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# RIEMANNIAN MANIFOLDS WITH BOUNDED CURVATURE RATIOS

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## 1. Introduction

The following classical result of Schur is well-known: If the sectional curvature of a riemannian manifold of dimension greater than two at every point is the same for every element of the grassmannian of tangent 2-planes, then the curvature is constant over the manifold. It is natural to ask what can be said if the ratios of sectional curvatures are close to one only. Gribkov [3] shows that not much can be said for open manifolds. He proves that even if the ratios of the curvatures are arbitrarily close to one, the variation of the curvature over the manifold can still be arbitrarily large.

In this paper we prove that if the riemannian manifold is compact, the sectional curvature is positive, and the curvature ratios are close to one, then the manifold is diffeomorphic to a spherical space form. This result is new even if we specialize to simply connected manifolds and assert the existence of a homeomorphism only. The well-known sphere theorem of Berger [1] and Klingenberg [4], while optimal in other respects, does not apply under the above local pinching assumption.

Basically, the result is a consequence of the second Bianchi equation and the Calderón-Zygmund inequality. We prove that the curvature $R$ is close to a certain local average $\bar{R}$ whose covariant derivative is small. Techniques similar to those of [5] provide a new metric connection $\nabla'$ on $M$ whose curvature $R'$ is close to $\bar{R}$ and whose torsion $T'$ is small. As a consequence, [5, Theorem 2] applies and yields the result.

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## 2. The result

Let $M$ denote a riemannian manifold, and $K$ its sectional curvature. We introduce the following concept of local pinching.

**Definition.** The sectional curvature $K$ is locally $\delta$-pinched if there exists a positive function $A: M \to \mathbf{R}$ such that at every point $x \in M$ the inequality $\delta A(x) < K < A(x)$ holds.

**Theorem.** *There exists $\delta = \delta(n)$ with $\frac{1}{4} < \delta < 1$, such that any compact locally $\delta$-pinched manifold of dimension $n$ is diffeomorphic to a spherical space form $\Gamma \setminus S^n$.*

The diffeomorphism constructed in the proof is in fact an almost isometry in the sense that the dilatation of the differential is controlled in terms of $(1 - \delta)$ —not just $(1 - \delta)A(x)$. The curvature of the model must be chosen to be the average of the scalar curvature of $M$. If $G$ is the isometry group of $M$, then $G$ is isomorphic to a closed subgroup of the isometry group of the spherical space form $\overline{M} = \Gamma \setminus S^n$ of the theorem, and the diffeomorphism is equivariant with respect to the actions of $G$ on $M$ and $\overline{M}$ respectively.

## 3. The proof

For convenience assume that the metric is normalized such that the sectional curvature $K$ satisfies $0 < K \leqslant 1$. As is well-known, the space of 4-tensors having all the symmetries of the riemannian curvature tensor $R$ decomposes into three subspaces which are irreducible under the action of the orthogonal group. The three components of $R$ correspond to scalar curvature, trace-less Ricci tensor, and Weyl tensor. Let $n(n - 1)S$ denote the trace of the Ricci tensor. The factor $n(n - 1)$ is chosen such that $S$ is the average of the sectional curvature. Let $g_{ij}$ and $R_{ij}$ denote the components of the riemannian metric and the trace-less Ricci tensor respectively.

The assumptions of the theorem imply that, for $1 - \delta$ sufficiently small, the estimate

$$|R_{ij}| < \varepsilon S \tag{1}$$

can be achieved for any $\varepsilon > 0$. The second Bianchi equation implies the following well-known equation relating scalar curvature and trace-less Ricci tensor

$$S_{;m} = \frac{1}{(n-1)(n-2)}\left(g^{ij}R_{im}\right)_{;j}, \tag{2}$$

where, in the usual tensor notation, the index after the semicolon denotes covariant derivative, and the Einstein summation convention is in effect.

