ in U_j, with $m \geqslant 1$ independent of j, satisfying the following requirements:

(1.1) $dZ_j^1,\cdots,dZ_j^m$ are linearly independent at each point of U_j (and therefore we must have $m \leqslant N$);

(1.2) if $U_j \cap U_k \neq \varnothing$, there are an open neighborhood $\mathcal{O}_j$ of $Z_j(U_j \cap U_k)$ in $\mathbf{C}^m$ and a holomorphic map F_k^j of $\mathcal{O}_j$ into $\mathbf{C}^m$ such that

$$Z_k = F_k^j \circ Z_j \quad \text{in } U_j \cap U_k.$$

We have used the notation $Z_j = (Z_j^1,\cdots,Z_j^m): U_j \to \mathbf{C}^m$. Henceforth we suppose that Ω is equipped with a hypo-analytic structure as defined in Definition 1.1; we refer to Ω as a *hypo-analytic manifold*. The integer $n = N - m$ will be called the *codimension* of the hypo-analytic structure of Ω.

Let p_0 be an arbitrary point of Ω, and f a complex-valued function defined in some neighborhood of p_0.

Definition 1.2. We say that f is hypo-analytic at the point p_0 if, for some (or, equivalently, for every) index j such that $p_0 \in U_j$, there is a holomorphic function $\tilde{f}_j$ in an open neighborhood of $Z_j(p_0)$ in $\mathbf{C}^m$ such that $f = \tilde{f}_j \circ Z_j$ in a neighborhood of p_0.

We shall say that a function defined in a subset S of Ω is *hypo-analytic* in S if it is hypo-analytic at every point of S; clearly it then extends as a hypo-analytic function in an open neighborhood of S.

Note that any hypo-analytic function in an open subset of Ω is of class C^q in that subset, and also that if the function in question does not vanish, its reciprocal is hypo-analytic there. The hypo-analytic functions in an open subset of Ω form an algebra over the complex numbers.

All the standard definitions in the analytic category extend routinely to the hypo-analytic one. Let us mention a few.

Let Ω' be another hypo-analytic manifold. A mapping $F: \Omega \to \Omega'$ is hypo-analytic if, given any open subset U' of Ω' and any hypo-analytic function $f: U' \to \mathbf{C}$, the compose $f \circ F$ is a hypo-analytic function in $F^{-1}(U')$.

A *hypo-analytic local chart* in Ω will be an $(m+1)$-tuple $(U, Z^1, \cdots, Z^m)$ consisting of an open subset U of Ω and of m hypo-analytic functions $Z^1, \cdots, Z^m$ whose differentials are linearly independent at every point of U. We shall write

$$Z = (Z^1, \cdots, Z^m) : U \to \mathbf{C}^m;$$

in general, the mapping Z is *not* a diffeomorphism, not even a local embedding (see Example 1.5 below).

Example 1.1. Suppose that Ω is a real-analytic manifold, and that $N = \dim \Omega = m$. The real-analytic structure of Ω is defined by local charts $(U, Z^1, \cdots, Z^m)$ where the Z^j are real-valued and real-analytic coordinates in U; it is a hypo-analytic structure.

Example 1.2. Let Z be a complex C^q function in $\mathbf{R}^1$ such that $dZ \neq 0$ at every point. The single local chart $(\mathbf{R}^1, Z)$ defines a hypo-analytic structure on $\mathbf{R}^1$. Suppose that $Z(\mathbf{R}^1)$ is *not* a real-analytic submanifold of the complex plane. Then it is clear that there will not be a covering of open intervals in each one of which there is defined a *real*-valued hypo-analytic function whose differential is nowhere zero, and thus the structure defined by Z is not a real-analytic structure.

Example 1.3. Suppose $N = 2m$, and Ω to be a complex analytic manifold. The complex structure of Ω is then a hypo-analytic structure, defined by local charts $(U, Z^1, \cdots, Z^m)$ where the Z^j are complex coordinates in U.

Example 1.4. Let $\tilde{\Omega}$ be a complex analytic manifold of dimension d, and assume that Ω is a C^q submanifold of $\tilde{\Omega}$. Thus $N \leqslant 2d$. Assume that the following holds:

(1.3) Given any complex analytic local chart $(\tilde{U}, z^1, \cdots, z^d)$ in $\tilde{\Omega}$, the pull-backs to $U = \tilde{U} \cap \Omega$ of the differentials $dz^1, \cdots, dz^d$ span an m-dimensional subspace of the complex cotangent space to U at every point of the latter set (with m independent of the local chart).

When (1.3) holds, Ω is called a CR submanifold of $\tilde{\Omega}$; Ω carries a natural hypo-analytic structure; basic hypo-analytic charts $(U, Z^1, \cdots, Z^m)$ consist of sets U like the one in (1.3) and of the restrictions to U of m independent holomorphic functions in an open subset $\tilde{U}$ of $\tilde{\Omega}$ such that $U = \tilde{U} \cap \Omega$. A hypo-analytic manifold such as Ω is called an *embedded* CR *manifold*.

Example 1.5. The function $z = x + iy^2/2$ defines a hypo-analytic structure on $\mathbf{R}^2$, called the *Mizohata structure*. Pre-images of points in $\mathbf{C}$ via Z are of the form $\{(x, y)\} \cup \{(x, -y)\}$.

Since the manifold structure of Ω is of class C^q with q possibly finite, we cannot in general deal with the distributions on Ω. But we shall do so when $q = +\infty$. Otherwise we shall mostly limit ourselves to dealing with C^1 functions (recalling that $q \geqslant 2$). Another concept (and notation) we shall borrow from the analytic theory is the following.

Definition 1.3. Let u be a function (or a distribution when $q = +\infty$) in an open subset Ω' of Ω. The complement in Ω' of the open set of points at which u is hypo-analytic will be called the hypo-analytic singular support of u and denoted by $\operatorname{sing\,supp}_{ha} u$.

Suppose that Ω is a real-analytic manifold. A hypo-analytic structure on Ω will then be called *real analytic* (or, in short, analytic) if every hypo-analytic function, for this structure, is real analytic. Note that *in order that the real-analytic hypo-analytic structure of* Ω *be identical to the real-analytic structure of* Ω, *it is necessary and sufficient that its codimension be zero.*

2. The structure bundle and the solutions

The hypo-analytic structure of Ω defines another structure on Ω, which we shall now briefly describe.

Indeed, if $(U, Z^1, \cdots, Z^m)$ is a hypo-analytic local chart in Ω, the differentials $dZ^1, \cdots, dZ^m$ span a complex vector subbundle of the complex cotangent bundle of Ω over U, T'_U (by virtue of (1.1)). And it follows from (1.2) that if $(U', Z'^1, \cdots, Z'^m)$ is another chart, T'_U and $T'_{U'}$ are equal on the intersection $U \cap U'$. In other words, the hypo-analytic structure of Ω defines a complex vector subbundle T' of $\mathbf{C}T^*\Omega$ (the complex cotangent bundle of Ω) whose fiber dimension is equal to m. Bundles of this kind were called *locally integrable* RC-*structures* in [16]. As T' is unambiguously defined, we shall refer to it as the "*structure bundle*". Note however that the same structure bundle might correspond to different hypo-analytic structures: take for instance the case where $m = N (= \dim \Omega)$. Then $T' = \mathbf{C}T^*\Omega$. But the choice of the hypo-analytic structure, *i.e.*, of the local hypo-analytic functions Z^j, might vary greatly, as shown in Examples 1.1, 1.2.

We shall recall some of the terminology of [16]: $T'^{\perp}$ will denote the orthogonal of T' for the duality between tangent and cotangent vectors; thus $T'^{\perp}$ is a complex vector subbundle of $\mathbf{C}T\Omega$, the complex *tangent* bundle of Ω, and its fiber dimension is equal to $n = N - m$, the codimension of the hypo-analytic structure of Ω.

The intersection

$$T^0 = T' \cap T^*\Omega$$

is *not*, in general, a vector bundle, but it is nevertheless an important object: it is the subset of the *real* cotangent bundle $T^*\Omega$ on which all the symbols of the sections of $T'^{\perp}$ vanish. In the terminology of partial differential equations it is the *characteristic set* of the differential operators defined by those sections. The structure is said to be *elliptic* when $T^0 = 0$, and *essentially real* when $T' = T^0 \otimes_{\mathbf{R}} \mathbf{C}$.

There is a class of submanifolds of Ω which will play a crucial role in the sequel. Let X be a C^q submanifold of Ω, and $\mathbf{C}N^*X$ its complex conormal bundle. Consider the quotient map

$$\pi_X : \mathbf{C}T^*\Omega|_X \to \mathbf{C}T^*\Omega|_X/\mathbf{C}N^*X \cong \mathbf{C}T^*X. \tag{2.1}$$

Definition 2.1. A C^q submanifold of Ω will be said to be maximally real if the quotient map π_X induces a bijection of $T'|_X$ onto $\mathbf{C}T^*X$.

If X is maximally real, then

$$\mathbf{C}T^*\Omega|_X = (T'|_X) \oplus \mathbf{C}N^*X \quad (\oplus : \text{direct sum}). \tag{2.2}$$

By duality, this is equivalent to

$$\mathbf{C}T\Omega|_X = (T'^{\perp}|_X) \oplus \mathbf{C}TX. \tag{2.3}$$

We see that the dimension of any maximally real submanifold is exactly equal to m.

If X is maximally real, and $(U, Z^1,\cdots,Z^m)$ is an arbitrary hypo-analytic local chart in Ω such that $U \cap X \neq \varnothing$, then the pull-backs to $U \cap X$ of $dZ^1,\cdots,dZ^m$ make up a basis of each complex cotangent space to $U \cap X$.

Let $(U, Z^1,\cdots,Z^m)$ be a hypo-analytic local chart in Ω. Possibly after contracting U about one of its points, p_0, we can find n C^q functions $u^1,\cdots,u^n$ in U such that $dZ^1,\cdots,dZ^m$, $du^1,\cdots,du^n$ span the whole cotangent space $\mathbf{C}T_p^*\Omega$ at every point p of U. Actually we may select, if we so wish, the functions u^i to be real-valued. We may then define N complex vector fields in U with C^{q-1} coefficients $L_1,\cdots,L_n$, $M_1,\cdots,M_m$ by the "orthonormality" conditions

$$\begin{aligned} &L_j Z^k = 0, \quad L_j u^i = \delta_j^i, \\ &M_l u^i = 0, \quad M_l Z^k = \delta_l^k, \quad i, j = 1,\cdots,n;\ k, l = 1,\cdots,m. \end{aligned} \tag{2.4}$$

Since $q \geqslant 2$ we have the right to consider the commutation brackets of the L's and of the M's. We derive at once from (2.4), for the same indices,

$$[L_i, L_j] = [L_j, M_k] = [M_k, M_l] = 0. \tag{2.5}$$

When the functions u^i are selected to be real-valued, we usually call them y^i (sometimes t^i). It is immediate that the submanifolds of U defined by the equations $y =$ const. are maximally real. Conversely, if X is any maximally real submanifold of Ω (of class C^q) passing through a point p_0, then C^q functions y^i can be defined in an open neighborhood U of p_0, as well as hypo-analytic functions $Z^1,\cdots,Z^m$, so that the preceding description applies.

Notice that besides the functions y^i we can also select m real-valued functions x^j in U, so that $(x^1,\cdots,x^m, y^1,\cdots,y^n)$ is a system of local coordinates in the open neighborhood U of p_0 (possibly contracted). Denote the Jacobian matrix of the Z^k's with respect to the x^j's by $\partial Z/\partial x$ or Z_x. Possibly after contraction of U, we may assume that

(2.6) *Z_x is nonsingular at every point of U.*

We shall customarily take the hypo-analytic functions Z^k and the local coordinates x^j, y^i to vanish at the "central" point p_0. Note then that

$$Z_x(0,0)^{-1}Z = x + W(x, y), \quad W(0,0) = 0, d_xW(0,0) = 0. \tag{2.7}$$

If we put $u^i = y^i$ in (2.4), we derive from those relations:

$$L_j = \frac{\partial}{\partial y^j} + \sum_{k=1}^{m} \lambda_j^k(x, y)\frac{\partial}{\partial x^k}, \quad j = 1,\cdots,n; \tag{2.8}$$

$$M_l = \sum_{k=1}^{m} \mu_l^k(x, y)\frac{\partial}{\partial x^k}, \quad l = 1,\cdots,m. \tag{2.9}$$

In (2.9) the matrix $(\mu_l^k)_{1\leq k,l\leq m}$ is the inverse of the Jacobian matrix Z_x. There are various expressions for the coefficients λ_j^k; for instance the one we could derive from the obvious equation

$$L_j = \frac{\partial}{\partial y^j} - \sum_{k=1}^{m} \frac{\partial Z^k}{\partial y^j}(x, y)M_k. \tag{2.10}$$

We shall be concerned with the C^1 functions (or with the distributions, when $q = +\infty$) in some open subset of Ω whose differential is a section (a distribution section) of T' over that subset. We shall refer to them as "*the solutions*". (In [16] they are called *RC-functions* or *RC-distributions.*) In a hypo-analytic local chart $(U, Z^1,\cdots,Z^m)$ like the one above, they are the solutions of the homogeneous equations

$$L_jh = 0, \quad j = 1,\cdots,n. \tag{2.11}$$

We recall the main properties of solutions, as stated and proved in [16, Chap. II]. Let $(U, Z^1,\cdots,Z^m)$ be a hypo-analytic local chart of the kind described above. Then U contains an open neighborhood U_0 of p_0 such that the following are true, whatever the C^1 solution h in U:

(2.12) *h is the limit, in $C^1(U_0)$, of a sequence of polynomials with complex coefficients in $Z^1, \cdots, Z^m$* (when $q = +\infty$ and different classes of solutions are considered, the convergence takes place in the "natural" distribution spaces, e.g., in the distribution sense when dealing with distribution solutions, in the C^k sense when dealing with C^k solutions, $0 \leqslant k \leqslant +\infty$, etc.).

(2.13) *There is a continuous function $\tilde{h}$ on $Z(U_0)$ such that $h = \tilde{h} \circ Z$ in U_0.*

Let now X be a maximally real C^q submanifold of Ω.

(2.14) *If the restriction of h to $X \cap U$ vanishes identically, then $h \equiv 0$ in an open neighborhood of $X \cap U$.*

When $q = +\infty$ more can be said about distribution solutions of (2.11). We recall the representation theorem of [2] ([16, Theorem 3.1, Chap. II]):

Now h is a distribution solution of (2.11) in U; U_0 is a suitable open neighborhood of p_0 contained in U. Then there are an integer $\nu \geqslant 0$ and a C^1 solution h_1 of (2.11) in U_0 such that, in U_0,

$$h = \left(\sum_{j=1}^{m} M_j^2 \right)^{\nu} h_1. \tag{2.15}$$

The vector fields M_j are the same as before (see (2.9)).

Still in the case $q = +\infty$, we note the following two results in [16, Chap. II, Theorem 2.3]:

(2.16) *If X is a maximally real C^∞ submanifold of Ω, and h a distribution solution in an open neighborhood of X, then the following hold*:

(2.17) *h is a C^∞ function transversally to X valued in the space of distributions with respect to the variables tangential to X*;

(2.18) *If the trace of h on X vanishes identically, then h itself vanishes identically in an open neighborhood of X.*

3. Local properties of hypo-analytic functions

Same notation as in the previous two sections. In particular, Ω is a hypo-analytic manifold of class C^q $(2 \leqslant q \leqslant +\infty)$ and dimension $N \geqslant 1$. It is obvious that any hypo-analytic function in an open set is a function of class C^q

there, and is also a solution. The following result characterizes hypo-analytic functions among solutions, and sheds some light on the meaning of hypo-analyticity. We reason in a hypo-analytic local chart $(U, Z^1, \cdots, Z^m)$ of the kind described in §2.

Theorem 3.1. *In order that a function* (*resp., a distribution, when* $q = +\infty$) *h in U be hypo-analytic in U it is necessary and sufficient that h be a C^q solution* (*of* (2.11)) *in U and that every point p_* in U have an open neighborhood $V_* \subset U$ such that, uniformly absolutely in V_*,*

$$h = \sum_{\alpha \in \mathbf{Z}_+^m} c_\alpha(p_*)\left[Z - Z(p_*)\right]^\alpha, \tag{3.1}$$

where the $c_\alpha(p_*)$ are uniquely determined complex numbers, depending on p_*.

Proof. The sufficiency is evident since (3.1) tells us that $h = \tilde{h} \circ Z$ where $\tilde{h}$ is the holomorphic function in an open neighborhood of $Z(p_*)$ defined by the property that

$$c_\alpha(p_*) = \left[(\partial/\partial z)^\alpha \tilde{h}\right]\left(Z(p_*)\right)/\alpha!, \quad \alpha \in \mathbf{Z}_+^m. \tag{3.2}$$

Conversely suppose h is hypo-analytic in U, and $\tilde{h}$ is a holomorphic function in an open neighborhood of $Z(p_*) = z_*$ such that $h = \tilde{h} \circ Z$ in a neighborhood of p_*. The series

$$\sum_{\alpha \in \mathbf{Z}_+^m} \left[Z(p) - z_*\right]^\alpha \left[(\partial/\partial z)^\alpha \tilde{h}\right](z_*)/\alpha!$$

converges absolutely uniformly in some open neighborhood $\mathcal{O}$ of $(p_*, Z(p_*))$ in $U \times \mathbf{C}^m$. By way of consequence it defines a continuous function $H(p, z_*)$ in $\mathcal{O}$, holomorphic with respect to z_*. Direct computation shows that the differential of $H(p, z_*)$ with respect to z_* is identically equal to zero, hence

$$H(p, z_*) = H(p, Z(p)) = \tilde{h}(Z(p)) = h(p).$$

This proves that (3.1) holds.

Remark 3.1. Let M_j be the vector fields defined by the orthonormality relations (2.4). In general, when $q < +\infty$, the monomials

$$M^\alpha = M_1^{\alpha_1} \cdots M_m^{\alpha_m}, \quad \alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbf{Z}_+^m,$$

are not necessarily well-defined. However, as differential operators acting on hypo-analytic functions, although these are only of class C^q, the M^α are well-defined. It suffices to observe that if $\tilde{h}$ is a holomorphic function in some open neighborhood of $Z(p_*)$ ($p_* \in U$), we have

$$M_j(\tilde{h} \circ Z) = \frac{\partial \tilde{h}}{\partial z^j} \circ Z. \tag{3.3}$$

Then, of course,

$$(3.4)\qquad M^{\alpha}(\tilde{h}\circ Z)=(\partial/\partial z)^{\alpha}\tilde{h}\circ Z,\quad \alpha\in \mathbf{Z}_{+}^{m}.$$

Note that if h is any hypo-analytic function in U, we have:

(3.5) *to every compact subset K of U there is a constant $C_K > 0$ such that*

$$\operatorname*{Max}_{K}|M^{\alpha}h|\leqslant C_K^{|\alpha|+1}\alpha!,\quad \forall\alpha\in \mathbf{Z}_{+}^{m}.$$

In a sense (rigorously when $q=+\infty$) one may say that the hypo-analytic functions in U are those solutions which are *analytic vectors* for the M_j. Also note that formula (3.2), in conjunction with (3.4), reads

$$(3.6)\qquad c_{\alpha}=M^{\alpha}h/\alpha!,\quad \alpha\in \mathbf{Z}_{+}^{m},$$

which shows, among other things, that the coefficients c_α in the "Taylor expansion" (3.1) are themselves hypo-analytic functions (of p_*) in the whole of U, a fact which is not immediately obvious in the proof of Theorem 3.1.

A function f in U is *flat* at a point p_* of U if to every integer $\nu\geqslant 0$ there is a constant $C_\nu > 0$ such that, for all points p in some neighborhood of p_* in U (independent of ν),

$$|f(p)-f(p_*)|\leqslant C_{\nu}\left[\operatorname{dist}(p,p_*)\right]^{\nu},$$

where the *distance* is defined by any reasonably smooth Riemannian metric on U. This means that $f-f(p_*)$ vanishes to infinite order at p_*, when $q=+\infty$ and the function f itself is C^∞.

Corollary 3.1. *If a hypo-analytic function h in U is flat at a point p_* of U, it is constant in some neighborhood of p_*.*

Proof. By virtue of the "Taylor expansion" (3.1) in order that h be flat at p_* it is necessary that $c_\alpha(p_*)=0$ for all $\alpha\neq 0$, whence the assertion by virtue of Theorem 3.1.

Theorem 3.2. *Let p_0 be any point of the hypo-analytic manifold Ω, and X any maximally real C^q submanifold of Ω passing through p_0. In order that a solution h in an open set containing X be hypo-analytic at p_0 it is necessary and sufficient that the trace h_X of h on X be hypo-analytic at p_0 (for the hypo-analytic structure on X induced by Ω).*

Proof. The necessity is obvious. Let us prove the sufficiency. For this assume there is a holomorphic function $\tilde{h}_X$ in some open neighborhood of $Z(p_0)$ in $\mathbf{C}^m$ (using a hypo-analytic local chart $(U, Z^1,\cdots,Z^m)$ whose domain U contains p_0) such that $h_X=\tilde{h}_X\circ Z$ on $U\cap X$. But clearly $\tilde{h}_X\circ Z$ is a hypo-analytic solution in an open neighborhood of p_0 in U. By the uniqueness result (2.14) (or (2.18) when $q=+\infty$), $h=\tilde{h}_X\circ Z$ in such a neighborhood.

Corollary 3.2. *Suppose that the manifold* Ω *is real-analytic and that so is the given hypo-analytic structure of* Ω. *Then in order that a distribution h in some open subset of* Ω *be hypo-analytic it is necessary and sufficient that it be an analytic solution in that open subset.*

Proof. The necessity follows trivially from the definition of a real-analytic hypo-analytic structure. Let us prove the sufficiency. Suppose h is an analytic solution in an open subset U of Ω containing a point p_0. Select arbitrarily a real-analytic maximally real submanifold X of Ω passing through p_0. The trace h_X of h on X is a real-analytic function in X, and therefore certainly hypo-analytic on X. It suffices then to apply Theorem 3.2.

4. A Fourier transform criterion of hypo-analyticity

In most of the present section we deal solely with a C^q manifold X equipped with a hypo-analytic structure of codimension zero (as before, $2 \leq q \leq +\infty$). The structure bundle of X is the whole complex cotangent bundle $\mathbf{C}T^*X$. The only maximally real (Definition 2.1) submanifold of X is X itself. We shall not talk of "solutions" in X in the sense of §2: as there are no vector fields L_j (see (2.4)) all C^1 functions, in fact all continuous (or even locally integrable) functions (and all distributions, when $q = +\infty$) are solutions.

