J. DIFFERENTIAL GEOMETRY
17 (1982) 523–529

CURVATURE OF AN ∞-DIMENSIONAL MANIFOLD RELATED TO HILL'S EQUATION

H. P. McKEAN

1. Introduction

Let C_+^∞ be the space of positive infinitely differentiable functions e_0 of period 1 with $\int_0^1 e_0^2 = 1$, and let M be the class of real infinitely differentiable functions q of period 1 such that the corresponding Hill's operator $Q = -D^2 + q$ has ground state $\lambda_0 = 0$, where D signifies differentiation with regard to $0 \leqslant x < 1$. The map $C_+^\infty \to M$ defined by $e_0''/e_0 = q$ is $1:1$ and onto, the ground state of Q being necessarily simple; in particular, M comes in one simply-connected piece. The purpose of this note is to study the curvature of M considered as immersed in the space C_1^∞ of all real infinitely differentiable functions of period 1; evidently, it is a surface of codimension 1 defined by the single relation $\lambda_0 = 0$, and since the gradient of the latter is $\nabla\lambda_0 = e_0^2 \neq 0$, M sits smoothly in C_1^∞.

The curvatures of 2-dimensional slices of M are found to be positive, the principal curvatures being proportional to the reciprocals of the excited periodic eigenvalues $0 < \bar{\lambda}_j$ ($j = 1, 2, 3, \cdots$) of the so-called *allied operator* $\bar{Q}$. The latter is the Hill's operator with ground state proportional to $e_0^{3/2}$ relative to the scale $d\bar{x} = (\int_0^1 e_0)^{-1} e_0\, dx$. The *maximal curvature* of a 2-dimensional slice is

$$m = 4\left(\int_0^1 e_0\right)^4 \left(\int_0^1 e_0^4\right)^{-1} \times (\bar{\lambda}_1\bar{\lambda}_2)^{-1},$$

while the *total curvature* is

$$k = 4\left(\int_0^1 e_0\right)^4 \left(\int_0^1 e_0^4\right)^{-1} \times \sum_{1 \leqslant i < j} (\bar{\lambda}_i\bar{\lambda}_j)^{-1}.$$

Received June 23, 1981. The support of the National Science Foundation under Grant NSF-MCS-76-0739 is gratefully acknowledged.

The latter may be expressed directly in terms of the ground state e_0:

$$k = 4\left(\int_0^1 e_0^{-2}\right)^2\left(\int_0^1 e_0^4\right)^{-2}\int\int\int e_0^4(x_1)e_0^4(x_2)e_0^4(x_3) \times \int_{x_1^*}^{x_2^*} e_0^{-2}\int_{x_2^*}^{x_3^*} e_0^{-2}\int_{x_3^*}^{x_1^*} e_0^{-2}\, d^3x,$$

in which x_1^*, x_2^*, x_3^* are the points x_1, x_2, x_3 arranged in their natural order around the circle. For example, at the place $q = 0$, $m = \frac{1}{4}\pi^{-4}$ and $k = 1/90$. The quantities m and k may be as large or as small as one pleases; for e_0 approximating $x^{-1/4}$ $(0 \leqslant x < 1)$, k is small, while for e_0^2 approximating a saw-tooth function of period $1/3$, m is large: in the first case, the potential approximates $(5/20)x^{-2}$, while in the second it has 6 poles of alternating signature.

A manifold M of different character is obtained by fixing the first excited eigenvalue of Q at $\lambda_1 = 0$, say. M comprises the functions q of class C_1^∞ expressible as e_1''/e_1, the function e_1 having just 2 simple roots per period. This is a more complicated manifold exhibiting some negative curvature; in fact, the second fundamental form has just one negative eigenvalue. The computations are similar and readily extended to the higher eigenvalues λ_2, λ_3, *etc.*