The main point in the proof is a study of the positive real valued function $S$. The main tool is the Calderón-Zygmund inequality. We refer to [2, Chapter 5: Potential theoretical approach] for background material. To state this inequality let

$$k(x) = \frac{\omega(x)}{|x|^n},$$

with $\omega(x)$ positive homogeneous of order zero on $\mathbf{R}^n$ be smooth for $x \neq 0$, $\int \omega(x) = 0$, where the integration is over the unit sphere in $\mathbf{R}^n$. A function $k(x)$ with the above properties is called a singular kernel. For $\Psi$ with compact support in $\mathbf{R}^n$ let

$$k * \Psi = \int k(x-y)\Psi(y)dy.$$

The Calderón-Zygmund inequality (compare [2]) states

$$\|k * \Psi\|_{L_q} \leqslant \text{const.}\|\Psi\|_{L_q}, \tag{3}$$

where const. is a constant depending on $\omega$, $1 < q < \infty$, the support of $\Psi$, and $\|\Psi\|_{L_q} = (\int |\Psi(x)|^q dx)^{1/q}$. In addition to using this inequality directly, we also use a consequence, the interior regularity theorem for the Laplace operator.

The general strategy of the proof is as follows. Because $R_{ij}$ is small, (2) indicates that $S$ is close to a constant function. The main work in the proof is to determine in which sense this is true. To do this we define an average $\bar{S}$ of $S$ over small balls. It turns out that $\bar{S}$ is nearly constant, and $\|S - \bar{S}\|_{L_q}$ is suitably small compared to $\sup \bar{S}$. We obtain this estimate by studying a function $Q(x) = P(S - \bar{S})(x)$, where $P$ is a parametrix for the Laplacian. It turns out that on one hand $Q$ solves $\Delta Q = S - \bar{S}$ up to a small error, and on the other hand that the second derivatives of $Q$ are suitably small in $L_q$. Both estimates combined yield that $\|S - \bar{S}\|_{L_q}$ is small. The main tool for the above estimates is the Calderón-Zygmund inequality

The estimate on $\|S - \bar{S}\|_{L_q}$ allows the definition of a new metric connection $\nabla'$ which satisfies the conditions of the comparison theorem [5, Theorem 2] with the sphere $S^n$ as model. The method here is essentially the same as in [5]. In addition to the estimate on $\|S - \bar{S}\|_{L_q}$, we need an upper bound for the diameter of $M$. This bound is obtained by Jacobi field estimates.

The estimates in the proof are done in terms of the following norms. Let $f$ be a real valued function on $M$, and

$$\|f\| = \sup|f(x)|, \tag{4}$$

where sup denotes the supremum over the manifold $M$. To define the Sobolev norm involving $s$ derivatives measured in $L_q$, let exp: $T_pM \to M$ denote the exponential map. We define

$$\|f\|_{s,q} = \sup_{p \in M} \left( \int_{B_1(0)} \sum_{|\alpha|=0}^{s} \left| \frac{\partial}{\partial x^\alpha} f(\exp y) \right|^q dy \right)^{1/q}, \tag{5}$$

where $\partial/\partial x^\alpha$, in standard multi-index notation, denotes a derivative of order $|\alpha|$, $\partial/\partial x^i$, $|i| = 1$, are euclidean coordinate vector fields in $T_pM$, and $B_1(0)$ is the ball with radius 1 and center $0 \in T_pM$.

Let $B_\pi(0) \subset T_pM$ denote the ball with radius $\pi$ and center 0. Because we are interested in local properties of $S$ we let $S$, without change in notation, denote the pull back of $S$ on $M$ via exp: $T_pM \to M$ to $B_\pi(0)$. Next we write (2) in terms of euclidean coordinate vector fields $\partial/\partial x^i$ and replace the covariant derivatives by their coordinate expressions. We obtain

$$\operatorname{grad} S = L\Phi, \tag{6}$$

where $L = L_1 + L_0$ is a first order differential operator with leading term $L_1 = \sum a_i(\partial/\partial x^i)$ whose matrix coefficients $a_i$ are constant, and $\Phi$ is the component matrix $g^{ij}R_{im}$ in terms of the euclidean coordinate system chosen in $T_pM$. Because of (2) the norm of each coefficient in $a_i$ is bounded by $1/(n-1)(n-2)$. If we restrict $L$ to a ball $B_2(0) \subset T_pM$ with radius 2, then the coefficients of $L_0$ as well, via Jacobi field estimates due to the normalization $0 < K \leqslant 1$ of the sectional curvature, are bounded by a constant depending on the dimension. Thus the following estimate holds:

$$|a_i| + |L_0| < c, \tag{7}$$

where $c$ in this inequality, as well as in the remainder of the paper, denotes a constant depending on the dimension of $M$ only. Similarily, due to (1) the component matrix $\Phi$ satisfies

$$|\Phi| < c\varepsilon S. \tag{8}$$

Because the symbol $S$ is used for a function on $M$ as well as its pull back to $B_\pi(0) \subset T_pM$, the definitions (4) and (5) are ambiguous. To avoid this problem we let $\|S\|_{s,q}$ denote either the norm (5) for $S$ on $M$ or the supremum over $p \in M$ of the norms of the pull backs of $S$ to $B_\pi(0) \subset T_pM$. These norms are equivalent.

Next we define a local average $\bar{S}$ of $S$ as a function on $B_\pi(0) \subset T_pM$ as follows:

$$\bar{S}(x) = \int_{B_\rho(x)} \eta(|x-y|)S(y)\,dy, \tag{9}$$

where $\eta$: $\mathbf{R} \to \mathbf{R}$ is a nonnegative function supported in $|t| < \rho$, $\eta \equiv$ const. in a neighborhood of zero, and $\int_{B_\rho(x)} \eta(|x-y|)dy = 1$. We will be interested in $\bar{S}(x)$ for $|x| \leq 2$ only and will choose the radius $\rho$ of the ball $B_\rho(x)$ with center $x \in T_pM$ small.

Integration by parts implies

$$\operatorname{grad} \bar{S}(x) = \int_{B_\rho(x)} \eta(|x-y|)\operatorname{grad} S(y)dy.$$

A fixed choice of $\eta$ yields $|d\eta| < c\rho^{-(n+1)}$ with $c$ depending on dim $M$ only. (1) and (6), together with inequalities (7) and (8), via integration by parts yield for $|x| \leq 1$,

$$|\operatorname{grad} \bar{S}(x)| < c\frac{\varepsilon}{\rho}\|\bar{S}\|. \tag{10}$$

To estimate $\bar{S} - S$ let $c_n$ be chosen, compare [2, p. 211], such that the identity

$$f(x) = c_n\Delta\int \frac{f(y)}{|x-y|^{n-2}}dy \tag{11}$$

holds, where $\Delta$ is the Laplace operator and $f$ is a real valued function with compact support in $\mathbf{R}^n$. In addition, let $\theta$: $\mathbf{R} \to \mathbf{R}$ denote a cut-off function with support in $|t| < 1$ and $\theta(t) = 1$ for $|t| < \frac{1}{2}$. For $|x| \leq 1$ we define

$$Q(x) = c_n\int \frac{\theta(|x-y|)}{|x-y|^{n-2}}(\bar{S} - S)(y)dy. \tag{12}$$

The following two lemmas state the main estimates of the proof. The functions $S$, $\bar{S}$ and $Q$ are as defined above and the sectional curvature of $M$ is normalized to $0 < K \leq 1$.

**Lemma 1.** *The gradient of $Q$ satisfies the inequality*

$$\|\operatorname{grad} Q\|_{1,q} < c\sqrt{\varepsilon}\left(\|\bar{S}\| + \|S\|_{0,q}\right). \tag{13}$$

**Lemma 2.** *$Q$ satisfies the Laplace equation $\Delta Q = \bar{S} - S$ up to an error estimated by inequality*

$$\|\Delta Q - (\bar{S} - S)\|_{0,q} < c\sqrt{\varepsilon}\,\|S\|_{0,q}. \tag{14}$$

Since $\Delta$ involves only second order derivatives, Lemmas 1 and 2 combined yield

$$\|\bar{S} - S\|_{0,q} < c\sqrt{\varepsilon}\left(\|\bar{S}\| + \|S\|_{0,q}\right).$$