For the sake of exposition we shall slightly strengthen the meaning of *hypo-analytic local chart* $(U, Z^1, \cdots, Z^m)$: we shall require that the mapping $Z = (Z^1, \cdots, Z^m) : U \to \mathbf{C}^m$ be a diffeomorphism (of class C^q) of U onto $Z(U)$, and also that U be the domain of local coordinates x^j $(1 \leq j \leq m)$ all vanishing at a "central point", to which we shall systematically refer as the *origin*, and which shall be denoted by 0. We shall also suppose $Z(0) = 0$. As before we denote by Z_x the Jacobian matrix of the Z^j with respect to the x^k. We shall always reason under the hypothesis

$$\forall x, y \in U, |\mathrm{Im}(Z(x) - Z(y))| \leq |\mathrm{Re}(Z(x) - Z(y))|/2. \tag{4.1}$$

Note that substitution of $Z_x(0)^{-1}Z(x)$ for $Z(x)$ and contraction of U about the origin can always bring us into a situation in which (4.1) is valid. We shall also assume that all derivatives of Z of order $< q + 1$ are bounded in U.

Definition 4.1. Let u be a compactly supported continuous function (distribution when $q = +\infty$) in U, and κ a number > 0. We shall write

$$F^\kappa(u; z, \zeta) = \int e^{-i\zeta \cdot Z(y) - \kappa\langle\zeta\rangle[z - Z(y)]^2} u(y)\, dZ(y),$$

where $z \in \mathbf{C}^m$, $\zeta \in \mathbf{C}_m$, $|\mathrm{Im}\,\zeta| < |\mathrm{Re}\,\zeta|$, $\langle\zeta\rangle = (\zeta_1^2 + \cdots + \zeta_m^2)^{1/2}$, and $[z]^2 = (z^1)^2 + \cdots + (z^m)^2$.

Note that by introducing the function $\tilde{u} = u \circ Z^{-1}$ on $Z(U)$, we have

$$F^{\kappa}(u; z, \zeta) = \int_{Z(U)} e^{-i\zeta\cdot z' - \kappa\langle\zeta\rangle[z-z']^2}\tilde{u}(z')\,dz'. \tag{4.2}$$

When u is a distribution, the integral defining F^{κ} must be understood as a duality bracket. As a matter of fact, let us take a closer look at the case where u is a compactly supported distribution ($q = +\infty$). Let M_j ($j = 1,\cdots,m$) be the vector fields in (2.4) and in (2.9) (keeping in mind that there are neither u^i's nor y^i's). Of course we may write

$$u = \sum_{|\alpha|\leq k} M^{\alpha}u_{\alpha}, \tag{4.3}$$

where $u_{\alpha} \in C_c^0(U)$. In the present situation, for any distribution f in U,

$$df = \sum_{j=1}^{m} M_j f\, dZ^j. \tag{4.4}$$

If then $u \in \mathcal{E}'(U)$, $\phi \in C^{\infty}(U)$,

$$\phi M_j u dZ^1 \wedge \cdots \wedge dZ^m = (-1)^{j-1}\big[d(\phi u\, dZ^1 \wedge \cdots \wedge \widehat{dZ^j} \wedge \cdots \wedge dZ^m) - ud(\phi dZ^1 \wedge \cdots \wedge \widehat{dZ^j} \wedge \cdots \wedge dZ^m)\big],$$

where the hatted factor must be omitted. In this manner we obtain the formula for integration by parts

$$\int_U \phi M_j u\, dZ = -\int_U u M_j \phi\, dZ. \tag{4.5}$$

Lemma 4.1. *Assume $q = +\infty$, $\kappa > 0$, $u \in \mathcal{E}'(U)$. Whatever $z \in \mathbf{C}^m$, $\zeta \in \mathbf{C}_m$, $|\operatorname{Im}\zeta| < |\operatorname{Re}\zeta|$, $\alpha \in \mathbf{Z}_+^m$, we have*

$$(\partial/\partial z)^{\alpha}(e^{i\zeta\cdot z}F^{\kappa}(u; z, \zeta)) = e^{i\zeta\cdot z}F^{\kappa}(M^{\alpha}u; z, \zeta). \tag{4.6}$$

Proof. In (4.5) put

$$\phi = \phi(z, y, \zeta) = \exp\{i\zeta\cdot[z - Z(y)] - \kappa\langle\zeta\rangle[z - Z(y)]^2\}.$$

Then $\partial\phi/\partial z^j = -M_j\phi$ with M_j acting in the variables y. This yields exactly (4.6) for $|\alpha| = 1$. Repeated use of the formula (4.6) when $|\alpha| = 1$ yields it for arbitrary α.

Corollary 4.1. *Same hypotheses as in Lemma 4.1. Then, for all $x \in U$, $\alpha \in \mathbf{Z}_+^m$, $\zeta \in \mathbf{C}_m$ with $|\operatorname{Im}\zeta| < |\operatorname{Re}\zeta|$,*

$$M^{\alpha}(e^{i\zeta\cdot Z(x)}F^{\kappa}(u; Z(x), \zeta)) = e^{i\zeta\cdot Z(x)}F^{\kappa}(M^{\alpha}u; Z(x), \zeta). \tag{4.7}$$

Follows at once from (3.3) (or (3.4)) and (4.6).

Remark 4.1. When $q \leq +\infty$, Formulas (4.6) and (4.7) can be applied to $u \in C_c^k(U)$ and $|\alpha| \leq k \leq q$.

Corollary 4.2. *Let u be given by* (4.3). *Then*

$$e^{i\zeta\cdot z}F^\kappa(u; z, \zeta) = \sum_{|\alpha|\leq k} (\partial/\partial z)^\alpha \{e^{i\zeta\cdot z}F^\kappa(u_\alpha; z, \zeta)\}. \tag{4.8}$$

Lemma 4.2. *Suppose that* (4.1) *holds. For any u in* $C_c^0(U)$ *we have, uniformly in U,*

$$u(x) = \lim_{\nu\to+\infty} (\nu/\pi)^{m/2} \int e^{-\nu[Z(x)-Z(y)]^2} u(y)\, dZ(y). \tag{4.9}$$

Proof. Put in the integrals at the right in (4.9), $y = x + t/\nu^{1/2}$. By availing ourselves of (4.1) we can easily show that the (uniform) limit of those integrals, as $\nu \to +\infty$, is equal to $u(x)$ multiplied by

$$\pi^{-m/2} \int e^{-St\cdot t} (\det S)^{1/2}\, dt, \tag{4.10}$$

where $S = {}^tZ_x(x)Z_x(x)$. Observe that the integral (4.10) is a holomorphic function of S in the region defined by (4.1). Since it is equal to one when all eigenvalues of S are strictly positive, it is equal to one everywhere.

Corollary 4.3. *Same hypotheses as in Lemma* 4.2. *We have*

$$u(x) = \lim_{\varepsilon\to+0} (2\pi)^{-m} \iint_{\xi\in\mathbf{R}_m} e^{i\xi\cdot[Z(x)-Z(y)]-\varepsilon|\xi|^2} u(y)\, dZ(y)\, d\xi. \tag{4.11}$$

Proof. It suffices to observe that, if $\varepsilon = 1/4\nu$,

$$e^{-\nu[Z(x)-Z(y)]^2} = (\varepsilon/\pi)^{m/2} \int_{\mathbf{R}_m} e^{i\xi\cdot[Z(x)-Z(y)]-\varepsilon|\xi|^2}\, d\xi.$$

Corollary 4.4. *Same hypotheses as in Lemma* 4.2. *Let* κ *be any number* > 0. *We have, uniformly in U,*

$$u(x) = (\kappa/4\pi^3)^{m/2} \lim_{\varepsilon\to+0} \int_{\substack{\xi\in\mathbf{R}_m\\ t\in\mathbf{R}^m}} e^{i\xi\cdot Z(x)-\varepsilon|\xi|^2} F^\kappa(u; t, \xi)\, |\xi|^{m/2}\, d\xi\, dt. \tag{4.12}$$

Proof. Apply Definition 4.1 and the fact that

$$\int e^{-\kappa|\xi|[t-Z(y)]^2}\, dt = (\pi/\kappa|\xi|)^{m/2}.$$

Then (4.11) and (4.12) are seen to be the same formula.

Corollary 4.5. *Suppose* $q = +\infty$, *and that* (4.1) *holds. Then* (4.12) *is valid for any* $u \in \mathcal{E}'(U)$ *with the limit in the sense of distributions.*

Proof. It suffices to apply (4.6) (with $z = \tau$) and use the fact that the action on $\exp(i\xi \cdot [Z(x) - \tau])$ of $\partial/\partial\tau^j$ is the same as that of $-M_j$.

Remark 4.2. For any $q \leqslant +\infty$, Formulas (4.9), (4.11), (4.12) can be applied to $u \in C_c^k(U)$, with $k \leqslant q$. The limits are then valid in $C^k(U)$. This follows at once from (4.5) and Remark 4.1.

Remark 4.3. We have only extended Formula (4.12) to compactly supported distributions (when $q + \infty$) but it is evident by the formula (4.5) for integration by parts that (4.9) and (4.11) also routinely extend (with the limits in the distribution sense).

We shall be interested in the following property of $u \in C_c^0(U)$ ($u \in \mathcal{E}'(U)$ when $q = \infty$):

(4.13)$_\kappa$ *There are an open neighborhood $V^{\mathbf{C}}$ of the origin in $\mathbf{C}^m$, an open cone Γ in $\mathbf{C}^m \backslash \{0\}$ containing $\mathbf{R}_m \backslash \{0\}$ and numbers R, $C > 0$ such that*

$$|F^\kappa(u; z, \zeta)| \leqslant Ce^{-|\zeta|/R}, \forall z \in V^{\mathbf{C}}, \zeta \in \Gamma.$$

Theorem 4.1. *Suppose that $u \in C_c^0(U)$ ($u \in \mathcal{E}'(U)$ when $q = \infty$) is hypo-analytic at the origin. Then, to every $\kappa > 0$ there is $c \geqslant 0$ such that if*

$$\forall x \in U, |\operatorname{Im} Z(x)| \leqslant c|\operatorname{Re} Z(x)| + \kappa |\operatorname{Re} Z(x)|^2/4, \tag{4.14}$$

then Condition (4.13)$_\kappa$ *is satisfied.*

Proof. We only give the proof when $q = \infty$ and $u \in \mathcal{E}'(U)$, the case $2 \leqslant q < \infty$ and $u \in C_c^0(U)$ will then be obvious. Note that the defintion of F^κ does not depend on the choice of the local coordinates in U. It is convenient to avail ourselves of (4.1) and take the $\operatorname{Re} Z^j$ ($j = 1, \cdots, m$) as coordinates. Then

$$Z^j = x^j + \sqrt{-1}\,\Phi^j(x), \quad j = 1, \cdots, m, \tag{4.15}$$

$$\forall x, y \in U, |\Phi(x) - \Phi(y)| \leqslant |x - y|/2. \tag{4.16}$$

Let $g \in C_c^\infty(U)$ have its support in an open neighborhood of the origin in which u is a hypo-analytic function. Assume furthermore that $0 \leqslant g \leqslant 1$ everywhere, and that $g \equiv 1$ in a neighborhood of the origin. We may use a representation of the kind (4.3) for $(1 - g)u$, and thus write

$$u = gu + \sum_{|\alpha| \leqslant K} M^\alpha v_\alpha,$$

where every $v_\alpha \in C_c^0(U)$ vanishes identically in an open neighborhood of the origin. We see that gu and the v_α all satisfy the hypotheses of Theorem 4.1. If we prove itş conclusion when each of these replaces u we derive the same result for u itself by linearity and Corollary 4.2 (and the Cauchy's inequalities). In other words we may, and shall, assume that u is a compactly supported continuous function. We shall write $u = \tilde{u} \circ Z$ with $\tilde{u} \in C_c^0(Z(U))$.

In the integral (4.2) we deform the domain of integration from $Z(U)$ to the image of $Z(U)$ under the mapping

(4.17) $$z' \mapsto w = z' - i\delta g(y)\zeta/|\zeta| \,.$$

where y is defined by the fact that $z' = Z(y)$, and δ is a small number > 0 chosen below, at any rate small enough that w remains in an open neighborhood of the origin in $\mathbf{C}^m$ to which $\tilde{u}$ extends holomorphically, and that the same is true of all the positive numbers $< \delta$, so that the deformation (4.17) is permitted.

We focus on the quantity

$$Q = \operatorname{Re}\{i\zeta \cdot w + \kappa\langle\zeta\rangle[z - w]^2\}/|\zeta|\,,$$

but actually only when $z = 0$ and $\zeta = \xi \in \mathbf{R}_m \backslash \{0\}$, in which case we call it Q_0. Writing $\dot{\xi} = \xi/|\xi|$ we observe that

$$\begin{aligned} Q_0 &= -\dot{\xi} \cdot \Phi(y) + \delta g(y) + \kappa\{|y|^2 - |\Phi(y) - \delta g(y)\dot{\xi}|^2\} \\ &\geqslant -|\Phi(y)| + \delta g(y) + \kappa\{|y|^2 - 2|\Phi(y)|^2 - 2\delta^2 g(y)^2\} \\ &\geqslant -c|y| + \frac{\kappa}{4}|y|^2 + \delta g(y)[1 - 2\delta\kappa g(y)], \end{aligned}$$

by the hypothesis (4.14), and by (4.15), (4.16). We first require

(4.18) $$4\kappa\delta < 1,$$

whence

(4.19) $$Q_0 \geqslant \frac{\kappa}{4}|y|^2 + \frac{\delta}{2}g(y) - c|y|\,.$$

Let $d > 0$ be such that $|y| \leqslant d$ implies $g(y) = 1$. Then $Q_0 \geqslant \frac{1}{2}\delta - cd$ for $|y| \leqslant d$. We shall therefore require

(4.20) $$4c < \delta/d.$$

On the other hand, for $|y| > d$, $Q_0 \geqslant (\frac{\kappa}{4}d - c)d$, and we require

(4.21) $$4c < \kappa d.$$

We reach the conclusion that whatever $y \in U$, $\xi \in \mathbf{R}_m \backslash \{0\}$, we have $Q_0 \geqslant 2/R > 0$. But then we shall also have $Q \geqslant 1/R$ for all z in $V^{\mathbf{C}}$ and ζ in Γ, if $V^{\mathbf{C}}$ is chosen small enough, and the cone Γ "thin" enough. From this the inequality in $(4.13)_\kappa$ follows easily. q.e.d.

We prove now a partial converse to Theorem 4.1. It will require that we strengthen our requirements on the map Z.

Theorem 4.2. *Suppose that there is a constant $B > 0$ such that*

(4.22) $$\forall x \in U, |\operatorname{Im} Z(x)| \leqslant B|\operatorname{Re} Z(x)|^2.$$

Then any function $u \in C_c^0(U)$ $(u \in \mathcal{E}'(U)$ *when* $q = \infty)$ *that satisfies* $(4.13)_\kappa$ *for some* $\kappa > 4B$ *is hypo-analytic at the origin.*

Proof. As in Theorem 4.1 we give the proof only when $q = \infty$ and $u \in \mathcal{E}'(U)$. We shall once again assume that the Z^j's are given by (4.15), with (4.16) being valid. Then (4.22) reads $|\Phi(x)| \leqslant B|x|^2$. The proof of Theorem 4.2 is based on Corollary 4.4.

We shall deform the domain of t-integration. For this we extend the map Z, which is to say, the map Φ, to the whole space $\mathbf{R}^m$, so that it continues to satisfy the hypotheses, namely (4.16) and (4.22) (this might require a slight shrinking of U); we shall take in fact Φ compactly supported in an arbitrary neighborhood of the closure of U. We shall then go from integration with respect to t over $\mathbf{R}^m$ to integration over the image of $\mathbf{R}^m$ via the map $t \mapsto Z(t)$. Then (4.12) reads

$$(4.23)\qquad \begin{aligned}&(4\pi^3/\kappa)^{m/2}u(x)\\ &\quad= \lim_{\varepsilon \to +0} \iint_{\substack{t \in \mathbf{R}^m \\ \xi \in \mathbf{R}_m}} e^{i\xi\cdot Z(x) - \varepsilon|\xi|^2} F^\kappa(u;\, Z(t), \xi)\, |\xi|^{m/2}\, dZ(t)\, d\xi.\end{aligned}$$

Notice that if $\kappa > 4B$, Property (4.22) implies (4.14) for $c = 0$. Let then $g \in C_c^\infty(U)$, $g \equiv 1$ in some open neighborhood $U_0 \subset U$ of the origin. Theorem 4.1 tells us that $(1 - g)u$ also satisfies $(4.13)_\kappa$. We have therefore the right to replace u by gu. In other words we have the right to assume that $\operatorname{supp} u$ is contained in as small an open neighborhood U_1 of the origin as we wish.

We shall first look at the integral

$$(4.24)\qquad I_d^{\kappa,\varepsilon}(z) = \int_{|Z(t)| \geqslant d} e^{i\xi\cdot z - \varepsilon|\xi|^2} F^\kappa(u;\, Z(t), \xi)\, |\xi|^{m/2}\, dZ(t)\, d\xi,$$

where $d > 0$ is at first arbitrarily chosen. We shall apply the following.

Lemma 4.3. *If the neighborhood* U_1 *containing* $\operatorname{supp} u$ *is small enough, the following is true, whatever* $\kappa > 0$.

(4.25) *To every* $d > 0$ *there is an open neighborhood* $V_{\kappa,d}^{\mathbf{C}}$ *of the origin in* $\mathbf{C}^m$ *such that, as* $\varepsilon \to +0$, *the entire functions* $I_d^{\kappa,\varepsilon}$ *converge uniformly in* $V_{\kappa,d}^{\mathbf{C}}$.

Proof of Lemma 4.3. We deform the domain of ξ-integration in (4.24), from $\mathbf{R}_m$ to the image of $\mathbf{R}_m$ under the map

$$(4.26)\qquad \xi \mapsto \zeta = \xi + 4iB|\xi|[z - Z(y)],$$

where y varies in U_1 and z in $V_{\kappa,d}^{\mathbf{C}}$. Our first requirement on these two neighborhoods is that they be small enough that, for $\xi \neq 0$,

$$(4.27)\qquad \forall y \in U_1,\ z \in V_{\kappa,d}^{\mathbf{C}},\quad |\operatorname{Im}\zeta| < |\operatorname{Re}\zeta|/3.$$

This is what enables us to perform the deformation (4.26). Furthermore it insures that

(4.28) $$\operatorname{Re}\langle\zeta\rangle \geqslant 2|\xi|/3, \quad |\operatorname{Im}\langle\zeta\rangle| \leqslant \operatorname{Re}\langle\zeta\rangle/3.$$

We have

(4.29) $$I_d^{\kappa,\varepsilon}(z) = \int_{|Z(t)|\geqslant d} e^{i\zeta\cdot[z-Z(y)]-\kappa\langle\zeta\rangle[Z(t)-Z(y)]^2-\varepsilon\langle\zeta\rangle^2}\langle\zeta\rangle^{m/2}u(y)\,dZ(y)\,dZ(t)\,d\zeta.$$

We look at the quantity

$$Q = -\operatorname{Re}\{i\zeta\cdot[z-Z(y)] - \kappa\langle\zeta\rangle[Z(t)-Z(y)]^2\}/|\xi|.$$

Actually we study it only when $z = 0$, in which case we call it Q_0. With the notation $\dot{\xi} = \xi/|\xi|$, we have

$$\begin{aligned} Q_0 &= -\dot{\xi}\cdot\Phi(y) + 4B(|y|^2 - |\Phi(y)|^2) \\ &\quad + \frac{\kappa}{|\xi|}\{(\operatorname{Re}\langle\zeta\rangle)(|t-y|^2 - |\Phi(t)-\Phi(y)|^2 \\ &\qquad + 2(\operatorname{Im}\langle\zeta\rangle)(t-y)\cdot(\Phi(t)-\Phi(y)))\} \\ &\geqslant -|\Phi(y)| + 3B|y|^2 + \frac{2}{9}\kappa|t-y|^2. \end{aligned}$$

We have availed ourselves of (4.28) and (4.16). Using then (4.22) yields

$$Q_0 \geqslant 2B|y|^2 + 2\kappa|t-y|^2/9.$$

But since $\kappa > 0$, the quantity at the right is bounded away from zero as $|Z(t)|$ remains $\geqslant d$. Therefore, if the neighborhood $V_{\kappa,d}^{\mathbf{C}}$ is small enough, there are positive constants c, c' such that

(4.30) $Q \geqslant c + c'|t|^2$, $\forall y \in U_1$, $z \in V_{\kappa,d}^{\mathbf{C}}$, $t \in \mathbf{R}^m$ *such that* $|Z(t)| \geqslant d$ *and* $\forall\xi \in \mathbf{R}_m\backslash\{0\}$.

This implies easily what we want. Suppose first that u is a continuous function. Then

(4.31) $$|I_d^{\kappa,\varepsilon}(z) - I_d^{\kappa,\varepsilon'}(z)| \leqslant c \iiint_{|Z(t)|\geqslant d} e^{-c|\xi|-c'|t|^2|\xi|}|e^{-\varepsilon\langle\zeta\rangle^2} - e^{-\varepsilon'\langle\zeta\rangle^2}||\xi|^{m/2}|u(y)|\,dy\,dt\,d\xi,$$

for some constant $C > 0$, whence the uniform convergence of the $I_d^{\kappa,\varepsilon}$ in $V_{\kappa,d}^{\mathbf{C}}$.

When u is an arbitrary distribution, belonging to $\mathcal{E}'(U_1)$, use a representation (4.3) with $u_\alpha \in C_c^0(U_1)$ for every α, and apply the preceding result to each

u_α. Then Corollary 4.2 in conjuction with the Cauchy's inequalities yields the desired convergence.

End of proof of Theorem 4.2. We select $d > 0$ so that the ball $|z| \leq d$ in $\mathbf{C}^m$ is contained in the neighborhood $V^{\mathbf{C}}$ of $(4.13)_\kappa$. From the inequality in $(4.13)_\kappa$ we derive that the integrals $J_d^{\kappa,\epsilon}(z)$, defined like $I_d^{\kappa,\epsilon}(z)$ except that the integration with respect to t is carried out on the ball $|Z(t)| < d$, satisfy an inequality analogous to (4.31), and therefore converge in $V_{\kappa,d}^{\mathbf{C}}$—perhaps after the latter neighborhood has been contracted about the origin. We conclude that the entire functions $I_d^{\kappa,\epsilon} + J_d^{\kappa,\epsilon}$ converge uniformly in $V_{\kappa,d}^{\mathbf{C}}$, to a holomorphic function $\tilde{u}$ whose restriction to $Z(U) \cap V_{\kappa,d}^{\mathbf{C}}$ must be, by (4.23), equal to

$$\left(4\pi^3/\kappa\right)^{m/2} u \circ Z^{-1}. \quad \text{q.e.d.}$$

The criteria provided by Theorems 4.1, 4.2 apply to distributions that are not necessarily compactly supported, after multiplication by cut-off functions which are equal to one in some neighborhood of the origin.

