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QUASI-INVARIANCE OF THE YANG-MILLS EQUATIONS UNDER CONFORMAL TRANSFORMATIONS AND CONFORMAL VECTOR FIELDS

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1. Introduction

It is well-known that the Yang-Mills equations on Minkowski space admit as an invariance group the 15-parameter group of *conformal*, or Lorentz angle-preserving transformations. We consider here what happens in the case of a conformal transformation h between two finite-dimensional oriented pseudoriemannian manifolds M and N of arbitrary dimension and signature.

The *Yang-Mills equations* give a nonlinear condition $y(A) = 0$ on a Lie algebra-valued one-form over M or N. *Quasi-invariance relations* give formulas for $y(h^*A)$, and thus measure the obstruction to h^*A satisfying the equations. This obstruction vanishes when $\dim M = 4$ or when h actually multiplies the metric tensor by a constant. Similar results hold for quasi-invariance of the linearized equations under conformal transformations and under Lie derivation with respect to conformal vector fields.

2. The Yang-Mills equations

Let M be a smooth (C^∞) oriented pseudoriemannian manifold, with metric tensor g of signature (k, q), $k + q = m = \dim M$. The inner product g_x on tangent spaces M_x given by g induces a nondegenerate inner product on cotangent spaces M_x^* upon identification of M_x with M_x^* through g_x. This in turn induces a nondegenerate inner product (also called g_x) on the exterior products $\Lambda^p(M_x^*)$, which may be characterized by

$$(2.1)\quad g_x(\omega^1 \wedge \cdots \wedge \omega^p, \eta^1 \wedge \cdots \wedge \eta^p) = \det\big(g_x(\omega^i, \eta^j)\big), \quad \omega^i, \eta^j \in M_x^*.$$

We extend g to the exterior algebra $\Lambda(M_x^*)$ by requiring that the inner product of forms of different order vanish.

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The orientation of M provides us with a distinguished connected component of the punctured line $\Lambda^m(M_x^*) - 0$, and thus an $E_x \in \Lambda^m(M_x^*)$ with $g_x(E_x, E_x) = (-1)^q$. The *Hodge operator* is the unique linear operator $*$ on $\Lambda(M_x^*)$ carrying $\Lambda^p(M_x^*) \to \Lambda^{m-p}(M_x^*)$ and satisfying

$$* E_x = (-1)^q, \tag{2.2}$$

$$g_x(\omega, \eta)E_x = *(\omega \wedge * \eta). \tag{2.3}$$

The right-hand side of each equation may be viewed as a real number because $\Lambda^0(M_x^*) \cong \mathbf{R}$ canonically. We also denote by $*$ the induced operator on section spaces of $\Lambda(T^*(M))$; in particular on smooth differential forms.

Both the Hodge $*$ and the exterior derivative d are "unchanged" in their action on forms which take their "values" in a real vector space V; that is, on sections of $V \otimes_{\mathbf{R}} \Lambda(T^*(M))$. Any choice of a basis $v_1, \cdots, v_n$ for V allows us to write

$$* \left(v_j \otimes \omega^j\right) = v_j \otimes * \omega^j \quad \text{(summation convention)},$$
$$d\left(v_j \otimes \omega^j\right) = v_j \otimes d\omega^j, \quad \omega^j \in \Lambda(M_x^*),$$

and these formulas are basis-independent.

If V is actually a Lie algebra $\mathfrak{g}$, we may generalize the wedge product of $\mathbf{R}$-valued forms to the *bracket* of $\mathfrak{g}$-valued forms. In the notation above,

$$\left[v_j \otimes \omega^j, v_k \otimes \eta^k\right] = \left[v_j, v_k\right]\omega^j \wedge \eta^k. \tag{2.4}$$

This product satisfies the $\mathbf{Z}_2$-graded anticommutativity law and Jacobi identity:

$$[\Xi, \Omega] = (-1)^{pq+1}[\Omega, \Xi], \tag{2.5}$$

$$(-1)^{pr}[\Xi, [\Omega, \Psi]] + (-1)^{qp}[\Omega, [\Psi, \Xi]] + (-1)^{rq}[\Psi, [\Xi, \Omega]] = 0,$$

$$\Xi \in \mathfrak{g} \otimes \Lambda^p(M_x^*), \quad \Omega \in \mathfrak{g} \otimes \Lambda^q(M_x^*), \quad \Psi \in \mathfrak{g} \otimes \Lambda^r(M_x^*).$$

We may also wedge a real-valued form with a $\mathfrak{g}$-valued form, this operation being characterized by the formula

$$\omega \wedge \left(v_j \otimes \eta^j\right) = v_j \otimes \left(\omega \wedge \eta^j\right); \; \omega, \eta^j \in \Lambda(M_x^*),$$

and satisfying

$$d(\omega \wedge \Omega) = d\omega \wedge \Omega + (-1)^p \omega \wedge d\Omega, \tag{2.6}$$

where ω is a smooth $\mathbf{R}$-valued p-form, and Ω is a smooth $\mathfrak{g}$-valued form.

