

ON THE LOWER BOUND FOR THE
INJECTIVITY RADIUS
OF 1/4-PINCHED RIEMANNIAN MANIFOLDS

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The purpose of this paper is to prove that if M is a compact simply connected riemannian manifold whose sectional curvature K satisfies $1/4 < K < 1$, then every closed geodesic has length $\geq 2\pi$, or equivalently, the injectivity radius of the exponential map is $\geq \pi$.

This result, which for example, is necessary for the proof of Berger's rigidity theorem [1], [2], is well known in the even dimensional case (assuming only $0 < K \leq 1$), and in the odd dimensional case [6] provided one assumes $1/4 < K \leq 1$. The case $1/4 \leq K \leq 1$ is treated in [7], but the argument seems to be somewhat unclear at several points. Our proof is in spirit a modification of the ideas in [7].

We will use some fairly general standard facts about lifting curves by the exponential map $\exp: TM \rightarrow M$. Our basic references are [2] and [4]. For $m \in M$, \exp_m is the restriction of \exp to the tangent space M_m of M at m . All curves under consideration will be continuous and parametrized on $[0, 1]$. Let P denote the set of all curves in M , and \tilde{P} the set of vertical curves \tilde{c} in TM emanating from the zero section, i.e., $c = \pi \circ \tilde{c}$ is constant, $\tilde{c}(0) = 0 \in M_{c(0)}$, where $\pi: TM \rightarrow M$ is the natural projection. Throughout this paper, we will use the topology of uniform convergence for P , \tilde{P} , and any subset.

Let $\text{Exp}: \tilde{P} \rightarrow P$ denote the continuous map induced by \exp , $\text{Exp}(\tilde{c}) = \exp \circ \tilde{c}$. The image of Exp is the set of *liftable* curves; any $\tilde{c} \in \tilde{P}$ is a *lift* of $c = \text{Exp}(\tilde{c})$. We call \tilde{c} a *regular lift* if \exp_m has maximal rank $n = \dim M$ at all points $\tilde{c}(t)$, $0 \leq t \leq 1$, $m = \pi \circ \tilde{c}(0)$, or equivalently, $\pi \times \exp$ has maximal rank $2n$ at all $\tilde{c}(t)$. Clearly, any $c \in P$ has a unique lift if it has a regular lift, but in general need not have a regular lift or any lift at all. All regular lifts form an open subset $\tilde{Q} \subset \tilde{P}$, and Exp imbeds \tilde{Q} homeomorphically onto an open subset $Q \subset P$. We are particularly interested in the set of closed curves $P_0 \subset P$. Lifts of closed curves, if they exist, need not be closed. P_0 certainly

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contains the Exp-image of the closed curves \tilde{P}_0 in \tilde{P} . Finally, let $\tilde{Q}_0 = \tilde{P}_0 \cap \tilde{Q}$ denote the set of closed regular lifts, which is open in \tilde{P}_0 . It follows from the above that Exp maps \tilde{Q}_0 homeomorphically onto an open subset Q_0 of P_0 . We conclude this discussion with two simple

Remarks. (1) Any geodesic $c \in P$ always has a canonical (radial) lift \tilde{c} , regular or not, which is never closed.

(2) Suppose for some $m \in M$ and $r > 0$, \exp_m is nonsingular on the open ball $U_r(m) = \{v \mid \|v\| < r\} \subset M_m$. Let $c \in P$ be piecewise differentiable, $c(0) = m$, of length $L(c) < r$. Then $c \in Q$, so c has a unique regular lift $\tilde{c} = \text{Exp}^{-1}(c)$, and $\tilde{c}(t) \in U_r$, $0 \leq t \leq 1$. In essence, this is a consequence of the Gauss lemma.

Since we have to employ some arguments from Morse theory, we will also work with the standard approximations of P_0 by the finite dimensional subspaces $\Omega^{\leq a}$ of all closed broken geodesics c of energy $E(c) \leq a$, with break point parameters chosen as usual, sufficiently fine and fixed for an energy level $b \geq a$ large enough in a given situation. Let $\Omega^{< a}$ denote the open subset of all curves in $\Omega^{\leq a}$ whose energy is strictly less than a . Observe that $\Omega^{\leq a}$ need not be the closure of $\Omega^{< a}$. The critical points of the energy function E on these free loop spaces are precisely smoothly closed geodesics in M .

We now prove a crucial fact about limits of closed lifts which is quite general. For any curve c and $0 \leq s \leq 1$, let c_s be the curve given by $c_s(t) = c(st)$, and let c^- denote the reversed curve, $c^-(t) = c(1 - t)$. By c_s^- we will mean $(c_s)^-$.