In dealing with a hypo-analytic manifold Ω whose structure has codimension ≥ 1 one may apply the criteria provided by Theorems 4.1, 4.2 to the traces of distribution solutions on maximally real submanifolds passing through the "central point" p_0. After this one can apply Theorem 3.2 to conclude that the distribution solution under study is hypo-analytic in a full neighborhood of p_0 in Ω.

II. MICROLOCAL HYPO-ANALYTICITY

1. Hypo-analytic wave-front set in hypo-analytic structures of codimension zero

In the whole Chapter II we shall assume that $q = \infty$. In the present section as well as the next one, we deal solely with a C^∞ hypo-analytic manifold X whose hypo-analytic structure has *codimension zero*. We assume that the local chart $(U, Z^1, \cdots, Z^m)$ and the local coordinates $x^1, \cdots, x^m$ satisfy the assumptions of §4, Chapter I. We also assume that U is convex in the x^j coordinates which will make the use of the remainder formula in the Taylor expansion easy.

We shall make use of the coordinates x^j to identify the portion of the tangent bundle $TX|_U$ of X which lies over U, to $U + i\mathbf{R}^m \subset \mathbf{C}^m$. Thus a tangent vector v to X at $x \in U$ is identified to the purely imaginary m-vector iv, and the pair (x, v) to $x + iv$.

We shall deal with *almost-analytic extensions* of the map Z. Such an extension is a C^∞ mapping $Z_\# : U + i\mathbf{R}^m \to \mathbf{C}^m$ such that $Z_\#(x) = Z(x)$ for every x in U, and

(1.1) $(\partial/\partial\bar{z}^j)Z^k_\#$ *vanishes to infinite order at* $\operatorname{Im} z = 0$ *for all* $j, k = 1,\cdots,m.$

It is seen at once that if $Z_\#$, $Z_{\#\#}$ are two almost-analytic extensions of Z the difference $Z_\# - Z_{\#\#}$ vanishes to infinite order at $\operatorname{Im} z = 0$.

In the forthcoming we deal with an *acute* and *open* (but never empty!) cone Γ in $\mathbf{R}^m\backslash\{0\}$ (a cone is a subset which is invariant under all dilations $v \mapsto \rho v$, $\rho > 0$; it is acute if it is contained in a strictly convex cone).

Let then A be any *open* subset of U, $\mathcal{O}$ any open subset in $U + i\mathbf{R}^m$ which contains A. We shall use the following notation:

$$\mathfrak{N}_{\mathcal{O}}(A, \Gamma) = \{Z_\#(x + iv) \in \mathbf{C}^m;\, x \in A, v \in \Gamma \cup \{0\}, x + iv \in \mathcal{O}\},$$

where $Z_\#$ is a given almost-analytic extension of Z. Of course $\mathfrak{N}_{\mathcal{O}}(A, \Gamma)$ depends on the choice of $Z_\#$ and the notation might have to be modified to indicate this fact if there is any risk of confusion. We shall denote by

$$\mathring{\mathfrak{N}}_{\mathcal{O}}(A, \Gamma)$$

the *interior* of the set $\mathfrak{N}_{\mathcal{O}}(A, \Gamma)$. We shall sometimes refer to sets like $\mathfrak{N}_{\mathcal{O}}(A, \Gamma)$, $\mathring{\mathfrak{N}}_{\mathcal{O}}(A, \Gamma)$ as *conoids*.

In the sequel we shall assume that all derivatives of the Z^j with respect to the x^k (of any order) are bounded in U. In particular, there is a constant $B \geqslant 0$ such that

(1.2) $\forall x, x^* \in U, |Z_x(x)^{-1}[Z(x) - Z(x^*)] - (x - x^*)| \leqslant B|x - x^*|^2.$

It will then be convenient to hypothesize

(1.3) $$\forall x, x^* \in U, B|x - x^*| < 1/2,$$

Lemma 1.1. *Let* $Z_\#$ *be an almost-analytic extension of* Z *to* $U + i\mathbf{R}^m$ *whose derivatives of any order are uniformly bounded in* $U + i\mathbf{R}^m$. *There is an open neighborhood* $U_\#$ *of* U *in* $U + i\mathbf{R}^m$ *which is mapped diffeomorphically by* $Z_\#$ *onto* $Z_\#(U_\#)$.

Proof. Write $Z_\#(x + iv) = f(x, v) + ig(x, v)$. Then, modulo $|v|^\infty$, we have $f_x - g_v \equiv f_v + g_x \equiv 0$, and therefore the Jacobian matrix of (f, g) with respect to (x, v) is congruent to $\left(\begin{smallmatrix} f_x & g_x \\ -g_x & f_x \end{smallmatrix}\right)$. Notice that the latter annihilates a $2m$-vector (a, b) if and only if $(f_x + ig_x)(a - ib) = 0$, and thus is nonsingular if and only if the complex $m \times m$ matrix $f_x + ig_x$ is nonsingular. But the latter is congruent $\operatorname{mod}|v|^\infty$ to Z_x, which is nonsingular by hypothesis. We reach the conclusion that we may select $U_\#$ close enough to U that $Z_\#$ has an injective differential at every point. We must show that $U_\#$ can be selected so that $Z_\#$ itself be injective.

Let ρ be an upper bound for $|\operatorname{Im} z|$ in $U_{\#}$. Suppose there are two points $x + iv, x^* + iv^*$ in $U_{\#}$ such that

$$Z_{\#}(x + iv) = Z_{\#}(x^* + iv^*).$$

We take a Taylor expansion of order two, about x and x^* respectively, of the two sides:

$$Z(x) + iZ_x(x)v + R(x, v)v \cdot v = Z(x^*) + iZ_x(x^*)v^* + R(x^*, v^*)v^* \cdot v^*,$$

where R is a matrix-valued C^∞ function in $U + i\mathbf{R}^m$. Because the derivatives of $Z_{\#}$ are bounded there is a constant $C > 0$ such that

$$\begin{aligned}
&|Z_x(x)^{-1}[Z(x) - Z(x^*)] + i(v - v^*)| \\
&= \left|i\left[Z_x(x)^{-1}Z_x(x^*) - I\right]v^* - Z_x(x)^{-1}[R(x, v)v \cdot v - R(x^*, v^*)v^* \cdot v^*]\right| \\
&\leq C\rho(|x - x^*| + |v - v^*|).
\end{aligned}$$

At this point we avail ourselves of (1.2) and (1.3). We get

$$|x - x^* + i(v - v^*)| \leq C\rho(|x - x^*| + |v - v^*|) + \tfrac{1}{2}|x - x^*|,$$

and it suffices to take $\rho < 1/4C$ to conclude that $x = x^*$, $v = v^*$.

Corollary 1.1. *Let $Z_{\#}$ be as in Lemma* 1.1. *Every open subset A of U has an open neighborhood $\mathcal{O}$ in $U + i\mathbf{R}^m$ such that $Z_{\#}$ is a diffeomorphism of $(A + i\Gamma) \cap \mathcal{O}$ onto $\mathfrak{N}_{\mathcal{O}}(A, \Gamma)$.*

Corollary 1.2. *Let $Z_{\#}$ be as in Lemma* 1.1. *Let A be a relatively compact open subset of U, and $\mathcal{O}$ an open neighborhood of A in $U + i\mathbf{R}^m$ such that $Z_{\#}$ is a diffeomorphism of $(A + i\Gamma) \cap \mathcal{O}$ onto $\mathfrak{N}_{\mathcal{O}}(A, \Gamma)$. Then there is a constant $C > 0$ such that*

$$|v|/C \leq \operatorname{dist}[Z_{\#}(x + iv), Z(A)] \leq C|v|$$

for all $x + iv$ in $(A + i\Gamma) \cap \mathcal{O}$.

Henceforth we assume that all the derivatives of any order of $Z_{\#}$ are bounded in $U + i\mathbf{R}^m$. Note, in passing, that the properties of $Z_{\#}$ far away from U do not really matter; $Z_{\#}$ might as well be supposed to vanish identically when $|\operatorname{Im} z|$ is "large".

Lemma 1.2. *Let $Z_{\#\#}$ be another almost-analytic extension of Z whose derivatives are all bounded in $U + i\mathbf{R}^m$. Let A be an open subset of U, and $\mathcal{O}$ an open neighborhood of A in the open set $U_{\#}$ of Lemma* 1.1. *Let A' be any relatively compact open subset of A, and Γ' any nonempty open cone in $\mathbf{R}^m \setminus \{0\}$ whose closure is contained in Γ. Then there is an open neighborhood $\mathcal{O}'$ of A' in $U + i\mathbf{R}^m$ such that*

$$\mathfrak{N}_{\mathcal{O}'}(A', \Gamma')_{\#\#} \subset \mathfrak{N}_{\mathcal{O}}(A, \Gamma)_{\#}.$$

The subscripts $\#$, $\#\#$ indicate which almost-analytic extension of Z, $Z_\#$ or $Z_{\#\#}$, is used in the definition of the conoid.

Proof. First we take $\mathcal{O}'$ so small that $Z_{\#\#}$ is a diffeomorphism on $\mathcal{O}' \cap (A + i\Gamma)$. By the same reason as in the first part of the proof of Lemma 1.1 we can apply the implicit function theorem to get

$$x = x(x', v'), \quad v = v(x', v'),$$

such that

$$Z_\#(x(x', v') + iv(x', v')) = Z_{\#\#}(x' + iv') \tag{1.4}$$

for $x' + iv' \in \mathcal{O}' \cap (A' + i\Gamma')$.

To complete the proof we observe that the differential of (1.4) with respect to v' at $v' = 0$ is

$$Z_x(x,0)\left[\frac{\partial x}{\partial v'}(x,0) + i\left(\frac{\partial v}{\partial v'}(x,0) - I\right)\right] \equiv 0 \quad (\text{mod}\,|v|^\infty). \tag{1.5}$$

It follows that

$$\frac{\partial v}{\partial v'}(x,0) \equiv I \quad (\text{mod}\,|v|^\infty).$$

So if $\mathcal{O}'$ is small enough the assertion will hold. q.e.d.

A similar argument will enable the reader to prove

Lemma 1.3. *Let A, $\mathcal{O}$, A', Γ' be as in Lemma 1.2. Then there is an open neighborhood $\mathcal{O}'$ of A' in $U + i\mathbf{R}^m$ such that*

$$\mathfrak{N}_{\mathcal{O}'}(A', \Gamma') \subset \{Z(x) + iZ_x(x)v;\, x + iv \in (A + i\Gamma) \cap \mathcal{O}\},$$

$$\{Z(x) + iZ_x(x)v;\, x + iv \in (A' + i\Gamma') \cap \mathcal{O}'\} \subset \mathfrak{N}_{\mathcal{O}}(A, \Gamma).$$

Henceforth, and otherwise specified, we make sole use of the extension $Z_\#$. We deal with the triplet $(A, \Gamma, \mathcal{O})$ dealt with so far.

Definition 1.1. We denote by $B'_{\mathcal{O}}(A, \Gamma)$ the space of holomorphic functions $\tilde{f}$ in $\mathring{\mathfrak{N}}_{\mathcal{O}}(A, \Gamma)$ which have the following property:

To every compact subset $\tilde{K}$ of $\mathfrak{N}_{\mathcal{O}}(A, \Gamma)$ there are an integer $k \geqslant 0$ and a constant $C > 0$ such that

$$|\tilde{f}(z)| \leqslant C(\text{dist}[z, Z(A)])^{-k} \tag{1.6}$$

for all z in $\tilde{K} \cap \mathring{\mathfrak{N}}_{\mathcal{O}}(A, \Gamma)$.

It is an easy consequence of Cauchy's inequalities that the partial derivatives $\partial/\partial z^j$ $(j = 1, \cdots, m)$ define linear maps of $B'_{\mathcal{O}}(A, \Gamma)$ into itself.

In the sequel we make sole use of open neighborhoods $\mathcal{O}$ of A such that $Z_\#$ maps diffeomorphically $(A + i\Gamma) \cap \mathcal{O}$ onto $\mathfrak{N}_{\mathcal{O}}(A, \Gamma)$. Let $\tilde{f}$ belong to $B'_{\mathcal{O}}(A, \Gamma)$ and write $f = \tilde{f} \circ Z_\#$. Then the following hold.

(i) *f is almost-analytic in* $(A + i\Gamma) \cap \mathcal{O}$. Explicitly, this means that if $A' \subset\subset A$, if the closure of the cone Γ' is contained in Γ, and if $|\operatorname{Im} z| \to 0$ while z remains in $A' + i\Gamma'$, then $(\partial f/\partial \bar{z})(z)$ tends to zero faster than any power of $|\operatorname{Im} z|$. This is a direct consequence of the holomorphy of $\tilde{f}$, the almost-analyticity of $Z_\#$, and Property (1.6).

(ii) If $\tilde{K}$ is a compact set as in (1.6), and we write $K = Z_\#^{-1}(\tilde{K})$, then, for a suitable $k \in \mathbf{Z}_+$ and $C > 0$,

$$|f(x + iv)| \leqslant C|v|^{-k}, \quad \forall x + iv \in K, v \neq 0. \tag{1.7}$$

This is a direct consequence of (1.6) and Corollary 1.2.

The preceding two properties enable us to define the *boundary value* (or trace) of f on A. We now describe how this is done.

Below, A' and Γ' are as in Property (i) above. Let $\delta > 0$ be such that

$$x \in A', v \in \Gamma', |v| < \delta \Rightarrow x + iv \in \mathcal{O}.$$

We denote by χ an arbitrary C^∞ function compactly supported in A'. Let then σ denote any measurable subset of the intersection of Γ' with the unit sphere $S^{m-1} \subset \mathbf{R}^m$. We define

$$F_\sigma(t) = |\sigma|^{-1} \iint_\sigma f(x + it\dot{v})\chi(x)\, dx\, d\dot{v},$$

where $d\dot{v}$ is the measure on S^{m-1} induced by the Lebesgue measure on $\mathbf{R}^m$, and $|\sigma|$ is the measure of the set σ.

Lemma 1.4. *The function F_σ defines a C^∞ function of t in the closed interval* $[0, \delta]$.

Proof. We have, for $t > 0$,

$$F_\sigma'(t) = |\sigma|^{-1} \iint_\sigma \dot{v} \cdot \frac{\partial f}{\partial y}(x + it\dot{v})\chi(x)\, dx\, d\dot{v}.$$

We take into account the almost-analyticity of f:

$$\frac{\partial f}{\partial y}(x + it\dot{v}) \equiv -D_x f(x + it\dot{v}), \quad D_x = -\sqrt{-1}\,\frac{\partial}{\partial x},$$

where $\equiv$ means *congruent modulo functions vanishing to infinite order at* $t = 0$. We obtain

$$F_\sigma'(t) \equiv |\sigma|^{-1} \iint_\sigma f(x + it\dot{v})(\dot{v} \cdot D_x\chi(x))\, dx\, d\dot{v}$$

by integration by parts, and

$$F_\sigma^{(j)}(t) \equiv |\sigma|^{-1} \iint_\sigma f(x + it\dot{v})(\dot{v} \cdot D_x)^j\chi(x)\, dx\, d\dot{v}, \quad \forall j \in \mathbf{Z}_+,$$

by iteration. Applying (1.7) gives

$$|F_\sigma^{(j)}(t)| \leqslant C_j t^{-k}, \quad j = 0, 1, \cdots, 0 < t \leqslant \delta.$$

If we integrate successively $(k+1)$ times from t to δ for every derivative $F_\sigma^{(j)}$, $j \geqslant k+1$, we reach easily the desired conclusion.

Lemma 1.5. *The value $F_\sigma(+0)$ does not depend on the measurable set σ, nor on the almost-analytic extension $Z_\#$ of Z.*

Proof. If τ is any other measurable subset of $\Gamma' \cap S^{m-1}$, we may write

$$F_\sigma(t) - F_\tau(t) = (|\sigma||\tau|)^{-1} \iiint_{\sigma\tau} [f(x + it\dot{v}) - f(x + it\dot{w})]\chi(x)\,dx\,d\dot{v}\,d\dot{w}.$$

But we also have

$$f(x + it\dot{v}) - f(x + it\dot{w}) = t(\dot{v} - \dot{w}) \cdot \int_0^1 \frac{\partial f}{\partial y}(x + it[\theta\dot{v} + (1-\theta)\dot{w}])\,d\theta,$$

which is congruent to

$$-t(\dot{v} - \dot{w}) \cdot D_x \int_0^1 f(x + it[\theta\dot{v} + (1-\theta)\dot{w}])\,d\theta,$$

modulo C^∞ functions of t vanishing to infinite order at $t = 0$. Thus

$$F_\sigma(t) - F_\tau(t) \equiv t\int_0^1 G_{\sigma,\tau}(t, \theta)\,d\theta, \tag{1.8}$$

where

$$G_{\sigma,\tau}(t) = (|\sigma||\tau|)^{-1} \iiint_{\sigma\tau} f(x + it[\theta\dot{v} + (1-\theta)\dot{w}])(\dot{v} - \dot{w})D_x\chi(x)\,dx\,d\dot{v}\,d\dot{w}.$$

At this point we take advantage of the hypothesis that Γ is acute. It implies that there is $c' > 0$ such that

$$|\theta\dot{v} + (1-\theta)\dot{w}| \geqslant c', \quad \forall \dot{v}, \dot{w} \in \Gamma' \cap S^{m-1}, 0 \leqslant \theta \leqslant 1.$$

But then exactly the same argument used in the proof of Lemma 1.4 works here, to show that $G_{\sigma,\tau}$ is a C^∞ function of (t, θ) in $[0, \delta] \times [0, 1]$. Thus (1.8) implies that $F_\sigma(+0) = F_\tau(+0)$.

Next let $Z_{\#\#}$ be another almost-analytic extension of Z in $U + i\mathbf{R}^m$. By applying Lemma 1.2 we see that if $\dot{v} \in \Gamma' \cap S^{m-1}$ and $0 < t \leqslant \delta'$ with $\delta' > 0$ small enough, then $Z_{\#\#}(x + it\dot{v})$ will remain in a compact subset of $\mathfrak{N}_0(A, \Gamma)_\#$ whatever x in A'. If $F_{\sigma\#}$ stands for what has been denoted F_σ so far, and $F_{\sigma\#\#}$ for the analogue when $Z_{\#\#}$ replaces $Z_\#$, we see that

$$F_{\sigma\#}(t) - F_{\sigma\#\#}(t) = |\sigma|^{-1} \iint_\sigma [\tilde{f}(Z_\#(x + it\dot{v})) - \tilde{f}(Z_{\#\#}(x + it\dot{v}))]\chi(x)\,dx\,d\dot{v}.$$

It suffices to apply the mean value theorem and to take advantage of the property that $Z_\# - Z_{\#\#}$ vanishes to infinite order at $\mathrm{Im}\, z = 0$, in order to reach the conclusion that $F_{\sigma\#}(+0) = F_{\sigma\#\#}(+0)$. q.e.d.

Thus we may define the boundary value bf of f on A by the formula

$$(1.9)\quad \int bf(x)\varphi(x)\, dZ(x) = \lim_{t\to+0} |\sigma|^{-1} \iint_\sigma f(x+it\dot{v})\varphi(x)\, dZ(x)\, d\dot{v},$$

where φ is an arbitrary element of $C_c^\infty(A)$, and σ is a relatively compact measurable subset of $\Gamma \cap S^{m-1}$ (apply what precedes to an open set $A' \subset\subset A$ containing supp φ^0, to a cone Γ' containing σ and whose closure is contained in Γ, and to $\chi = \varphi \det Z_x$).

Note however that f depends on the choice of $Z_\#$ whereas $\tilde{f}$ does not, and that we shall be interested in holomorphic extensions to conoids in $\mathbf{C}^m$ of the transfer of bf to $Z(A)$ via Z. Because of this for us the more important concept is the *boundary value* $b\tilde{f} = bf$ of $\tilde{f}$ on A. Note that A is not a subset of the space in which $\tilde{f}$ is defined; $b\tilde{f}$ is really the pull-back via Z of the boundary value of $\tilde{f}$ on $Z(A)$. But the terminology will be somewhat simplified if we make this small abuse of language.

The boundary value $\tilde{f} \to b\tilde{f}$ defines a linear map from $B'_{\mathcal{O}}(A, \Gamma)$ into $\mathcal{D}'(A)$.

Remark 1.1. Observe that in (1.9) we may let σ range over the net of open neighborhoods of a fixed unit vector $\dot{v}$. Actually, the requirement that $\dot{v}$ be a unit vector is irrelevant, and we may deal with any vector v in Γ. We obtain

$$(1.10)\quad \int bf(x)\varphi(x)\, dZ(x) = \lim_{t\to+0} \int \tilde{f}(Z_\#(x+itv))\varphi(x)\, dZ(x).$$

Actually, by combining (1.10) with the argument used in the first part of the proof of Lemma 1.5, one could also show that $\int bf\varphi\, dZ$ is the limit of the integrals

$$\int \tilde{f}(Z_\#(x+iv))\varphi(x)\, dZ(x)$$

as v ranges over any net in Γ, which converges to 0 in such a way that $x + iv$ remains in a compact subset of $(A + i\Gamma) \cap \mathcal{O}$ when x remains in supp φ.

