J. DIFFERENTIAL GEOMETRY
14 (1979) 621–628

# CONVEX MANIFOLDS OF NONNEGATIVE CURVATURE

STEPHEN KRONWITH

During the past decade, exciting breakthroughs have occurred in the study of complete open (i.e., noncompact) manifolds of nonnegative curvature. Cheeger and Gromoll [3] have shown that such a manifold $M$ contains a compact totally geodesic submanifold $S$, the *soul* of $M$, whose normal bundle is diffeomorphic to $M$ itself. The soul also has the property of being convex in the sense of definition 1.1. In fact, the existence of large convex sets with boundary is of basic importance in the construction of the soul. It is mainly because of this that we undertake to better understand their structure, though most questions relating to convexity seem to be important also in their own right.

We first study compact convex sets in the abstract, what we call convex manifolds. We show that such a manifold of nonnegative curvature has a complete metric of nonnegative curvature on its interior. This is used to answer the fundamental question: Can a compact convex manifold of nonnegative curvature be isometrically imbedded into a complete open manifold of nonnegative curvature?–a converse to the procedure of Cheeger and Gromoll. For the most part, but not always, we find the answer to be yes, the most general results being obtained in the case of surfaces.

This paper is, in a large part, taken from the author's doctoral dissertation at the State University of New York at Stony Brook. The author would like to thank Detlef Gromoll for the many helpful discussions we had which gave the author deeper insight into the delicate aspects of convexity.

## 1. Convex manifolds

Given a Riemannian manifold $N$ (without boundary) and a set $C \subset N$, $C$ is said to be *convex* if, for any point $p \in \bar{C}$, there is a number $e(p)$ with $0 < e(p) < r(p)$ such that $C \cap B_{e(p)}(p)$ has the property that between any

Received March 2, 1978.

two points there is a unique minimal geodesic in $N$, completely contained in $C \cap B_{e(p)}(p)$, which joins these points. The set $C \cap B_{e(p)}(p)$ is said to be strongly convex.

Here, and hereafter, $r(p)$ denotes the convexity radius of $N$ at $p$, $\bar{C}$ is the closure of $C$ in $N$, and $B_\delta(p)$ denotes the open ball of radius $\delta$ about $p$ in $N$. For other basic facts and notations which will be used without comment, we refer the reader to [2] and [4].

Using this definition, it can be shown that a closed convex set $C \subset N$ is an imbedded topological manifold with smooth totally geodesic interior $\text{int}(C)$, and possibly nonsmooth boundary $\partial C$ (which may be empty) [3]. If $\partial C$ is smooth, then $C$ is a smooth submanifold with boundary. Accordingly, a smooth compact convex subset $C \subset N$ will be called a convex submanifold of $N$.

Let us give a somewhat more intrinsic definition.

**Definition 1.1.** A compact Riemannian manifold $M^n$ of dimension $n$ with smooth boundary $\partial M \neq \varnothing$ is said to be *convex* if there are a Riemannian manifold $W^n$ without boundary and isometric imbedding $i$: $M \to W$ such that $i(M)$ is a convex subset of $W$.

It is straightforward to see that any compact convex submanifold $C \subset N$ is a convex manifold in the above sense. Just exponentiate the outward unit normal field of $\partial C$ in $C$ into $N$ to obtain a submanifold $W$ of $N$ with $\partial W = \varnothing$ and $\dim W = \dim C$.

Two important facts about convex manifolds are standard [1]:

**1.a.** If $C \subset N$ is convex and smooth with $p \in \partial C$ and $v \in (\partial C)_p$, where $X_p$ denotes the tangent space of $X$ at $p$, then the geodesic $\alpha$: $(-\delta, \delta) \to M$ with $\alpha(0) = p$, $\dot{\alpha}(0) = v$, locally stays to the outside of $\text{int}(C)$. That is, for sufficiently small $\varepsilon$, $\alpha(t) \notin \text{int}(C)$ for all $t \in (-\varepsilon, \varepsilon)$.