2. The second fundamental form

Let $e_n (n \geqslant 0)$ be the full set of periodic eigenfunctions of Q corresponding to the eigenvalues $\lambda_0 < \lambda_1 \leqslant \lambda_2 < \lambda_3 \leqslant \lambda_4 <$ *etc.* The unit normal to M at q is $n = (\int_0^1 e_0^4)^{-1/2} e_0^2$, and with the aid of the inverse operator

$$Q^{-1}: f \to \sum_{n=1}^{\infty} \lambda_n^{-1} e_n (f, e_n) = \int_0^1 Q_{xy}^{-1} f(y)\, dy,$$

mapping the annihilator of e_0 into itself, it is a simple matter to compute

$$\frac{\partial e_0(x)}{\partial q(y)} = -Q_{xy}^{-1} e_0(y),$$

$$\frac{\partial n(x)}{\partial q(y)} = -2\left(\int_0^1 e_0^4\right)^{-1/2} e_0(x) Q_{xy}^{-1} e_0(y) + 2\left(\int e_0^4\right)^{-3/2} e_0^2 \otimes e_0 Q^{-1} e_0^3,$$

and, finally, *the second fundamental form*:

$$J_{ab} = \int_0^1\int_0^1 a(x)\frac{\partial n(x)}{\partial q(y)} b(y)\, dx dy$$

$$= -2\left(\int_0^1 e_0^4\right)^{-1/2}\int_0^1\int_0^1 a(x) e_0(x) Q_{xy}^{-1} e_0(y) b(y)\, dx dy$$

for directions a and b tangent to M at q, $\int_0^1 ae_0^2 = \int_0^1 be_0^2 = 0$. This makes the second portion of $\partial n/\partial q$ drop out. Now let a and b form a unit perpendicular frame: $\int_0^1 a^2 = \int_0^1 b^2 = 1$, $\int_0^1 ab = 0$. They define a 2-dimensional slice of M with curvature

$$K_{ab} = J_{aa}J_{bb} - J_{ab}^2.$$

This number is necessarily positive, J being strictly negative on the tangent space:

$$J_{cc} = -2\left(\int_0^1 e_0^4\right)^{-1/2} \sum_{n=1}^{\infty} \lambda_n^{-1}(e_n, e_0 c)^2 < 0 \quad \text{if } c \neq 0.$$

3. The allied operator

The form J is closely connected to the so-called *allied operator* $\bar{Q}$. Introduce the new scale

$$\bar{x} = \left(\int_0^1 e_0\right)^{-1} \int_0^x e_0,$$

and view

$$\bar{e}_0 = \left(\int_0^1 e_0^4 dx\right)^{-1/2} \left(\int_0^1 e_0 dx\right)^{1/2} e_0^{3/2}$$

as a function of $0 \leqslant \bar{x} < 1$, noticing that $\int_0^1 (\bar{e}_0)^2 \, d\bar{x} = 1$. $\bar{Q}$ is now defined to be the Hill's operator with ground state $\bar{e}_0$ relative to the scale $\bar{x}$, and with the notation (the discrepancy between this notation and $\bar{e}_0$ will not prove troublesome):

$$\bar{f}(\bar{x}) = \left(\int_0^1 e_0 \, dx\right)^2 e_0^{-1/2}(x) f(x),$$

direct computation provides the identity

$$\bar{Q} e_0^{1/2} Q^{-1} e_0 f = \bar{f},$$

in which the necessary condition of perpendicularity $[\int_0^1 e_0^2 f \, dx = 0]$ for the existence of $Q^{-1} e_0 f$ is satisfied if and only if $\int_0^1 \bar{e}_0 \bar{f} \, d\bar{x}$ also vanishes. Then $\bar{Q}^{-1}\bar{f}$ exists, and the upshot is that $e_0^{1/2} Q^{-1} e_0 f = \bar{Q}^{-1}\bar{f}$. This permits a simplified expression of the second fundamental form:

$$J_{ab} = -2\left(\int_0^1 e_0^4\right)^{-1/2} \left(\int_0^1 e_0\right)^{-1} \int \bar{a} \bar{Q}^{-1} \bar{b} \, d\bar{x}.$$

Notice

$$\int_0^1 \bar{a}\bar{b}\,d\bar{x} = \left(\int_0^1 e_0\right)^2 \int_0^1 abe_0^{-1}\left(\int_0^1 e_0\right)^{-1} e_0\,dx = \left(\int_0^1 e_0\right)^3 \int_0^1 abdx,$$

so that $ab \to \bar{a}\bar{b}$ maintains perpendicularity. The point of all this computation is