The triangle inequality implies

$$\|\bar{S} - S\|_{0,q} < c\sqrt{\varepsilon}\left(\|\bar{S}\| + \|\bar{S}\|_{0,q} + \|\bar{S} - S\|_{0,q}\right),$$

and for $\varepsilon$ small enough we obtain the Main Estimate

$$\|\bar{S} - S\|_{0,q} < c\sqrt{\varepsilon}\ \|\bar{S}\|. \tag{15}$$

*Proof of Lemma* 1. (12) by integration by parts gives

$$\begin{aligned} \operatorname{grad} Q(x) &= c_n \int \frac{\theta(|x-y|)}{|x-y|^{n-2}} \operatorname{grad} \bar{S}(y)\,dy \\ &\quad - c_n \int \frac{\theta(|x-y|)}{|x-y|^{n-2}} \operatorname{grad} S(y)\,dy \\ &= I_1(x) + I_2(x). \end{aligned}$$

The smoothing property of the kernel $|x-y|^{2-n}$ together with (10), for $\rho = \sqrt{\varepsilon}$ yields

$$\|I_1\|_{1,q} < c\sqrt{\varepsilon}\ \|\bar{S}\|.$$

(6) and integration by parts imply

$$\begin{aligned} I_2(x) &= -c_n \int \frac{\theta(|x-y|)}{|x-y|^{n-2}} (L\Phi)(y)\,dy \\ &= c_n \int L_1\left(\frac{\theta(|x-y|)}{|x-y|^{n-2}}\right) \Phi(y)\,dy \\ &\quad - c_n \int \frac{\theta(|x-y|)}{|x-y|^{n-2}} (L_0\Phi)(y)\,dy \\ &= I_2'(x) + I_2''(x). \end{aligned}$$

The second integral, by (7), (8) and the smoothing property of the kernel $|x-y|^{2-n}$, satisfies the inequality

$$\|I_2''\|_{1,q} < c\varepsilon\|S\|_{0,q}.$$

To obtain an estimate for $\|I_2'\|_{1,q}$ we recall that $L_1$ is a first order differential operator with constant coefficients. To obtain first derivatives of $I_2'$ we have to differentiate $\dfrac{\theta(|x-y|)}{|x-y|^{n-2}}$ a second time. It turns out that $\dfrac{\partial}{\partial x^j}\dfrac{\partial}{\partial x^i}\dfrac{\theta(|x-y|)}{|x-y|^{n-2}}$ for $|x-y| < \frac{1}{2}$ is a singular kernel. Thus the Calderón-Zygmund inequality (3) applies and yields, together with (7) and (8),

$$\|I_2'\|_{1,q} < c\varepsilon\|S\|_{0,q}.$$

Now the above estimates for $\|I_1\|_{1,q}$, $\|I_2'\|_{1,q}$, and $\|I_2''\|_{1,q}$ give

$$\|\operatorname{grad} Q\|_{1,q} < c\left(\sqrt{\varepsilon}\ \|\bar{S}\| + \varepsilon\|S\|_{0,q}\right),$$

which implies Lemma 1 in case $\varepsilon < 1$.

To prove Lemma 2 we observe that, because of (11), $Q$ satisfies the equation $\Delta Q = \bar{S} - S$ up to an error caused by the derivatives of the cut-off function $\theta$ up to second order. The error is of the form

$$E(x) = \int f(x, y)(\bar{S} - S)(y)dy,$$

where $f$ is supported in $\frac{1}{2} \leqslant |x - y| \leqslant 1$, and the gradient of $f$ can be estimated in terms of a constant depending on the dimension of $M$ only.

To estimate $E$ we make use of the fact that $\bar{S}$ is an average of $S$ over small balls. Let

$$E(x) = E_1(x) + E_2(x) = \int f(x, y)\bar{S}(y)dy - \int f(x, y)S(y)dy.$$

To compare the two summands we write

$$E_2(x) = -\int f(x, y)\int \eta(|z - y|)S(y)dz\,dy,$$

which holds because $\int \eta = 1$, and define $g$ by $f(x, y) = f(x, z) + g(x, y, z)$. We have

$$E_2(x) = -\int f(x, z)\bar{S}(z)dz - \iint g(x, y, z)\eta(|z - y|)S(y)dy\,dz.$$

The above estimate on the gradient of $f$ yields, for $|y - z| < \rho = \sqrt{\varepsilon}$, $|g(x, y, z)| < c\sqrt{\varepsilon}$, which implies

$$\|E\|_{0,q} < c\sqrt{\varepsilon}\ \|S\|_{0,q},$$

which is the statement of Lemma 2.