**1.b.** A compact Riemannian manifold $M$ with boundary $\partial M$ is convex if and only if the second fundamental form of the outward pointing unit normal field along $\partial M$ is positive semidefinite. We call a point $p \in \partial(M)$ where the form is positive definite, a point of *strict convexity*.

The main result of this section is the following.

**Theorem 1.2.** *Let $M$ be a convex manifold with metric $\langle\, , \rangle$ and distance function $\rho$. Assume that the curvature $K$ of $M$ is nonnegative (respectively positive). Then for any $\varepsilon > 0$ sufficiently small, there exists a complete metric of nonnegative (respectively positive) curvature on* $\text{int}(M)$ *such that this new metric agrees with the old one off the one-sided $\varepsilon$-tubular neighborhood of the boundary.*

*Proof.* Let $f$: $\text{int}(M) \to R$ be the function $f(x) = \rho(x, \partial M)$. It is known that $f$ is continuously convex [3]. But since $\partial M$ is smooth, there is a one-sided tubular neighborhood of the boundary which will exclude any points of the cut locus of $\partial M$ and on which $f$ will be smoothly convex, in that the hessian form $h_f$ of $f$ will be negative semidefinite. Call this neighborhood $B_\varepsilon(\partial M)$, that is, $B_\varepsilon(\partial M) = \{x \in \operatorname{int}(M) | \rho(x, \partial M) < \varepsilon\}$.

Let $\chi = 1/f$. $\chi$ is $C^\infty$ on $B_\varepsilon(\partial M)$ and $\lim_{x\to\partial M} \chi(x) = \infty$. For $p \in B_\varepsilon(\partial M)$, we have $h_\chi(v_p, v_p) = h_{1/f}(v, v) = \langle \nabla_v \nabla(1/f), v\rangle = (2v(f))^2/f^3 - 1/f^2(h_f(v, v)) \geqslant 0$ where $v_p \in M_p$, $\nabla$ is the Levi-Civita connection on $M$, and $\nabla(1/f)$ is the gradient of $1/f$.

Now define $g: R \to R$ by

$$g(t) = \begin{cases} \displaystyle\int_{1/\varepsilon}^{t} e^{-1/(s^2 - 1/\varepsilon^2)}\, ds, & t > 1/\varepsilon, \\ 0, & \text{else.} \end{cases}$$

$g$ is $C^\infty$ with $d^n g/dt^n_{|1/\varepsilon} = 0$ for all $n$.

Let $\tilde{\chi}$: $\operatorname{int}(M) \to R$ be defined by

$$\tilde{\chi}(x) = \begin{cases} \chi(x), & x \in B_\varepsilon(\partial M), \\ \gamma(x), & \text{else,} \end{cases}$$

where $\gamma$ is any $C^\infty$ extension of $\chi$ outside of $B_\varepsilon(\partial M)$.

Finally, let $\Gamma = g \circ \tilde{\chi}$: $\operatorname{int}(M) \to R$. We have that $\lim_{x\to\partial M} \Gamma(x) = \infty$, $\Gamma$ is $C^\infty$, and in $B_\varepsilon(\partial M)$, $h_\Gamma(v, v) = (g' \circ \tilde{\chi})h_{\tilde{\chi}}(v, v) + (g'' \circ \tilde{\chi})(v(\tilde{\chi}))^2$.

In $B_\varepsilon(\partial M)$, $h_{\tilde{\chi}}$ is nonnegative, and $h_\Gamma(v, v) \geqslant 0$ since $g'$ and $g''$ are also nonnegative. If $p \in M - B_\varepsilon(\partial M)$, then $g' = g'' = 0$, so $h_\Gamma = 0$. Thus by continuity, $h_\Gamma(v, v) \geqslant 0$ for all $p \in \overline{M - B_\varepsilon(\mu M)}$. So $\Gamma$ is (nonstrictly) concave.

