J. DIFFERENTIAL GEOMETRY
18 (1983) 317–328

THREE EXOTIC $\mathbf{R}^4$'S AND OTHER ANOMALIES

ROBERT E. GOMPF

Abstract

We show that there are at least four smooth structures on $\mathbf{R}^4$, and that there are at least four oriented diffeomorphism types of Casson handles. This is also an expository paper concerning the theorems of Freedman and Donaldson; these theorems interact to demonstrate anomalous behavior for 4-manifolds.

0. Introduction

There have recently been two quantum jumps in our understanding of 4-manifolds. Mike Freedman's work [4] has shown that topological 4-manifolds behave much like higher dimensional manifolds, at least in the simply connected case (with hope for the nonsimply connected case as well). In sharp contrast, Simon Donaldson's theorem [2] shows that smooth 4-manifolds behave in a radically different way. In particular, Freedman's main results, surgery and h-cobordism theorems, have counter-examples in the smooth category. The most dramatic example of this pathology is the existence of exotic smooth structures on $\mathbf{R}^4$, in particular, smooth manifolds which are homeomorphic but not diffeomorphic to $\mathbf{R}^4$.

These exotic $\mathbf{R}^4$'s are surprising for several reasons. First, it is a standard fact that for $n \neq 4$, exotic $\mathbf{R}^n$'s cannot exist. More fundamentally, exotic $\mathbf{R}^4$'s show the failure in the smooth category of several major theories of higher dimensional manifolds. We see the breakdown of surgery theory (even in the simply connected case) during the construction of an exotic $\mathbf{R}^4$ (see §1). There can be no 5-dimensional proper h-cobordism theorem, as it is easy to construct a smooth h-cobordism from any exotic $\mathbf{R}^4$ to the standard one. (This cannot be a smooth product $\mathbf{R}^4 \times I$, as one boundary component is not diffeomorphic to $\mathbf{R}^4$.) Finally, we see a major breakdown of higher dimensional smoothing theory, as we will now explain.

In dimensions five and up, the number of smooth structures on a given manifold is determined by certain cohomology groups [6]. For example,

Received January 19, 1983.

$S^3 \times \mathbf{R}^n$ $(n \geqslant 2)$ has two smoothings, up to "isotopy." (These are actually diffeomorphic for $n > 2$, but the diffeomorphism is, in some sense, exotic.) The exotic structures are "stable" under products with $\mathbf{R}^m$, meaning that if $(S^3 \times \mathbf{R}^n)_\Sigma$ denotes the exotic structure, then $(S^3 \times \mathbf{R}^n)_\Sigma \times \mathbf{R}^m$ is the exotic structure on $S^3 \times \mathbf{R}^{n+m}$. In general, smooth structures of manifolds in these dimensions always exhibit such stability.

Only one exotic smooth structure on a 4-manifold was known prior to Donaldson's theorem; this was $(S^3 \times \mathbf{R})_F$, the exotic $S^3 \times \mathbf{R}$ of Freedman [3]. This structure has the expected stability; $(S^3 \times \mathbf{R})_F \times \mathbf{R}$ is the exotic structure on $S^3 \times \mathbf{R}^2$. Exotic $\mathbf{R}^4$'s, however, break this pattern. They are all unstable, as $\mathbf{R}^5$ only admits one smooth structure (up to isotopy). In fact, higher dimensional smoothing theory predicts that there should only be one structure on $\mathbf{R}^4$, as it has no cohomology.

There are now many examples of unstable smoothings (and resulting unusual h-cobordisms). There are at least four oriented smooth manifolds homeomorphic to $\mathbf{R}^4$. From these, thirteen unstable $S^3 \times \mathbf{R}$'s can easily be constructed. Casson handles, the fundamental tools of Freedman's theory, provide additional examples. Freedman's main theorem says that these are all homeomorphic to an open 2-handle, but Donaldson's theorem shows that they are not all diffeomorphic to it. In fact, they represent at least four oriented diffeomorphism types.

In §1 we will construct an exotic $\mathbf{R}^4$. In §2 we will modify this construction to exhibit four distinct structures on $\mathbf{R}^4$, as well as the thirteen $S^3 \times \mathbf{R}$'s. Finally, in §3 we will use similar methods to construct four diffeomorphism classes of Casson handles. We will use the following notation: if M is an oriented manifold, $\overline{M}$ will denote the manifold obtained from M by reversing orientation. We will, unless otherwise noted, work in the category of oriented smooth manifolds and orientation-preserving diffeomorphisms. Thus $\mathbf{C}P^2$ and $\overline{\mathbf{C}P^2}$ will be considered different manifolds, as there is no orientation-preserving diffeomorphism between them. All homeomorphisms and imbeddings will also be assumed to preserve orientation.