The Yang-Mills equations may be stated as follows. If A is a $\mathfrak{g}$-valued one-form on M, the *covariant derivative* of a $\mathfrak{g}$-valued p-form Ω with respect to A is

$$d_A\Omega = d\Omega - e_p[A, \Omega],$$

where e_p is a nonzero *coupling constant* depending on p. Choosing $e_2 = 2e_1$ results in the *Bianchi identity* $d_A d_A A = 0$. Here we assume only $e_2 = 2e_1 \equiv e'$, and define $e_{m-1} \equiv e$.

The *Yang-Mills equations* are

$$F = d_A A, \quad d_A * F = 0.$$

The one-form A is called the *connection* (in geometry) or *potential* (in physics); F is called the *curvature form* or *field strengths*.

3. Conformal transformations and vector fields

The following definitions and lemmas are contained in [3].

Definition 3.1. (a) Let M and N be pseudoriemannian manifolds of signature (k, q) equipped with pseudometrics g_M and g_N respectively. A diffeomorphism h: $M \to N$ is a *conformal transformation* if $h^* g_N = \gamma g_M$ for some positive $\gamma \in C^\infty(M, \mathbf{R})$, where h^* is the pullback of covariant tensors under h. A *conformal transformation on* M is a conformal transformation $M \to M$.

(b) A smooth vector field X on M is *conformal* if $\theta(X) g_M = \rho g_M$ for some $\rho \in C^\infty(M, \mathbf{R})$. Here $\theta(X)$, the *Lie derivative*, is the unique type-preserving derivation on the mixed tensor algebra $\mathfrak{D}(M)$ which extends $f \mapsto Xf$ on functions and $Y \mapsto [X, Y]$ on vector fields, and which commutes with contractions [2].

(c) A conformal vector field X is *locally integrable to a local one-parameter group of conformal transformations* if for each $x \in M$ there are an open set U_x containing x and a local one-parameter group h_t of conformal transformations "on U_x" (between open subsets of U_x, the domain set always containing x) with *generator* X in the sense that X_x is tangent to $t \mapsto h_t(x)$ at $t = 0$.

Remark 3.2. (a) The set of conformal transformations on M forms a group under composition.

(b) Let h be a conformal transformation $M \to N$. Since h is a diffeomorphism, $h^*(g_N)_{h(x)}$ is necessarily nondegenerate on M_x; furthermore, it has signature (k, q), the same as $(g_N)_{h(x)}$. Thus the hypothesis $\gamma > 0$ is superfluous unless m is even and $k = q = m/2$.

(c) In the situation of part (c) of Definition 3.1, the action of $\theta(X)$ on covariant tensors (real or vector-valued) is given by

$$(3.1)\qquad (\theta(X)\Omega)_x = \frac{d}{dt} h_t^*\Omega_{h_t(x)}\Big|_{t=0}.$$

If $h_t^* g = \gamma_t g$, application of (3.1) with $\Omega = g$ yields $\theta(X)g = \rho g$, where

$$(3.2)\qquad \rho(x) = \frac{d}{dt}\gamma_t(x)\Big|_{t=0}.$$

(d) In most applications, the manifolds M and N are open subsets of such manifolds as Minkowski space or its conformal compactification [5].

The properties which are crucial to the quasi-invariance relations for the Yang-Mills equations describe the behavior of the Hodge $*$ relative to conformal transformations and vector fields. We let $\mathfrak{D}_p(M, \mathfrak{g})$ denote the space of smooth $\mathfrak{g}$-valued p-forms on M.