Up to now $\bar{S}$ was defined in $B_2(0) \subset T_pM$ for arbitrary $p \in M$. For the next step of the proof we need a local average of $S$ defined on $M$, and set

$$\bar{S}(p) = \bar{S}(0), \text{ where } 0 \in T_pM.$$

In the second half of the proof we utilize the estimate on $\|\bar{S} - S\|_{0,q}$ to construct a new metric connection $\nabla'$ on $M$ which satisfies the conditions of [5, Theorem 2]. In contrast to [5], where we solved the equation Cartan curvature $= \bar{\Omega} = 0$ exactly, here we are only interested in solving the corresponding equation up to an error bounded by $c\sqrt{\varepsilon}\ \|\bar{S}\|$. It suffices therefore to deal with a parametrix instead of a Green's function for the partial differential equation in question. We proceed with a few definitions.

The exterior derivative $d^{\nabla}$ associated to the Levi-Civita connection $\nabla$ acting on differential forms with values in a tensor bundle associated to the tangent bundle $TM$ is defined by

$$(d^{\nabla}\alpha)(X_0,\cdots,X_p) = \sum_{i=0}^{p} (-1)^i(\nabla_{X_i}\alpha)(X_0,\cdots,\hat{X}_i,\cdots,X_p).$$

The adjoint operator $\delta^{\nabla}$ of $d^{\nabla}$ with respect to the metrics induced by the riemannian metric is

$$(\delta^{\nabla}\alpha)(X_2,\cdots,X_p) = -\sum_{i=1}^{n} (\nabla_{e_i}\alpha)(e_i, X_2,\cdots,X_p),$$

where $\{e_i\}$ is an orthonormal basis.

To obtain the new metric connection $\nabla'$ we define

$$\nabla' = \nabla + \delta^{\nabla}\beta, \tag{16}$$

and set

$$\beta(p) = c_n\int_{B_1(0)} \frac{\theta(|y|)}{|y|^{n-2}}\tau(y)(\bar{S}(\exp y)I - R(\exp y))dy, \tag{17}$$

where $B_1(0)$ is the ball with radius 1 and center $0 \in T_pM$, exp: $T_pM \to M$ is the exponential map, $\tau(y)$ is parallel translation along the geodesic $\exp ty$ from $\exp y$ to $\exp 0 = p$, $I$: $V \wedge V \to V \wedge V$ with $V = T_{\exp y}M$ is the identity, $R$ is the riemannian curvature tensor, and $c_n$ is the constant of (11).

To estimate the curvature $R'$ of $\nabla'$ defined in (16) we observe that (17) is a parametrix for the operator $\Delta^{\nabla} = d^{\nabla}\delta^{\nabla} + \delta^{\nabla}d^{\nabla}$. This is true because the coefficients of the second order term of $\Delta^{\nabla}$, in terms of a canonical coordinate system in $0 \in T_pM$, coincide with the coefficients of the Laplace operator $\Delta$ on the component functions. As in [5], the curvature $R'$ of $\nabla'$ is

$$R' = R + d^{\nabla}\delta^{\nabla}\beta + \delta^{\nabla}\beta \wedge \delta^{\nabla}\beta, \tag{18}$$

where $\delta^{\nabla}\beta \wedge \delta^{\nabla}\beta$ is a certain quadratic expression in $\delta^{\nabla}\beta$ whose exact definition does not matter here.