Lemma 1.6. *Let $\dot{v}$ be any unit vector in Γ, and φ any element of $C_c^\infty(A)$. Then*

$$(1.11)\quad \int b\tilde{f}\varphi\, dZ = \lim_{t\to+0} \int \tilde{f}(Z(x) + itZ_x(x)\dot{v})\varphi(x)\, dZ(x).$$

Proof. Introduce the complex m-vector

$$\Delta(x,t) = Z(x) + itZ_x(x)\dot{v} - Z_\#(x+it\dot{v}).$$

If x belongs to supp φ and $0 < t \leqslant \delta$ with δ suitably small, we have $|\Delta(x,t)| \leqslant$ const. t^2. If δ is small enough, the polydisk centered at $Z_\#(x+it\dot{v})$ with radii

$|\Delta^j(x,t)|(j=1,\cdots,m)$ will be contained in a fixed compact subset of $\mathcal{N}_{\mathcal{O}}(A,\Gamma)$, and the distance from any point in that polydisk to $Z(A)$ will be of the order of t. We use a finite Taylor expansion

$$\tilde{f}(Z(x)+itZ_x(x)\dot{v}) = \sum_{|\alpha|\leq J} \Delta(x,t)^\alpha[(\partial/\partial z)^\alpha\tilde{f}](Z_\#(x+it\dot{v}))/\alpha! + \mathcal{R}_J(x,t). \tag{1.12}$$

From (1.6) it follows that

$$|[(\partial/\partial z)^\alpha\tilde{f}](Z_\#(x+it\dot{v}))| \leq \text{const.}\, t^{-k-|\alpha|}. \tag{1.13}$$

We may take an expansion of $\Delta(x,t)/t^2$ in powers of t, to order $J+k$. In this manner we get easily, for $0<|\alpha|\leq J$,

$$\frac{1}{\alpha!}\Delta(x,t)^\alpha = t^{2|\alpha|}\sum_{j=0}^{J+k} C_j^\alpha(x)t^j + 0(t^{J+k+3}). \tag{1.14}$$

Putting this into (1.12) yields easily

$$\begin{aligned}\int[\tilde{f}(Z(x)+itZ_x(x)\dot{v}) - \tilde{f}(Z_\#(x+it\dot{v}))]\varphi(x)\,dZ(x) \\ = \sum_{0<|\alpha|\leq J}\sum_{j=0}^{J+k} t^{2|\alpha|+j}\int \tilde{f}^{(\alpha)}(Z_\#(x+it\dot{v}))C_j^\alpha(x)\varphi(x)\,dZ(x) \\ + \int \mathcal{R}_J(x,t)\varphi(x)\,dZ(x) + 0(t^3).\end{aligned} \tag{1.15}$$

We also use the classical expression of the remainder in the Taylor series and the inequality similar to (1.13) when $Z_\#(x+it\dot{v})$ is replaced by any point in the polydisk centered at $Z_\#(x+it\dot{v})$ with radii $|\Delta^j(x,t)|$. We obtain at once that

$$|\mathcal{R}_J(x,t)| \leq \text{const.}\ t^{2J-(k+J+1)}.$$

We select $J \geq k+2$. In this manner we see that the left-hand side in (1.15) differs from the double sum (on α and j) on the right-hand side by a quantity whose absolute value is bounded by constant $\times\, t$. Lastly, we apply formula (1.9) to each term in the sum in question, we replace $\tilde{f}$ by $\tilde{f}^{(\alpha)}$ and take $\chi = C_j^\alpha \varphi \det Z_x$. Thus we conclude that each integral

$$\int \tilde{f}^{(\alpha)}(Z_\#(x+it\dot{v}))C_j^\alpha(x)\varphi(x)\,dZ(x)$$

converges as t goes to zero. But since it is multiplied in (1.15) by t raised to a power no less than two, we see that the right-hand side in (1.15) converges entirely to zero. Formula (1.11) hence follows from this fact and (1.9).

Corollary 1.3. *Let $x \mapsto v(x)$ be a C^∞ map of A into Γ. Then, whatever $\varphi \in C_c^\infty(A)$,*

$$(1.16)\qquad \int b\tilde{f}\,\varphi\, dZ = \lim_{t\to+0} \int \tilde{f}(Z(x) + itZ_x(x)v(x))\,\varphi(x)\, dZ(x).$$

Proof. It suffices to prove (1.16) when $\operatorname{supp}\varphi$ is contained in an open set $V \subset A$ as small as we wish, for then we may patch up the various limit formulas by means of a partition of unity. If V is small enough we may select a diffeomorphism $\hat{x} \mapsto x = x(\hat{x})$ of an open subset $\hat{V}$ of $\mathbf{R}^m$ onto V such that $\hat{v} = (\partial x/\partial\hat{x})^{-1}v$ is constant in V. Set $\hat{Z}(\hat{x}) = Z(x(\hat{x}))$; we have $Z_x v = \hat{Z}_{\hat{x}}\hat{v}$, and there obviously is an open cone $\hat{\Gamma}$ containing $\hat{v}$ such that $\tilde{f}$ is holomorphic in the conoid $\{z \in \mathbf{C}^m;\ z = \hat{Z}(\hat{x}) + i\hat{Z}_{\hat{x}}(\hat{x})\hat{v},\ \hat{x} \in \hat{V},\ \hat{v} \in \hat{\Gamma},\ |\hat{v}| \text{ small}\}$. By changing variables from x to $\hat{x}$ in the integral in (1.16), and applying (1.11) with Z and x hatted we obtain at once (1.16). q.e.d.

Formula (1.16) may be viewed as the stepping stone to a completely invariant definition of the boundary values of holomorphic functions in certain conoids (functions which satisfy inequalities such as the one in (1.6)). First of all, instead of dealing with a "constant" cone Γ, we should deal with a variable cone $\Gamma(x)$. More accurately we should replace $A \times \Gamma$ by a conic open subset $\tilde{\Gamma}$ of $TX\backslash 0|_A$ whose base projection is equal to A. We define then $\mathfrak{N}_{\mathcal{O}}(\tilde{\Gamma})$ as the image of $\tilde{\Gamma}$ via the mapping

$$(1.17)\qquad TX|_A \ni (x, v) \mapsto Z(x) + iZ_x(x)v \in \mathbf{C}^m,$$

where $\mathcal{O}$ would be regarded as an open neighborhood of the *zero section* in $TX|_A$. The definition of $B'_{\mathcal{O}}(\tilde{\Gamma})$ would be entirely similar to Definition (1.1), and formula (1.16) would still be valid if we take $v(x)$ to be a smooth section of $\tilde{\Gamma}$ over A. We leave the details to the reader.

Definition 1.2. Let u be a distribution in U, and (x, ξ) a point in $U \times (\mathbf{R}_m\backslash\{0\})$. We say that u is hypo-analytic at (x, ξ) if there are an open neighborhood $V \subset U$ of x, an open neighborhood $\mathcal{O}$ of V in $\mathbf{C}^m$ and a finite collection of nonempty acute open cones Γ_k in $\mathbf{R}^m\backslash\{0\}$ $(k = 1,\cdots,\nu)$, such that the following hold:

$$(1.18)\qquad \text{for every } k = 1,\cdots,\nu \text{ and every } v \in \Gamma_k,\quad \langle \xi, v\rangle < 0;$$

(1.19) for each $k = 1,\cdots,\nu$, there is $\tilde{f}_k \in B'_{\mathcal{O}}(V, \Gamma_k)$ (Def. 1.1) such that in V

$$u = b\tilde{f}_1 + \cdots + b\tilde{f}_\nu.$$

The consideration which precedes Definition 1.2 should make it quite clear that (x, ξ) (in Definition 1.2) is a point in the cotangent bundle T^*X, and that if (x, ξ) is thus interpreted, Definition 1.2 is independent of the local coordinates $x^1, \cdots, x^m$ used.

Suppose now that we change the basic hypo-analytic functions $Z^1, \cdots, Z^m$, *i.e.*, that we make use of a holomorphism $z \mapsto H(z)$ of an open neighborhood of $Z(U)$ in $\mathbf{C}^m$ onto an open subset of $\mathbf{C}^m$. We may then avail ourselves of the limit formula (1.10), and replace $Z_\#$, $\tilde{f}$ and $\varphi(x)$ respectively by $H \circ Z_\#$, $\tilde{f} \circ H^{-1}$ and

$$\varphi(x)\left[\det \frac{\partial H}{\partial z}(Z(x))\right]^{-1}.$$

This actually shows that the boundary value $b\tilde{f}$ is independent of the choice of the Z^k's, when regarded as a distribution in an open subset of U.

Remark 1.2. In all this we are regarding distributions as *currents of degree zero* (see [14, Chapter IX]) on the manifold X. They are functionals on the space of compactly supported C^∞ m-forms, which can always be put in the form $\varphi\, dZ$.

The independence of Definition 1.2 from the choice of the local coordinates and the hypo-analytic functions $Z^1, \cdots, Z^m$ allows us to "globalize" Definition 1.2. Thus given an arbitrary distribution u in an open subset Y of X, we may consider the set of points (x, ξ) in $T^*X\backslash 0\,|_Y$ at which u is hypo-analytic. This in turn allows us to introduce the following defintion.

Definition 1.3. The complement in $T^*X\backslash 0\,|_Y$ of the set of points at which the distribution u in Y is hypo-analytic will be called the hypo-analytic wave-front set of u and denoted by $WF_{ha}(u)$.

Since by Definition 1.2 the set of points $(x, \xi) \in T^*X\backslash 0\,|_Y$ at which a distribution u in Y is hypo-analytic is evidently open and conic, $WF_{ha}(u)$ is a *closed conic* subset of $T^*X\backslash 0\,|_Y$.

The hypo-analytic wave-front set is invariant under hypo-analytic isomorphism (this is a hypo-analytic map which is a diffeomorphism, and whose inverse is hypo-analytic).

It is evident that the base projection of the hypo-analytic wave-front set of a distribution is contained in its hypo-analytic singular support (Def. 1.3, Chap. I). Actually, as we are going to see soon, that projection is identical to the hypo-analytic singular support of the distribution.

2. A Fourier transform criterion of microlocal hypo-analyticity in hypo-analytic structures of codimension zero

Our aim in the present section is to establish the microlocal analogues of the results of §4 of Chapter I. We shall make use of the concepts and notation introduced there, and primarily of the "Fourier integral" F^κ (Def. 4.1, *loc cit.*)

We carry out the analysis in a hypo-analytic local chart $(U, Z^1,\cdots,Z^m)$ of the manifold X satisfying all the conditions in §4, *loc. cit.*, in particular (4.1). We begin by stating the microlocal version of Property $(4.13)_\kappa$, *ibid.* Below, and until otherwise specified, u is an element of $\mathcal{E}'(U)$. By ξ^0 we denote a nonzero *real m*-vector, *i.e.*, $\xi^0 \in \mathbf{R}_m\backslash\{0\}$. The reader may think of ξ^0 as a cotangent vector to X at the origin (the central point of U). Thus the property we are interested in reads:

$(2.1)_\kappa$ *There exist an open neighborhood $V^{\mathbf{C}}$ of the origin in $\mathbf{C}^m$, an open cone $\mathcal{C}^0$ in $\mathbf{C}_m\backslash\{0\}$ containing ξ^0, and numbers R, $C>0$ such that*

$$|F^\kappa(u;\, z, \zeta)| \leqslant Ce^{-|\zeta|/R}, \quad \forall z \in V^{\mathbf{C}}, \zeta \in \mathcal{C}^0.$$

The microlocal version of Theorem 4.1 of Chapter I is

Theorem 2.1. *Suppose that* $(0, \xi^0) \in T_0^*X\backslash 0$ *does not belong to the hypo-analytic wave-front set of* $u \in \mathcal{E}'(U)$. *Then to every* $\kappa > 0$ *there is* $c > 0$ *such that if the following holds with* $\dot{\xi}^0 = \xi^0/|\xi^0|$:

(2.2) $$\forall x \in U, \quad \dot{\xi}^0 \cdot \operatorname{Im} Z(x) \leqslant c\,|\operatorname{Re} Z(x)| + \kappa\,|\operatorname{Re} Z(x)|^2/4,$$

then Condition $(2.1)_\kappa$ *is satisfied.*

Proof. Since F^κ does not depend on the choice of local coordinates, we take these to be the functions $\operatorname{Re} Z^j$ $(j = 1,\cdots,m)$, as we may thank to (4.1) of Chapter I. Then (4.15), (4.16), *ibid.*, hold.

Let V, Γ_k, $\tilde{f}_k$, be as in Definition 1.2. Select $g \in C_c^\infty(V)$ equal to one in an open neighborhood $V_0 \subset V$ of the origin. It will then suffice to prove the estimate in $(2.1)_\kappa$ when each $b\tilde{f}_k$ is substituted for u. We shall therefore omit the subscript k and assume that $u = g\,b\tilde{h}$ for some $\tilde{h} \in B'_{\mathcal{O}}(V, \Gamma)$, where Γ is a nonempty strictly convex cone in $\mathbf{R}^m\backslash\{0\}$ such that $\xi^0 \cdot v < 0$ for every $v \in \Gamma$. For the sake of simplicity we write

$$\tilde{f}(z^*) = e^{-i\zeta\cdot z^* - \kappa\langle\zeta\rangle[z-z^*]^2}\tilde{h}(z^*).$$

Note that

(2.3) $$\int b\tilde{f}\, g\, dZ = F^\kappa(u;\, z, \zeta).$$

We shall make use of the following immediate consequence of (1.11):

$$\int b\tilde{f}\, g\, dZ = \lim_{t\to+0}\int \tilde{f}[Z(x) + itZ_x(x)v]\,g(x)\,d[Z(x) + itZ_x(x)v],$$

where v is an arbitrary vector in Γ, kept fixed in the sequel. For any $t > 0$ call $\mathfrak{S}_t$ the image of V under the map

$$x \mapsto z^* = Z(x) + itZ_x(x)v.$$

Let A be a relatively compact open subset of V containing supp g. There is $\delta > 0$ such that if (x, t) varies in $A \times [0, \delta]$, z^* remains in a compact subset of $\mathfrak{N}_\Theta(V, \Gamma)$. Furthermore, if t stays fixed, $0 < t \leq \delta$, the image of A under the map $x \mapsto z^*$ is a totally real m-dimensional (*i.e.*, a maximally real) C^∞ submanifold of $\mathring{\mathfrak{N}}_\Theta(A, \Gamma)$. Let us write $\tilde{g}(z^*, t) = g(x)$. Then the above formula reads

$$\text{(2.4)} \qquad \int b\tilde{f}\, g\, dZ = \lim_{t \to +0} \int_{\mathfrak{S}_t} \tilde{f}(z^*)\tilde{g}(z^*, t)\, dz^*.$$

We shall now further deform the domain of integration with respect to z^*. We select a function $g_0 \in C_c^\infty(V_0)$, $g_0 \equiv 1$ in an open ball $\{x; |x| < d\}$, $0 \leq g_0 \leq 1$ everywhere. We restrict the variation of t to an interval $0 \leq t \leq \delta'$ $< \delta$ and select a number $s > 0$ such that $s + \delta' < \delta$; s will be more precisely determined below. Then we deform the domain of integration in the integral at the right in (2.4), from $\mathfrak{S}_t$ to the image of V under the map

$$\text{(2.5)} \qquad x \mapsto z^* = Z(x) + i[t + sg_0(x)]\, Z_x(x)v.$$

Since the deformation occurs in the region where $f(z^*)g(z^*, t)$ is equal to $f(z^*)$ and is holomorphic, Stokes' theorem yields

$$\int_{\mathfrak{S}_t} \tilde{f}(z^*)\tilde{g}(z^*, t)\, dz^*$$

$$= \int \tilde{f}[Z(x) + i(t + sg_0(x))Z_x(x)v]\, g(x) d[Z(x) + i(t + sg_0(x))Z_x(x)v].$$

By duplicating the proof of Lemma 1.4, from the above equation one can derive

$$\text{(2.6)} \qquad \begin{aligned} &\left(\frac{d}{dt}\right)^j \int_{\mathfrak{S}_t} \tilde{f}(z^*)\tilde{g}(z^*, t)\, dz^* \\ &\quad = \int \tilde{f}[Z(x) + i(t + sg_0(x))Z_x(x)v]\, P_j(t, x, D_x)g(x)\, dz^*(x) \end{aligned}$$

where $P_j(t, x, D_x)$ is a linear partial differential operator with respect to x, of order $\leq j$, whose coefficients are C^∞ functions of (t, x) in $[0, \delta'] \times V$. Indeed, and provided δ is small enough, the Jacobian matrix of the map (2.5) with respect to the x^j is nonsingular and, as a consequence, there is a complex vector field $L(t, x, D_x)$ with C^∞ coefficients such that

$$\begin{aligned} &\frac{\partial}{\partial t}\tilde{f}[Z(x) + i(t + sg_0(x))Z_x(x)v] \\ &\qquad = L(t, x, D_x)\tilde{f}[Z(x) + i(t + sg_0(x))Z_x(x)v]. \end{aligned}$$

Integration by parts and iteration then yields (2.6).

Suppose now that there are an integer $k \geqslant 0$ and a constant $K > 0$ such that, for a suitable choice of s, $0 < s < \delta - \delta'$,

$$|\tilde{f}[Z(x) + i(t + sg_0(x))Z_x(x)v]| \leqslant Kt^{-k} \tag{2.7}$$

for all x in V, and t in $[0, \delta']$. The exponent k in (2.7) is determined by our hypothesis that, for all $(x, t) \in A \times [0, \delta']$, and all s, $0 \leqslant s \leqslant \delta - \delta'$,

$$|\tilde{h}[Z(x) + i(t + sg_0(x))Z_x(x)v]| \leqslant \text{const. } t^{-k}.$$

Putting (2.7) into (2.6) implies that, for some $C_j > 0$ and all t in $[0, \delta']$, we have

$$\left|(d/dt)^j \int_{\mathfrak{S}_t} \tilde{f}(z^*)\tilde{g}(z^*, t)\, dz^*\right| \leqslant C_j K t^{-k}.$$

We take $j = k + 2$ and integrate $k + 2$ times the left-hand side in (2.6) from δ' to t. We get, with a suitable $C > 0$,

$$\left|\int_{\mathfrak{S}_t} \tilde{f}(z^*)\tilde{g}(z^*, t)\, dz^*\right| \leqslant CK,$$

and

$$|F^\kappa(u; z, \zeta)| \leqslant CK, \tag{2.8}$$

after letting t go to zero, and taking (2.3) and (2.4) into account. We must therefore determine a bound K in (2.7), which will yield the inequality in $(2.1)_\kappa$.

Keeping in mind what is the expression of $\tilde{f}$ it is obvious that the nature of the bound K is determined by a lower bound for the quantity

$$Q = \operatorname{Re}\{i\zeta \cdot z^* + \kappa\langle\zeta\rangle[z - z^*]^2\}/|\zeta|,$$

when z^* is given by (2.5) (and $x \in \operatorname{supp} g$).

By hypothesis there is $c_0 > 0$ such that

$$\operatorname{Re}\langle\zeta, v\rangle < -c_0, \tag{2.9}$$

when $\zeta = \xi^0$. Our first requirement on the cone $\mathcal{C}^0$ will be that (2.9) must hold for all ζ in $\mathcal{C}^0$. This said we begin by looking at Q when $z = 0$, $\zeta = \xi^0$ in which case we write Q_0 rather than Q. For the sake of simplicity we take ξ^0 and v to be unit covectors. Putting z^* as given by (2.5), and recalling that $Z(x) = x + i\Phi(x)$, we get

$$\begin{aligned} Q_0 &= -\xi^0 \cdot \Phi(x) - (t + sg_0(x))\xi^0 \cdot v \\ &\quad + \kappa\{|x - (t + sg_0(x))\Phi_x(x)v|^2 - |\Phi(x) + (t + sg_0(x))v|^2\} \\ &\geqslant -c|x| - \kappa|x|^2/4 + c_0(t + sg_0(x)) \\ &\quad + \kappa\left\{\frac{4}{5}|x|^2 - \frac{6}{5}|\Phi(x)|^2\right\} - 7\kappa(t + sg_0(x))^2. \end{aligned}$$

We have taken advantage of (2.2), (2.9) and (4.16) of Chapter I.

By (4.16) of Chapter I, we know that $|\Phi(x)| \leqslant |x|/2$. We then select δ' and s in such a way as to have

$$14(\delta' + s) < c_0/\kappa.$$

We obtain

$$Q_0 \geqslant -c|x| + \kappa|x|^2/4 + c_0(t + sg_0(x))/2$$
$$\geqslant \kappa|x|^2/4 - c|x| + c_0 sg_0(x)/2.$$

We shall then put the following conditions on the number c in (2.2):

$$(2.10) \qquad 0 < c < \mathrm{Min}(\kappa d/8, \kappa s^2/d).$$

We recall that d is the positive number such that $|x| \leqslant d$ implies $g_0(x) = 1$. If $|x| > d$, we have $Q_0 \geqslant (\kappa d/4 - c)d \geqslant \kappa d^2/8$. If $|x| \leqslant d$, then $Q_0 \geqslant c_0 s/2 - cd \geqslant 3\kappa s^2 - cd \geqslant 2\kappa s^2$.

This yields a lower bound $Q \geqslant 2/R > 0$ for all x in supp g when $z = 0$ and $\zeta = \xi^0$. But the estimate $Q \geqslant 1/R$ will then hold if we allow z and $\zeta/|\zeta|$ to vary in complex neighborhoods of the origin and ξ^0 respectively. In this manner we conclude that (2.7) does indeed hold with

$$K = \text{const.}\ \exp(-|\zeta|/R),$$

and thus (2.8) is shown to be the same as the inequality in $(2.1)_k$. q.e.d.

Next we prove the microlocal version of Theorem 4.2 of Chapter I.

Theorem 2.2. *Suppose that*

$$(2.11) \qquad \mathrm{Im}\, Z_x(0) = 0.$$

Then there is a number $\kappa_* > 0$ such that, given any cotangent vector $\xi^0 \neq 0$ to X at the origin and any compactly supported distribution u in U, if $(2.1)_k$ holds for some $\kappa > \kappa_*$, then $(0, \xi^0) \notin WF_{ha}(u)$.

Proof. Hypothesis (2.11) allows us to require, first of all,

$$\kappa_* \geqslant 4 \sup_{x \in U} \{|\mathrm{Im}\, Z(x)|/|\mathrm{Re}\, Z(x)|^2\},$$

which insures that if $\kappa > \kappa_*$, then Condition (4.14) of Chapter I is satisfied whatever $c > 0$. We select $g \in C_c^\infty(U)$, $g = 1$ in a neighborhood of the origin and apply Theorem 4.1., *loc cit.*, to $(1 - g)u$. We conclude that $(1 - g)u$ satisfies $(2.1)_\kappa$. Assuming that u also does, the same is true of gu. In other words we may assume that supp u is contained in an open neighborhood $U_1 \subset U$ of the origin as small as we wish.

We shall assume, once again, that $Z(x) = x + \sqrt{-1}\,\Phi(x)$, with all the hypotheses in Theorem 4.1, *ibid.*, fulfilled. As we did in the proof of Theorem 4.2 of Chapter I, we extend Φ, and therefore Z, to the whole of $\mathbf{R}^m$ in such a way that our basic hypotheses be satisfied. Our starting point will then be the limit formula (4.23), *ibid.*, and we shall avail ourselves of Lemma 4.3 *ibid.* The number $d > 0$ will be chosen so that the closed ball $\{z \in \mathbf{C}^m; |z| \leq d\}$ be contained in the open neighborhood $V^{\mathbf{C}}$ in $(2.1)_\kappa$. We reach thus the conclusion that, modulo a holomorphic function in some open neighborhood of the origin in $\mathbf{C}^m$, we have

$$\tilde{u} \equiv (\kappa/4\pi^3)^{m/2} \lim_{\varepsilon \to +0} J_d^{\kappa,\varepsilon}, \tag{2.12}$$

where $\tilde{u}$ is the transfer of u via Z from U to $Z(U)$, and

$$J_d^{\kappa,\varepsilon}(z) = \iiint_{|Z(t)|\leq d} e^{i\xi\cdot[z-Z(y)]-\kappa|\xi|[Z(t)-Z(y)]^2-\varepsilon|\xi|^2} |\xi|^{m/2} u(y)\, dZ(y)\, dZ(t)\, d\xi.$$

In the remainder of the proof we assume that $|Z(x)| \leq d$ implies $x \in U$ and that $|x| < \frac{1}{2}$ for all x in U. We deform the domain of ξ-integration from $\mathbf{R}_m$ to the image of $\mathbf{R}_m$ under the map

$$\xi \mapsto \zeta = {}^tZ_x(t)^{-1}\xi + 4iB'|\xi|[z - Z(t)], \tag{2.13}$$

where B' is a positive number such that

$$\left\|\mathrm{Re}\left[Z_x(t)^{-1}Z_x(x)\right] - I\right\| \leq B'|t||x - t|, \tag{2.14}$$

$$\left|\mathrm{Im}\, Z_x(t)^{-1}[Z(x) - Z(t)]\right| \leq B'|x - t|^2. \tag{2.15}$$

That B' exists is a consequence of (2.11).