The author wishes to thank Mike Freedman for his time and support.

1. An exotic $\mathbf{R}^4$

We now give a standard construction of an exotic $\mathbf{R}^4$, which we will call $\mathbf{R}^4_{DF}$, the *Donaldson-Freedman* $\mathbf{R}^4$. Its existence was first observed by Freedman. The construction follows easily from three theorems which we state below.

We first state Donaldson's theorem [2] in its most convenient form for our purposes.

Theorem 1.1 (*Donaldson*). *If a smooth closed simply connected* 4-*manifold has a negative definite intersection form* (*over* $\mathbf{Z}$), *then this form must be isomorphic* (*over* $\mathbf{Z}$) *to* $\oplus_k\langle -1\rangle$, *i.e., it has a basis* $e_1,\cdots,e_k$ *of* k *elements* ($k \geqslant 0$) *with* $e_i \cdot e_j = -\delta_{ij}$.

In particular, there is no such smooth manifold with form $E_8 \oplus E_8$, where E_8 is the unique symmetric form over $\mathbf{Z}$ which has rank 8, is even, and is negative definite.

It is well known that the *Kummer surface* K, the zero set of $z_1^4 + z_2^4 + z_3^4 + z_4^4$ in $\mathbf{C}P^3$, is a smooth closed simply connected 4-manifold which realizes the form $E_8 \oplus E_8 \oplus H$, where H is the hyperbolic form of rank 6, i.e., the sum of three subspaces each with matrix $\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$. Notice that H is also the intersection form of the compact manifold $X = (\#_3 S^2 \times S^2) - (\text{open 4-ball})$.

Theorem 1.2 (*Freedman*). *The subspace* H *of the intersection form of* K *is represented by a topological imbedding* $i\colon X \to K$. *This imbedding is "smoothly equivalent to" an imbedding* $j\colon X \to \#_3 S^2 \times S^2$.

The latter statement means that there are neighborhoods U of $i(X)$ and V of $j(X)$ which are diffeomorphic, with the following diagram commuting:

$$\begin{tikzcd} & U \subset K \arrow[dd, leftrightarrow, "\text{diffeomorphism}"] \\ X \arrow[ur, "i"] \arrow[dr, "j"'] & \\ & V \subset \#_3 S^2 \times S^2 \end{tikzcd}$$

In particular, $U - i(X)$ is diffeomorphic to $V - j(X)$. It is easy to arrange that $U - i(X)$ be homeomorphic to $S^3 \times \mathbf{R}$ (although it will not be diffeomorphic), since the boundary of X is S^3.

Theorem 1.2 is a special case of Freedman's topological (simply connected) surgery [4]. Notice that $K - i(X)$ is an open manifold whose end is collared by $U - i(X)$, which is homeomorphic to $S^3 \times \mathbf{R}$. Thus we may topologically cap this end by gluing on a 4-ball, obtaining a closed topological manifold with intersection form $E_8 \oplus E_8$. In other words, we have "surgered out" the summand H from the intersection form.

We now see why Donaldson's theorem shows the failure of smooth surgery, for if we could smoothly surger out H we would obtain an impossible smooth manifold representing $E_8 \oplus E_8$. In particular, $U - i(X)$ cannot be diffeomorphic to $S^3 \times \mathbf{R}$, i.e., it is an exotic smoothing of $S^3 \times \mathbf{R}$. In fact, it cannot contain any smooth S^3 separating its two ends; otherwise we could trim off the end of $K - i(X)$ by cutting along this S^3, and then cap the new boundary smoothly with a 4-ball.

Theorem 1.2 is proven by using a theorem of Casson [1] to construct six Casson handles in K representing a basis for H. We will discuss a similar proof in the next section.

Theorem 1.3 (*Freedman* [4, *Corollary* 1.2]). *Any open* 4-*manifold* M *with* $\pi_1(M) = 0$, $H_2(M) = 0$ *and end collared* (*topologically*) *by* $S^3 \times \mathbf{R}$ *is homeomorphic to* $\mathbf{R}^4$.

