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ON THE EXISTENCE OF INFINITELY MANY ISOMETRY-INVARIANT GEODESICS

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Among various problems about geodesics, one of the most prominent questions is whether or not there exist infinitely many closed geodesics on an arbitrary compact Riemannian manifold. The main interest lies in the question of whether it is possible to estimate the number of closed geodesics in terms of topological properties of the manifold only. In 1969 Gromoll and Meyer succeeded to find such a criterion [4]. They obtained the following result:

Theorem. *Let M be a compact and simply connected Riemannian manifold. Then M has infinitely many closed geodesics if the sequence of the Betti numbers for the space, with the compact-open topology, of all continuous maps $S^1 \to M$ is unbounded.*

Let us note that the spheres do not satisfy the above topological condition, though clearly the standard one has infinitely many closed geodesics. About ten years after their proof W. Klingenberg published a lecture note [10] which offers a proof of the claim that there exist infinitely many closed geodesics on any compact Riemannian manifold with finite fundamental groups. However the proof seems to need much improvement. In the same spirit as the problem of closed geodesics, one might ask if there are topological restrictions that ensure the existence of infinitely many isometry-invariant goedesics. Here a geodesic c: $R \to M$ is said to be invariant under an isometry A on M (or A-invariant) if $A(c(t)) = c(t+1)$ for all real t. Letting id_M be the identity map on M, an id_M-invariant geodesic is simply a closed geodesic of period 1 and vice versa. Thus the theory of isometry-invariant geodesics contains that of closed geodesics. There are however some essential differences. For example a rotation on a flat torus has no invariant geodesic, although any compact Riemannian manifold has at least one closed geodesic (see e.g. [5]). While there are infinitely many closed geodesics on the standard sphere, a rotation on it has only finitely many invariant geodesics. Therefore it would be very reasonable to ask if there are infinitely many A-invariant geodesics under the

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assumption of unboundedness of the sequence of the Betti numbers for $C^0(M, A)$. Here M denotes a compact and simply connected Riemannian manifold, A an isometry on M, and $C^0(M, A)$ the space, with the compact-open topology, of all continuous maps $x: [0, 1] \to M$ satisfying $A(x(0)) = x(1)$. Note that $C^0(M, A_1)$ and $C^0(M, A_2)$ are homotopy equivalent when A_1 and A_2 are homotopic isometries. In 1974 K. Grove solved the above problem affirmatively in the case of involutive isometries [7]. Following that the author did it in the case of any isometry with prime power order [19], [20]. In 1976 K. Grove and the author solved the problem affirmatively in the case of any isometry of finite order [9]. The purpose of the present paper is to prove the following optimal result:

Main theorem. *Let M be a compact and simply connected Riemannian manifold and let A be an arbitrary isometry on M. Then there exist infinitely many A-invariant geodesics if the sequence of Betti numbers of the space $C^0(M, A)$ is unbounded.*

As in the theory of closed geodesics, we shall make use of a certain path space $\Lambda(M, A)$ of A-invariant curves, to estimate the number of A-invariant geodesics. As usual the energy function E^A is defined on $\Lambda(M, A)$ and any A-invariant geodesic is characterized as a positive valued critical point of E^A. One should note that A-invariant geodesics do not correspond bijectively to critical points, but rather either to noncompact critical orbits or to towers of compact critical orbits. This makes it difficult to estimate the number of isometry-invariant geodesics. The difficulty occurs even in the case of $A = \mathrm{id}_M$. Let us outline the proof of our theorem. The proof will be done by obtaining a bound for the Betti numbers of $C^0(M, A)$ under the assumption that there exist only finitely many A-invariant geodesics. If the assumption holds, then all the A-invariant geodesics are closed. This essential observation was done by Grove [6]. Therefore let c be a closed A-invariant geodesic, and say that A acts rationally (resp. irrationally) on c if the least period α of c is rational (resp. irrational). To show boundedness of the Betti numbers of $C^0(M, A)$, it is sufficient to prove that there are only a finite number of different characteristic invariants among those of $c^{m\alpha+1}$, $m \in Z$, the critical tower defined by c. We use the notation of characteristic invariants as treated by Gromoll and Meyer [3], which is very useful to handle degenerate critical orbits. If all the critical orbits of $c^{m\alpha+1}$, $m \in Z$, are nondegenerate, then the finiteness is easily shown. In case A acts rationally on c, the finiteness will be done by reducing it to the case when A is of finite order. Otherwise it will be done by approximations of E^A by other energy functions in a topological sense. In both cases formulas of nullities of $c^{m\alpha+1}$, $m \in Z$, are very crucial. These formulas will tell us which energy functions are suitable approximations of E^A. Moreover from these formulas we will obtain a crucial fact that the null space at a sufficiently many iterated A-invariant closed geodesic consists of periodic Jacobi fields only. This was actually conjectured by K. Grove in 1976 in his personal letter to the author, where he gave a proof of it in case the closed geodesic is fixed by A. We refer to [2], [3], [5], [10] and [14] for fundamental knowledge of path spaces and Hilbert manifolds and to [5], [6] and [9] for basic facts of isometry-invariant geodesics. In fact K. Grove initiated and developed the theory of them in [5], [6] and [7]. Finally the author wishes to thank K. Grove for helpful conversations during the author's visit to Copenhagen in 1979/80.