Lemma 1 (Lifting lemma). *Suppose, for some $r > 0$ and all $m \in M$, \exp_m is nonsingular on the open ball $U_r(m) \subset M_m$. Let $c \in P_0$ be in the closure of $\text{Exp}(\tilde{P}_0) \cap \Omega^{\leq 4r^2}$. Then either $c \in Q_0$ (and thus is not a closed geodesic if nonconstant), or both $c_{1/2}$ and $c_{1/2}^-$ are geodesics of length r with conjugate end points. Furthermore, $C = Q_0 \cap \Omega^{\leq 4r^2}$ is a connected component of $\Omega^{\leq 4r^2}$.*

Remarks. (3) $\Omega^{\leq 4r^2}$ is a manifold, so its connected components are precisely its pathwise connected components. But $\Omega^{\leq 4r^2}$ need not be a manifold (with boundary) if $4r^2$ is a critical value of the energy E . To avoid difficulties, we will mostly work with connected, rather than pathwise connected components.

(4) If we replace the assumption $\exp_m|_{U_r(m)}$ nonsingular for all m by the assumption $\exp_{c(0)}|_{U_r(c(0))}$ is nonsingular and leave the rest of the hypothesis unchanged, then the conclusions of Lemma 1 continue to hold for the loop space at $c(0)$. But this is not sufficient for our purposes.

Proof. We can assume that at least one of the two branches $c_{1/2}$, $c_{1/2}^-$ (say $c_{1/2}$) is not a geodesic, or if so, is free of conjugate points. Note that

$L(c_{1/2}) \leq \sqrt{E(c_{1/2})} \leq r$, with equality iff $c_{1/2}$ is a geodesic of length r . Therefore, by Remarks 1 and 2, $c_{1/2} \in Q$. But Q is open, so in fact $c_s \in Q$ for some $1/2 < s \leq 1$. Then, again by Remark 2, $c_{1-s}^- \in Q$. By hypothesis, there exists a sequence $\tilde{c}_i \in \tilde{P}_0$ with $\text{Exp}(\tilde{c}_i) = c_i$ converging to c . Now $(\text{Exp}|_{\tilde{Q}})^{-1}$ is well-defined and continuous, so it follows that $\text{Exp}^{-1}(c_{i,s}) = \tilde{c}_{i,s} \rightarrow \tilde{c}_s$, and $\text{Exp}^{-1}(c_{i,1-s}^-) = \tilde{c}_{i,1-s}^- \rightarrow \tilde{c}_{1-s}^-$.

Since $\tilde{c}_{i,s}(1) = \tilde{c}_{i,1-s}^-(1)$ for all i , we have $\tilde{c}_s(1) = \tilde{c}_{1-s}^-(1)$. Thus \tilde{c}_i converges to some $\tilde{c} \in \tilde{Q}_0$, and $\text{Exp}(\tilde{c}) = c \in Q_0$. $C = Q_0 \cap \Omega^{<4r^2}$ is open, and we have just shown it is also closed in $\Omega^{<4r^2}$. Therefore C is a union of connected components of $\Omega^{<4r^2}$. If C_1 is any component of C , then C_1 contains a closed geodesic on which E takes its minimum value. Since C contains no nontrivial closed geodesics, it follows that $C = C_1$ is connected.

We need the following result from Morse theory.

Lemma 2. *Let f be a smooth function on the differentiable manifold X of dimension k , and p a possibly degenerate critical point of index ≥ 2 (or a regular point) with $f(p) = a$. Then there exists a neighborhood N of p such that $N \cap X^{<a}$ is (pathwise) connected and dense in $N \cap X^{<a}$.*

Proof. We may assume $X = \mathbf{R}^k$, $p = 0$, $f(0) = a = 0$. If 0 is a regular point, our claim is trivial. Otherwise, according to the generalized Morse Lemma in [5], we have, after a change of coordinates near the origin in $\mathbf{R}^k = \mathbf{R}^2 \times \mathbf{R}^{k-2}$,

$$f(x, y) = -\|x\|^2 + g(y),$$

on some neighborhood U , where g is a smooth function in \mathbf{R}^{k-2} . Now choose $d, r > 0$ such that $(x, y) \in U$ and $g(y) < d^2$ on