We strengthen our requirements on d and on the neighborhood $V^{\mathbf{C}}$ of the origin in which z is allowed to vary. We shall require that they be small enough that if $z \in V^{\mathbf{C}}$ and $|Z(t)| \leq d$, then the complex vector ζ in (2.13) satisfy the following:

$$\forall \xi \in \mathbf{R}_m \backslash \{0\}, \quad 4|\mathrm{Im}\langle\zeta\rangle| < \tfrac{1}{4}|\xi| < \mathrm{Re}\langle\zeta\rangle; \tag{2.16}$$

(2.17) if ξ stays in an acute and open cone $\underline{C}^0 \subset \mathbf{R}_m\backslash\{0\}$ containing ξ^0, then ζ remains in the cone $\mathcal{C}^0$ of $(2.1)_\kappa$.

We then select a finite collection of acute and open cones $\underline{C}^j$ $(j = 1, \cdots, \nu)$ in $\mathbf{R}_m\backslash\{0\}$ endowed with the following properties:

(2.18) $\underline{C}^j \cap \underline{C}^k = \varnothing$ if $j \neq k$, $j, k = 0, 1, \cdots, \nu$;
$\mathbf{R}_m\backslash(\underline{C}^0 \cup \underline{C}^1 \cdots \cup \underline{C}^\nu)$ *has measure zero*;

(2.19) *If* $j = 1, \cdots, \nu$, *the closed convex hull of* $\underline{C}^j$ *does not contain* ξ^0.

Regarding ζ in (2.13) as a function of $\xi \in \mathbf{R}_m$ we set

$$J_{d,j}^{\kappa,\varepsilon}(z) = \iiint_{\substack{|Z(t)|\leq d \\ \xi\in\underline{C}^j}} e^{i\zeta\cdot[z-Z(y)]-\kappa\langle\zeta\rangle[Z(t)-Z(y)]^2-\varepsilon\langle\zeta\rangle^2} \langle\zeta\rangle^{\frac{m}{2}} u(y)\, dZ(y)\, dZ(t)\, d\zeta.$$

Observe that

$$J_{d,0}^{\kappa,\varepsilon}(z) = \iint\limits_{\substack{|Z(t)|\leqslant d\\ \xi\in \underline{C}^0}} e^{i\zeta\cdot z}F^{\kappa}(u; Z(t),\zeta)e^{-\varepsilon\langle\zeta\rangle^2}\langle\zeta\rangle^{m/2}dZ(t)\,d\zeta.$$

We may therefore take advantage of $(2.1)_\kappa$, where we put $z = Z(t)$ and ζ is given by (2.13) with $\xi \in \underline{C}^0$. We conclude at once that when $\varepsilon \to +0$, $J_{d,0}^{\kappa,\varepsilon}$ converges uniformly in a neighborhood of the origin in $\mathbf{C}^m$. This shows that (2.12) can be reduced to

$$\tilde{u} \equiv (\kappa/4\pi^3)^{m/2}\sum_{j=1}^{\nu}\lim_{\varepsilon\to+0} J_{d,j}^{\kappa,\varepsilon}. \tag{2.20}$$

Thus we must look at $J_{d,j}^{\kappa,\varepsilon}$ for arbitrary j, $1 \leqslant j \leqslant \nu$. We shall take

$$z = Z(x) + iZ_x(x)v, \quad x \in U,$$

where v is a vector which will vary in a certain strictly convex open cone $\Gamma_j \subset \mathbf{R}^m\backslash\{0\}$ chosen below. With this choice of z we look at the quantity

$$Q = -\mathrm{Re}\{i\zeta\cdot[z - Z(y)] - \kappa\langle\zeta\rangle[Z(t) - Z(y)]^2\}/|\xi|.$$

We have, by (2.13),

$$\begin{aligned} Q = \dot{\xi}\cdot \mathrm{Im}\{Z_x(t)^{-1}[Z(x) - Z(y) + Z_x(x)v]\}\\ +4B'\,\mathrm{Re}\{[Z(x) - Z(t) + iZ_x(x)v]\cdot[Z(x) - Z(y) + iZ_x(x)v]\}\\ +\kappa\,\mathrm{Re}\left\{\frac{\langle\zeta\rangle}{|\xi|}[Z(t) - Z(y)]^2\right\}, \quad \dot{\xi} = \xi/|\xi|. \end{aligned}$$

In the first two terms we write

$$Z(x) - Z(y) = [Z(x) - Z(t)] + [Z(t) - Z(y)].$$

First we make use of (2.14) and (2.15):

$$\begin{aligned} \dot{\xi}\cdot \mathrm{Im}\, Z_x(t)^{-1}[Z(x) - Z(y) + iZ_x(x)v]\\ \geqslant \dot{\xi}\cdot v - B'(|x-t|^2 + |y-t|^2 + |x-t||v|)\\ \geqslant \dot{\xi}\cdot v - B'(2|x-t|^2 + |y-t|^2 + |v|^2). \end{aligned}$$

This yields a lower bound for the first term in the above expression of Q. In order to obtain one for the second term we use the fact that $Z(x) = x + i\Phi(x)$ and $|\Phi(x) - \Phi(y)| \leqslant |x-y|/2$. One can check easily that

$$\begin{aligned} \mathrm{Re}\{[Z(x) - Z(t) + iZ_x(x)v]\cdot[Z(x) - Z(t) - Z(y) + Z(t) + iZ_x(x)v]\}\\ \geqslant \tfrac{1}{2}|x-t|^2 - 10|y-t|^2 - 10|v|^2. \end{aligned}$$

Last, by using (2.16), we get

$$\operatorname{Re}\left\{\frac{\langle\zeta\rangle}{|\xi|}[Z(t)-Z(y)]^2\right\} \geqslant |t-y|^2/8.$$

Gathering all of these we obtain

$$Q \geqslant \dot{\xi}\cdot v + (\kappa/8 - 41B')|y-t|^2 - 41B'|v|^2.$$

We require $\kappa > \kappa_*$ with

$$\kappa_* \geqslant 328B', \tag{2.21}$$

and reach the conclusion that

$$Q \geqslant \dot{\xi}\cdot v - 41B'|v|^2.$$

At this point we make our choice of the cone Γ_j. We avail ourselves of (2.19). This enables us to select Γ_j so that

$$v\cdot\xi^0 < 0 \textit{ for all } v \textit{ in } \Gamma_j; \tag{2.22}$$

(2.23) *there is a constant* $c_0 > 0$ *such that*

$$v\cdot\xi \geqslant c_0|v||\xi| \quad \textit{for all } v\in\Gamma_j,\ \xi\in \underline{C}^j.$$

We then require

$$|v| \leqslant c_0/(82B'), \tag{2.24}$$

so that

$$Q \geqslant \tfrac{1}{2}c_0|v|,\ x\in U, y\in U_1, |Z(t)|\leqslant d, \quad v\in\Gamma_j, \xi\in\underline{C}^j. \tag{2.25}$$

We take then for V any open neighborhood of 0 whose closure is compact and contained in U. We apply Lemma 1.3 after choosing the neighborhood $\mathcal{O}$ of V in $U + i\mathbf{R}^m$ so as to take (2.24) into account. We conclude that, as ε goes to zero, the entire functions $J_{d,j}^{\kappa,\varepsilon}$ converge uniformly on compact subsets of $\mathring{\mathfrak{N}}_{\mathcal{O}}(V,\Gamma_j)$, and thus have there a limit which is a holomorphic function $\tilde{J}_{d,j}^{\kappa}$.

If u is a compactly supported *continuous function* in U_1, we have

$$\left|\tilde{J}_{d,j}^{\kappa}[Z(x)+iZ_x(x)v]\right| \leqslant \text{const.}|v|^{-\mu}, \tag{2.26}$$

with $\mu = 3m/2$, as one sees by looking at the integrals $J_{d,j}^{\kappa,\varepsilon}$. When u is a compactly supported distribution in U_1, we can use a representation of the kind (4.3), Chapter I, and integrate by parts. We find that one can take $\mu = k + 3m/2$. In all cases this proves that $\tilde{J}_{d,j}^{\kappa} \in B'_{\mathcal{O}}(V,\Gamma_j)$ (Def. 1.1) and therefore, by virtue of (2.20), that $(0,\xi^0)\notin WF_{ha}(u)$ (see Def. 1.2).

Remark 2.1. It is worthwhile to underline the fact that κ_* is submitted to conditions (2.21) and $\kappa_* > 4B$ with B given in (4.22), Chapter I. The factor

328, needless to say, has no particular significance. The important fact is that κ_* is proportional to the constant B' satisfying conditions (2.14), (2.15) (with a factor that is large compared to one). Now, if all second derivatives of Φ^j, $j = 1,\cdots,m$, vanish at the origin, by suitably contracting U about 0 we can obtain that B' be as small as we wish. This is important in forthcoming applications of the criteria provided by Theorems 2.1, 2.2.

Note that we may always select the functions Z^j so that all the derivatives of order two of the Φ^j vanish at the origin. If this is not already so (but assuming that Φ and $d\Phi$ vanish at the origin) it suffices to replace Z^j by

$$Z^j - \frac{i}{2}\sum_{k,l=1}^{m} \frac{\partial^2\Phi^j}{\partial x^k \partial x^l}(0) Z^k Z^l. \tag{2.27}$$

Remark 2.2. If a distribution u in U is not compactly supported one can apply Theorems 2.1, 2.2 to gu where $g \in C_c^\infty(U)$ is equal to one in a neighborhood of the origin.

Inspection of the proof of Theorem 2.2 shows that the conclusion can be made more precise. In fact, combining the argument there with Theorem 2.1 enables us to generalize a result of [6]:

Theorem 2.3. *Let $\Gamma_1,\cdots,\Gamma_\nu$ be acute and open cones in $\mathbf{R}^m\backslash\{0\}$. For any distribution u in U the following two properties are equivalent:*

$$T_0^*X \cap WF_{ha}(u) \subset \bigcup_{j=1}^{\nu} \Gamma_j^0; \tag{2.28}$$

(2.29) *Given, for each $j = 1,\cdots,\nu$, a nonempty open cone $\tilde{\Gamma}_j$ whose closure is contained in Γ_j, there are an open neighborhood $V \subset U$ of the origin, an open neighborhood $\mathcal{O}$ of V in $U + i\mathbf{R}^m$ and functions $\tilde{f}_j \in B'_{\mathcal{O}}(V, \tilde{\Gamma}_j)$ $(1 \leqslant j \leqslant \nu)$ such that, in V,*

$$u = b\tilde{f}_1 + \cdots + b\tilde{f}_\nu. \tag{2.30}$$

We have used the following notation, for any acute and open cone $\Gamma(\neq \varnothing)$ in $\mathbf{R}^m\backslash\{0\}$:

$$\Gamma^0 = \{\xi \in \mathbf{R}_m\backslash\{0\}; \forall v \in \Gamma, \xi\cdot v \geqslant 0\}.$$

Γ^0 is called the *polar* of Γ. Note that Γ^0 is a strictly convex and closed cone, and is identical to the set of covectors ξ such that $\xi \cdot v > 0$ for all $v \in \Gamma$.

Proof. (2.28) implies (2.29). Let $Z = (Z^1,\cdots,Z^m)$ satisfy (2.11). Therefore if $\kappa > \kappa_*$, the number in Theorem 2.2 (see Remark 2.1), the hypothesis (2.2) will also be satisfied, whatever $\xi^0 \in \mathbf{R}_m\backslash\{0\}$. Let $g \in C_c^\infty(U)$ be equal to one in a neighborhood of the origin. By virtue of Theorem 2.1 Property (2.28) has the following implication:

$(2.31)_\kappa$ *Given any closed cone $\Gamma' \subset \mathbf{R}_m \backslash (\Gamma_1^0 \cup \cdots \cup \Gamma_\nu^0)$ there are an open neighborhood of the origin $V^{\mathbf{C}}$ in $\mathbf{C}^m$, a conic open neighborhood $\mathcal{C}'$ of Γ' in $\mathbf{C}_m \backslash \{0\}$ and constants C, $R > 0$ such that*

$$|F^\kappa(gu; z, \zeta)| \leqslant Ce^{-|\zeta|/R}, \quad \forall z \in V^{\mathbf{C}}, \zeta \in \mathcal{C}'.$$

The proof that $(2.31)_k$ entails (2.29) follows from inspection of the proof of Theorem 2.2, where it is shown that (2.30) holds not exactly but modulo a hypo-analytic function at the origin, if we take

$$\tilde{f}_j = (\kappa/4\pi^3)^{m/2} \tilde{J}^\kappa_{d,j}.$$

Call K_j the intersection of the closure of $\tilde{\Gamma}_j$ with the unit sphere S^{m-1}, and K_j' the intersection of Γ_j^0 with S_{m-1}. The function $v \cdot \xi$ is everywhere > 0 in $K_j \times K_j'$, hence there is $c_0 > 0$ such that $v \cdot \xi \geqslant 2c_0$ for all $v \in K_j$, $\xi \in K_j'$. Define

$$\underline{\tilde{C}}^j = \{\xi \in \mathbf{R}_m \backslash \{0\}; \forall v \in Cl \tilde{\Gamma}_j, v \cdot \xi \geqslant c_0 |v||\xi|\}.$$

The complement in $\mathbf{R}_m \backslash \{0\}$ of $\underline{\tilde{C}}^1 \cup \cdots \cup \underline{\tilde{C}}^\nu$ is a closed cone $\underline{C}^0$ contained in $\mathbf{R}_m \backslash (\Gamma_1^0 \cup \cdots \cup \Gamma_\nu^0)$. For each $j = 1, \cdots, \nu$, define $\underline{C}^j \subset \underline{\tilde{C}}^j$ in such a way that (2.18) and (2.19) are true. We see at once that (2.23) holds if we substitute $\tilde{\Gamma}_j$ for Γ_j, and by the same argument as at the end of the proof of Theorem 2.2 we conclude that $J^\kappa_{d,j} \in B'_{\mathcal{O}}(V, \tilde{\Gamma}_j)$.

(2.29) implies (2.28). If ξ^0 does not belong to $\Gamma_1^0 \cup \cdots \cup \Gamma_\nu^0$, then for each j there is $v_j \in \Gamma_j$ such that $\xi^0 \cdot v_j < 0$, and therefore there is a strictly convex open cone $\tilde{\Gamma}_j$ containing v_j such that $\xi^0 \cdot v < 0$ for all $v \in \tilde{\Gamma}_j$. Hence it suffices to apply (2.29) and Definition 1.2. q.e.d.

It is sometimes useful to know when u can be represented (in a neighborhood of a point) as the boundary value of a *single* holomorphic function. This is related to the possibility of extending holomorphically the transfer $\tilde{u}$ of u via Z. The following immediate corollary of Theorem 2.3 yields a criterion.

Corollary 3.2. *Let Γ be a nonempty acute and open cone in $\mathbf{R}^m \backslash \{0\}$. For any $u \in \mathcal{D}'(U)$ the following two properties are equivalent:*

(2.32) $$T_0^* X \cap WF_{ha}(u) \subset \Gamma^0;$$

(2.33) *given any nonempty open cone Γ_* whose closure is contained in Γ, there are an open neighborhood $V \subset U$ of the origin, an open neighborhood $\mathcal{O}$ of V in $U + i\mathbf{R}^m$ and a function $\tilde{f} \in B'_{\mathcal{O}}(V, \Gamma_*)$ such that $u = b\tilde{f}$ in V.*

Remark 2.3. The neighborhood V in (2.33) will generally depend on the cone Γ_*. We may take V *decreasing* as Γ_* increases and fills Γ. The holomorphic functions $\tilde{f}$, each defined in the corresponding conoid $\mathring{\mathfrak{N}}_{\mathbb{C}}(V, \Gamma_*)$, are necessarily equal on overlaps, by the "uniqueness" of the extension (a property we have not established), and thus define a holomorphic function in the union of these conoids.

We conclude this section with the following consequence of Theorem 2.1.

Theorem 2.4. *Assume that the hypo-analytic structure of X has codimension zero. Then, given any distribution u in an open subset of X, the base projection of $WF_{ha}(u)$ is equal to the hypo-analytic singular support of u (Definition 1.3, Chapter I).*

Proof. As noted at the end of §1 it suffices to prove that the base projection of the hypo-analytic wave-front set contains the hypo-analytic singular support. We may reason in a hypo-analytic local chart $(U, Z^1, \cdots, Z^m)$ in which Hypothesis (2.11) is satisfied. Let u be a distribution in U whose hypo-analytic wave-front set does not intersect T_0^*X. Then we want to show that $0 \notin \operatorname{sing\,supp}_{ha} u$. Neither the hypothesis nor the conclusion is modified if we replace u by gu with $g \in C_c^\infty(U)$, $g \equiv 1$ in some neighborhood of 0. We may therefore suppose that $\operatorname{supp} u \subset\subset U$. If then we apply Theorem 2.1 provided $\kappa > 4B$ we reach the conclusion that Property $(4.13)_\kappa$, Chapter I, is satisfied. But then Theorem 4.2, Chapter I, implies that u is hypo-analytic at the origin.

3. Hypo-analytic structures of arbitrary codimension: characteristic points and traces of distribution solutions on maximally real submanifolds

In the present section we return to a hypo-analytic manifold Ω whose hypo-analytic structure has codimension n. As before we write $\dim \Omega = N = m + n$. We shall reason in a hypo-analytic local chart $(U, Z^1, \cdots, Z^m)$, centered at a point which we continue to call the origin. We assume that U is the domain of local coordinates $x^1, \cdots, x^m, y^1, \cdots, y^n$, all vanishing at the origin. It will be convenient to take U in the form of a product,

$$U = V \times W,$$

where V (resp., W) is an open ball in x-space $\mathbf{R}^m$ (resp., in y-space $\mathbf{R}^n$), centered at the origin. The dual coordinates will be denoted by $\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_n$, and $T^*\Omega|_U$ will be identified to $U \times \mathbf{R}_{m+n}$.

We shall also suppose that the functions Z^j all vanish at the origin. Furthermore we shall assume that

$$(3.1)\qquad |\mathrm{Im}\,[Z(x, y) - Z(x^*, y)]| \leq |\mathrm{Re}[Z(x, y) - Z(x^*, y)]|/2, \quad \forall x, x^* \in V, y \in W.$$

We shall also assume that the mapping $(x, y) \mapsto (Z(x, y), y)$ is a diffeomorphism of U onto a C^∞ submanifold Σ of $\mathbf{C}^m \times \mathbf{R}^n$. This is equivalent to saying that, for each y in W, the map $x \mapsto Z(x, y)$ is a diffeomorphism of V onto a maximally real submanifold of $\mathbf{C}^m$, which we denote by Σ_y. Of course $\Sigma_y = \{z \in \mathbf{C}^m; (z, y) \in \Sigma\}$.

Remark 3.1. As indicated in §2 of Chapter I, if X is any maximally real submanifold of Ω passing through a point p_0, the local chart $(U, x^1,\cdots,x^m, y^1,\cdots,y^n)$ can be "adapted" to X in the sense that $X \cap U$ is defined by the equation $y = 0$, and that the coordinates x^j, y^k and the hypo-analytic functions Z^l retain all the properties described above (relative to p_0 as the origin).

In what follows, although our concern is basically with arbitrary distribution solutions, mostly we shall restrict our attention to C^1 solutions. All the reasoning and conclusions will extend routinely by exploiting representation formulas of the kind (4.3), Chapter I.

Let t be an arbitrary point in W. We shall call X_t the maximally real submanifold of U defined by $y = t$, and h_t the trace on X_t of an arbitrary distribution solution h in U (§2, Chapter I). Notice that Σ_t is the image of X_t under the map $x \mapsto Z(x, t)$.

Let $g \in C_c^\infty(V)$. We define (cf. Definition 4.1, Chapter I)

$$F_t^\kappa(gh; z, \zeta) = \int_V e^{-i\zeta\cdot Z(x,t) - \kappa\langle\zeta\rangle[z - Z(x,t)]^2} g(x) h(x, t)\, dZ(x, t).$$

Suppose, for the sake of simplicity, that h is a C^1 solution. Possibly after contracting U we may write $h = \tilde{h} \circ Z$ with $\tilde{h}$ a continuous function on $Z(U)$ ((2.13), Chapter I). Since the m-form $h dZ$ ($dZ = dZ^1 \wedge \cdots \wedge dZ^m$) is closed in U, its push-forward via the diffeomorphism $(x, y) \mapsto (Z(x, y), y)$ is a closed m-form $\tilde{h}\, dz$ on Σ. Actually we shall associate with h the following closed m-form on Σ:

$$(3.2)\qquad H = H^\kappa(z^*; z, \zeta) = e^{-i\zeta\cdot z^* - \kappa\langle\zeta\rangle[z - z^*]^2} h(z^*)\, dz^*.$$

The integration variable is now denoted by z^* to distinguish it from the parameter z as it enters in $F_t^\kappa(gh; z, \zeta)$. If then we write $\tilde{g}(z^*, y) = g(x)$ when $z^* = Z(x, y)$, we obtain

$$(3.3)\qquad F_t^\kappa(gh; z, \zeta) = \int_{\Sigma_t} \tilde{g}(z^*, t) H^\kappa(z^*; z, \zeta).$$

As already pointed out all this continues to make sense when h is a distribution solution.

Lemma 3.1. *Let κ, d be numbers > 0. Assume that $K_d = \{x \in V; |\mathrm{Re}\, Z(x,0)| \leq d\}$ is a compact subset of V. There are open neighborhoods of the origin, $W' \subset W$, $V^{\mathbf{C}} \subset \mathbf{C}^m$ and a number $\varepsilon > 0$ such that the following is true:*

Let $\xi \neq 0$ be any vector in $\mathbf{R}_m$ such that with $\dot{\xi} = \xi/|\xi|$

$$\text{(3.4)} \quad \forall x \in V, \quad |\dot{\xi} \cdot \mathrm{Im}\, Z(x,0)| \leq \frac{\kappa}{4} |\mathrm{Re}\, Z(x,0)| (|\mathrm{Re}\, Z(x,0)| + d),$$

and call $\mathcal{C}$ the cone in $\mathbf{C}^m \backslash \{0\}$ defined by $|\dot{\zeta} - \dot{\xi}| < \varepsilon$.