Corollary 1. *The principal curvatures of M at q, i.e., the eigenvalues of the second fundamental form J, are simply*

$$-2\left(\int_0^1 e_0^4\right)^{-1/2}\left(\int_0^1 e_0\right)^{-1}\left(\int_0^1 e_0\right)^3 \times \textit{the eigenvalues of } \bar{Q}^{-1}$$

$$= -2\left(\int_0^1 e_0^4\right)^{-1/2}\left(\int_0^1 e_0\right)^2 \times \textit{the reciprocals of the excited eigenvalues of } \bar{Q}.$$

The latter are written $0 < \bar{\lambda}_1 \leqslant \bar{\lambda}_2 \leqslant \bar{\lambda}_3 \leqslant \bar{\lambda}_4 <$ etc.

Corollary 2. *The maximal curvature of a 2-dimensional slice of M at q is*

$$m = 4\left(\int_0^1 e_0\right)^4\left(\int_0^1 e_0^4\right)^{-1} \times (\bar{\lambda}_1\bar{\lambda}_2)^{-1}.$$

Corollary 3. *The total curvature of M at q is*

$$k = 4\left(\int_0^1 e_0\right)^4\left(\int_0^1 e_0^4\right)^{-1} \times \sum_{1 \leqslant i < j} (\bar{\lambda}_i\bar{\lambda}_j)^{-1}.$$

The rest of the paper is devoted to the investigation of these numbers.

Proof of Corollary 2. The curvature of the general slice may be expressed as the product of $4(\int_0^1 e_0)^4(\int_0^1 e_0^4)^{-1}$ and

$$\sum \frac{a_i^2}{\bar{\lambda}_i} \sum \frac{b_j^2}{\bar{\lambda}_j} - \left(\sum \frac{a_i b_i}{\bar{\lambda}_i}\right)^2 = \sum_{i<j} \frac{(a_i b_j - a_j b_i)^2}{\bar{\lambda}_i\bar{\lambda}_j},$$

with $\sum a_i^2 = \sum b_j^2 = 1$ and $\sum a_i b_i = 0$. The final sum is over-estimated by the product of $(\bar{\lambda}_1\bar{\lambda}_2)^{-1}$ and $\sum_{i<j}(a_i b_j - a_j b_i)^2 = 1$.

Amplification 1. Let $\bar{e}$ be an excited eigenfunction of $\bar{Q}$ with eigenvalue $\bar{\lambda}$. Then $e = (\int_0^1 e_0^{-2})e_0^{1/2}\bar{e}$ satisfies $(\int_0^1 e_0)^2 e_0^{-1} Q e_0^{-1} e = \bar{\lambda} e$ and vice versa. Now Q can be expressed as $-e_0^{-1}De_0^2De_0^{-1}$, so $\bar{\lambda}$ is an eigenvalue of $-(\int_0^1 e_0)^2 e_0^{-2}De_0^2De_0^{-2}$ which is similar to $-(\int_0^1 e_0)^2 e_0^{-4}De_0^2D$ and so also to $-(\int_0^1 e_0)^2(\int_0^1 e_0^{-2})^{-2}e_0^{-6}D^2$, in which the differentiation is now with regard to the new scale $(\int_0^1 e_0^{-2})^{-1}\int_0^x e_0^{-2}$. This remark will be helpful in §5.

Amplification 2. $\bar{Q}$ can be any Hill's operator with $\bar{\lambda}_0 = 0$.

Proof. Let $\bar{Q}$ be the general Hill's operator relative to the fixed scale $\bar{x}$, and $\bar{e}_0$ its ground state; it is required to prove that $\bar{Q}$ is allied to some Q. Define $e_0^{3/2}(x) = a\bar{e}_0(\bar{x})$ with a new scale x specified by $dx = be_0^{-1}d\bar{x} = c(\bar{e}_0)^{-3/2}d\bar{x}$, the constants a, b, c being chosen to make $x = 1$ at the end and $\int_0^1 e_0^2\,dx = 1$. This can be done:

$$1 = x(1) = b\int_0^1 e_0^{-1}dx = ba^{-2/3}\int_0^1 (\bar{e}_0)^{-3/2}d\bar{x},$$

$$c = ba^{-2/3}, \quad 1 = \int_0^1 e_0^2\,dx = a^{2/3}b\int_0^1 (\bar{e}_0)^{3/2}\,d\bar{x}.$$