1. Preliminaries and basic formulas

Let (M, g) be a compact and connected Riemannian manifold with a Riemannian metric tensor g and let A be an isometry on (M, g). A continuous map $x\colon R \to M$ is said to be of class H^1 if it is absolutely continuous and its tangent vector field $\dot{x}$ is locally summable. H^1-vector fields along x are defined analogously. Let $\Lambda(M, A)$ be the Hilbert manifold of all H^1-maps x with $A(x(t)) = x(t+1)$ for all $t \in R$. The tangent space $T_x\Lambda(M, A)$ to $\Lambda(M, A)$ at x, is identified by the vector space of all H^1-vector fields X along x with $A_*(X(t)) = X(t+1)$. Here A_* denotes the differential map of A. $\Lambda(M, A)$ carries a natural complete Riemannian metric $\langle\, , \rangle_1$ induced from g. If X and Y are tangent vectors to $\Lambda(M, A)$ at x, then $\langle\, , \rangle_1$ is defined by

$$\langle X, Y\rangle_1 = \int_0^1 (g(X(t), Y(t)) + g(X'(t), Y'(t)))\, dt,$$

where X', Y' denote the covariant derivatives along x of X, Y respectively. The energy function E^A on the manifold is defined by

$$E^A(x) = \frac{1}{2}\int_0^1 g(\dot{x}(t), \dot{x}(t))\, dt.$$

A critical point c of E^A is either a constant map in $\mathrm{Fix}(A)$, the fixed point set of A, or an A-invariant geodesic [5]. The Hessian $H_c(E^A)$ of E^A at a critical point c, is given by

$$H_c(E^A)(X, Y) = \int_0^1 (g(X', Y') - g(R(X, \dot{c})\dot{c}, Y))\, dt,$$

where R denotes the Riemannian curvature tensor of M. If X is C^∞, the Hessian is expressed as

$$H_c(E^A)(X, Y) = \int_0^1 g(LX, Y)\, dt,$$

where $-LX = X'' + R(X, \dot{c})\dot{c}$.

There exists a continuous R-action on $\Lambda(M, A)$ by isometries [6] defined by

$$T_u(x)(t) = x(t+u) \quad \text{for any } t, u \in R \text{ and } x \in \Lambda(M, A).$$

Moreover A induces an isometry on $\Lambda(M, A)$, which we also denote by A. We will say that A acts irrationally on $x \in \Lambda(M, A)$ if no positive integer s satisfies $A^s x = x$. Otherwise we say that A acts rationally on x. For each positive number u and $x \in \Lambda(M, A)$, x^u denotes the curve defined by $x^u(t) = x(ut)$ for any real t. Let c be a closed A-invariant geodesic with least period α. Then $c^{m\alpha+1}$, $m \in Z$, are critical points of E^A and $\mathrm{orb}(c^{m\alpha+1})$ denotes the critical orbit containing $c^{m\alpha+1}$, i.e.,

$$\mathrm{orb}(c^{m\alpha+1}) = \{T_u(c^{m\alpha+1}) \mid u \in R\}.$$

Let $\lambda(c, E^A)$ (resp. $\nu(c, E^A)$) be the index (resp. nullity) of $\mathrm{orb}(c)$. When $\nu(c, E^A)$ is zero, $\mathrm{orb}(c)$ is said to be nondegenerate [14]. In order to compute $\lambda(c^{m\alpha+1}, E^A)$ and $\nu(c^{m\alpha+1}, E^A)$, we introduce the vector space V_c of all C^∞ vector fields along c which are orthogonal to $\dot{c}$, and consider the differential operator L defined above on V_c. The map A_* (resp. T_u) induces a bijective linear map of V_c onto $V_{T_1(c)}$ (resp. of V_c onto $V_{T_u(c)}$). Then the index and nullity of $\mathrm{orb}(c^{m\alpha+1})$ are given by

$$(1.1)\qquad \begin{aligned} \lambda(c^{m\alpha+1}, E^A) &= \sum_{\mu<0} \dim\{X \in V_c \mid LX = \mu X, A_* X = T_{m\alpha+1} X\}, \\ \nu(c^{m\alpha+1}, E^A) &= \dim\{X \in V_c \mid LX = 0, A_* X = T_{m\alpha+1} X\}. \end{aligned}$$

Since L is an elliptic differential operator, each eigenspace of L is finite dimensional and it has discrete eigenvalues $\mu_1 < \mu_2 < \mu_3 < \cdots \mu_j < \cdots$ which are bounded from below, and μ_j goes to infinity as j tends to infinity. Therefore the index and nullity of $\mathrm{orb}(c^{m\alpha+1})$ are finite. For each real μ, set $J(\mu) = \{X \in V_c \mid LX = \mu X\}$. Define a semipositive definite bilinear form ω_m on V_c by

$$\omega_m(X, Y) = \int_0^{m\alpha+1} g(X(t), Y(t))\, dt.$$

Then $T_1^{-1} \circ A_*$ is an orthogonal transformation on the vector space $\{X \in J(\mu) \mid A_* X = T_{m\alpha+1} X\}$ with inner product ω_m. In order to compute the index and the nullity of $\mathrm{orb}(c^{m\alpha+1})$ complexify $J(\mu)$, which will be denoted by $J(\mu) \otimes \mathbf{C}$. L, A_* and T_u denote also the $\mathbf{C}$-linear extensions of L, A_* and T_u respectively. Let $S[\mu, m\alpha + 1, zA]$ ($z \in \mathbf{C}$) be the linear space $\{X \in J(\mu) \otimes \mathbf{C} \mid zA_* X = T_{m\alpha+1} X\}$ with Hermitian inner product $\tilde{\omega}_m$, the Hermitian extension of ω_m to $J(\mu) \otimes \mathbf{C}$.

Lemma 1.2. *For each real μ, $S[\mu, m\alpha + 1, A]$ has a direct sum decomposition of finitely many of its subspaces;*

$$S[\mu, m\alpha + 1, A] = \bigoplus_{|z|=1} \bigoplus_{\rho^m = z} \{X \in S[\mu, 1, z^{-1}A] \mid T_\alpha X = \rho X\}.$$

Here $|z|$ denotes the absolute value of z.