$$N = \{(x, y) \mid \|x\|^2 \leq d^2, \|y\|^2 \leq r^2\}.$$

Let $q_0 = (x_0, y_0)$, $q_1 = (x_1, y_1) \in N \cap X^{<0}$. It suffices to construct a continuous path $\tau: [0, 1] \rightarrow N \cap X^{<0}$ from q_0 to q_1 so that $\tau(t) \in N \cap X^{<0}$ whenever $0 < t < 1$. For $x \in \mathbf{R}^2$, let $h(x) = dx/\|x\|$ if $x \neq 0$, $h(0) = (d, 0)$. Notice that $f(h(x), y) < 0$ for all $(x, y) \in N$. The path τ can be chosen as the composition of the following four simple curves: First, move (x_0, y_0) linearly to $(h(x_0), y_0)$, then $(h(x_0), y_0)$ to $(h(x_1), y_0)$ through a rotation on the circle $\|x\| = d$ keeping y_0 fixed (here we are using index ≥ 2), then $(h(x_1), y_0)$ linearly into $(h(x_1), y_1)$, and finally $(h(x_1), y_1)$ linearly to (x_1, y_1) .

Lemmas 1 and 2 have the following consequence.

Lemma 3. *Assume the hypothesis of Lemma 1, and furthermore that any smoothly closed geodesic $c \in Q_0 \cap \Omega^{<4r^2}$ (necessarily of length $2r$) has index ≥ 2 . Then $\bar{Q}_0 \cap \Omega^{<4r^2}$ is the closure of $Q_0 \cap \Omega^{<4r^2}$, and a connected component of $\Omega^{<4r^2}$.*

Proof. Let $p \in \bar{Q}_0 \cap \Omega^{<4r^2}$, and let N be a neighborhood of p in $X = \Omega^{<b}$ for some $b > 4r^2 = a$ as in Lemma 2. We have $N \cap Q_0 \cap \Omega^{<4r^2} \neq \emptyset$. But $N \cap \Omega^{<4r^2}$ is dense in $N \cap \Omega^{<4r^2}$ and Q_0 open, so that also $N \cap Q_0 \cap \Omega^{<4r^2} \neq \emptyset$. Now $N \cap \Omega^{<4r^2}$ is connected, and by the lifting lemma, $Q \cap \Omega^{<4r^2}$ is a connected component of $\Omega^{<4r^2}$. Therefore $N \cap \Omega^{<4r^2} \subset Q_0$, and thus $N \cap \Omega^{<4r^2}$ is contained in the closure of $Q_0 \cap \Omega^{<4r^2}$. This implies that $\bar{Q}_0 \cap \Omega^{<4r^2}$ is the closure of the (connected) set $Q_0 \cap \Omega^{<4r^2}$, and is relatively open (and closed) in $\Omega^{<4r^2}$, which completes the argument.

The following fact is basically standard.

Lemma 4 (*Connectedness lemma*). *Let f be a smooth proper function on a finite dimensional manifold X . Suppose, for some regular value b , all critical points of f in $X^{<b} - X^{<a}$ have index ≥ 2 (but are possibly degenerate). Let C_1, \dots, C_N be the connected components of $X^{<b}$. Then $C_1 \cap X^{<a}, \dots, C_N \cap X^{<a}$ are the connected components of $X^{<a}$. In particular, if $X^{<b}$ is connected, so is $X^{<a}$.*

Remark. (5) If in addition, all critical points in $f^{-1}(a)$ have index > 2 , then $X^{<b}$ (path) connected implies that $X^{<a}$ is also path connected, since by Lemma 2, $X^{<a}$ is locally path connected.

Proof. Choose a decreasing sequence $b > a_k > a$, $\lim_{k \rightarrow \infty} a_k = a$, of regular values of f . By standard Morse theory, we can approximate f on $X^{<b}$ by a nondegenerate Morse function f_k which agrees with f on $X^{<a_k} \cup f^{-1}(b)$. If f_k is sufficiently close to f , all critical points of f_k in $X^{<b} - X^{<a_k}$ will have index ≥ 2 . Then it follows from the Morse inequalities that $H_i(X^{<b}, X^{<a_k}) = 0$ for $i = 0, 1$. Thus $C_1 \cap X^{<a_k}, \dots, C_N \cap X^{<a_k}$ are the path connected components of $X^{<a_k}$. Since the intersection of a decreasing sequence of compact connected sets is connected, the lemma is proved.