Given any distribution solution h in U and any function $g \in C_c^\infty(V)$ equal to one in K_d, there is $C > 0$ such that

$$\text{(3.5)} \quad |F_t^k(gh; z, \zeta) - F_0^k(gh; z, \zeta)| \leq C e^{-\kappa d^2 |\zeta|/8}, \qquad \forall t \in W', z \in V^{\mathbf{C}}, \zeta \in \mathcal{C}.$$

Proof. It is based on (3.3). Let $l(t)$ denote the straight-line segment in W which joins 0 to t, and call $\mathbf{c}(t)$ the $(m+1)$-chain on Σ, which is the image of $V \times l(t)$ under the map $(x, y) \mapsto (Z(x, y), y)$. Notice that $\mathrm{supp}\, \tilde{g}$ intersects the boundary of $\mathbf{c}(t)$ only on its "horizontal faces", Σ_0 and Σ_t. Therefore by Stokes' theorem we have

$$\text{(3.6)} \qquad \int_{\Sigma_t} \tilde{g} H - \int_{\Sigma_0} \tilde{g} H = \int_{\mathbf{c}(t)} d\tilde{g} \wedge H.$$

We must then look at the quantity

$$Q_0 = \mathrm{Re}(i\dot{\xi} \cdot z^*) + \kappa\, \mathrm{Re}[z^*]^2,$$

where $z^* = Z(x,0)$, $x \in \mathrm{supp}\, dg$. We apply (3.1) with $x^* = 0, y = 0$, and (3.4) to obtain

$$Q_0 > \frac{\kappa}{2} |\mathrm{Re}\, Z(x,0)| (|\mathrm{Re}\, Z(x,0)| - d/2) \geq \kappa d^2/4,$$

since $g \equiv 1$ when $|\mathrm{Re}\, Z(x,0)| \leq d$. But then, if W' and $V^{\mathbf{C}}$ are small enough, and the cone $\mathcal{C}$ "thin" enough, we shall have

$$\mathrm{Re}\{i\zeta \cdot z^* + \kappa \langle \zeta \rangle [z - z^*]^2\}/|\zeta| \geq \kappa d^2/8, \qquad \forall t \in W', z \in V^{\mathbf{C}}, z^* \in \mathbf{c}(t) \cap \mathrm{supp}\, d\tilde{g}, \zeta \in \mathcal{C}.$$

Putting this into the right-hand side of (3.6) yields what we wanted.

Corollary 3.1. *Let κ and d be as in Lemma 3.1. Suppose that we have*

$$\text{(3.7)} \qquad \forall x \in V, |\mathrm{Im}\, Z(x,0)| \leq \frac{\kappa}{4} |\mathrm{Re}\, Z(x,0)| (|\mathrm{Re}\, Z(x,0)| + d).$$

Then there are open neighborhoods of the origin, $W' \subset W$, $V^{\mathbf{C}} \subset \mathbf{C}^m$, and an open cone $\mathcal{C}$ in $\mathbf{C}_m \backslash \{0\}$ containing $\mathbf{R}_m \backslash \{0\}$ such that, given any distribution solution h in U and any function $g \in C_c^\infty(V)$ equal to one in K_d, there is $C > 0$ such that (3.5) *holds for all t in W', z in $V^{\mathbf{C}}$ and all ζ in $\mathcal{C}$.*

Proof. It suffices to observe that the "sizes" of $V^{\mathbf{C}}$, W', $\mathcal{C}$ in Lemma 3.1 are solely determined by κ and d, and are independent of ξ.

Remark 3.2. If in addition to (3.1) we make the hypothesis

$$\forall x \in V, |\operatorname{Im} Z(x,0)| \leqslant B |\operatorname{Re} Z(x,0)|^2, \tag{3.8}$$

Condition (3.7) is satisfied whatever $d > 0$, as soon as $\kappa \geqslant 4B$.

Let X be any maximally real submanifold of Ω. Let us denote by π_X the natural quotient map $T^*\Omega|_X \to T^*X$, and by $\pi_X^{\mathbf{C}}$ the analogue for the complexified cotangent bundles. By definition of "maximally real" (Definition 2.1, Chapter I) $\pi_X^{\mathbf{C}}$ induces a bijection of $T'|_X$ onto $\mathbf{C}T^*X$. As a consequence π_X induces an *injection* of $T^0 = T' \cap T^*\Omega$ into T^*X. We recall that $T^0 \backslash 0$ is the characteristic set of all the sections of $T'^{\perp}$.

Definition 3.1. We say that a point of $T^*X \backslash 0$ is characteristic if it belongs to $\pi_X(T^0|_X)$, noncharacteristic otherwise.

When the codimension of the hypo-analytic structure of Ω is zero, we have $T^0 = T^*\Omega$, and *all* points off the zero section are characteristic.

In order to obtain a handy description of characteristic, and noncharacteristic points, we shall reason in a hypo-analytic local chart of the kind introduced at the beginning of the present section, now adapted to the maximally real submanifold X (Remark 3.1). Actually we shall somewhat strengthen our hypotheses on the hypo-analytic functions Z^j, and assume that we have, in U,

$$\begin{aligned} &Z^j = x^j + \sqrt{-1}\, y^j, \quad j = 1,\cdots,r, \\ &Z^j = x^j + \sqrt{-1}\, \Phi^j(x, y), \quad j = r+1,\cdots,m, \end{aligned} \tag{3.9}$$

where the Φ^j are real-valued, and

$$\Phi^j(0,0) = 0, d\Phi^j(0,0) = 0, j = r+1,\cdots,m. \tag{3.10}$$

As we have done earlier we write X_t for the submanifold $y = t$ of U; thus $X_0 = X \cap U$. We shall systematically identify the cotangent bundle of X_t to $V \times \mathbf{R}_m$ and π_{X_t} to the coordinate projection $U \times \mathbf{R}_{m+n} \to V \times \mathbf{R}_m$. A point $(x, y, \xi, \eta) \in U \times \mathbf{R}_{m+n}$ belongs to T^0 if there are m complex numbers c_j such that

$$\sum_{j=1}^m \xi_j dx^j + \sum_{k=1}^n \eta_k dy^k = \sum_{j=1}^m c_j dZ^j. \tag{3.11}$$

From (3.9)–(3.10) we get, at the origin,

$$dZ^j = dx^j + idy^j, \quad j = 1,\cdots,r,$$
$$dZ^j = dx^j, \quad j = r+1,\cdots,m. \tag{3.12}$$

Then (3.11) demands $c_j = \xi_j$ for all j, and

$$\xi_j = 0 \text{ if } j = 1,\cdots,r;\ \eta_k = 0,\ k = 1,\cdots,n.$$

It is then convenient to use the notation $x' = (x^1,\cdots,x^r)$, $x'' = (x^{r+1},\cdots,x^m)$, $t' = (t^1,\cdots,t^r)$, $t'' = (t^{r+1},\cdots,t^n)$. In this notation we see that

$$T_0^0 = \{(\xi,\eta) \in T_0^*\Omega;\ \xi' = 0, \eta = 0\},$$
$$\pi_X(T_0^0) = \{\xi \in T_0^*X;\ \xi' = 0\}. \tag{3.13}$$

Theorem 3.1. *Let h be an arbitrary distribution solution in some open subset of Ω containing the maximally real submanifold X. Then the hypo-analytic wave-front set of the trace of h on X is entirely contained in the set of characteristic points of T^*X.*

Proof. Of course we assume that h is defined in the open set U where the preceding description applies. Note that if $r = 0$, then all points in $T_0^*X\backslash 0$ are characteristic, and it is obvious that

$$WF_{ha}(h) \cap T_0^*X \subset \pi_X(T_0^0). \tag{3.14}$$

Our purpose is to prove that (3.14) holds even when $r > 0$. For this we go back to the integral $F_t^\kappa(gh;\, z, \xi)$, and look at the quantity

$$Q = \mathrm{Re}\{i\zeta \cdot Z(x,t) + \kappa\langle\zeta\rangle[z - Z(x,t)]^2\}/|\zeta|,$$

where however we take $\zeta = \xi \in \mathbf{R}_m\backslash\{0\}$, $z = 0$, in which case we call it Q_0. We have

$$Q_0 = -\dot{\xi}' \cdot t' - \dot{\xi}'' \cdot \Phi''(x,t) + \kappa\big[|x|^2 - |t'|^2 - |\Phi''(x,t)|^2\big],$$

where $\Phi'' = (\Phi^{r+1},\cdots,\Phi^m)$, and $(\dot{\xi}', \dot{\xi}'') = \xi/|\xi|$.

We make now the assumption that $(0, \xi)$ is noncharacteristic. According to (3.13) this means that $\dot{\xi}' \neq 0$. We can therefore select t so that

$$\dot{\xi}' \cdot t' < -c_0|t| \tag{3.15}$$

for some $c_0 > 0$. Note that (3.15) remains true if we replace t by ρt with $\rho > 0$ arbitrary.

By (3.10) we know that there is $B > 0$ such that

$$|\Phi''(x,y)| \leq B(|x|^2 + |y|^2), \tag{3.16}$$

whence

$$\begin{aligned} Q_0 &\geqslant c_0|t| - B\left(|x|^2+|t|^2\right) + \kappa|x|^2 - \kappa B^2\left(|x|^2+|t|^2\right)^2 - \kappa|t|^2 \\ &\geqslant \left(c_0 - \kappa|t| - B|t| - 2\kappa B^2|t|^3\right)|t| + \left(\kappa - B - 2B^2\kappa|x|^2\right)|x|^2. \end{aligned}$$

Using Remark 2.1 we may assume that $\operatorname{Im} Z_x(0,0) = 0$. Let us now take $\kappa > \kappa_*$, the number in Theorem 2.2, and $\kappa > 4B$ with B the number in (3.16). Notice that (3.16) implies (3.8) and that as a consequence Remark 3.2 applies. Define $V' \subset V$ by the requirement that for $x \in V'$, $2B|x| < 1$. If $|t|$ is small enough, we shall have $Q_0 \geqslant c_0|t|/2$ and consequently there will be an open neighborhood $V^{\mathbf{C}}$ of the origin in $\mathbf{C}^m$ and an open cone $\mathcal{C}$ in $\mathbf{C}^m \backslash \{0\}$ containing ξ such that

$$Q \geqslant c_0|t|/4, \quad \forall x \in V', z \in V^{\mathbf{C}}, \zeta \in \mathcal{C}. \tag{3.17}$$

We then select $g \in C_c^\infty(V')$ equal to one in some neighborhood of the origin. From (3.17) we derive that, for suitable constants C, $R > 0$,

$$|F_t^\kappa(gh; z, \zeta)| \leqslant Ce^{-|\zeta|/R}, \quad \forall z \in V^{\mathbf{C}}, \zeta \in \mathcal{C}. \tag{3.18}$$

We apply the version of Lemma 3.1 expounded in Remark 3.2 (it is clear that we may take t in W'). Thus we obtain an inequality similar to (3.18) but for $t = 0$. The sought conclusion follows then from Theorem 2.2.

Theorem 3.2. *Let h be a distribution solution in U such that* $(0, \xi^0)$ *does not belong to the hypo-analytic wave-front set of the trace h_0 of h on X_0. Then there is an open neighborhood $W_0 \subset W$ of the origin such that* $(0, \xi^0)$ *does not belong to the hypo-analytic wave-front set of the trace h_t of h on X_t whatever t in W_0.*

Proof. We continue to use the functions Z^j of (3.9)–(3.10). Possibly after contracting V and W we may assume that

$$\begin{aligned} &\left|\operatorname{Im}\left\{Z_x(0,t)^{-1}[Z(x,y) - Z(x^*,y)]\right\}\right| \\ &\quad \leqslant \tfrac{1}{2}\left|\operatorname{Re}\left\{Z_x(0,t)^{-1}[Z(x,y) - Z(x^*,y)]\right\}\right| \\ &\qquad\qquad \textit{for all } x, x^* \textit{ in } V, y, t \textit{ in } W. \end{aligned} \tag{3.20}$$

Also, for a suitable $B > 0$,

$$|Z_x(0,t)^{-1}[Z(x,t) - Z(0,t)] - x| \leqslant B|x|^2, \quad \forall x \in V. \tag{3.21}$$

This will allow us at the end of the ongoing argument to apply Theorem 2.2 with $X = X_t$, $u = gh_t$ and

$$Z(x) = Z_x(0,t)^{-1}[Z(x,t) - Z(0,t)]. \tag{3.22}$$

According to Remark 2.1 the number κ_* can be chosen independently of $t \in W$. Let $d > 0$ be such that $\{x; |\operatorname{Re}(Z_x(0,t)^{-1}Z(x,t))| \leqslant d\}$ is contained in a fixed compact subset of V for all $t \in W$. Then we assume $\kappa \geqslant \kappa_*$ and

$\kappa > 4B$, the constant in (3.21). Notice that if we put $t = 0$ in (3.21), then we obtain (3.8) and thus, by Remark 3.2, that Condition (3.7) is satisfied. This remains true if we replace $Z(x, 0)$ by $Z_x(0, t)^{-1}Z(x, 0)$ provided t remains in a sufficiently small open neighborhood $W_0 \subset W$ of the origin. We may then apply Corollary 3.1 with $Z_x(0, t)^{-1}Z(x, y)$ in the place of $Z(x, y)$.

We want to combine the information provided by (3.5) in the present set-up with that provided by our hypothesis, namely that $(0, \xi^0) \notin WF_{ha}(h_0)$. We apply Theorem 2.1 with $X = X_0$, $Z(x) = Z_x(0, t)^{-1}Z(x, 0)$ and $u = gh_0$, and select the number $c > 0$ as required in the proof of Theorem 2.1. Note that the upper bound on c in (2.10) only depends on κ, d and the number s. The latter in turn depends on κ, δ', and the constant c_0 in (2.9). It follows that we may choose c independent of $t \in W$. Recalling that $\kappa > 4B$ we see that Condition (2.2) is certainly satisfied for $Z(x) = Z_x(0, t)^{-1}Z(x, 0)$ when $t = 0$, and therefore also when t remains in W_0 provided W_0 is small enough.

We conclude that $(2.1)_\kappa$ holds. But then, possibly after some further contracting of W_0 about 0, we derive an inequality similar to that in $(2.1)_\kappa$ but for the function

$$e^{i\zeta \cdot Z_x(0,t)^{-1}Z(0,t)} F^\kappa\left(gh_t;\, z + Z_x(0, t)^{-1}Z(0, t) \cdot \zeta\right), \tag{3.23}$$

with F^κ defined as in Definition 4.1, Chapter I, but with $Z(x) = Z_x(0, t)^{-1}Z(x, t)$. Of course, in order that such an inequality be valid, we must contract about the origin both the neighborhood $V^{\mathbf{C}}$ in $\mathbf{C}^m$ in which z varies and, as already said, the one, W_0, in which t varies. The smallness of W_0 must also insure that the absolute value of (3.23) decay exponentially as $|\zeta| \to +\infty$ (with ζ in an open cone in $\mathbf{C}^m \backslash \{0\}$ containing ξ^0), despite the presence of the exponential factor in front of F^κ.

In order to conclude the proof of Theorem 3.2 it suffices to observe that (3.23) is nothing but $F_t^\kappa(gh; z, \zeta)$ with the choice (3.22), and to apply Theorem 2.2 as announced.

4. Hypo-analytic wave-front set in hypo-analytic structures of arbitrary codimension

We continue to deal with a hypo-analytic manifold Ω whose hypo-analytic structure has arbitrary codimension n. We use the notation of §3. In particular we continue to reason in the hypo-analytic local chart $(U, Z^1, \cdots, Z^m)$ in which (3.9) and (3.10) hold. Observe that in order that T_0^0 be $\neq 0$ we must have

$$r < m. \tag{4.1}$$

Until otherwise specified we make the hypothesis that this is indeed the case. In passing note that when $r = m$, which requires $m \leqslant n$, the structure T' is elliptic in some open neighborhood of the origin (and thus $T^0 = 0$ over that entire neighborhood).

The next statement aims at describing all the maximally real submanifolds which pass through the origin.

Proposition 4.1. *If Y is any maximally real C^∞ submanifold of U containing the origin there are an invertible $r \times r$ complex matrix S and a C^∞ map from an open neighborhood $V_\# \subset V$ of the origin to $\mathbf{R}^n$, $f_\#$, with $f_\#(0) = 0$, such that, if we make the change of coordinates*

$$x'_\# + iy'_\# = Sz',\ x''_\# = x'',\ y''_\# = y'', \tag{4.2}$$

then Y is defined in an open neighborhood $U_\# \subset U$ of the origin by the equations

$$y_\# = f(x_\#). \tag{4.3}$$

Conversely, any system of equations (4.2), (4.3) *defines a maximally real C^∞ submanifold in some open neighborhood of the origin, passing through that point.*

Proof. Suppose that, in an open neighborhood $U' \subset U$ of the origin, Y is defined by equations

$$f^j(x, y) = 0, \quad j = 1,\cdots,n, \tag{4.4}$$

such that $df^1,\cdots,df^n$ are linearly independent. Then $dZ^1,\cdots,dZ^r$, $dx^{r+1},\cdots,dx^m$, $df^1,\cdots,df^n$ make up a basis of the cotangent space $\mathbf{C}T_0^*\Omega$. Indeed, the first m differentials $dZ^1,\cdots,dx^m$ have linearly independent restrictions to $\mathbf{C}T_0Y$, while $df^1,\cdots,df^n$ span the orthogonal of $\mathbf{C}T_0Y$.

We know, on the other hand, that

$$dZ^1,\cdots,dZ^r,\ dx^{r+1},\cdots,dx^m,\ d\bar{Z}^1,\cdots,d\bar{Z}^r,\ dy^{r+1},\cdots,dy^n$$

also make up a basis of $\mathbf{C}T_0^*\Omega$. It follows from this that there is a $\mathbf{C}$-linear bijection of the $\mathbf{C}$-linear span of $(d\bar{Z}^1,\cdots,d\bar{Z}^r,\ dy^{r+1},\cdots,dy^n)$ onto that of $(df^1,\cdots,df^n)$. After an $\mathbf{R}$-linear substitution of the f^j's we may assume that the matrix

$$S = 2i\left(\frac{\partial f^j}{\partial z^k}(0)\right)_{1 \leqslant j,\, k \leqslant r}$$

is nonsingular. Define then $x_\#$, $y_\#$ as in (4.2). It is checked at once that if we set $f^j_\#(x_\#, y_\#) = f^j(x, y)$ $(j = 1,\cdots,n)$, then

$$\frac{\partial f^j_\#}{\partial x^k_\#}(0,0) = 0,\ \frac{\partial f^j_\#}{\partial y^k_\#}(0,0) = \delta^j_k,\quad j, k = 1,\cdots,r,$$

where δ_k^j is the Kronecker index. In turn this requires that the Jacobian determinant at the origin of the $f_\#^j$ with respect to the $y_\#^k$, j, $k = r+1,\cdots,n$, be invertible, and thus that $Df_\#/Dy_\# \neq 0$ at the origin. But then we may apply the implicit function theorem, and solve (4.4) with respect to $y_\#$, thus obtaining (4.3).

The "converse" part in Proposition 4.1 has an easy proof, left to the reader.

In the sequel Y will continue to denote a maximally real C^∞ submanifold of $U_\# \subset U$ defined by (4.2) and (4.3).

We define in some reasonable manner the powers S^t of the matrix S for all real t (for instance by means of a contour integral over a smooth simple curve winding once around the spectrum of S and entirely contained in the complement of a closed half-line issuing from 0). We then consider the following C^∞ map from $\mathbf{R}^m \times \mathbf{R}^1$ to $\mathbf{C}^r \times \mathbf{R}^{m+n-2r}$ which is identified with $\mathbf{R}^{m+n}$:

$$(4.5) \qquad (\tilde{x}, t) \mapsto z' = S^{-t}\tilde{x}', x'' = \tilde{x}'', y'' = 0.$$

As usual identifying $U = V \times W$ to an open neighborhood of the origin in $\mathbf{R}^{m+n}$, we select an open ball $\tilde{V}$ centered at 0 in $\mathbf{R}^m$ with radius small enough that the image of $\tilde{V} \times]-2,2[$ under the mapping (4.5) be contained in U. We then call $\tilde{Z}^j$ the pull-back of Z^j ($1 \leq j \leq m$) via (4.5). We have

$$(4.6) \qquad \begin{aligned} \tilde{Z}^j &= (S^{-t}\tilde{x}')^j, \quad j = 1,\cdots,r; \\ \tilde{Z}^j &= \tilde{x}^j + i\tilde{\Phi}^j(\tilde{x}, t), \quad j = r+1,\cdots,m. \end{aligned}$$

Write $\tilde{\Phi}'' = (\tilde{\Phi}^{r+1},\cdots,\tilde{\Phi}^m)$. By virtue of (3.10) and (4.5) we have, for some $B > 0$,

$$(4.7) \qquad |\tilde{\Phi}''(\tilde{x}, t)| \leq B|\tilde{x}|^2, \quad \forall \tilde{x} \in \tilde{V}, t \in]-2,2[.$$

The $\tilde{Z}^j$ define a hypo-analytic structure on $\tilde{V} \times]-2,2[$. For each t, $|t| < 2$, $\tilde{X}_t = \tilde{V} \times \{t\}$ is a maximally real submanifold in this structure, and (4.5) induces a hypo-analytic isomorphism of $\tilde{X}_t$ onto an open neighborhood of the origin in the maximally real submanifold of U,

$$X_t = \{(x, y) \in U; \operatorname{Im}(S^t z') = 0, y'' = 0\}.$$

Next we define another C^∞ mapping of an open neighborhood of the origin in $\mathbf{R}^{m+1}$ into U. We use now the coordinates $x_\#^j$, $y_\#^k$ of (4.2), and assume that the neighborhood $V_\#$ in Proposition 4.1 and an open ball $W_\#$ in $\mathbf{R}^n$ centered at the origin are small enough that the following two properties hold:

(i) $V_\# \times W_\# \subset U$; (ii) for all t, $|t| < 2$, $tf_\#(V_\#) \subset W_\#$.