Proof. Since $T_1^{-1} \circ A_*$ is a unitary transformation on $S[\mu, m\alpha + 1, A]$ with the inner product $\tilde{\omega}_m$, $S[\mu, m\alpha + 1, A]$ has a direct sum decomposition of eigenspaces of $T_1^{-1} \circ A_*$, i.e.,

$$S[\mu, m\alpha + 1, A] = \bigoplus_{|z|=1} S[\mu, m\alpha + 1, A] \cap S[\mu, 1, z^{-1}A].$$

Furthermore each eigenspace of $T_1^{-1} \circ A_*$ is invariant under T_α. Hence each one also has a decomposition:

$$\begin{aligned} &S[\mu, m\alpha + 1, A] \cap S[\mu, 1, z^{-1}A] \\ &\quad = \bigoplus_{\rho} \{X \in S[\mu, m\alpha + 1, A] \cap S[\mu, 1, z^{-1}A] \mid T_\alpha X = \rho X\} \\ &\quad = \bigoplus_{\rho^m = z} \{X \in S[\mu, 1, z^{-1}A] \mid T_\alpha X = \rho X\}, \end{aligned}$$

which completes the proof.

From (1.1) and the above lemma we have

Corollary 1.3.

$$\lambda(c^{m\alpha+1}, E^A) = \sum_{|z|=1} \sum_{\rho^m = z} \Lambda^z(\rho),$$

$$\nu(c^{m\alpha+1}, E^A) = \sum_{|z|=1} \sum_{\rho^m = z} N^z(\rho),$$

where $\Lambda^z(\rho) = \Sigma_{\mu<0} \dim_{\mathbf{C}}\{X \in S[\mu, 1, z^{-1}A] \mid T_\alpha X = \rho X\}$ *and* $N^z(\rho) = \dim_{\mathbf{C}}\{X \in S[0, 1, z^{-1}A] \mid T_\alpha X = \rho X\}$ *which are considered as functions on the unit circle* $\{\rho \in \mathbf{C} \mid |\rho| = 1\}$ *for each z with absolute value* 1.

Remark. Compare [16] for different index formulas for isometry-invariant geodesics. The functions Λ^z, N^z have the following crucial properties (cf. [1], [9] and [19]).

(1.4) Λ^z and N^z are identically zero except for a finite number of z's.

(1.5) For each z, $N^z(\rho) = 0$ except for at most $2(\dim M - 1)$ points ρ, which will be called L^z Poincaré points.

(1.6) For each z, Λ^z is locally constant except possibly at the L^z Poincaré points.

(1.7) For each z, the inequality $\lim_{\rho \to \rho_0} \Lambda^z(\rho) \geqslant \Lambda^z(\rho_0)$ holds for any ρ_0.

As in [9] we obtain from Corollary 1.3, (1.6) and (1.7) the following important growth-estimate.

Lemma 1.8. *Either* $\lambda(c^{m\alpha+1}, E^A) = 0$ *for all nonnegative integers* m *or there exist positive numbers* ε *and* a *such that*

$$\lambda(c^{m_1\alpha+1}, E^A) - \lambda(c^{m_2\alpha+1}, E^A) \geqslant (m_1 - m_2)\varepsilon - a$$

for all integers $m_1 \geqslant m_2 \geqslant 0$.

Remark. It is also possible to derive the above lemma from a general index theorem by Klingmann [11] together with Lemma 2.8 in [9] or Lemma 1 in [4].

Next let us recall the definition and some properties of the local homological and characteristic invariants as defined by Gromoll and Meyer [3]. Let orb(c) be an isolated critical (compact) submanifold in $\Lambda(M, A)$. Hence c is a closed geodesic by Theorem 2.4 in [6]. The definition of local homological invariant for this orbit is similar to the one in the case of isometries with finite orders [9]. Let $\mathfrak{N}$ be the normal bundle of orb(c) and let $\Psi\colon \mathfrak{N} \to \Lambda(M, A)$ be an arbitrary R-equivariant smooth (C^∞) map which is the identity on the zero-section and of maximal rank there. The image by Ψ of a sufficiently small disc bundle of $\mathfrak{N}$ defines an equivariant tubular neighborhood $\mathfrak{D} = \bigcup_{u\in R} T_u(\mathfrak{D}_c)$ of orb(c), where $\mathfrak{D}_c$ is the fiber over c. From the construction, $T_u(\mathfrak{D}_c) = \mathfrak{D}_{T_u(c)}$ holds for any u. For $\delta > 0$ and $0 < \rho_0 < \rho_1$, let $d_\delta\colon \mathfrak{D} \to R$ be the function defined by

$$d_\delta(x) = (2\delta)^{-1}(\rho_1^2 - \rho_0^2)E(x) + \langle \Psi^{-1}x, \Psi^{-1}x\rangle_1,$$

where $E = E^A - E^A(c)$. Then for sufficiently small positive δ, ρ_0, ρ_1

$$W = E^{-1}[-\delta, \delta] \cap d_\delta^{-1}\left(-\infty, (\rho_0^2 + \rho_1^2)/2\right] \quad \text{and } W^- = E^{-1}(-\delta) \cap W$$

is a pair of so called admissible regions and

$$\mathfrak{H}(E^A, \mathrm{orb}(c)) = H_*(W, W^-)$$

is a well-defined local homological invariant of orb(c). Here we take singular homology with coefficient in a field. This local homological invariant has the following crucial property which can be proven by the excision theorem and the standard deformation by integral curves of the gradient vector field of E^A [3]:

Lemma 1.9. *If* b *is the only one critical value of* E^A *in* $[b - \varepsilon, b + \varepsilon]$ *for some* $\varepsilon > 0$, *and if there exist only finitely many (compact) critical orbits* orb$(c_1), \cdots,$ orb(c_k) *with* E^A*-valued* b, *then*

$$H_*\left(\Lambda(M, A)^{b+\varepsilon}, \Lambda(M, A)^{b-\varepsilon}\right) = \bigoplus_{i=1}^{k} \mathfrak{H}(E^A, \mathrm{orb}(c_i)),$$

where $\Lambda(M, A)^a = (E^A)^{-1}[0, a]$.