Lemma 3.3. (a) *If h is a conformal transformation $M \to N$, $h^*(g_N) = \gamma g_M$, then*

$$(3.3)\qquad * h^*\Omega = \pm\gamma^{-(m-2p)/2} h^*(*\,\Omega), \quad \Omega \in \mathfrak{D}_p(N, \mathfrak{g}),$$

the plus sign taken if h is orientation-preserving *($h^* E_N = \delta E_M$, $\delta \in C^\infty(M, \mathbf{R})$ with $\delta > 0$), and the minus if h is* orientation-reversing *($\delta < 0$).*

(b) *If X is a conformal vector field on M, $\theta(X)g_M = \rho g_M$, which is locally integrable to a local one-parameter group of conformal transformations, then*

$$(3.4)\qquad * \,\theta(X)\Omega = \theta(X) *\Omega - \tfrac{1}{2}(m-2p)\rho *\Omega, \quad \Omega \in \mathfrak{D}_p(M, g).$$

Proof. (a) It is clearly enough to prove (3.3) with a real-valued p-form ω in place of Ω.

If φ is a real-valued one-form on N, the identification of tangent and cotangent spaces given by g_M identifies $h^*\varphi$ with $\gamma(dh^{-1})X_\varphi$, where X_φ is identified with φ through g_N. Thus

$$\begin{aligned} g_M(h^*\varphi, h^*\psi) &= \gamma^2 g_M\big((dh^{-1})X_\varphi, (dh^{-1})X_\psi\big) \\ &= \gamma(h^* g_N)\big((dh^{-1})X_\varphi, (dh^{-1})X_\psi\big) \\ &= \gamma g_N(X_\varphi, X_\psi) \circ h \\ &= \gamma g_N(\varphi, \psi) \circ h, \end{aligned}$$

where $\varphi, \psi \in \mathfrak{D}_1(N, \mathbf{R})$. Now if $\omega, \eta \in \mathfrak{D}_p(N, \mathbf{R})$, then (2.1) gives

$$(3.5)\qquad g_M(h^*\omega, h^*\eta) = \gamma^p g_N(\omega, \eta) \circ h.$$

In particular,

$$g_M(h^* E_N, h^* E_N) = \gamma^m(-1)^q,$$

so that $h^* E_N = \pm \gamma^{m/2} E_M$. Thus taking h^* of both sides of (2.3) in the form

$$g_N(\omega, \eta) E_N = \eta \wedge *\omega$$

yields

$$\begin{aligned} &\pm \gamma^{(m-2)/2} h^*\eta \wedge *h^*\omega \\ &= \left[\gamma^{-p} g_M(h^*\omega, h^*\eta)\right]\left(\pm \gamma^{m/2} E_M\right) \\ &= h^*\eta \wedge h^*(*\omega). \end{aligned}$$

Because an $(m-p)$-form on M is determined by its wedge products with elements of $\mathfrak{D}_p(M, \mathbf{R})$ and thus by its wedge with the $h^*\eta$, (3.3) follows.

(b) Let h_t be the local one-parameter group of conformal transformations generated by X, so that $h_t^* g_M = \gamma_t g_M$. Since h_0 is the identity, continuity implies that all h_t preserve orientation. If $\Omega \in \mathfrak{D}_p(M, \mathfrak{g})$, then (3.1), (3.2), and (3.3) give

$$\begin{aligned} (\theta(X)^*\Omega)_x &= \frac{d}{dt} h_t^*(*\Omega)_{h_t(x)}\Big|_{t=0} \\ &= \frac{d}{dt}\left(\gamma_t(x)^{(m-2p)/2} * h_t^*\Omega_{h_t(x)}\right)\Big|_{t=0} \\ &= *\left(\frac{d}{dt} h_t^*\Omega_{h_t(x)}\Big|_{t=0} + \frac{1}{2}(m-2p)\left(\frac{d}{dt}\gamma_t(x)\Big|_{t=0}\right)\Omega_x\right) \\ &= *\left((\theta(X)\Omega)_x + \frac{1}{2}(m-2p)\rho(x)\Omega_x\right), \end{aligned}$$

which is equivalent to (3.4).

We note finally that the relations

$$\begin{aligned} h^*(\omega \wedge \eta) &= h^*\omega \wedge h^*\eta, \\ \theta(X)(\omega \wedge \eta) &= \omega \wedge \theta(X)\eta + \theta(X)\omega \wedge \eta \end{aligned}$$

for real-valued differential forms imply the relations

$$\begin{aligned} h^*[\Xi, \Omega] &= [h^*\Xi, h^*\Omega], \\ \theta(X)[\Xi, \Omega] &= [\Xi, \theta(X)\Omega] + [\theta(X)\Xi, \Omega] \end{aligned} \tag{3.6}$$

for $\mathfrak{g}$-valued forms.