Next we define $H$: $\operatorname{int}(M) \times R \to R$ by $H(x, t) = \Gamma(x) - t$. $H$ is a regular map, and by the implicit function theorem $H^{-1}(0) = \tilde{M} =$ graph of $\Gamma$ is a Riemannian submanifold of $\operatorname{int}(M) \times R$, with the metric induced from the product metric on $\operatorname{int}(M) \times R$; by the nature of the behavior of $\Gamma$ at the boundary and the compactness of $M$, the metric is complete. Then by the Gauss equations, we see that the graph has nonnegative (positive) curvature.

Finally, let $G$: $\operatorname{int}(M) \to M$ be defined by $G(x) = (x, \Gamma(x))$. If we give $\operatorname{int}(M)$ the induced metric from $M$, it is readily seen that $G$ is an isometry. So $\operatorname{int}(M)$ is given a complete metric of nonnegative (respectively positive) curvature which agrees with the original metric off $B_\varepsilon(\partial M)$.

The last theorem implies, of course, that any compact convex submanifold of $\operatorname{int}(M)$ is isometrically contained in a complete manifold of nonnegative (respectively positive) curvature. In particular, we have

**Corollary 1.3.** *Let $M$ a convex manifold of nonnegative (respectively, positive) curvature. Then the convex manifolds $M^\varepsilon = \{x \in M | \rho(x, \partial M) > \varepsilon\}$ can be isometrically imbedded in a complete manifold of nonnegative (respectively, positive) curvature of the same dimension for arbitrarily small $\varepsilon > 0$.*

We do not know if Corollary 1.3 is always true also for $\varepsilon = 0$, i.e., for $M^0 = M$ itself. But we easily obtain a positive answer in general for the following case.

**Corollary 1.4.** *Let $M$ be a strictly convex manifold of positive curvature. Then $M$ can be isometrically imbedded in a complete open manifold of positive curvature and same dimension.*

*Proof.* By definition, $M^n \subset W^n$. Since $K(M) > 0$ and the second fundamental form of the boundary is positive definite, by continuity, $W$ contains a convex manifold $M'$ of positive curvature such that $\operatorname{int}(M') \supset M$. Then by applying Theorem 1.2 to $M'$ we get our result.

(With a little extra work we can prove Theorem 1.2 if $\partial M$ is merely continuous. for a discussion of the procedure, see [5]).

## 2. Convex surfaces

In this section we take up the imbedding problem for convex surfaces. After possibly passing to an orientable double cover, we may assume all surfaces are orientable. According to [3] we can also restrict attention to the case where $\partial M$ is connected. Otherwise, $M$ is isometrically a product of a circle and an interval, and our imbedding problem has a trivial solution. So $M$ will have to be a disc topologically.

Choose an orientation of the surface $M$ such that $\dot\gamma(\gamma = \partial M)$ and the global outward pointing unit normal field $N$ along $\gamma$ form an oriented base along $\gamma$. We assume $\gamma$ to be parametrized by arc length, and then define the *geodesic curvature* of $\gamma$ by $k = \langle \nabla_{\dot\gamma}\dot\gamma, N\rangle$. By Remark 1.b we see that $k \leqslant 0$.

It is well known that we can choose a global Fermi-coordinate system $(x, y)$ in a strip about our compact boundary such that $ds^2 = dy^2 + g dx^2$ and the $y$-coordinate curves are arc-length geodesics perpendicular to $\gamma$. Simple computations show that in these coordinates, $K = -(1/\sqrt{g})\partial^2\sqrt{g}/\partial y^2$ and the geodesic curvature of any $y$-coordinate curve is given by $-(1/2g)\partial g/\partial y$.