Remark. In [6] it is proven that c is a closed A-invariant geodesic if $\operatorname{orb}(c)$ is an isolated critical submanifold. Hence $\operatorname{orb}(c)$ is compact if it is isolated.

It follows from the construction of the pair of admissible regions that $T_u(W_c) = W_{T_u(c)}$ and $T_u(W_c^-) = W_{T_u(c)}^-$ hold for any real u. Here we put $W \cap \mathfrak{D}_c = W_c$ and $W^- \cap \mathfrak{D}_c = W_c^-$. As in [9] choose open intervals I_1 and I_2 of $\operatorname{orb}(c)$ with $\operatorname{orb}(c) = I_1 \cup I_2$, and let $\pi\colon W \to \operatorname{orb}(c)$ denote the natural projection. Define $X_j = \pi^{-1}(I_j)$ and $A_j = X_j \cap W^-$, $j = 1, 2$. The Mayer-Vietoris sequence [17] for pairs (X_1, A_1) and (X_2, A_2) gives

$$(1.10)\quad \dim \mathcal{H}_k(E^A, \operatorname{orb}(c)) \leq 2\big(\dim H_{k-1}(W_c, W_c^-) + \dim H_k(W_c, W_c^-)\big)$$

for all integers k.

Moreover $H_*(W_c, W_c^-)$ is the local homological invariant which Gromoll and Meyer defined in [3] for an isolated critical point c in $\mathfrak{D}_c$ for the function $E \mid \mathfrak{D}_c$, i.e.,

$$(1.11)\qquad \mathcal{H}(E^A \mid \mathfrak{D}_c, c) = H_*(W_c, W_c^-).$$

It follows from the shifting theorem in [3] that

$$(1.12)\qquad \mathcal{H}_{k+\lambda}(E^A \mid \mathfrak{D}_c, c) = \mathcal{H}_k^0(E^A, c) \quad \text{for all } k,$$

where λ is the index of c, and $\mathcal{H}_k^0(E^A, c)$ denotes the characteristic invariant of c, i.e., the local homological invariant of $E^A \mid \mathfrak{D}_c$ restricted to a characteristic submanifold [3]. Note that the definitions of characteristic invariants may depend on the choice of $\mathfrak{D}_c$. Since the dimension of a characteristic submanifold is equal to the nullity of $\operatorname{orb}(c)$ ($\leq 2\dim M - 2$), we have

$$(1.13)\qquad \mathcal{H}_k^0(E^A, c) = 0 \quad \text{for all } k \geq 2\dim M - 1.$$

If we set $B_k(c, E^A) = \dim \mathcal{H}_k(E^A, \operatorname{orb}(c))$ and $B_k^0(c, E^A) = \dim \mathcal{H}_k^0(E^A, c)$, then

$$(1.14)\qquad B_k(c, E^A) \leq 2\big(B_{k-\lambda}^0(c, E^A) + B_{k-\lambda-1}^0(c, E^A)\big)$$

holds for all k, where $\lambda = \lambda(c, E^A)$.

2. Rational cases

In this section $c \in \Lambda(M, A)$ denotes an A-invariant closed geodesic with least period α, and let us suppose that A acts rationally on c. Then $\operatorname{orb}(c^{m\alpha+1})$, $m \in Z^+$, are critical submanifolds for E^A, and there exist unique relatively prime positive integers m_0 and s_0 with $\alpha = s_0/m_0$. Let us examine the formulas for the nullities of $\operatorname{orb}(c^{m\alpha+1})$ in detail. From Corollary 1.3 we have

$$\nu(c^{m\alpha+1}, E^A) = \sum_{|z|=1} \sum_{\rho^m = z} N^z(\rho),$$

where $N^z(\rho) = \dim_{\mathbf{C}} S[0, 1, z^{-1}A] \cap S[0, \alpha, \rho\, \mathrm{id}_M]$. Obviously $A_*^{s_0}X = \rho^{m_0} z^{s_0} X$ holds for any $X \in S[0, 1, z^{-1}A] \cap S[0, \alpha, \mathrm{id}_M]$. On the other hand, since $A_*^{s_0}\colon \{X \in V_c \otimes \mathbf{C} \mid LX = 0\} \to \{X \in V_c \otimes \mathbf{C} \mid LX = 0\}$ is a unitary transformation there exist at most $2(\dim M - 1)$ eigenvalues of $A_*^{s_0}$, $\exp(2\pi i\vartheta_1), \cdots, \exp(2\pi i\vartheta_k)$. Here $0 \leqslant \vartheta_1 < \vartheta_2 < \cdots < \vartheta_k < 1$. If ϑ_i is rational, choose positive integers p_i and q_i which are relatively prime and satisfies $p_i/q_i = \vartheta_i$. If all ∂_i's are irrational, define $s = s_0$, otherwise let s/s_0 be the least common multiple of the q_i's with ϑ_i rational.

Lemma 2.1. *There exists a positive integer k_0 such that*

$$\nu(c^{m\alpha+1}, E^A) = \nu(c^{m\alpha+1}, E^f)$$

for any $m \geqslant k_0$, where f is the restriction of A to $\mathrm{Fix}(A^s)$.

Remark. Since $\mathrm{Fix}(A^s)$ is a collection of closed totally geodesic submanifolds of M, one can consider the Hilbert manifold $\Lambda(\mathrm{Fix}(A^s), f)$ and the energy function E^f and so on. Note that any critical point of E^f is one of E^A.

Proof. From Corollary 1.3, (1.4) and (1.5) it follows that there exist positive integers k_0 and r such that

$$\nu(c^{m\alpha+1}, E^A) = \sum_{z^r=1} \sum_{\rho^m=z} N^z(\rho)$$

for all $m \geqslant k_0$.