Finally, we need

Lemma 5 (*Index lemma*). *Let M be odd dimensional with sectional curvature $0 < K \leq 1$. Then any nonconstant smoothly closed geodesic $c \in \bar{Q}_0 \cap \Omega^{<4\pi^2}$ (necessarily of length 2π) has¹ index ≥ 2 .*

Proof. Recall that by standard index form comparison, $K \leq 1$ implies that for any geodesic in M of length π , at most the end points can be conjugate, and if this is the case, any Jacobi field vanishing at the end points looks like a Jacobi field on the Euclidean sphere of curvature 1, i.e., is a multiple of a parallel field. Therefore, in our situation, the hypothesis of Lemma 1 is satisfied for $r = \pi$, and the geodesic c has precisely two conjugate points for $t = 1/2$ and $t = 1$, at length π and 2π respectively.

¹ Some further information has been communicated to us by T. Sakai.

We can find broken Jacobi fields J_{\pm} along c , not identically zero, where J_+ , J_- are Jacobi on $[0, 1/2]$, $[1/2, 1]$ respectively, and vanish otherwise. Now J_+ and J_- span a 2-dimensional space V on which the index form I is zero. If V does not intersect the null space N of I nontrivially, then the orthogonal projection of V on the negative eigenspace of I must be an injection which implies index $(c) \geq 2$. Otherwise, there is $0 \neq J \in V \cap N$. We are working in the free loop space, so N consists of periodic Jacobi fields. By the above comparison argument, we have $J(t) = \sin 2\pi t \cdot E(t)$, where E is a closed parallel field along c . Using the Synge argument, since M is odd dimensional and $K > 0$, M is orientable, and we conclude that there exists a second closed parallel field F normal to c . Then E and F span a 2-dimensional space on which I is negative definite.

We can now derive our main result.

Theorem 6. *Let M be a simply connected compact riemannian manifold of odd dimension n . Suppose the sectional curvature satisfies $1/4 \leq K \leq 1$. Then*

- (a) $\Omega^{<4\pi^2}$ is the closure of $\Omega^{<4\pi^2}$, and is connected;
- (b) $\Omega^{<4\pi^2}$ is contained in Q_0 , and is (pathwise) connected;
- (c) $c \in \Omega^{<4\pi^2}$ implies either $c \in Q_0$, or both $c_{1/2}$ and $c_{1/2}^-$ are geodesics of length π with conjugate end points;
- (d) any nonconstant smoothly closed geodesic in M has length $> 2\pi$ and index ≥ 2 .

Proof. As described in the proof of Lemma 5, $K \leq 1$ implies that the hypothesis of Lemma 1 is satisfied for $r = \pi$. By the same comparison technique, since $1/4 \leq K$, any geodesic in M of length $> 2\pi$ has index $n - 1 \geq 2$.

We argue first that for any $a > 4\pi^2$, $\Omega^{<a}$ is connected. If not, let c_1, c_2 be curves in different connected components of $\Omega^{<a}$. Since M is simply connected, c_1 and c_2 are homotopic. After suitable refinement of the break point subdivision of the parameter interval $[0, 1]$, we can therefore join c_1 and c_2 by a continuous path in $\Omega^{<b}$ for some $b \geq a$. So c_1 and c_2 belong to the same connected component of $\Omega^{<b}$ for some regular value $b > a$, contradicting Lemma 4.

Using Lemma 4 again, we conclude that $\Omega^{<4\pi^2}$ is connected. Now we apply Lemmas 5 and 3 to obtain that $\bar{Q}_0 \cap \Omega^{<4\pi^2} = \Omega^{<4\pi^2}$ is the closure of $Q_0 \cap \Omega^{<4\pi^2}$, which completes the proof of (a). In particular, $Q_0 \cap \Omega^{<4\pi^2}$ is dense, and by Lemma 1, connected and closed in $\Omega^{<4\pi^2}$. Thus $Q_0 \cap \Omega^{<4\pi^2} = \Omega^{<4\pi^2}$ is connected, which proves (b). The last two statements (c) and (d) are an immediate consequence of (a), (b), and Lemmas 1 and 5.

We conclude our discussion with some remarks. The result in (d) that all

nonconstant smoothly closed geodesics have index ≥ 2 is rather surprising. One has as immediate consequence (which is not the strongest conclusion that can be drawn) that for any $a \geq 0$, the loop spaces $\Omega^{<a}$ (and $\Omega^{<a}$ as well) are pathwise connected; cf. also Remark 5. The last statement holds also in even dimensions under the much weaker assumption $0 < K \leq 1$, since by the Synge Lemma, every nonconstant smoothly closed geodesic has index ≥ 1 . Clearly, Theorem 6 holds in that case except that in (d), the index is only ≥ 1 . Using the preceding remark, just Lemma 1 is needed for the proof. Finally, both in odd and even dimensions, Theorem 6(b) provides an obvious direct argument for the fact that the injectivity radius of the exponential map is $> \pi$.

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