The following two lemmas serve to prove the estimate

$$\|R' - \bar{S}I\| < c\|R' - \bar{S}I\|_{1,q} < c\sqrt{\varepsilon}\ \|\bar{S}\|, \tag{19}$$

where the Sobolev norm for tensor fields on $M$ is defined analogous to (5). The only difference is that the tensors are translated parallelly along $\exp ty$. We will show later that $\bar{S}$ in this estimate can be replaced by $\|\bar{S}\|$. Thus modified, (19) together with a corresponding estimate on the torsion $T'$ of $\nabla'$ will prove that [5, Theorem 2] is applicable.

**Lemma 3.** *Under the assumptions of the theorem and the normalization* $0 < K \leqslant 1$, *the differential form* $\beta$ *defined in* (17) *satisfies the inequality*

$$\|\beta\|_{2,q} < c\sqrt{\varepsilon}\ \|\bar{S}\|. \tag{20}$$

**Lemma 4.** *Under the assumptions of the theorem and the normalization* $0 < K \leqslant 1$, $\beta$ *defined in* (17) *satisfies the inequality*

$$\|\delta^{\nabla} d^{\nabla}\beta\|_{1,q} < c\sqrt{\varepsilon}\ \|\bar{S}\|. \tag{21}$$

The pinching constant $\delta$ of the theorem and the number $\varepsilon$ of the above lemmas are related as in (1).

The above lemmas imply (19) as follows. Lemma 3, via the Sobolev inequality for $q > \dim M$, gives

$$\|\delta^{\nabla}\beta\| < c\sqrt{\varepsilon}\ \|\bar{S}\|. \tag{22}$$

This implies immediately that the quadratic term $\delta^{\nabla}\beta \wedge \delta^{\nabla}\beta$ of (18) is within the bounds claimed in (19). It suffices therefore to deal with the term $R + d^{\nabla}\delta^{\nabla}\beta$ of (18). By Lemma 4 we can replace $d^{\nabla}\delta^{\nabla}\beta$ by $\Delta^{\nabla}\beta$ without exceeding the error allowed in (19). Now (17) is a parametrix for $\Delta^{\nabla}$, therefore $\Delta^{\nabla}\beta = \bar{S}I - R$ except for an error which is a smoothing of $\bar{S}I - R$. To estimate this error we observe that analogous to (1), $|SI - R| < c\varepsilon S$ holds for any $\varepsilon > 0$ if $1 - \delta$ is suitably small. The Main Estimate (15), together with the triangle inequality, implies

$$\|\bar{S} - IR\|_{0,q} < c\sqrt{\varepsilon}\ \|\bar{S}\|. \tag{23}$$

A smoothing of $\bar{S}I - R$ is therefore within the bound allowed in (19). Finally, modulo the errors accounted for, $R' \sim R + d^{\nabla}\delta^{\nabla}\beta \sim R + (\bar{S}I - R) = \bar{S}I$, which proves (19).

To apply [5, Theorem 2] we also need an estimate on the torsion $T'$ of $\nabla'$. In view of definition (16) the estimate (22) and the fact that $\nabla$ has no torsion yield immediately

$$\|T'\| < c\sqrt{\varepsilon}\ \|\bar{S}\|. \tag{24}$$

*Proof of Lemma* 3. The main observation in the proof is that the second derivatives of $\theta(|y|)/|y|^{n-2}$ for $|y| < \frac{1}{2}$ are singular kernels in the sense introduced at the beginning of this section. Therefore the Calderón-Zygmund inequality (3) applies to the second derivatives of $\beta$ with $\Psi = \bar{S}I - R$, except for a contribution to the integral due to the derivative of $\tau(y)$ in the formula (17). To estimate the first derivative of $\tau$ we estimate the difference between the identity and the parallel translation along geodesic triangles. This difference is well-known to be bounded by the area of the triangle times the norm of the curvature. Since $|K| \leqslant 1$, the norm of the derivative of $\tau$ is bounded by $c|y|$ which cancels one of the factors $|y|$ in the denominator, and the contribution of the derivative of $\tau$ to the derivative of $\beta$ is actually smoother than the main term. Consequently, (23) and (3) yield Lemma 3.