To see this note that there are at most finitely many integers m with $N^z(\rho) \neq 0$ and $\rho^m = z$ in case $\arg(z)/2\pi i$ is irrational. If $N^z(\rho)$ is positive for some z and ρ, then $\rho^{m_0} z^{s_0}$ is an eigenvalue of $A_*^{s_0}$ on $\{X \in V_c \otimes \mathbf{C} \mid LX = 0\}$. Therefore $\bigoplus_{z^r=1} \bigoplus_{\rho^m=z} S[0, 1, z^{-1}A] \cap S[0, \alpha, \rho\, \mathrm{id}_M]$ is contained in $\{X \in S[0, m\alpha + 1, a] \mid A_*^{s_0}X = X\}$. This implies that $\nu(c^{m\alpha+1}, E^A) \leqslant \nu(c^{m\alpha+1}, E^f)$ for all $m \geqslant k_0$. On the other hand $\nu(c^{m\alpha+1}, E^f)$ is not greater than $\nu(c^{m\alpha+1}, E^A)$ from the defintion of nullities.

Proceeding exactly as in the proof of Proposition 3.5 in [9] we get from Lemma 2.1 the following important consequence.

Lemma 2.2. *For all $m \geqslant k_0$*

$$\mathcal{H}^0(E^A, c^{m\alpha+1}) = \mathcal{H}^0(E^f, c^{m\alpha+1})$$

if $\mathrm{orb}(c^{m\alpha+1})$ *is an isolated critical submanifold in* $\Lambda(M, A)$.

As in [9], now in particular from this we get

Corollary 2.3. *There exists a constant B such that*

$$B_k^0(c^{m\alpha+1}, E^A) \leqslant B$$

for all k and $m \geqslant 0$.

3. Irrational cases

In this section suppose that A acts irrationally on c. Hence the least period α of c is irrational. Let us introduce various path spaces for technical reasons. For each nonnegative integer m, let $\Lambda^{m\alpha+1}$ be the Hilbert manifold consisting of all H^1 maps $x\colon R \to M$ with $Ax = T_{m\alpha+1}x$, and let ${}^{m\alpha+1}E$ be the energy function on $\Lambda^{m\alpha+1}$, i.e.,

$$ {}^{m\alpha+1}E(x) = \frac{1}{2}\int_0^{m\alpha+1} g(\dot{x}(t), \dot{x}(t))\,dt $$

for any $x \in \Lambda^{m\alpha+1}$.

The Riemannian metric $\langle\, , \rangle_{m\alpha+1}$ on $\Lambda^{m\alpha+1}$ is defined by

$$ \langle X, Y\rangle_{m\alpha+1} = \int_0^{m\alpha+1}(g(X, Y) + g(X', Y'))\,dt $$

for $X, Y \in T_x\Lambda^{m\alpha+1}$. For each positive integer ξ, $\Lambda^{\xi\alpha}$ denotes the Hilbert manifold of all H^1 maps $x\colon R \to M$ with $T_{\xi\alpha}x = x$. Let ${}^{\xi\alpha}E$ be the energy function on $\Lambda^{\xi\alpha}$. If we regard $\Lambda^{m\alpha+1} \cap \Lambda^{\xi\alpha}$ as the fixed point set of $T_{\xi\alpha}\colon \Lambda^{m\alpha+1} \to \Lambda^{m\alpha+1}$, then $\Lambda^{m\alpha+1} \cap \Lambda^{\xi\alpha}$ can be understood as a totally geodesic submanifold of $\Lambda^{m\alpha+1}$. On the other hand it can also be understood as a totally geodesic submanifold of $\Lambda^{\xi\alpha}$. However these two manifolds are diffeomorphic. Let ${}^{m\alpha+1}E^{\xi\alpha}$ (resp. ${}^{\xi\alpha}E^{m\alpha+1}$) denote the restriction of ${}^{m\alpha+1}E$ (resp. ${}^{\xi\alpha}E$) to $\Lambda^{m\alpha+1} \cap \Lambda^{\xi\alpha}$. As in the proof of Lemma 2.1 it follows from Corollary 1.3, (1.4) and (1.5) that there exist positive integers m_0 and s such that

$$ \nu(c^{m\alpha+1}, E^A) = \sum_{z^s=1}\ \sum_{\rho^m=z} N^z(\rho) $$

for any $m \geqslant m_0$. Note that of course $\lambda(c^{m\alpha+1}, E^A) = \lambda(c, {}^{m\alpha+1}E)$ and $\nu(c^{m\alpha+1}, E^A) = \nu(c, {}^{m\alpha+1}E)$ for any integer m, since $(m\alpha + 1)E^A = {}^{m\alpha+1}E \circ \psi_m$ on $\Lambda(M, A)$. Here $\psi_m(x)(t) = x(t/(m\alpha + 1))$ for any $x \in \Lambda(M, A)$ and real t.

Lemma 3.1. *There exist positive integers $k_1, \cdots, k_q$ and sequences $\{m_j^i\}$, $i > 0, j = 1, \cdots, q$, such that $m_j^i k_j$ are mutually distinct, $\{m_j^i k_j\} = Z^+$ and*

$$ \bigoplus_{z^s=1}\ \bigoplus_{\rho^{m_j^i k_j}=z} J(\rho, z) = \bigoplus_{z^{s_j^i}=1}\ \bigoplus_{\rho^{k_j}=z} J(\rho, z), $$

where s_j^i is the maximal integer relatively prime to m_j^i dividing s, and $J(\rho, z) = S[0, 1, z^{-1}A] \cap S[0, \alpha, \rho\,\mathrm{id}_M]$.