In the sequel we call $U_\#$ the product $V_\# \times W_\#$ (this is compatible with Proposition 4.1). The mapping from $V_\# \times]-2,2[$ to U (or to $U_\#$) we shall be interested in is the following one:

$$(\tilde{x}_{\#}, t) \mapsto x_{\#} = \tilde{x}_{\#}, y_{\#} = tf_{\#}(\tilde{x}_{\#}). \tag{4.8}$$

We call $Z_{\#}$ the pull-back of (SZ', Z'') via the mapping (4.8). We have

$$\begin{aligned} \tilde{Z}^j_{\#} &= \tilde{x}^j_{\#} + itf^j_{\#}(\tilde{x}_{\#}), \quad j = 1,\cdots,r, \\ \tilde{Z}^j_{\#} &= \tilde{x}^j_{\#} + i\tilde{\Phi}^j_{\#}(\tilde{x}_{\#}, t), \quad j = r+1,\cdots,m. \end{aligned} \tag{4.9}$$

Here also we avail ourselves of (3.10). Because $f_{\#}(0) = 0$ (Proposition 4.1) we have, for a suitable $B > 0$,

$$|\tilde{\Phi}''_{\#}(\tilde{x}_{\#}, t)| \leq B|\tilde{x}_{\#}|^2, \quad \tilde{x}_{\#} \in V_{\#}, |t| < 2. \tag{4.10}$$

The $Z^j_{\#}$ define a hypo-analytic structure on $V_{\#} \times]-2, 2[$. The image via (4.8) of the maximally real submanifold $Y_{\#t} = V_{\#} \times \{t\}$ is an open neighborhood of the origin in the maximally real submanifold Y_t of $U_{\#}$ defined by the equation $y_{\#} = tf_{\#}(x_{\#})$.

Since S is a constant matrix (SZ', Z'') define on U the same hypo-analytic structure as Z. Thus the hypo-analytic structures defined on $\tilde{V} \times]-2, 2[$ and $V_{\#} \times]-2, 2[$ respectively are both the pull-back of one and the same structure on U. In particular the pull-back of any distribution solution in U is a distribution solution in each of the hypo-analytic structures so defined.

Another important remark is the following: Both mappings (4.5), (4.8) preserve x'' and therefore any form $\xi'' \cdot dx''$: the pull-back of $\xi'' \cdot dx''$ via (4.5) is $\xi'' \cdot d\tilde{x}''$, which via (4.8) is $\xi'' \cdot d\tilde{x}''_{\#}$. Thus, if $(0, \xi'', 0) \in T^*_0\Omega$ is a characteristic point (see (3.13)), its pull-back to the maximally real submanifolds $\tilde{X}_t$ via (4.5) and to $Y_{\#t}$ via (4.8) is represented by $(0, \xi'')$ in the coordinates $\tilde{x}^j$ and $\tilde{x}^j_{\#}$ respectively.

Actually we are going to change the coordinates $\tilde{x}^j$, $j \leq r$, in $\tilde{V} \times]-2, 2[$. Since we shall not modify the coordinates $\tilde{x}^j$ for $j > r$, our change of coordinates will have no effect on the remark just made. We proceed as follows: we cover the closed interval $[0, 1]$ with a finite number of open intervals $J_1, \cdots, J_\nu$, all contained in $]-2, 2[$ and such that if we call t_λ the central point in J_λ, the matrix

$$\operatorname{Re} S^{t_\lambda - t}$$

is invertible for any t in some neighborhood of the closure of J_λ. Then in $\tilde{V} \times J_\lambda$ we use the coordinates $\operatorname{Re}(S^{t_\lambda - t}\tilde{x}')^j$, $j = ,\cdots, r$, which we shall call $\tilde{x}'^j$ for the sake of simplicity. We then have (cf. (4.6))

$$\tilde{Z}^j = \tilde{x}'^j + i\tilde{\Phi}^j(\tilde{x}', t), \quad j = 1,\cdots,r, \tag{4.11}$$

such that if we write $\tilde{\Phi}' = (\tilde{\Phi}^1, \cdots, \tilde{\Phi}^r)$, we have

$$|\tilde{\Phi}'(\tilde{x}', t)| \leq \text{const.}|t - t_\lambda||\tilde{x}'|, \quad \tilde{x} \in \tilde{V}, t \in J_\lambda. \tag{4.12}$$

As a result of what precedes we see that in both cases, the one corresponding to use of the map (4.5) and the one to use of (4.8), we find ourselves in a situation that can be described as follows.

Let V be an open ball centered at the origin in $\mathbf{R}^m$, and J an open interval in the real line. On $V \times J$ we are given a hypo-analytic structure defined by the following m C^∞ functions:

$$Z^j = x^j + i\Phi^j(x, t), \quad j = 1, \cdots, m, \tag{4.13}$$

with

$$\Phi^j \textit{ real-valued}, \Phi^j(0, t) \equiv 0 \textit{ in } J \ (1 \leqslant j \leqslant m). \tag{4.14}$$

Moreover there is $B > 0$ such that

$$|\Phi''(x, t)| \leqslant B|x|^2, \quad x \in V, t \in J, \tag{4.15}$$

where we have used the notation $\Phi'' = (\Phi^{r+1}, \cdots, \Phi^m)$.

Lemma 4.1. *Assume that* (4.13), (4.14), (4.15) *hold. Let h be a distribution solution in $V \times J$, and t_0 any point in J. If* $(0, \xi''^0) \in T_0^*V$ *does not belong to the hypo-analytic wave-front set of the trace h_{t_0} of h on $V \times \{t_0\}$, then it does not belong to that of h_{t_1} whatever $t_1 \in J$.*

Proof. It suffices to show that to every closed subinterval K of J there is a number $\delta > 0$ such that the assertion is true if $t_0, t_1 \in K$ and $|t_0 - t_1| < \delta$.

We shall begin by modifying the Z^j for $j \leqslant r$. Let $L^j(x, t)$ denote the linear part of the Taylor expansion of Z^j with respect to x about $x = 0$. Notice that by (4.14), (4.15) we have

$$|Z^j - L^j(0, Z'', t_1)| \leqslant \text{const.}\left(|x'| + |x|^2 + |t - t_1|^2\right).$$

Thus after substitution of $Z^j - L^j(0, Z'', t_1)$ for Z^j we may assume that $L^j(x, t_1)$ is independent of x''. But this means that, possibly after decreasing V and increasing B, we have, whatever t_1 in K,

$$\left|\frac{\partial Z'}{\partial x'}(0, t_1)^{-1} Z'(x, t_1) - x'\right| \leqslant B|x|^2, \quad x \in V.$$

As a consequence, if we substitute

$$\frac{\partial Z'}{\partial x'}(0, t_1)^{-1} Z'(x, t)$$

for $Z'(x, t)$ and take (4.15) into account, we may as well suppose

$$|\Phi(x, t_1)| \leqslant B|x|^2, \quad x \in V. \tag{4.16}$$

Our first requirement on δ will then be that, after contraction of V about the origin and for all t in J, $|t - t_1| \leqslant \delta$,

$$\forall x, y \in V, \quad |\Phi(x, t) - \Phi(y, t)| < |x - y|/2. \tag{4.17}$$

We wish to apply Theorems 2.1, 2.2 and Lemma 3.1. First of all we select a number $d > 0$ such that the closed ball $\{x; |x| \leq d\}$ is contained in V, and a function $g \in C_c^\infty(V)$ equal to one in that ball.

We shall apply Theorem 2.1 to the trace of gh on $V \times \{t_0\}$, gh_{t_0}. To be able to do this we must know that Condition (2.2) is satisfied. Notice that in the present situation this means that $\xi''^0 \cdot \Phi''(x, t_0) \leq c|x| + \kappa |x|^2/4$ for some $c > 0$ and all x in V. But by (4.15) this is automatically true *whatever* $c > 0$ provided $\kappa > 4B$.

We shall apply Theorem 2.2 to gh_{t_1}. For this all we need to know is that Condition (2.11) is satisfied. But this is precisely what (4.16) insures. We shall take $\kappa > \kappa_*$, the number in Theorem 2.2.

Finally we apply Lemma 3.1 after a translation $t \mapsto t - t_1$ (thus what is $t = 0$ in the statement of Lemma 3.1 now becomes $t = t_1$). Here condition (3.4) must be satisfied, and so it is whatever the value of $d > 0$, exactly for the same reason that (2.2) is satisfied whatever the value of c: as a consequence of (4.15). The neighborhood W' in Lemma 3.1, which here becomes a neighborhood $W'(t_1)$ of t_1, only depends on d and on $\kappa > \sup(4B, \kappa_*)$. The latter constant may be taken to be independent of $t_1 \in K$; this was noted earlier insofar as B is concerned and is quite clear for κ_* (cf. Remark 2.1). We determine δ by the additional requirement that the interval $]t_1 - \delta, t_1 + \delta[$ be contained in $W'(t_1)$, again whatever the point t_1 in K.

Theorem 2.1 tells us that gh_{t_0} has Property $(2.1)_\kappa$. Lemma 3.1 then implies that the same is true of gh_{t_1}. Theorem 2.2 deduces from this fact that $(0, \xi''^0) \notin WF_{ha}(gh_{t_1})$. q.e.d.

Lemma 4.1 is a result on the propagation of (microlocal) singularities in certain hypo-analytic structures (of codimension one; see Chapter I). We derive from it:

Theorem 4.1. *Let h be a distribution solution in an open neighborhood U of a point p_0 of Ω, $\theta^0 \neq 0$ a characteristic cotangent vector to Ω at p_0, X, Y two maximally real C^∞ submanifolds of U passing through p_0, and h_X and h_Y the traces of h on X and on Y respectively. If $\pi_X(\theta^0)$ does not belong to the hypo-analytic wave-front set of h_X, then $\pi_Y(\theta^0)$ does not belong to that of h_Y.*

Proof. We assume that $(U, Z^1, \cdots, Z^m)$ is a hypo-analytic local chart centered at p_0 (henceforth called the origin), adapted to X, as in §3. We assume that (3.9), (3.10) hold, and that X is defined by the equation $y = 0$. On the other hand we take Y to be defined (in $U_\#$) by (4.2), (4.3). If then we return to the definitions of X_t and Y_t in the construction which follows Proposition 4.1, we observe the following

$$X = X_0,\ U_\# \cap X_1 = Y_0,\ Y_1 = Y \cap U_\#.$$

We first apply Lemma 4.1 to the hypo-analytic manifold $\tilde{V} \times J_\lambda$, whose hypo-analytic structure is defined by the functions (4.6) and (4.11). Since the intervals J_λ, $\lambda = 1, \cdots, \nu$, make up an open covering of $[0, 1]$, we reach the conclusion that if the hypo-analytic wave-front set of the trace of a distribution solution $\tilde{h}$ on $\tilde{V} \times \{0\}$ does not contain $\pi_Y(\theta^0) = \pi_X(\theta^0) = (0, \xi''^0)$, the same is true of that of its trace on $\tilde{V} \times \{1\}$.

Next we apply Lemma 4.1 to the hypo-analytic manifold $V_\# \times]-2, 2[$, whose structure is defined by the functions (4.9). Here we conclude that if the hypo-analytic wave-front set of the trace of a distribution solution $h_\#$ on $V_\# \times \{0\}$ does not contain $(0, \xi''^0)$, the same is true of that of its trace on $V_\# \times \{1\}$.

We take h to be the pull-back via (4.5) of a distribution solution $\tilde{h}$ in U, and $h_\#$ to be the pull-back via (4.8) of the same distribution solution. Since (4.5) induces an isomorphism of $\tilde{X}_0$ onto an open neighborhood of the origin in $X_0 = X$, and (4.8) induces one of $Y_\#$ onto an open neighborhood of the origin in $Y_1 = Y \cap U_\#$ (and that these isomorphisms "preserve" $(0, \xi''^0)$ as we have seen), the assertion in Theorem 4.1 is proved. q.e.d.

Let h be a distribution solution in an open subset Ω' of Ω, (p_0, θ^0) a point of $T^*\Omega$ with $p_0 \in \Omega'$ and $\theta^0 \neq 0$, X a maximally real C^0 submanifold of Ω' passing through p_0, and h_X the trace of h on X.

If θ^0 is noncharacteristic, *i.e.*, if $\theta^0 \notin T^0_{p_0}$, then either $\pi_X(\theta^0) = 0$ or else $\pi_X(\theta^0) \notin WF_{ha}(h_X)$, by Theorem 3.1. Actually, if we keep in mind that the hypo-analytic wave-front set never intersects the zero-section, we see that $\pi_X(\theta^0) \notin WF_{ha}(h_X)$ in all cases. Suppose now that $\theta^0 \in T^0_{p_0}$. Then Theorem 4.1 tells us that $\pi_X(\theta^0) \in WF_{ha}(h_X)$ if and only if $\pi_Y(\theta^0) \in WF_{ha}(h_Y)$ whatever the maximally real submanfold Y of Ω' passing also through p_0. In other words, this "if and only if" property is valid whether θ^0 is characteristic or not. This allows us to introduce the following.

Definition 4.1. The point $(p_0, \theta^0) \in T^*\Omega$, with $p_0 \in \Omega'$ and $\theta^0 \neq 0$, will be said to belong to the hypo-analytic wave-front set of h if its natural projection on the cotangent bundle of any (or, equivalently, of every) maximally real submanifold X of Ω' passing through p_0 belongs to the hypo-analytic wave-front set of the trace of h on X.

When the codimension of the hypo-analytic structure of Ω is equal to zero, we have $T^0 = T^*\Omega$, and Definition 4.1 agrees with Definition 1.2.

When there is not risk of confusion we shall denote by $WF_{ha}(h)$ the hypo-analytic wave-front set (for the structure of Ω) of a distribution solution h in $\Omega' \subset \Omega$. Note that, by Theorem 3.1,

$$WF_{ha}(h) \subset T^0|_{\Omega'}. \tag{4.18}$$

Also note that by Definitions 1.2 and 4.1 the hypo-analytic wave-front set is a hypo-analytic "invariant".

Remark 4.1. The hypo-analytic wave-front set of an arbitrary distribution in $\Omega' \subset \Omega$ has *not* been defined.

Theorem 4.2. *The base projection of the hypo-analytic wave-front set of a distribution solution is equal to its hypo-analytic singular support* (*Definition* 1.3, *Chapter* I).

Proof. Combine Theorem 3.2, Chapter I with Theorem 2.4 of present chapter. q.e.d.

It is not immediately apparent that $WF_{ha}(h)$ is a closed subset of $T^0|_{\Omega'}$. But so it is:

Theorem 4.3. *The hypo-analytic wave-front set of a distribution solution h in an open subset Ω' of Ω is a closed conic subset of $T^*\Omega \backslash 0\,|_{\Omega'}$.*

Proof. Since T^0 is closed in $T^*\Omega$, it suffices to show that $WF_{ha}(h)$ is relatively closed in $T^0|_{\Omega'}$. Let us go back to the usual hypo-analytic local chart $(U, Z^1, \cdots, Z^m)$ centered at a point (called the origin) of Ω', in which (3.9), (3.10), and all the other standard properties are valid. Let us use the same notation as in the proofs of Theorems 3.1, 3.2.

Let $(0, \xi^0)$ be an arbitrary point in $T_0^0|_{X_0} \backslash 0$ (cf. (3.13)). We know that the hypo-analytic wave-front set of the trace h_0 of h on X_0 (defined by $y = 0$) is closed. Therefore if $(0, \xi^0) \notin WF_{ha}(h_0)$, we shall have $(x, \xi) \notin WF_{ha}(h_0)$ for all x in a suitably small open neighborhood $V' \subset V$ of the origin, and for all ξ in an open conic neighborhood Γ of ξ^0. According to Theorem 3.2 to each such point (x, ξ) there is an open neighborhood $W_0 \subset W$ of the origin such that $(x, \xi) \notin WF_{ha}(h_t)$ for all t in W_0. All that remains to be checked is that W' can be chosen independently of (x, ξ) provided V' is small enough and Γ "thin" enough. Inspection of the proof of Theorem 3.2 shows that W_0 is determined by the neighborhood W' in Lemma 3.1, which only depends on κ and a number $d > 0$ which can be taken to be the same for all the relevant (x, ξ), and on the number c in Theorem 2.1. It is true that the latter depends on ξ^0 through its dependence on the number c_0 in (2.9). But of course this number can be taken to be the same if ξ^0 is replaced by points ξ which are close enough, "conically speaking", to ξ^0. By Definition 4.1 we conclude that none of the points $(x, y, \xi, 0)$, with $x \in V', y \in W_0, \xi \in \Gamma$, belong to $WF_{ha}(h)$. They obviously make up a neighborhood of $(0, 0, \xi^0, 0)$ in T^0.

Remark 4.2. Suppose Ω is a real analytic manifold of dimension $m + n$, and let n linearly independent analytic complex vector fields $L_1, \cdots, L_n$ satisfying the bracket condition, be given in Ω. Then one may want to study the *analytic* wave-front set $WF_a(h)$ of the solutions h of the equations $L_j h = 0$

$(j = 1, \cdots, m)$. In our terminology this would be the hypo-analytic wave-front set of h for the initial hypo-analytic structure $\mathcal{C}_0$ of Ω, its real-analytic structure, whose codimension is zero. A priori it has little to do with the hypo-analytic structure $\mathcal{C}_1$ defined on Ω by the vector fields L_j (codim $\mathcal{C}_1 = n$). But note that if X is any maximally real analytic submanifold of Ω for $\mathcal{C}_1$, both $\mathcal{C}_0$ and $\mathcal{C}_1$ induce on X the same structure, which is real analytic.

Let h be a solution in Ω of the equations $L_j h = 0$, $j = 1, \cdots, n$, h_X its trace on X, and let $(p_0, \theta^0) \in T^*\Omega \backslash 0$ be a characteristic point (for the system $L_1, \cdots, L_n$) such that $p_0 \in X$. By Definition 3.2 (p_0, θ^0) does not belong to the hypo-analytic wave-front set of h (for $\mathcal{C}_1$) if $\pi_X(\theta^0) \notin WF_{ha}(h_X)$. But the "true" microlocal Holmgren theorem (see [15]) states that the latter is equivalent to $(p_0, \theta^0) \notin WF_{ha}(h)$. Thus one sees that $WF_{ha}(h) = WF_a(h)$.

5. Standardized local charts and the Levi form

Let $(U, Z^1, \cdots, Z^m)$ be our usual hypo-analytic chart centered at the origin. As before we suppose that U is the domain of local coordinates $x^1, \cdots, x^m$, $y^1, \cdots, y^n$, and that (3.9)–(3.10) hold. In the present section the analysis will be focused on a conic neighborhood of a characteristic covector at the origin $\theta^0 \in T_0^0 \backslash \{0\}$. As noted in §3 we have $\theta^0 = (0, \xi'', 0)$ with $\xi'' = (\xi_{r+1}, \cdots, \xi_m)$ $\neq 0$. We shall perform right away a linear change of the variables $x'' = (x^{r+1}, \cdots, x^m)$, which brings us into the situation where $\xi'' = (1, 0, \cdots, 0)$, that is to say,

$$\theta^0 = dx^{r+1}|_0. \tag{5.1}$$

Our aim in the first part of the present section is to use appropriate manipulations of the basic hypo-analytic functions Z^l and the coordinates x^j, y^k, to obtain an expression for Φ^{r+1} which will serve us well in the subsequent arguments.

Let us spell out in detail what are the "manipulations" which are allowed. We have the right to change variables but only under the proviso that the hypotheses (3.9)–(3.10), and (5.1), always be valid. We also have the right to perform holomorphic substitutions of $Z^1, \cdots, Z^m$—under the same proviso. With this in mind we shall make use of the following notation: if f and g are two *real* quadratic forms in the x^j, y^k, we say that they are equivalent, and write $f \sim g$ if there is a holomorphic quadratic form h in the Z^l's such that

$$f - g = \operatorname{Re} h(Z) + 0(|x|^3 + |y|^3).$$

Note that if this is so, then

$$\Phi^{r+1} - (f - g) = \operatorname{Im}\{Z^{r+1} - ih(Z)\} + O(|x|^3 + |y|^3),$$

and consequently up to third order terms the substitution of $Z^{r+1} - ih(Z)$ for Z^{r+1} has the effect of substituting $\Phi^{r+1} - f + g$ for Φ^{r+1} (and the expression of the former might be more suited to our purposes than that of the latter). It is true that it also modifies $\operatorname{Re} Z^{r+1}$. But then we may take $\operatorname{Re}\{Z^{r+1} - ih(Z)\}$ as new coordinate x^{r+1}. These substitutions have no effect on our basic hypotheses (3.9), (3.10) and (5.1).

With this notation we have

$$x^j x^k \sim y^j y^k,\ x^j y^k \sim -x^k y^j \quad \text{if } j, k \leqslant r; \tag{5.2}$$

$$x^j x^k \sim x^j y^k \sim 0 \quad \text{if } j > r,\ k \leqslant r; \tag{5.3}$$

$$x^j x^k \sim 0 \quad \text{if } j, k > r. \tag{5.4}$$

Lemma 5.1. *The hypo-analytic functions $Z^1, \cdots, Z^m$ and the local coordinates $x^1, \cdots, x^m$, $y^1, \cdots, y^n$ (all vanishing at the origin) in U can be chosen in such a way that hypotheses* (3.9), (3.10) *and* (5.1) *are satisfied, and that furthermore the following hold*:

There are s real numbers λ_j ($j = 1, \cdots, s$) with $r \leqslant s \leqslant n$, such that $\lambda_j \neq 0$ for every $j > r$, and complex numbers a_{jk} ($j = s + 1, \cdots, n$, $k = 1, \cdots, r$) such that

$$\begin{aligned}\Phi^{r+1}(x, y) = {} & \sum_{j=1}^{s} \lambda_j (y^j)^2 + \sum_{j=s+1}^{n} y^j \operatorname{Re}\left(\sum_{k=1}^{r} a_{jk} z^k \right) \\ & + 0\left(\sum_{j=s+1}^{n} |y^j||x''| \right) + 0(|x|^3 + |y|^3).\end{aligned} \tag{5.5}$$

Proof. We look at the homogeneous part of degree two in the Taylor expansion of Φ^{r+1} about the origin:

$$\Phi_2^{r+1} = \sum_{j,k=1}^{m} P_{jk} x^j x^k + 2 \sum_{j=1}^{m} \sum_{k=1}^{n} Q_{jk} x^j y^k + \sum_{j,k=1}^{n} R_{jk} y^j y^k,$$

with $R_{jk} = R_{kj}$. From (5.3) and (5.4) we derive

$$\begin{aligned}\Phi_2^{r+1} \sim {} & \sum_{j,k=1}^{r} \left(P_{jk} x^j x^k + 2Q_{jk} x^j y^k + R_{jk} y^j y^k \right) \\ & + 2 \sum_{j=r+1}^{n} \left(\sum_{k=1}^{m} Q_{kj} x^k + \sum_{k=1}^{r} R_{kj} y^k \right) y^j + \sum_{j,k=r+1}^{n} R_{jk} y^j y^k.\end{aligned}$$

We perform an **R**-linear change of variables $y'' = (y^{r+1},\cdots,y^n)$ so as to transform the last quadratic form into

$$\sum_{j=r+1}^{s} \lambda_j (y^j)^2, \quad \lambda_j \neq 0.$$

If the quadratic form in question happens to vanish identically, we take $s = r$, and the present step in the argument can be skipped. Otherwise we make the following change of variables:

$$y^j \mapsto \tilde{y}^j = y^j + \frac{1}{\lambda_j}\left(\sum_{k=1}^{m} Q_{kj} x^k + \sum_{k=1}^{r} R_{kj} y^k \right), \quad j = r+1,\cdots,s.$$

Note that, by virtue of (5.3), (5.4),

$$\left(\sum_{k=1}^{m} Q_{kj} x^k + \sum_{k=1}^{r} R_{kj} y^k \right)^2 \sim \left(\sum_{k=1}^{r} Q_{kj} x^k + R_{kj} y^k \right)^2,$$

and thus for suitable constants A_{jk}, B_{jk}, $C_{jk} \in \mathbf{C}$ with $C_{jk} = \overline{C}_{kj}$ we have

$$\Phi_2^{r+1} \sim \sum_{j,k=1}^{r} \operatorname{Re}\left(B_{jk} z^j z^k\right) + \sum_{j,k=1}^{r} C_{jk} z^j \bar{z}^k + \sum_{j=r+1}^{s} \lambda_j (\tilde{y}^j)^2$$
$$+ \sum_{j=s+1}^{n} \operatorname{Re}\left(\sum_{k=1}^{r} A_{jk} z^k \right) y^j + 2 \sum_{j=s+1}^{n} \sum_{k=r+1}^{m} Q_{kj} x^k y^j.$$

But the first sum is trivially equivalent to zero, and the second sum can be brought into the form of a sum of squares, after a **C**-linear change of the variables $z' = (z^1,\cdots,z^r)$. Noting that, by (5.2),

$$\sum_{j=1}^{r} \lambda_j |z^j|^2/2 \sim \sum_{j=1}^{r} \lambda_j (y^j)^2, \tag{5.6}$$

we obtain (5.5) (after deleting the tildas).