This is basically an application of Freedman's proper h-cobordism theorem.

We now exhibit $\mathbf{R}^4_{DF}$. Consider the imbedding $j: X \to \#_3 S^2 \times S^2$ of Theorem 1.2. It is easy to check that the open manifold $(\#_3 S^2 \times S^2) - j(X)$ must satisfy the hypotheses of Theorem 1.3; this is our $\mathbf{R}^4_{DF}$. (Consider $\#_3 S^2 \times S^2$ as the union of the two sets V and $\mathbf{R}^4_{DF}$. These intersect in $V - j(X)$, which is topologically $S^3 \times \mathbf{R}$. The hypotheses of Theorem 1.3 now follow from the Seifert-Van Kampen and Mayer-Vietoris theorems.) We must show that $\mathbf{R}^4_{DF}$ is not diffeomorphic to $\mathbf{R}^4$. Note that the end of $\mathbf{R}^4_{DF}$ is collared by $V - j(X)$. This is diffeomorphic to $U - i(X)$, the exotic $S^3 \times \mathbf{R}$ described above. It has no smooth S^3 separating its ends; hence $\mathbf{R}^4_{DF}$ has "no S^3's near infinity." Specifically, there is a compact set in $\mathbf{R}^4_{DF}$ (namely $\mathbf{R}^4_{DF} - V$) which cannot be enclosed by any smooth S^3. This behavior is impossible in the standard $\mathbf{R}^4$.

From our construction it is clear that $\mathbf{R}^4_{DF}$ imbeds smoothly in $\#_3 S^2 \times S^2$. With care, we could have obtained a sharper result. In particular, by considering the three $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ factors of H separately, we could have constructed our exotic $\mathbf{R}^4$ in $S^2 \times S^2$.

2. Three exotic $\mathbf{R}^4$'s

By suitably modifying the construction of $\mathbf{R}^4_{DF}$, we will obtain $\mathbf{R}^4_\Gamma$, an exotic $\mathbf{R}^4$ imbedded in $\mathbf{C}P^2$ rather than $S^2 \times S^2$. This will enable us to show that $\mathbf{R}^4_\Gamma$ has no orientation-reversing self-diffeomorphisms. Hence $\overline{\mathbf{R}}^4_\Gamma$ is distinct from $\mathbf{R}^4_\Gamma$. We can then easily construct a third exotic structure $\mathbf{R}^4_\Delta$ which has such a self-diffeomorphism. It is an open question whether $\mathbf{R}^4_{DF}$ can equal $\mathbf{R}^4_\Gamma$ or $\mathbf{R}^4_\Delta$ or even whether these manifolds are uniquely defined by the given constructions.

We obtain the necessary modification by replacing the Kummer surface with the manifold $M = \mathbf{C}P^2 \# (\#_9 \overline{\mathbf{C}P^2})$. The intersection form of M has rank 10, and can be naturally written as $\langle 1\rangle \oplus (\oplus_9 \langle -1\rangle)$ with corresponding basis $e_0, e_1, \cdots, e_9$. We will surger out the rank 1 subspace generated by the element $x = 3e_0 + e_1 + \cdots + e_8$. Note that $x \cdot x = 1$; hence this subspace has form $\langle 1\rangle$. Also, its orthogonal complement is $E_8 \oplus \langle -1\rangle$. (To see this, consider x in span $\{e_0, \cdots, e_8\} = \langle 1\rangle \oplus (\oplus_8 \langle -1\rangle)$. It is easy to check that for each y in this subspace $x \cdot y \equiv y \cdot y \pmod 2$. It follows that in this subspace the orthogonal complement of x is even (with rank $= 8$, signature $= -8$) and hence E_8). Now by Donaldson's theorem 1.1, $E_8 \oplus \langle -1\rangle$ cannot be realized by a smooth simply-connected manifold. Thus the required surgery cannot be accomplished in the smooth category.

We now give the appropriate modification of Theorem 1.2.

Theorem 2.1. *The element $x \in H_2(M)$ is represented by an open set W homeomorphic to $\mathbf{C}P^2$ − point. There is a smooth imbedding $j: W \to \mathbf{C}P^2$. (All maps preserve orientation.)*

W replaces U in Theorem 1.2; $j(W)$ replaces V. X will be replaced by S, a topological 2-sphere in W coming from $\mathbf{C}P^1 \subset \mathbf{C}P^2$ − point.