Proof. The proof is analogous to that of Lemma 2.9 in [9]. For each $z = \exp(2\pi iu/v)$ with $(u, v) = 1$, let Q^z denote

$$ Q^z = \{q \in Z^+ \,|\, {}^{\exists}b \in Z^+ \text{ s.t. } (b, qv) = 1, N^z(\exp(2\pi ib/qv)) > 0\}. $$

If we set $Q = \bigcup_{z^s=1} Q^z \cup \{1\}$, Q is a finite set by (1.5). For each $D \subset Q$ let $k(D)$ be the least common multiple of all elements in D. Choose distinct numbers $k_1, \cdots, k_q$ such that $\{k_1, \cdots, k_q\} = \{k(D) \mid D \subset Q\}$. For each k_j, we select from the sequence mk_j, $m \in Z^+$, the greatest subsequence $m_j^i k_j$ with the property that $p \in Q$ divides k_j whenever p divides $m_j^i k_j$. Then the numbers $m_j^i k_j$ are mutually distinct and $\{m_j^i k_j \mid i > 0, j = 1, \cdots, q\} = Z^+$. Let us check that $k_1, \cdots, k_q$ and the sequences m_j^i have the required property. If $N^z(\rho)$ is positive for some $z = \exp(2\pi i u/v)$ with $(u, v) = 1$, $z^s = 1$ and ρ with $\rho^{m_j^i k_j} = z$, then there exists positive integers b and q satisfying $(b, qv) = 1$ and $\rho = \exp(2\pi i b/(qv))$. Hence q is an element of Q^z. Since $m_j^i k_j b/q \equiv u \bmod v$, q divides $m_j^i k_j$. From the construction of $m_j^i k_j$, q divides k_j. Since u and v are relatively prime, so are m_j^i and v. Let s_j^i be the integer defined in the lemma. Then v divides s_j^i, because v also divides s and is relatively prime to m_j^i. This implies the claim.

Let us interpret the meaning of the equality in the above lemma.

Lemma 3.2. *For $m \geqslant m_0$, the null space of $H_c({}^{m\alpha+1}E)$ is equal to that of $H_c({}^{m\alpha+1}E^{\xi(m)\alpha})$, where $m = m_j^i k_j$ and $\xi(m) = s_j^i k_j$. Thus*

$$\mathcal{H}^0({}^{m\alpha+1}E, c) = \mathcal{H}^0({}^{m\alpha+1}E^{\xi(m)\alpha}, c),$$

if $\operatorname{orb}(c^{m\alpha+1})$ *is an isolated critical orbit in* $\Lambda(M, A)$.

Proof. It is sufficient to prove the first claim if we show the complexified null spaces are equal. Since ${}^{m\alpha+1}E^{\xi(m)\alpha}$ is the restriction of ${}^{m\alpha+1}E$ to $\Lambda^{\xi(m)\alpha} \cap \Lambda^{m\alpha+1}$, the Hessian of ${}^{m\alpha+1}E^{\xi(m)\alpha}$ at c is the restriction of $H_c({}^{m\alpha+1}E)$ to $T_c(\Lambda^{m\alpha+1} \cap \Lambda^{\xi(m)\alpha})$. Let A_c be the bounded linear operator defined by

$$H_c({}^{m\alpha+1}E)(X, Y) = \langle A_c X, Y \rangle_{m\alpha+1}$$

for $X, Y \in T_c\Lambda^{m\alpha+1}$. Then A_c maps $T_c(\Lambda^{m\alpha+1} \cap \Lambda^{\xi(m)\alpha})$ into itself, because $A_{T_u(c)} \circ T_u = T_u \circ A_c$ holds for any real u. Hence the null space of $H_c({}^{m\alpha+1}E^{\xi(m)\alpha})$ is given by

$$\{X \in T_c(\Lambda^{m\alpha+1} \cap \Lambda^{\xi(m)\alpha}) \mid H_c({}^{m\alpha+1}E)(X, Y) = 0 \text{ for any } Y \in T_c\Lambda^{m\alpha+1}\}.$$

This implies that the complexified orthogonal space of $\dot{c}$ in the null space of $H_c({}^{m\alpha+1}E^{\xi(m)\alpha})$ is $S[0, m\alpha + 1, A] \cap S[0, \xi(m)\alpha, \mathrm{id}_M]$. On the other hand, the equality in Lemma 3.1 implies that $S[0, m\alpha + 1, A]$ is a subspace of $S[0, \xi(m)\alpha, \mathrm{id}_M]$. Thus we have

$$S[0, m\alpha + 1, A] \cap S[0, \xi(m)\alpha, \mathrm{id}_M] = S[0, m\alpha + 1, A].$$

Noting that $S[0, m\alpha + 1, A]$ is the complexified orthogonal space of $\dot{c}$ in the null space of $H_c({}^{m\alpha+1}E)$, one gets that the null spaces of $H_c({}^{m\alpha+1}E)$ and $H_c({}^{m\alpha+1}E^{\xi(m)\alpha})$ are equal. Let $\mathfrak{D}^{m\alpha+1, \xi(m)\alpha}$ be an equivariant tubular neighborhood of $\operatorname{orb}(c)$ in $\Lambda^{m\alpha+1} \cap \Lambda^{\xi(m)\alpha}$ and let D be an equivariant one of