4. Quasi-invariance of the Yang-Mills equations

For a nonlinear differential equation, three types of quasi-invariance relations are relevant:

(1) quasi-invariance of the equations under conformal transformations;

(2) quasi-invariance of the linearized equations under conformal transformations;

(3) quasi-invariance of the linearized equations under Lie derivation with respect to conformal vector fields.

We set $y(A) = d_A * d_A A$ for $A \in \mathfrak{D}_1(M, \mathfrak{g})$; that is, y is the nonlinear function on $\mathfrak{D}_1(M, g)$ whose zeros are solutions of the Yang-Mills equations. As for the linearized equations, we make the following definition.

Definition 4.1. Let V and W be real vector spaces, and let

$$M_j\colon \underbrace{V \times \cdots \times V}_{j \text{ times}} \to W$$

be a j-linear function for $0 \leqslant j \leqslant N$. The *linearization* of the equation

$$\sum_{j=0}^{N} M_j(v, \cdots, v) = 0$$

at $v \in V$ is the equation

$$\sum_{j=0}^{N} \left[\sum_{i=1}^{j} M_j(v, \cdots, \underset{\substack{\uparrow \\ i\text{-th place}}}{X}, \cdots, v) \right] = 0$$

as a condition on $X \in V$.

Thus the linearization of the Yang-Mills system

$$F = d_A A = dA - \frac{e'}{2}[A, A],$$
$$0 = d_A * F = d * F - e[A, * F],$$

at $A \in \mathfrak{D}_1(M, \mathfrak{g})$ is

$$f = da - e'[A, a] \quad \text{(by (2.5))},$$
$$0 = d * f - e[a, * F] - e[A, * f]$$
$$= d_A * f - e[a, * F], \quad F = d_A A,$$

as a condition on $a \in \mathfrak{D}_1(M, \mathfrak{g})$. We define the linear function $Y_A\colon \mathfrak{D}_1(M, \mathfrak{g}) \to \mathfrak{D}_{m-1}(M, \mathfrak{g})$ by

$$Y_A a = d_A * f - e[a, * F],$$
$$f = da - e'[A, a], \quad F = d_A A.$$

Theorem 4.2. *Let $A \in \mathfrak{D}_1(M, \mathfrak{g})$ and $F = d_A A$.*

(a) *If h is a conformal transformation $M \to N$, $h^* g_N = \gamma g_M$, then*

$$y(h^*A) = \pm \left(\gamma^{(4-m)/2} h^* y(A) - \tfrac{1}{2}(m - 4)\gamma^{(2-m)/2} d\gamma \wedge h^*(* F)\right), \tag{4.1}$$

$$\begin{aligned} Y_{h^*A} h^* a &= \pm \left(\gamma^{(4-m)/2} h^* Y_A a - \tfrac{1}{2}(m - 4)\gamma^{(2-m)/2} d\gamma \wedge h^*(* f)\right), \\ f &= da - e'[A, a]. \end{aligned} \tag{4.2}$$

As usual, we take the plus sign if h preserves orientation, and the minus sign if h reverses orientation.

(b) *If X is a conformal vector field on M, $\theta(X)g_M = \rho g_M$, which is locally integrable to a local one-parameter group of conformal transformations which fix A, then*

$$(4.3)\qquad \begin{aligned} Y_A\theta(X)a &= \theta(X)Y_Aa - \tfrac{1}{2}(m-4)\{d_A(\rho * f) - e\rho[a, * F]\},\\ f &= da - e'[A, a]. \end{aligned}$$

Proof. (a) We calculate

$$y(h^*A) = d_{h^*A} * F',$$
$$F' = d_{h^*A}h^*A = dh^*A - \frac{e'}{2}[h^*A, h^*A] = h^*F.$$

By (3.3),

$$\begin{aligned} y(h^*A) &= d_{h^*A}(\pm\gamma^{(4-m)/2}h^*(* F))\\ &= \pm\left(d(\gamma^{(4-m)/2}h^*(* F)) - e\gamma^{(4-m)/2}[h^*A, h^*(* F)]\right)\\ &= \pm\left(\gamma^{(4-m)/2}h^*d_A * F - \tfrac{1}{2}(m-4)\gamma^{(2-m)/2}d\gamma \wedge h^*(* F)\right)\\ &= \pm\left(\gamma^{(4-m)/2}h^*y(A) - \tfrac{1}{2}(m-4)\gamma^{(2-m)/2}d\gamma \wedge h^*(* F)\right). \end{aligned}$$