Proof. A theorem of Casson ([1] or [4, Theorem 3.1]) gives conditions under which Casson handles can be constructed in a simply-connected manifold. In our case, it gives the following: let B be a small 4-ball in M, with C an unknotted circle in its boundary. Then there is a Casson handle $\mathrm{CH} \subset M$ with $\mathrm{CH} \cap B = \partial^- \mathrm{CH}$ = tubular neighborhood of C in ∂B, such that $\mathrm{CH} \cup B$ represents the class $x \in H_2(M)$. To get this, we need to check one algebraic condition: there must be an element $\beta \in H_2(M)$ with $x \cdot \beta = 1$ and $\beta \cdot \beta$ even. In fact $\beta = -e_1 + e_9$ works.

Now let $W = \mathrm{CH} \cup \operatorname{int} B$. By Freedman's theorem [4, Theorem 1.1] any Casson handle is homeomorphic to an open 2-handle; hence W is homeomorphic to the total space of an $\mathbf{R}^2$-bundle over S^2. This is actually a Hopf bundle, since $H_2(W)$ is generated by x, and $x \cdot x = +1$. Thus W is (oriented) homeomorphic to $\mathbf{C}P^2$ − point, the total space of the Hopf bundle.

We next construct the map $j: W \to \mathbf{C}P^2$. Now $\mathbf{C}P^2$ can be decomposed as 0-handle $\cup$ 2-handle $\cup$ 4-handle, where the 2-handle is glued to an unknotted circle in the boundary of the 0-handle, with one right-hand twist in the framing, forming a closed Hopf bundle. A Casson handle imbeds smoothly in a 2-handle (preserving the attaching region); in fact it is the complement of a certain "generalized cone on a Whitehead continuum" ([1] or [4, §5]). We can now define j on CH by this imbedding CH $\to$ 2-handle $\subset \mathbf{C}P^2$. Since CH and the 2-handle are both glued to $+1$-framed unknots, we can extend j to all of W by sending $\operatorname{int} B$ onto int (0-handle). This completes the proof of Theorem 2.1. q.e.d.

The construction of $\mathbf{R}^4_\Gamma$ is now analogous to that of $\mathbf{R}^4_{DF}$. Let $\mathbf{R}^4_\Gamma = \mathbf{C}P^2 - j(S)$ where S is the image in W of any $\mathbf{C}P^1$ under the homeomorphism $\mathbf{C}P^2 - \text{point} \approx W$. Then $\mathbf{R}^4_\Gamma$ is homeomorphic to $\mathbf{R}^4$ by Theorem 1.3, and its end is collared by $j(W - S)$ which is diffeomorphic to $W - S$, a topological $\mathbf{C}P^2 - \mathbf{C}P^1 - \text{point} = S^3 \times \mathbf{R}$. But $W - S$ collars $M - S$. Since we cannot cap this end by Donaldson's theorem, it follows that $\mathbf{R}^4_\Gamma$ has no S^3's near infinity.

Theorem 2.2. *$\mathbf{R}^4_\Gamma$ has no orientation-reversing self-diffeomorphisms.*

Since $\mathbf{R}^4_\Gamma \subset \mathbf{C}P^2$ by construction, this theorem follows immediately from the next lemma.

Lemma 2.3. *$\mathbf{R}^4_\Gamma$ does not imbed in $\overline{\mathbf{C}P^2}$ (or in any negative definite smooth simply connected closed 4-manifold).*

Proof. We suppose there is an imbedding $h: \mathbf{R}^4_\Gamma \to \overline{\mathbf{C}P^2}$. This will enable us to cap the end of $M - S$ with a negative definite manifold to obtain a closed manifold with form $E_8 \oplus \langle -1 \rangle \oplus$ (negative definite) contradicting Donaldson's theorem. Let Y be the compact set $\mathbf{R}^4_\Gamma - j(W - S)$. Then $\overline{\mathbf{C}P^2} - h(Y)$ is an open manifold with form $\langle -1 \rangle$, and end collared by $hj(W - S)$. (*Note.* The two ends of $W - S$ are very different from each other. We have exposed the end which was not exposed in $\mathbf{R}^4_\Gamma$.) Now use the map hj to glue this collar of $\overline{\mathbf{C}P^2} - h(Y)$ onto the collar $W - S$ of $M - S$, obtaining the desired closed manifold. q.e.d.