Then $\overline{Q}$ is allied to the Hill's operator Q with ground state e_0 relative to the scale x; indeed, $\bar{e}_0 = (\int_0^1 e_0^4)^{-1/2}(\int_0^1 e_0)^{1/2}e_0^{3/2}$, as it should be, in view of

$$1 = \int_0^1 (\bar{e}_0)^2\,d\bar{x} = \frac{1}{a^2 b}\int_0^1 e_0^4\,dx, \quad b = b\int_0^1 d\bar{x} = \int_0^1 e_0\,dx.$$

This fact will be helpful in §4.

4. Maximal Curvature

The purpose of this section is to prove that the maximal curvature m can be made as large as you please; in the next section, it is shown that the total curvature can be made as small as you please, so *anything can happen*.

Proof. m can be expressed as the reciprocal of $(\int_0^1(\bar{e}_0)^{-2/3}\,d\bar{x})^3 \times \bar{\lambda}_1\bar{\lambda}_2$ in the notation of §3, and as $\overline{Q}$ can be any Hill's operator at all, so it is required to prove that $(\int_0^1 e_0^{-2/3}\,dx)^3\lambda_1\lambda_2$ can be made small by choice of Q. Now

$$\lambda_1 = \int_0^1 e_1 Q e_1 = \int_0^1 \left|\left(\frac{e_1}{e_0}\right)'\right|^2 e_0^2$$

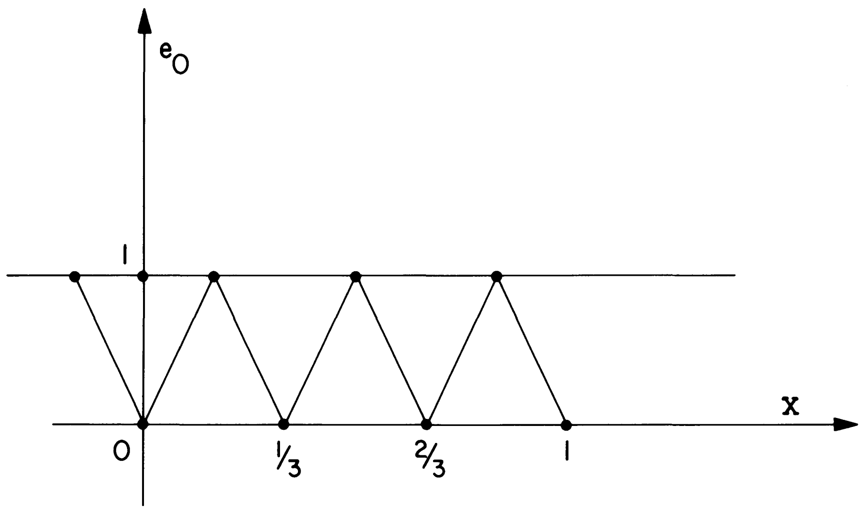
can be expressed as the minimum of the ratio of $\int_0^1(f')^2 e_0^2$ to $\int_0^1 f^2 e_0^2$ for f of class C_1^∞ with $\int_0^1 f e_0^2 = 0$; moreover, $\lambda_1 = \lambda_2$ if q is of period $1/3$, Borg [1], so it suffices to make

$$I = \left(\int_0^1 e_0^{-2/3}\right)^{3/2}\int_0^1 (f')^2 e_0^2\left(\int_0^1 f^2 e_0^2\right)^{-1}$$

small for even e_0 of period $1/3$ and odd f. Choose e_0 to approximate the saw-tooth function of Fig. 1 and let the odd function f be $\pm e_0^p$. Then I is closely approximated by a fixed multiple of

$$\frac{\int_0^{1/6} p^2 x^{2p-2} x^2\,dx}{\int_0^{1/6} x^{2p} x^2\,dx} = 36p^2\frac{2p+3}{2p+1}$$

and is small for $p = 0+$.