$\Lambda^{m\alpha+1} \cap \Lambda^{\xi(m)\alpha}$ in $\Lambda^{m\alpha+1}$. Then $\mathfrak{D}^{m\alpha+1} = D \,|\, \mathfrak{D}^{m\alpha+1,\xi(m)\alpha}$ is an equivariant tubular neighborhood of orb(c) in $\Lambda^{m\alpha+1}$. Since $T_u((\mathrm{grad}^{m\alpha+1}E)_x) = (\mathrm{grad}^{m\alpha+1}E)_{T_u x}$ for any real u and $x \in \Lambda^{m\alpha+1}$, $\mathrm{grad}^{m\alpha+1}E$ is tangent to $\Lambda^{m\alpha+1} \cap \Lambda^{\xi(m)\alpha}$ where $\mathrm{grad}^{m\alpha+1}E$ denotes the gradient vector field of ${}^{m\alpha+1}E$. Thus $\mathrm{grad}({}^{m\alpha+1}E \,|\, \mathfrak{D}_c^{m\alpha+1})$ is tangent to $\mathfrak{D}_c^{m\alpha+1,\xi(m)\alpha}$, where $\mathfrak{D}_c^{m\alpha+1}$ and $\mathfrak{D}_c^{m\alpha+1,\xi(m)\alpha}$ denote the fiber over c of $\mathfrak{D}^{m\alpha+1}$, $\mathfrak{D}^{m\alpha+1,\xi(m)\alpha}$ respectively. If m is not less than m_0, then the null space of the Hessian of ${}^{m\alpha+1}E \,|\, \mathfrak{D}_c^{m\alpha+1}$ at c is contained in $T_c\mathfrak{D}_c^{m\alpha+1,\xi(m)\alpha}$. Note that any element of the null space of $H_c({}^{m\alpha+1}E \,|\, \mathfrak{D}_c^{m\alpha+1})$ is pointwise orthogonal to $\dot{c}$. Hence by Lemma 7 in [3], $\mathcal{H}^0({}^{m\alpha+1}E, c) = \mathcal{H}^0({}^{m\alpha+1}E^{\xi(m)\alpha}, c)$.

Lemma 3.3. *For an isolated critical orbit* orb$(c^{m\alpha+1})$ *in* $\Lambda(M, A)$

$$\mathcal{H}^0\left({}^{m\alpha+1}E^{\xi(m)\alpha}, c\right) = \mathcal{H}^0\left({}^{\xi(m)\alpha}E^{m\alpha+1}, c\right)$$

holds if $m \geq m_0$, *where* $m = m_j^i k_j$ *and* $\xi(m) = s_j^i k_j$ *as defined in Lemma* 3.2.

Proof. At first let us note that $g(\dot{x}(t), \dot{x}(t))$ is independent of t for each smooth curve $x \in \Lambda^{m\alpha+1} \cap \Lambda^{\xi(m)\alpha}$, since $g(\dot{x}(t), \dot{x}(t))$ has $\xi(m)\alpha$ and $m\alpha + 1$ as periods, and $\xi(m)\alpha/m\alpha + 1$ is irrational. Therefore

$${}^{m\alpha+1}E^{\xi(m)\alpha}(x) = \frac{m\alpha + 1}{\xi(m)\alpha}\, {}^{\xi(m)\alpha}E^{m\alpha+1}(x)$$

holds for any smooth curve $x \in \Lambda^{m\alpha+1} \cap \Lambda^{\xi(m)\alpha}$. By the standard method in analysis any element in $\Lambda^{m\alpha+1} \cap \Lambda^{\xi(m)\alpha}$ can be approximated by smooth curves in $\Lambda^{m\alpha+1} \cap \Lambda^{\xi(m)\alpha}$. This implies

$${}^{m\alpha+1}E^{\xi(m)\alpha} = \frac{m\alpha + 1}{\xi(m)\alpha}\, {}^{\xi(m)\alpha}E^{m\alpha+1} \quad \text{on } \Lambda^{m\alpha+1} \cap \Lambda^{\xi(m)\alpha}.$$

Hence we are done.

Corollary 3.4. *Suppose that all critical orbits* orb$(c^{m\alpha+1})$, $m \in Z^+ \cup \{0\}$, *are isolated ones in* $\Lambda(M, A)$. *Then there exists a constant* B *such that* $B_k^0(c^{m\alpha+1}, E^A) \leq B$ *for all* k *and* $m \geq 0$.

Proof. Since $\psi_m^{-1}(\mathfrak{D}^{m\alpha+1})$ is an equivariant tubular neighborhood of orb$(c^{m\alpha+1})$ in $\Lambda(M, A)$ with the fiber $\psi_m^{-1}(\mathfrak{D}_c^{m\alpha+1})$ over $c^{m\alpha+1}$, ψ_m gives the isomorphism between $\mathcal{H}^0(E^A, c^{m\alpha+1})$ and $\mathcal{H}^0({}^{m\alpha+1}E, c)$. From Lemmas 3.1 and 3.2 there exists a positive integer $\xi(m) \leq s \cdot \max\{k_j \,|\, 1 \leq j \leq q\}$ such that $\mathcal{H}^0({}^{m\alpha+1}E, c) = \mathcal{H}^0({}^{m\alpha+1}E^{\xi(m)\alpha}, c)$ for $m \geq m_0$. From Lemma 3.3, $\mathcal{H}^0({}^{m\alpha+1}E^{\xi(m)\alpha}, c) = \mathcal{H}^0({}^{\xi(m)\alpha}E^{m\alpha+1}, c)$. Since $\Lambda^{m\alpha+1} \cap \Lambda^{\xi(m)\alpha} = \Lambda^{\eta(m)\alpha+1} \cap \Lambda^{\xi(m)\alpha}$ for some integer $\eta(m) \leq \xi(m)$ as Hilbert manifolds, there are only finitely many kinds of energy functions ${}^{\xi(m)\alpha}E^{m\alpha+1}$ for any $m \geq m_0$. Therefore the set $\{\mathcal{H}^0(E^A, c^{m\alpha+1}) \,|\, m \geq 0\}$ is finite. In particular there exists a constant B such that $B_k^0(c^{m\alpha+1}, E^A) \leq B$ for all k and $m \geq 0$.

4. Existence of infinitely many invariant geodesics

Let M be a compact and simply connected Riemannian manifold and A an isometry on M. Then the space $C^0(M, A)$ defined in the introduction has finite kth Betti number for each k [16].