To prove (4.2), set $f = da - e'[A, a]$, and calculate

$$Y_{h^*A}h^*a = d_{h^*A} * f' - e[h^*a, * h^*F],$$
$$f' = dh^*a - e'[h^*A, h^*a] = h^*f.$$

By (3.3),

$$\begin{aligned} Y_{h^*A}h^*a &= \pm\left(d_{h^*A}(\gamma^{(4-m)/2}h^*(* f)) - e\gamma^{(4-m)/2}h^*[a, * F]\right)\\ &= \pm\left(\gamma^{(4-m)/2}h^*d_A * f - \tfrac{1}{2}(m-4)\gamma^{(2-m)/2}d\gamma \wedge h^*(* f)\right.\\ &\qquad\qquad \left. - e\gamma^{(4-m)/2}h^*[a, * F]\right)\\ &= \pm\left(\gamma^{(4-m)/2}h^*Y_Aa - \tfrac{1}{2}(m-4)\gamma^{(2-m)/2}d\gamma \wedge h^*(* f)\right). \end{aligned}$$

(b) Let h_t be the one-parameter group generated by X, so that $h_t^*g_M = \gamma_t g_M$. Since the h_t fix A, (3.1) implies that $\theta(X)A = 0$, and the field strength perturbation f' associated to $\theta(X)a$ is

$$f' = d\theta(X)a - e'[A, \theta(X)a] = \theta(X)f$$

by (3.6) and the fact that d commutes with $\theta(X)$. Thus

$$\begin{aligned} Y_A\theta(X)a &= d_A * \theta(X)f - e[\theta(X)a, * F] \\ &= d_A\big(\theta(X) * f - \tfrac{1}{2}(m-4)\rho * f\big) - e[\theta(X)a, * F] \\ &= d\theta(X) * f - e[A, \theta(X) * f] - \tfrac{1}{2}(m-4)d_A(\rho * f) \\ &\quad - e[\theta(X)a, * F] \\ &= \theta(X)d_A * f - \tfrac{1}{2}(m-4)d_A(\rho * f) - e[\theta(X)a, * F] \\ &= \theta(X)d_A * f + \tfrac{1}{2}(m-4)d_A(\rho * f) - e\theta(X)[a, * F] \\ &\quad + e[a, \theta(X) * F]. \end{aligned}$$

Now $\theta(X) * F = * \theta(X)F + \frac{1}{2}(m-4)\rho * F$, which simplifies to $\frac{1}{2}(m-4)\rho * F$ as $\theta(X)F = \theta(X)(dA - \frac{1}{2}e'[A, A]) = d\theta(X)A - e'[A, \theta(X)A] = 0$. This makes the above

$$\theta(X)Y_Aa - \tfrac{1}{2}(m-4)\{d_A(\rho * f) - e\rho[a, * F]\}.$$

Remark 4.3. (a) The Theorem points up the importance of dimension 4 in the Yang Mills theory as $m = 4$ reduces (4.1)–(4.3) to

(4.4) $$y(h^*A) = h^*y(A),$$

(4.5) $$Y_{h^*A}h^*a = h^*Y_Aa,$$

(4.6) $$Y_A\theta(X)a = \theta(X)Y_Aa.$$

The signature (k, q) of the pseudometric is irrelevant to these formulas; in particular, it may be (4, 0) as in the case of *Euclidean* Yang-Mills (studied by Atiyah, Singer, et al), or (3, 1) as in the case of the equations in their original physical (hyperbolic) form, as studied by Segal.

(b) In *any* dimension, the Yang-Mills equations and their linearizations are invariant under *uniform dilations* ($h^*g_N = ag_M$, $a > 0$ constant), and in particular, under isometries ($a = 1$), since for such h, $d\gamma = 0$ in (4.1) and (4.2). For isometries, we again have (4.4) and (4.5). If a conformal vector field X integrates to a local one-parameter group of uniform dilations, the ρ in $\theta(X)g_M = \rho g_M$ is constant by (3.1), so that (4.3) becomes

$$Y_A\theta(X)a = \{\theta(X) - \tfrac{1}{2}(m-4)\rho\}Y_Aa,$$

and we have invariance. If X integrates to a local one-parameter group of isometries, $\theta(X)g_M = 0$ and we again have (4.6).

(c) For (4.2) and (4.3), it was not necessary to assume that the "background" potential A satisfy the Yang-Mills equations.

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