Next we construct $\mathbf{R}^4_\Delta$, an exotic $\mathbf{R}^4$ which has an orientation-reversing involution. It is an "end-connected sum" of $\mathbf{R}^4_\Gamma$ and $\overline{\mathbf{R}}^4_\Gamma$. Specifically, choose a smooth proper imbedding of a ray, $\gamma: [0, \infty) \to \mathbf{R}^4_\Gamma$. A tubular neighborhood of γ is diffeomorphic to $[0, \infty) \times \mathbf{R}^3$. Glue this onto the subset $[0, \frac{1}{2}) \times \mathbf{R}^3$ of $[0, 1] \times \mathbf{R}^3$, preserving the $\mathbf{R}^3$ fibers. Now take another copy of $\mathbf{R}^4_\Gamma$ and glue the same subset to $(\frac{1}{2}, 1] \times \mathbf{R}^3$ (preserving fibers; hence reversing orientation). We have now attached $\mathbf{R}^4_\Gamma$ to $\overline{\mathbf{R}}^4_\Gamma$, using $I \times \mathbf{R}^3$ like a piece of scotch tape. The resulting manifold $\mathbf{R}^4_\Delta$ is homeomorphic to $\mathbf{R}^4$, and has a reflection interchanging the two factors. It is not standard since it contains both $\mathbf{R}^4_\Gamma$ and $\overline{\mathbf{R}}^4_\Gamma$, so it cannot imbed in any definite manifold (negative or positive) as in Lemma 2.3.

We have now established that $\mathbf{R}^4_{\text{standard}}$, $\mathbf{R}^4_\Gamma$, $\overline{\mathbf{R}}^4_\Gamma$ and $\mathbf{R}^4_\Delta$ are all distinct oriented smooth structures on $\mathbf{R}^4$. It seems a reasonable conjecture that we can obtain many more structures via end-connected summing as above.

At this point, we can easily exhibit thirteen (unstable) oriented smooth structures on $S^3 \times \mathbf{R}$. Let $\mathbf{R}^4_\Sigma$ and $\mathbf{R}^4_{\Sigma'}$ each be any of the four structures listed above. Then $\mathbf{R}^4_\Sigma \# \mathbf{R}^4_{\Sigma'}$ is homeomorphic to $S^3 \times \mathbf{R}$ (in fact, it is diffeomorphic in the case where $\mathbf{R}^4_\Sigma$ and $\mathbf{R}^4_{\Sigma'}$ are both the standard structure). There are ten choices of $\{\mathbf{R}^4_\Sigma, \mathbf{R}^4_{\Sigma'}\}$, and these all yield distinct structures on $S^3 \times \mathbf{R}$. To see this, it suffices to check that the ends of the four $\mathbf{R}^4$'s all exhibit different behavior. For example the end of $\mathbf{R}^4_\Gamma$ cannot be capped with any negative definite manifold as in the proof of Lemma 2.3, but it can be capped by something positive definite (since $\mathbf{R}^4_\Gamma$ imbeds in $\mathbf{C}P^2$). The end of $\mathbf{R}^4_\Delta$, however, cannot be capped with either sort of manifold.

The three other $S^3 \times \mathbf{R}$'s occur as sufficiently narrow collars of the ends of the three exotic $\mathbf{R}^4$'s (such as the manifold $W - S$ defined earlier). These differ from the ten structures given above, as they admit no smooth S^3's separating their ends. (In fact, the inside ends of these $S^3 \times \mathbf{R}$'s differ from each other and from the ends of exotic $\mathbf{R}^4$'s. This may be seen by examining which manifolds they can be the ends of; e.g., not $\mathbf{R}^4$ with any smoothing.)

All of the exotic $S^3 \times \mathbf{R}$'s described here are unstable, that is, if $(S^3 \times \mathbf{R})_\Sigma$ is any one of these, then $(S^3 \times \mathbf{R})_\Sigma \times \mathbf{R}$ is diffeomorphic to the standard $S^3 \times \mathbf{R}^2$. This follows from the observation that each of these $S^3 \times \mathbf{R}$'s has Rohlin invariant zero.

3. Four diffeomorphism types of Casson handles

A Casson handle is a certain type of smooth manifold which comes with a fixed decomposition into kinky handles; see [1], [4] or [5] for details. By Freedman's theorem [4, Theorem 1.1], they are all homeomorphic to the standard open 2-handle $B^2 \times \operatorname{int} B^2$. It is well known that Donaldson's theorem shows that some Casson handles are not diffeomorphic to the standard 2-handle, but further results about their diffeomorphism types are scarce. We will show that there are at least four oriented diffeomorphism types of Casson handles.