5. Total curvature

The total curvature k can be expressed in the following compact form:

$$k = 4\left(\int_0^1 e_0^{-2}\right)^2\left(\int_0^1 e_0^4\right)^{-2}\int\int\int e_0^4(x_1)e_0^4(x_2)e_0^4(x_3) \times \int_{x_1^*}^{x_2^*} e_0^{-2}\int_{x_2^*}^{x_3^*} e_0^{-2}\int_{x_3^*}^{x_1^*} e_0^{-2}\, d^3x,$$

mentioned in §1.

Proof. The author owes the idea of this proof to a remark of G. Segal. The periodic spectrum of $\overline{Q}$ may be described [2] as the roots of $\overline{\Delta}(\lambda) = +2$, $\overline{\Delta}$ being the discriminant of $\overline{Q}$. $\overline{\Delta}$ is now expressed with the aid of the similar operator of Amplification 1 of §3:

$$-D_bD_a, \quad da = \left(\int_0^1 e_0^{-2}\right)^{-1} e_0^{-2}dx, \quad db = \left(\int_0^1 e_0\right)^{-2}\left(\int_0^1 e_0^{-2}\right)e_0^4\, dx.$$

The formula is

$$\overline{\Delta}(\lambda) = [y_1(1, \lambda) + y_2'(1, \lambda)].$$

The prime signifies differentiation with respect to a,

$$y_1(x, \lambda) = 1 + \lambda\int_0^x da\int_0^{x_1} db + \lambda^2\int_0^x da\int_0^{x_1} db\int_0^{x_2} da\int_0^{x_3} db + \text{etc.},$$

$$y_2(x, \lambda) = a(x) + \lambda\int_0^x da\int_0^{x_1} a\, db + \lambda^2\int_0^x da\int_0^{x_1} db\int_0^{x_2} da\int_0^{x_3} a\, db + \text{etc.},$$

and from the product $c\lambda\prod_{n=1}^{\infty}(1-\lambda/\bar{\lambda}_n)$ for $\bar{\Delta}(\lambda)-2$ is obtained

$$6\sum_{i<j}\left(\bar{\lambda}_i\bar{\lambda}_j\right)^{-1}=\left(\frac{d\bar{\Delta}}{d\lambda}\right)^{-1}\frac{d^3\bar{\Delta}}{d\lambda^3}\quad\text{evaluated at }\lambda=0$$

$$=2\left(\int_0^1 e_0\right)^2\left(\int_0^1 e_0^{-2}\right)^{-1}\left(\int e_0^4\right)^{-1}\times 3\left(\int_0^1 e_0\right)^{-6}$$

$$\times\left[\int_0^1 e_0^{-2}\int_0^{x_1}e_0^4\int_0^{x_2}e_0^{-2}\int_0^{x_3}e_0^4\int_0^{x_4}e_0^{-2}\int_0^{x_5}e_0^4\,d^6x\right.$$

$$\left.+\int_0^1 e_0^4\int_0^{x_1}e_0^{-2}\int_0^{x_2}e_0^4\int_0^{x_3}e_0^{-2}\int_0^{x_4}e_0^4\int_0^{x_5}e_0^{-2}\,d^6x\right].$$

This expression is inserted into

$$k=4\left(\int_0^1 e_0\right)^4\left(\int_0^1 e_0^4\right)^{-1}\times\sum_{i<j}\left(\bar{\lambda}_i\bar{\lambda}_j\right)^{-1},$$

and the result is reduced to the stated form by exchange of integrals.

The formula is applied to confirm that k can be made as small as one pleases: it suffices to let e_0 approximate x^p with $1/2>p>-1/4$ and to estimate

$$k\leqslant 24(1+4p)(1-2p)^{-5}2^{1-2p}\qquad\text{as } p\downarrow -1/4.$$

References

[1] G. Borg, *Eine Umkehrung der Sturm-Liouvillieschen Eigenwertaufgabe*, Acta Math. **78** (1945) 1–96.

[2] W. Magnus & W. Winkler, *Hill's equation*, Wiley-Interscience, New York, 1966.

NEW YORK UNIVERSITY