*Proof of Lemma* 4. The main point in this lemma is that we gain one more derivative than expected. The reason for this is the second Bianchi equation $d^{\nabla}R = 0$. By integration by parts we throw the derivatives involved in the definition of $d^{\nabla}$ on $(\bar{S}I - R)$. Because of (10) with $\rho = \sqrt{\varepsilon}$, $(\bar{S}I - R)$ satisfies the Bianchi equation up to an error bounded by $c\sqrt{\varepsilon}\ \|\bar{S}\|$. Now except for the contribution of the derivative of $\tau$, which is estimated as in the proof of Lemma 3, the Calderón-Zygmund inequality (3) yields Lemma 4.

It remains to be shown that (19) and (24) imply that $\nabla'$ satisfies the hypothesis of [5, Theorem 2] if $1 - \delta$, and hence $\varepsilon$, is sufficiently small. For convenience let $g' = \|\bar{S}\| g$, where $g$ is the original metric on $M$. Estimates (19) and (24) in terms of the new metric $g'$ yield

$$\left\| R' - \frac{\bar{S}}{\|\bar{S}\|} I \right\| \leqslant c\sqrt{\varepsilon}\,, \qquad \|T'\| < c\sqrt{\varepsilon}\ \sqrt{\|\bar{S}\|} \leqslant c\sqrt{\varepsilon}\,, \tag{25}$$

where the last inequality is valid because of the previous normalization $0 < K \leqslant 1$, which implies $\|\bar{S}\| \leqslant 1$. As long as the ratio $\bar{S}/\|\bar{S}\|$ does not differ much from 1 on the manifold $M$, (25) implies that the assumptions of [5, Theorem 2] are satisfied. This is so because the Cartan curvature of type $S^n$ is composed of the difference between the curvature form of $M$ and that of $S^n$, and the torsion form of $M$. These forms are naturally defined on the bundle of orthonormal frames over $M$. To show that $\bar{S}/\|\bar{S}\|$ is close to one it suffices, by (10) with $\rho = \sqrt{\varepsilon}$, to prove that the diameter $d'(M)$ of $M$, measured in terms of $g'$, is a priori bounded. We will show that $d'(M) < B$ can be achieved for any $B > \pi$ provided that $1 - \delta$ and hence $\varepsilon$ are sufficiently small.

To estimate $d'(M)$ let $\gamma$ be a geodesic. It suffices to show that the first conjugate point is at a distance of at most $B$. Let $Y$ be a Jacobi field along $\gamma$ with initial condition $Y(0) = 0$, $|\dot{Y}(0)| = 1$, where the dot denotes covariant derivative. $Y$ is a solution of the differential equation

$$\ddot{Y} + (T'(Y, \dot{\gamma}))^{\cdot} + R'(Y, \dot{\gamma})\dot{\gamma} = 0,$$

where $R'$ and $T'$ are curvature and torsion respectively of the metric connection $\nabla'$. Let $Z(t) = Y(t) + \int_0^t T'(Y, \dot{\gamma})ds$. Then $Z$ satisfies the equation

$$\ddot{Z} + R'(Z, \dot{\gamma})\dot{\gamma} = \varphi(t),$$

where $\varphi(t) = R'(\int_0^t T'(Y, \dot{\gamma})ds, \dot{\gamma})\dot{\gamma}$. For $|Y| < 2$ and $|t| \leqslant B < 4$, $\varphi(t)$ satisfies the estimate

$$|\varphi(t)| < c\sqrt{\varepsilon}\,.$$

Since $\|T'\| < c\sqrt{\varepsilon}$, the roots of $Y$ and $Z$ are close to each other. In addition, by well-known theorems on approximate solutions of ordinary differential equations, the roots of $Z$ and $X$, defined by

$$\ddot{X} + R'(X, \dot{\gamma})\dot{\gamma} = 0, \quad X(0) = 0, \quad \dot{X}(0) = \dot{Z}(0),$$

are close to each other as well. Now let $\gamma(0) = p$ be a point where $\bar{S}(p) = \|\bar{S}\|$. Because of (10) and (25) the first root of $X$ beyond $t = 0$ is close to $\pi$. Therefore the first conjugate point of $p = \gamma(0)$ along $\gamma$ is at a distance roughly equal to $\pi$, provided $\varepsilon$ is small enough. This concludes the proof.

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