Theorem 4.1. *Assume that M is a compact and connected Riemannian manifold and that A is an isometry on M. If there are at most finitely many A-invariant geodesics on M, then the sequence of* $\dim H_k(C^0(M, A), F)$, $k \geqslant 2\dim M$, *is bounded for any field F as a coefficient of the homology.*

Proof. Since the inclusion $\Lambda(M, A) \subset C^0(M, A)$ is a homotopy equivalence [5] it is sufficient to prove the theorem for $\Lambda(M, A)$ instead of $C^0(M, A)$. Since A has only finitely many invariant geodesics, all of them must be closed [6] and hence there exist finitely many A-invariant closed geodesics $c_1, \cdots, c_r$ such that any A-invariant geodesic lies in a critical orbit $\mathrm{orb}(c_i^{m\alpha_i+1})$, where m is a nonnegative integer, and α_i denotes the least period of c_i. Since all the critical orbits are isolated, we can apply the results obtained in the previous paragraphs. By Corollaries 2.3 and 3.4, there is a constant B such that $B_k^0(c_i^{m\alpha_i+1}, E^A) \leqslant B$ for all k and $i = 1, \cdots, r$. Thus from (1.14), $B_k(c_i^{m\alpha_i+1}, E^A) \leqslant 4B$ for any k, i and $m \geqslant 0$. It follows from (1.13) and Lemma 1.8 that the number of orbits with $B_k(c_i^{m\alpha_i+1}, E^A) \neq 0$ is bounded by a constant C for each $k \geqslant 2\dim M$. Thus from Lemma 1.9 and the exact sequence argument we get (Morse inequalities [12]) that for all regular values $0 < a < b$

$$\dim H_k\left(\Lambda(M, A)^b, \Lambda(M, A)^a\right) \leqslant 4BC$$

for $k \geqslant 2\dim M$. For a sufficiently small positive a, $\mathrm{Fix}(A) \subset \Lambda(M, A)$ is a strong deformation retract of $\Lambda(M, A)^a$, [5]. Since furthermore the dimension of any connected component of $\mathrm{Fix}(A)$ is not greater than that of M, we see that $\dim H_k(\Lambda(M, A)^b) \leqslant 4BC$ for all $k \geqslant 2\dim M$ and all regular values b. Fix now a $k \geqslant 2\dim M$ and choose b so large that $B_k(c, E^A) = B_{k+1}(c, E^A) = 0$ for all critical orbits $\mathrm{orb}(c)$ with $E^A(c) > b$. Then by Lemma 1.9 and an exact sequence argument $\dim H_k(\Lambda(M, A)) = \dim H_k(\Lambda(M, A)^b)$. Hence $\sup\{\dim H_k(\Lambda(M, A)) \mid k \geqslant 2\dim M\} \leqslant 4BC$.

Corollary 4.2. *Let M be a compact Riemannian manifold which has the same homotopy type as a compact symmetric space of rank* > 1, *and let A be an isometry on M which is homotopic to* id_M. *Then A has infinitely many invariant geodesics.*

Proof. Since A is homotopic to id_M, $\Lambda(M, A)$ has the same homotopy type as $\Lambda(M, \mathrm{id}_M)$. By Ziller [21], the sequence of the Betti numbers for $\Lambda(M, \mathrm{id}_M)$ is unbounded. Hence A has infinitely many invariant geodesics from Theorem 4.1.

Let us discuss the topological assumptions in the main theorem. In [8] a necessary and sufficient condition on A and on M for $C^0(M, A)$ to have a unbounded sequence of rational Betti numbers is given. From the result and our main theorem, we get

Corollary 4.3. *Let M be a compact 1-connected Riemannian manifold and let A be an isometry on M. Then*

$$\dim \pi_*^{\text{even}}(M)^{A\#} \otimes \mathbf{Q} \leq \dim \pi_*^{\text{odd}}(M)^{A\#} \otimes \mathbf{Q} \leq 1,$$

if A has only finitely many invariant geodesics, where $\pi_*(M)^{A\#}$ denotes the homotopy of M fixed by the induced map $A\#: \pi_*(M) \to \pi_*(M)$.

For further discussion of this see [8]. There are still interesting open problems on the existence of isometry-invariant geodesics. If an isometry A on a 1-connected compact Riemannian manifold has more than one fixed point or no fixed points, then A has at least one invariant geodesic. But the case when A has just one fixed point is still open. Consult [5] and [6] for more details.

Problem A. Does any isometry on a 1-connected compact Riemannian manifold have an invariant geodesic?

In what follows M denotes a compact connected Riemannian manifold, and d denotes the distance function on M induced from the Riemannian metric of M. If an isometry A on M is a small displacement without fixed point, i.e., $d(p, Ap)$ is less than the injectivity radius at p for each point p on M, then A has more than one invariant geodesic. The proof is easy if one notes that the function $p \in M \mapsto d(p, Ap) \in R$ is smooth on M and that any critical points of the function correspond to A-invariant geodesics [15]. As a generalization of this property, we may consider

Problem B. Suppose A is an isometry without fixed points on M. Do there exist more than one A-invariant geodesic if A is homotopic to id_M?

In [17], Serre proves that for any two points on M, there exist infinitely many geodesic segments connecting these two points. But his way of counting geodesic segments is not geometrical, i.e., whenever two points lie on a closed geodesic, there exist infinitely many geodesic segments connecting these two points by his way of counting. Hence it would be more natural and interesting to count them in a geometrical way. Let Ω be the loop space of M with a base point.

Problem C. Let M be a compact simply connected Riemannian manifold. For any given two points on M, do there exist infinitely many geodesic segments connecting these two points if the sequence of the Betti numbers of Ω is unbounded?

Remark. In case two points are nonconjugate along any geodesic segments connecting them, this problem can be solved positively by analogous technique used in proving Theorem 4.1. The difficultiy in solving this problem would lie in nonperiodicity of the Jacobi fields which form the null space of the energy function on the Hilbert manifold $\{x; [0, 1] \to M \mid x(0) = p, x(1) = q$ and x is of class $H^1\}$ at a critical point.

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