Freedman [4, §5] gives notation for Casson handles, representing a kinky handle decomposition by a signed tree with base point. The vertices of the tree correspond (bijectively) to kinky handles. Every vertex has one edge leaving it for each kink (self-plumbing) of the corresponding kinky handle. The edges are each labelled $(+)$ or $(-)$, representing the self-intersection numbers of the associated kinks. The first stage kinky handle corresponds to the base point. Each second stage kinky handle caps off some first stage kink; its vertex is the endpoint of the corresponding edge. Continuing this pattern with higher stages specifies the tree. We have established a bijection between kinky handle structures of Casson handles and locally finite signed trees with base points for which every edge path extends indefinitely away from the base point.

For convenience, we will extend this notation to allow branches which terminate. A vertex with no edges leaving it represents a "kinky handle with no kinks," i.e., a 2-handle glued to the appropriate framed curve in place of a kinky handle. Thus a tree with some finite branches represents a manifold with a decomposition into kinky handles and 2-handles. We will not consider these to be kinky handle decompositions of Casson handles; instead we call the space associated with any signed tree a *generalized Casson handle* (GCH). Note that capping a kink with a 2-handle is the same, up to diffeomorphism, as removing the kink. Thus, if an edge has no edges leaving its endpoint, it may be deleted without changing the diffeomorphism class of the corresponding GCH. In particular, a finite GCH is diffeomorphic to a 2-handle, and an infinite GCH is diffeomorphic to some Casson handle.

We next define our invariants for distinguishing Casson handles. For $m = 0, 1, 2, \cdots$ let $M_m = \mathbf{C}P^2 \# (\#_{m+9} \overline{\mathbf{C}P^2})$. Thus M_0 is the manifold M used in §2; the others are obtained from M_0 by connected sums with $\overline{\mathbf{C}P^2}$. If $e_0, \cdots, e_{m+9}$ is the natural basis for $H_2(M_m)$, let $x_m = 3e_0 + e_1 + \cdots + e_8$. This is essentially the element x of §2. Note that the orthogonal complement of x is $E_8 \oplus (\oplus_{m+1} \langle -1 \rangle)$ (see §2); so by Donaldson's theorem we may not smoothly surger out x_m.

Definitions. A GCH has *positive polarity* if for some m it has a smooth orientation-preserving imbedding in M_m representing x_m (attached to an unknot in the boundary of a small 4-ball as in the proof of Theorem 2.1). A GCH has *negative polarity* if reversing its orientation gives a GCH with positive polarity.

Note that reversing the orientation of a GCH corresponds to reversing all signs of the associated signed tree. There is an equivalent definition of negative polarity in terms of imbedding in $\overline{M}_m$. We will also refer to positive or negative polarity of a signed tree, meaning that of the corresponding GCH.

These polarities are clearly (oriented) diffeomorphism invariants. They determine four possible *states*: both positive and negative polarity (a *dipolar* GCH), neither (a *nonpolar* GCH), or exactly one (a positively or negatively *monopolar* GCH). Our goal is now to show that all four states are realized by Casson handles. This will follow axiomatically from the following five facts.

Fact 3.1. *There exists a Casson handle with positive polarity.*

In fact, we constructed one in M_0 while proving Theorem 2.1.

Fact 3.2. *The standard* 2*-handle is nonpolar.*

If it had positive polarity, we could imbed it in some M_m representing x_m. This would give us an open set W diffeomorphic to $\mathbf{C}P^2$ − point, representing x_m (compare with Theorem 2.1). We could use this to smoothly surger out x_m, contradicting Donaldson's theorem. The case of negative polarity is now trivial; see Fact 3.5.

Fact 3.3. *If Q is a signed tree with positive polarity, then any signed tree Q' containing Q (with the same base point) also has positive polarity.*

If we imbed the corresponding manifold GCH_Q in M_m as in the definition of positive polarity, we can enlarge it to obtain $\mathrm{GCH}_{Q'}$ in M_m by adding tiny kinks and kinky handles within small coordinate patches. Alternatively, using the dual picture of a Casson handle as a 2-handle minus a generalized cone on a Whitehead continuum, we get $\mathrm{GCH}_{Q'} \subset \mathrm{GCH}_Q$ whenever $Q \subset Q'$.