Lemma 5.2. *Suppose that* (3.9), (3.10) *and* (5.1) *hold, and that* Φ^{r+1} *is given by* (5.5) *with at least one coefficient* $a_{jk} \neq 0$. *Then the* Z^l, x^j, y^k *can be further modified so that all the preceding conditions are met, and that moreover* $\lambda_j < 0$ *for some* j, $1 \leq j \leq r$.

Proof. In (5.5) we make the change of variables

$$y^j \to \tilde{y}^j = y^j + K \sum_{k=1}^{r} \operatorname{Re}\left(a_{jk} z^k\right), \quad j = s+1,\cdots,n.$$

where K is a large positive constant. Again by (5.3), (5.4), we get from (5.5):

$$\Phi_2^{r+1} \sim \sum_{j=1}^{r} \lambda_j (y^j)^2 - K \sum_{j=s+1}^{n} \left(\sum_{k=1}^{r} \operatorname{Re}\left(a_{jk} z^k\right) \right)^2 + \sum_{j=r+1}^{s} \lambda_j (y^j)^2$$
$$+ \sum_{j=s+1}^{n} \operatorname{Re}\left(\sum_{k=1}^{r} a_{jk} z^k \right) \tilde{y}^j + 0\left(\sum_{j=s+1}^{n} |\tilde{y}^j||x''| \right).$$

By the argument already used to see that

$$\sum_{j=1}^{r} \lambda_j (y^j)^2 - K \sum_{j=s+1}^{n} \left(\sum_{k=1}^{r} \operatorname{Re}\left(a_{jk} z^k\right) \right)^2 \sim \sum_{j,k=1}^{r} C'_{jk} z^j \bar{z}^k,$$

with $C'_{jk} = \overline{C}'_{kj}$. If not all the a_{jk} are equal to zero, and K is large enough, then not all the eigenvalues of the hermitian form at the right will be $\geqslant 0$, as one sees by letting $\partial^2/\partial z^j \partial \bar{z}^k$ $(1 \leqslant j, k \leqslant r)$ act on the left-hand side. Reducing once again that hermitian form to a sum of squares and applying (5.6) yields the sought result. q.e.d.

We are now going to bring the Levi form into the picture. Let ω be an arbitrary point of Ω, $\theta \in T^0_\omega$, v_1, v_2 two complex vectors in $T'^{\perp}_\omega$, and V_1, V_2 two C^∞ sections of $T'^{\perp}$ such that $V_j|_\omega = v_j$ for $j = 1, 2$. The Levi form at the point $(\omega, 0)$ evaluated at (v_1, v_2) is defined by

$$\mathfrak{L}_{(\omega,\theta)}(v_1, v_2) = \frac{1}{2i}\left\langle \theta, \left[V_1, \overline{V}_2\right]_\omega \right\rangle. \tag{5.7}$$

We have denoted by $\langle\,,\rangle$ the bracket for the duality between tangent and cotangent vectors. One should show that the left-hand side is independent of the choice of the vector fields V_j. In order to prove this it suffices to show that if either v_1 or v_2 is zero, the expression (5.7) vanishes. Suppose for instance $v_2 = 0$. Let $L_1, \cdots, L_n$ be a basis of $T'^{\perp}$ in an open neighborhood Ω' of ω, and write

$$V_2 = c_1 L_1 + \cdots + c_n L_n \text{ with } c_j \in C^\infty(\Omega').$$

The functions $c_1, \cdots, c_n$ must vanish at ω. This clearly implies that the right-hand side of (1.1) is zero.

Thus $\mathfrak{L}_{(\omega,\cdot)}$ is a function on T^0_ω valued in the space of hermitian forms on $T'^{\perp}_\omega$. We can then define the quadratic form $\mathfrak{L}_{(\omega,\theta)}(v)$ on $T'^{\perp}_\omega$ by

$$\mathfrak{L}_{(\omega,\theta)}(v) = \mathfrak{L}_{(\omega,\theta)}(v, v). \tag{5.8}$$

Customarily we also refer to the quadratic form (5.8) as the *Levi form*. We shall make use of the customary basis of $T'^{\perp}$ over U, $L_1, \cdots, L_n$, defined by the conditions (cf. §2, Chapter I)

$$L_j Z^l = 0,\ L_j y^k = \delta_j^k \quad (l = 1, \cdots, m, j, k = 1, \cdots, n). \tag{5.9}$$

We call L_j^0 the value of the vector field L_j at the origin. An immediate computation based on (3.9)–(3.10) shows that

$$(5.10)\qquad \begin{aligned} L_j^0 &= -2i\frac{\partial}{\partial \bar{z}^j}, \quad i = \sqrt{-1},\ j = 1,\cdots,r;\\ L_j^0 &= \frac{\partial}{\partial y^j}, \quad j = r+1,\cdots,n. \end{aligned}$$

It follows from the second set of relations (5.9) that $[L_j, \bar{L}_k]y^l = 0$ for all j, k, $l = 1,\cdots,n$. On the other hand,

$$\left[L_j, \bar{L}_k\right]x^h = \tfrac{1}{2}\left[L_j, \bar{L}_k\right](Z^h + \bar{Z}^h) = \tfrac{1}{2}\left(L_j\bar{L}_kZ^h - \bar{L}_kL_j\bar{Z}^h\right) \quad (1 \leqslant h \leqslant m),$$

and therefore

$$\left[L_j, \bar{L}_k\right] = \tfrac{1}{2}\sum_{h=1}^{m}\left(L_j\bar{L}_kZ^h - \bar{L}_kL_j\bar{Z}^h\right)\frac{\partial}{\partial x^h}.$$

But $L_j\bar{Z}^h = L_j(\bar{Z}^h - Z^h) = -2iL_j(\operatorname{Im} Z^h)$. Returning to (3.9)–(3.10) we see that

$$L_j\bar{Z}^h = -2iL_jy^h \text{ if } h \leqslant r,\ L_j\bar{Z}^h = -2iL_j\Phi^h \text{ if } h > r.$$

But this of course means that $L_j\bar{L}_kZ^h - \bar{L}_kL_j\bar{Z}^h = 0$ if $h \leqslant r$. Also, by (3.10) we have at the origin

$$L_j\bar{L}_k\Phi^h = L_j^0\bar{L}_k^0\Phi^h \quad \text{if } h > r,$$

and therefore

$$(5.11)\qquad \frac{1}{2i}\left[L_j, \bar{L}_k\right]\big|_0 = \sum_{l=r+1}^{m}\left(L_j^0\bar{L}_k^0\Phi^l\right)(0,0)\frac{\partial}{\partial x^l} \quad (j, k = 1,\cdots,n).$$

Recalling Hypothesis (5.1) we reach the conclusion that

$$(5.12)\qquad \mathfrak{L}_{(0,\theta^0)}\left(L_j^0, L_k^0\right) = \frac{1}{2i}\sigma\left(\left[L_j, \bar{L}_k\right]\right)(0,\theta^0) = \left(L_j^0\bar{L}_k^0\Phi^{r+1}\right)(0,0).$$

Let us compute the numbers (5.12) when Φ^{r+1} has the form (5.5). We find that

$$(5.13)\qquad \left(L_j^0\bar{L}_k^0\Phi^{r+1}\right)(0,0) = \begin{cases} 0 & \text{if } j \neq k, j, k \leqslant s;\\ 2\lambda_j & \text{if } j = k \leqslant s;\\ ia_{jk} & \text{if } j > s,\ k \leqslant r;\\ 0 & \text{if } j > s,\ k > r. \end{cases}$$

Call $\mathfrak{L}_0$ for short the matrix with generic entry (5.12). Among its eigenvalues we definitely find $2\lambda_{r+1},\cdots,2\lambda_s$, but not necessarily $2\lambda_1,\cdots,2\lambda_r$. Nevertheless the following can be asserted:

Lemma 5.3. *Suppose that* (3.9), (3.10) *and* (5.1) *hold, and that* Φ^{r+1} *is given by* (5.5). *Possibly after further modification of the* Z^l, x^j, y^k *which does not modify those hypotheses, the following two properties are equivalent*:

(5.14) *at least one of the eigenvalues of* $\mathfrak{L}_0$ *is* < 0;

(5.15) *at least one of the real numbers* λ_j *in* (5.5) *is* < 0.

Proof. If all the a_{jk} in (5.5) are equal to zero, the eigenvalues of $\mathfrak{L}_0$ are exactly equal to $2\lambda_1,\cdots,2\lambda_s$, and the equivalence of (5.14) and (5.15) is trivial. If at least one of the numbers a_{jk} is nonzero, we modify the Z^l, x^j, y^k so that the conclusion in Lemma 5.2 is valid. Then (5.15) is automatically valid, and we must show that (5.14) is also valid. But (5.14) is equivalent to the property that the associated quadratic form $\mathfrak{L}_0(v)$ on $\mathbf{C}^n$ takes at least one strictly negative value. This is evident if one notes that when $v^{s+1} = \cdots = v^n = 0$, we have, according to (5.13),

$$\frac{1}{2}\mathfrak{L}_0(v) = \sum_{j=1}^{s} \lambda_j |v^j|^2.$$

Lemma 5.4. *Same hypotheses as in Lemma* 5.3. *The matrix* $\mathfrak{L}_0$ *is positive definite if and only if* $s = n$, *and if all the numbers* λ_j $(j = 1,\cdots,n)$ *are* > 0.

Proof. The conditions are sufficient by (5.13). We know that if some a_{jk} is nonzero, possibly after modification of the Z^l, x^j, y^k (it is easy to see that this does not have any effect on the signature of the Levi matrix) one of the numbers λ_j will be < 0, and therefore so will be one of the eigenvalues of $\mathfrak{L}_0$. If the latter is positive definite, then all a_{jk} must vanish. It suffices once again to apply (5.13) to see that we must have $s = n$, and $\lambda_j > 0$ for all $j = 1,\cdots,n$.

6. A criterion of microlocal hypo-analyticity based on the Levi form

We continue to use the same notation as in §5. In particular we reason in an open neighborhood U of a point of Ω which we call the origin, and denote by 0. By θ^0 we denote an arbitrary characteristic covector at the origin, *i.e.* $\theta^0 \in T_0^0\backslash 0$. As in §5 we denote by $\mathfrak{L}_{(0,\theta^0)}(v)$ the *Levi form* at that point. In the proofs below we shall always assume that U is the domain of independent hypo-analytic functions $Z^1,\cdots,Z^m$, and of local coordinates $x^1,\cdots,x^m$, $y^1,\cdots,y^n$, all vanishing at the origin, such moreover that (3.9)–(3.10) and (5.1) hold.

Theorem 6.1. *Suppose that there is* $v \in T_0'^{\perp}$ *such that* $\mathfrak{L}_{(0,\theta^0)}(v) < 0$. *Then* $(0, \theta^0)$ *does not belong to the hypo-analytic wave-front set of any distribution solution defined near* 0.

Proof. We select the Z^l, x^j, y^k so as to be able to apply Lemma 5.3. Of course the hypothesis says that Condition (5.14) holds. Thus Condition (5.15) holds. We shall assume that $\lambda_j < 0$ for some $j \leqslant r$. The proof in the case where $\lambda_j < 0$ for some j, $r < j \leqslant s$, is essentially identical, in fact somewhat simpler, and we leave it to the reader. After a permutation in the z' variables, and a dilation, we may and shall assume that $\lambda_1 = -1$.

Then according to (5.5) we can find a number $\delta_0 > 0$ such that whatever δ, $0 < \delta < \delta_0$, if $|x| < 8\delta$, then

$$\Phi^{r+1}(x,\delta,0) \leqslant -\delta^2/2, \quad |\Phi(x,\delta,0)| \leqslant 2\delta, \tag{6.1}$$

where, as usual, $\Phi = (\Phi^1,\cdots,\Phi^m)$. We suppose tacitly that

$$|x| \leqslant 8\delta_0, |y| \leqslant \delta_0 \text{ implies } (x, y) \in U.$$

It is convenient to take U in the form of a product $V \times W$ with V (resp., W) an open ball centered at the origin in x-space $\mathbf{R}^m$ (resp., in y-space $\mathbf{R}^n$).

In what follows we apply the results of §§1 to 4. We deal with an arbitrary distribution h in U, and denote by h_X its trace on the maximality real submanifold $V \times \{0\}$, which we call X. In order to prove Theorem 6.1 it will suffice to prove that $\pi_X(\theta^0) = (0, \xi'') \in T_0^*X$, with $\xi'' = (1,0,\cdots,0)$, does not belong to the hypo-analytic wave-front set of h_X (Definition 4.1 and Theorem 4.1). To this end we shall apply Theorem 2.2. But we want to avail ourselves to Remark 2.1. Note first that, by virtue of (5.5),

$$\Phi^{r+1}(x,0) = 0(|x|^3). \tag{6.2}$$

Suppose then that for each $j = r+2,\cdots,m$, we substitute

$$Z^j_{\#}(x, y) = Z^j(x, y) - \frac{i}{2}\sum_{k,l=1}^{m} \frac{\partial^2\Phi^j}{\partial x^k \partial x^l}(0,0)Z^k(x, y)Z^l(x, y),$$

for $Z^j(x, y)$. We set $x^j_{\#} = \operatorname{Re} Z^j_{\#}(x, y)$, $j = r+2,\cdots,m$, and then delete the subscripts $\#$. This modification of the Z^j has no effect on (5.5). It does however have the effect that now all second derivatives of the functions $Z^j(x,0)$ ($j = 1,\cdots,m$) vanish at the origin. Therefore by Remark 2.1 we can select the number σ_0 small enough that if the open ball $V_0 = \{x \in \mathbf{R}^m; |x| < 8\delta_0\}$ is taken as the neighborhood U in Theorem 2.2, then the corresponding number κ_* can be taken $\leqslant 1/16$. We shall thus take

$$\frac{1}{16} < \kappa < \frac{1}{8}, \tag{6.3}$$

and show that $(2.1)_\kappa$ holds when $\xi^0 = (0, \xi'')$, $\xi'' = (1,0,\cdots,0)$.

In order to obtain the latter we shall first apply Lemma 3.1 with $V = V_0$ and $d = 7\delta$. Because of (6.2), if $\delta_0 > 0$ is small enough, $|x| < 8\delta_0$, and (6.3) is true, then $|\Phi^{r+1}(x,0)| \leqslant \kappa |x|^2/4$, which implies (3.4) when $\xi = (0, \xi'')$. We thus select $g \in C_c^\infty(V_0)$, $g = 1$ if $|x| \leqslant 7\delta$, and reach the conclusion that (3.5) is valid. We shall in fact apply (3.5) with $t = (\delta, 0, \cdots, 0)$, $0 < \delta \leqslant \delta_0$ suitably small.

In order to estimate $F_t^\kappa(gh; z, \zeta)$ we estimate from below, away from zero, the quantity

$$Q = \mathrm{Re}\{i\zeta \cdot Z(x,t) + \kappa\langle\zeta\rangle[z - Z(x,t)]^2\}/|\zeta|,$$

for $z \in V^{\mathbf{C}}$, a suitable open neighborhood of the origin in $\mathbf{C}^m$, and ζ in some open cone in $\mathbf{C}_m \backslash \{0\}$ containing ξ^0. As usual it suffices to have a lower bound for $Q_0 = Q$ when $z = 0$, $\zeta = \xi^0$. The properties (6.1) imply

$$Q_0 = -\Phi^{r+1}(x,t) + \kappa\big(|x|^2 - |\Phi(x,t)|^2\big) \geqslant \tfrac{1}{2}(1 - 8\kappa)\delta^2,$$

which, in view of (6.3), implies all we wanted. (Cf. proof of Theorem 3.1.)

Corollary 6.1. *Suppose that to every $\theta^0 \in T_0^0\backslash 0$ there is $v \in T_0'^{\perp}$ such that $\mathfrak{L}_{(0,\theta^0)}(v) < 0$. Then any distribution solution is hypo-analytic at the origin.*

Corollary 6.2. *Suppose the hypo-analytic structure is real-analytic. If the hypothesis in Theorem 6.1 is satisfied, the point $(0, \theta^0) \in T^*\Omega\backslash 0$ does not belong to the analytic wave-front set of any distribution solution in some open neighborhood of the origin.*

The proof of Corollary 6.2 follows from Theorem 6.1 and Remark 4.2.

Corollary 6.3. *Suppose that the hypo-analytic structure is real-analytic and that to every $\theta^0 \in T_0^0\backslash 0$ there is $v \in T_0'^{\perp}$ such that $\mathfrak{L}_{(0,\theta^0)}(v) < 0$. Then every distribution solution in an open neighborhood of the origin is an analytic function in a smaller neighborhood.*

We prove next a weak converse of Theorem 6.1.

Theorem 6.2. *Suppose that the Levi form at the point $(0, \theta^0) \in T^0\backslash 0$ is positive definite. Then there exist an open neighborhood U of the origin and a C^1 solution h in U whose hypo-analytic wave-front set is exactly the ray $\{(0, \rho\theta^0);\ \rho > 0\}$.*

Proof. By Lemma 5.4 we know that if Z^l, x^j, y^k are as in Lemma 5.1 such that (3.9)–(3.10) and (5.1) hold, and if (5.5) holds, then $s = n$ and all the numbers λ_j are > 0. A dilation in each variable z^j enables us to suppose $\lambda_j = 2$ for every $j = 1, \cdots, n$. Thus we have, by (5.5),

$$\Phi^{r+1}(x,y) = 2|y|^2 + 0\big(|x|^3 + |y|^3\big), \tag{6.4}$$

and if we set

$$f = Z^{r+1} + i[Z]^2, \quad [Z]^2 = \sum_{j=1}^{m} (Z^j)^2,$$

then

$$\text{(6.5)} \qquad \operatorname{Im} f(x, y) = |x|^2 + |y'|^2 + 2|y''|^2 + 0\big(|x|^3 + |y|^3\big).$$

Let U be an open ball centered at the origin such that $\operatorname{Im} f > 0$ in $U \backslash \{0\}$. We take $h = f^{3/2}$ in U, where the square root is the main branch of that function. We are going to show that h fulfills our requirements.

For one thing h is a holomorphic function of f off the origin, and therefore the hypo-analytic wave-front set of h must be contained in T_0^0 (Theorem 3.1 and Definition 4.1). By virtue of Theorem 6.1 and our hypothesis (that $\lambda_j > 0$ for all j) we know that $-dx^{r+1} \notin WF_{ha}(h)$. Since obviously h is *not* hypo-analytic at the origin, it will suffice to prove that

$$(0, \theta) \notin WF_{ha}(h)$$

for every $\theta \in T_0^0 \backslash 0$ which is not a multiple of dx^{r+1}.

As usual let X be the maximally real submanifold of U defined by $y = 0$. We are going to show that $\pi_X(\theta) = (0, \xi'')$ does not belong to the hypo-analytic wave-front set of the trace h_X of h on X. It is convenient to assume that X is an open ball centered at the origin in x-space. Note also that

$$\text{(6.6)} \qquad Z(x, 0) = x + \sqrt{-1}\,\Phi(x, 0),$$

with

$$\text{(6.7)} \qquad |\Phi(x, 0)| \leq B|x|^2, \quad x \in X,$$

for a suitable $B > 0$.

Let us write

$$\tilde{f}(z) = z^{r+1} + i[z]^2, \tilde{h}(z) = \tilde{f}(z)^{3/2}.$$

Let us now consider the following C^∞ functions in $X \times J$, where J is the open interval $\{t \in \mathbf{R}^1; |t| < 2\}$:

$$\text{(6.8)} \qquad Z_\#^j(x, t) = x^j + \sqrt{-1}\, t\Phi^j(x, 0), \quad j = 1, \cdots, m.$$

Set $f_\#(x, t) = \tilde{f}(Z_\#(x, t))$. We have

$$\operatorname{Im} f_\#(x, t) = t\Phi^{r+1}(x, 0) + |x|^2 - t^2 \sum_{k=r+1}^{m} \big(\Phi^k(x, 0)\big)^2$$
$$= |x|^2 + 0\big(t|x|^3\big).$$

Thus, provided X is small enough, we shall have $\operatorname{Im} f_\# > 0$ in $(X \backslash \{0\}) \times J$, and this certainly allows us to deal with $h_\#(x, t) = \tilde{h}(Z_\#(x, t))$. The latter is C^1 in $X \times J$, and is a solution for the structure defined by (6.8). Properties (6.6)–(6.7) imply that the hypotheses in Lemma 4.1 are satisfied. Its conclusion enables us to derive that $(0, \xi'')$ does not belong to the hypo-analytic wave-front set of the trace of $h_\#$ on $X \times \{1\}$, trace which can be identified to h_X via the map $(x, 1) \mapsto x$, from the fact that it does not belong to the hypo-analytic wave-front set of the trace of $h_\#$ on $X \times \{0\}$.

Since $Z_\#^j(x, 0) = x^j, j = 1, \cdots, m$, we must show that $(0, \xi'')$ does not belong to the *analytic* wave-front set of the function

$$h_0 = \left(x^{r+1} + i|x|^2\right)^{3/2},$$

a well-known fact, whose proof follows directly from Definition 1.2, and which we leave as an exercise to the reader.

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