Fact 3.4. *If Q has positive polarity, then any tree Q' obtained from Q by "pruning" a finite number of $(-)$ labelled edges also has positive polarity.*

This means the following: take the signed graph Q and delete the specified edges. The component of the resulting graph which contains the base point is the desired Q'. We have thus "pruned off" some branches. We will prove this fact below.

Fact 3.5. *We obtain analogues of the above facts for negative polarity by reversing all signs.*

This follows immediately from the definition of negative polarity.

Proof of Fact **3.4.** We will prove the case where Q' is obtained from Q by pruning one edge; the general case will follow by induction. Imbed GCH_Q in some M_m as in the definition of positive polarity. Let $p \in \mathrm{GCH}_Q \subset M_m$ be the point of intersection we wish to remove; two sheets of the core C of some kinky handle k meet here transversely with intersection number -1. We will remove this intersection by "blowing up a $\overline{\mathbf{C}P}^2$," i.e., carefully forming the connected sum $M_m \# \overline{\mathbf{C}P}^2 = M_{m+1}$ to obtain the desired $\mathrm{GCH}_{Q'}$ imbedded in M_{m+1}.

Let S_1 and S_2 be two unoriented 2-spheres in $\overline{\mathbf{C}P}^2$, obtained by forgetting the orientations of two $\mathbf{C}P^1$'s. These 2-spheres intersect transversely in one point p'. If S_1 and S_2 were oriented to represent the same generator of $H_2(\overline{\mathbf{C}P}^2)$, their intersection number $S_1 \cdot S_2$ would be -1; if they represented opposite generators, it would be $+1$.

Now let B be a small 4-ball in M_m, centered at p, intersecting the core disk C in a standard pair of unknotted disks D_1 and D_2 with $D_1 \cap D_2 = \{p\}$. Let B' be a similar ball about p' in $\overline{\mathbf{C}P}^2$, intersecting S_1 and S_2 at D_1' and D_2' respectively. There is an orientation-reversing diffeomorphism $\phi: B \to B'$ sending D_1 onto D_1' and D_2 onto D_2'. We define the connected sum $M_m \# \overline{\mathbf{C}P}^2$ as follows: remove int B and int B' from the respective manifolds, then glue the resulting boundaries via $\phi \mid \partial B$. This connected sum respects orientation since ϕ reverses it. Note that removing int B from M has punctured C twice, by deleting int D_1 and int D_2. We plug these holes by the disjoint disks $S_1 -$ int D_1' and $S_2 -$ int D_2', respectively, in $\overline{\mathbf{C}P}^2$. We have thus eliminated the desired kink from C.

Suppose, for the moment, that our kinky handle k is the first stage of the GCH. In this case we have succeeded in imbedding $\mathrm{GCH}_{Q'}$ in M_{m+1}. We must now show that $\mathrm{GCH}_{Q'}$ represents x_{m+1} as in the definition of polarity, i.e., that the homology class represented by $k \cup$ (small 4-ball) is not changed by the "blowing up" process. Our construction has altered C by "connected summing" with S_1 and S_2. Thus an orientation of C induces orientations on S_1 and S_2. In fact, these orientations are related so that $\phi | D_i \colon D_i \to D_i'$ reverses orientation for $i = 1, 2$. Now all three maps ϕ, $\phi | D_1$, and $\phi | D_2$ reverse orientation, so the fact that the kink at p had intersection number $D_1 \cdot D_2 = -1$ implies that $S_1 \cdot S_2 = D_1' \cdot D_2' = +1$. It follows that S_1 and S_2 are oriented to represent opposite generators of $H_2(\overline{\mathbf{C}P^2})$. Thus our construction modified the homology class of C only by adding two terms $[S_1]$ and $[S_2]$ which cancelled each other. This completes the proof if k is the first stage kinky handle.

If k is in the nth stage, $n > 1$, our newly constructed subspace of M_{m+1} automatically represents x_{m+1}, as this is determined by the first stage. The difficulty in this case is in showing that we have not altered the framing by which our kinky handle attaches to the previous stage. If we have, our new object will not be a GCH (or even a topological 2-handle). This framing can be defined by a relative self-intersection number of C in M_m − int(first $n - 1$ stages of GCH_Q) (see [5, §2.2.2], or [1]). The above argument now shows that blowing up the $\overline{\mathbf{C}P^2}$ does not affect this relative homology class, hence preserves the framing. (Alternatively, note that this number $C \cdot C$ equals $\chi(\nu) + 2$ self C, where $\chi(\nu)$ is the relative normal Euler class, and self C is the number of self-intersections of C counted with sign. Our construction decreases $\chi(\nu)$ by two (from running twice over $\overline{\mathbf{C}P^2}$) but increases self C by one (from removing a (-1)-intersection). Thus there is no net change.) q.e.d.

It is now easy to prove our main result:

Theorem 3.6. *Each of the four polarity states is realized by a Casson handle.*

Proof. There is a Casson handle CH_+ with positive polarity and only $(+)$ kinks at the first stage (or in fact at the first n stages for any fixed n). To get CH_+, take any CH with positive polarity (Fact 3.1) and prune out all $(-)$ kinks from the first stage (or first n stages). Now CH_+ is positively monopolar, for if it had negative polarity we could prune away the entire first stage (Fact 3.4 with reversed orientation) to get the standard 2-handle with negative polarity, contradicting Fact 3.2. Clearly $\mathrm{CH}_- = \overline{CH}_+$ is negatively monopolar.

To get a dipolar Casson handle, take the signed trees Q_+ (for CH_+) and Q_- (for CH_-) and identify their base points. The resulting tree contains both Q_+ and Q_-; hence it is dipolar by Facts 3.3 and 3.5. (This operation is equivalent to boundary-connected summing the first stage kinky handle cores, or gluing the attaching regions $\partial B^2 \times \text{int } B^2$ of the Casson handles together along $I \times \text{int } B^2$.)

Any tree with only (+) edges at the first stage and only (–) edges at the second stage is nonpolar. If it had positive polarity we could prune out the whole second stage and get a finite tree with positive polarity, contradicting Fact 3.2. Similarly it cannot have negative polarity. q.e.d.

There may be a simple procedure for determining the polarity of a Casson handle by looking at its tree: let CH_0 be the simplest Casson handle, with only one positive kink at each stage and no negative kinks. Let Q_0 be the corresponding tree. Notice that any tree with positive polarity must contain Q_0, for otherwise we could obtain a finite tree by pruning a finite number of (–) edges.

Conjecture. CH_0 *has positive polarity.*

If this is true, then a tree has positive polarity if and only if it contains Q_0 (with the same base point). Negative polarity will be similarly determined by $\overline{Q}_0$.

This conjecture becomes more likely if we generalize the definition of positive polarity. For example:

New Definition. A GCH has *positive polarity* if it imbeds in the (noncompact) infinite connected sum $M_\infty = CP^2 \# (\#_\infty \overline{CP}^2)$, representing $x_\infty = 3e_0 + e_1 + \cdots + e_8$.

This definition enlarges (strictly?) the class of GCH's with positive polarity. For example, we can now prune an infinite number of (–) edges as in Fact 3.4, so there is a CH_+ with positive polarity and no negative kinks at all. The proof of Theorem 3.6 is essentially unaltered by this change of definition. (The standard 2-handle is still nonpolar (see Fact 3.2) because the set W required in the proof can be taken to be a small neighborhood of a topological 2-sphere which, by compactness, actually lies in M_m for some finite m.)

The author has explicitly constructed an imbedding of the first eight stages of CH_0 in M_∞ as required. Although he has found no way to continue the construction indefinitely, this does seem to suggest that the conjecture may be true.

References

[1] A. Casson, *Three lectures on new constructions in 4-dimensional manifolds*, notes prepared by L. Guillou, Prépublications Orsay 81T06.

[2] S. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Differential Geometry **18** (1983) 279–315.

[3] M. H. Freedman, *A fake* $S^3 \times \mathbf{R}$, Ann. of Math **110** (1979) 177–201.

[4] ______, *The topology of four-dimensional manifolds*, J. Differential Geometry **17** (1982) 357–453.

[5] R. E. Gompf & S. Singh, *On Freedman's reimbedding theorems*, in *Four manifold theory*, Amer. Math. Soc. Summer Conference, University of New Hampshire, 1982; Amer. Math. Soc. Contemporary Math., to appear.

[6] R. Kirby & L. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, Annals of Math. Studies No. 88, Princeton University Press, Princeton, 1977.

UNIVERSITY OF CALIFORNIA, BERKELEY