**Lemma 2.1.** *Let $M$ be a strictly convex surface of nonnegative curvature which can be $C^r$ $(r \geqslant 2)$ imbedded in some surface without boundary of nonnegative curvature. Then $M$ can be $C^r$ imbedded in a complete open surface and also in a compact surface (without boundary), both of nonnegative curvature.*

*Proof.* Choosing Fermi-coordinates as above, we see that by hypothesis and continuity of $k$, we can extend $g$ in an outsided strip $\partial M \times [0, \varepsilon]$, of $\partial M$ to a $C^\infty$ functon $\bar g$ where $g$ and $\bar g$ agree up to $r$th order on $\partial M(y = 0)$ and $k$ remains nonnegative as does $K$. Now in the strip $\partial M \times [0, y_0]$, $y_0 < \varepsilon$, define

$\bar{\bar{g}}$ by $\bar{\bar{g}}(x, y) = \bar{g}(x, P(y))$ where $P$ is a $C^\infty$ extension of $f(y) = y$ whose maximum value occurs before $y = y_0$ and has properties that $P(0) = 0$, $P'(0) = 1$, $P^j(0) = 0$ for $j > 1$ and $P^j(y_0) = 0$ for $j = 1, 2, \cdots, r$. Then it can be easily checked that $g$ and $\bar{\bar{g}}$ agree up to $r$th order, the curvature in the new metric $ds^2 = dy^2 + \bar{\bar{g}}dx^2$ remains nonnegative, and the geodesic curvature of the $y = y_0$ curve is zero. Therefore, if $T = \partial M \times [0, y_0]$, the first conclusion of the lemma now follows from applying Theorem 1.2 to $M \cup T$, and the second conclusion from taking the double of $M \cup T$, which, by the vanishing of the first $r$ partials of $\bar{\bar{g}}$ with respect to $y$, will be a $C^r$ compact manifold of nonnegative curvature. q.e.d.

It remains to show that the hypothesis "$r \geqslant 2$" can be satisfied easily enough for $r = 2$.

**Lemma 2.2.** *Let $M$ be a strictly convex surface with $K \geqslant 0$. Then $M$ can be $C^2$ imbedded in a surface of nonnegative curvature.*

Let us note first that a $C^2$ imbedding is, in general, the best we can hope for. For if $p \in \partial M$ and $K(p) = 0$ with $\partial K/\partial y|_p < 0$, we cannot extend our metric with $C^3$ agreement along the boundary since $\partial K/\partial y$ is basically determined by $\partial^3 g/\partial y^3$, and having the third derivative continuous would force $K$ negative in any extension.

*Proof.* Just define $\bar{g}$ on any small enough strip by $\bar{g}(x, y) = g(x, 0) + (\partial g/\partial y)(x, 0)y + (\partial^2 g/\partial y^2)(x, 0)(y^2/2)$.

Combining our lemmas we get

**Theorem 2.3.** *Let $M$ be a strictly convex surface of nonnegative curvature. Then $M$ can be $C^2$ imbedded in a complete open surface and also in a compact surface (without boundary), both of nonnegative curvature.*

If we drop the hypothesis of strict convexity, we run into trouble; for, using our techniques, a convex extension of $M$ cannot be guaranteed, and at times does not exist as we will show in the next section. However, if we examine the calculations in Lemma 2.1, we come up with the following.

**Corollary 2.4.** *If $M$ is a convex surface of nonnegative curvature with $K|_{\partial M} = 0$, then all the results of Theorem 2.3 hold.*

## 3. $H$-Convex extensions

In this section we provide a more geometric approach to our imbedding problem. Recall first that with $M$ compact and convex there exists an $r_0 > 0$ such that each ball $B_{r_0}(p)$ is strongly convex for each $p \in M$.

A convex surface is said to be *H-convex* if its boundary consists of a piecewise smooth geodesic.

**Lemma 3.1.** *Let $M$ be a convex surface. If $K > 0$ (respectively, $\geqslant 0$) and there exists a point $p_0 \in \partial M$ such that $k(p_0) < 0$, then $M$ can be imbedded (respectively, $C^2$ imbedded) in an $H$-convex surface of positive (respectively, nonnegative) curvature. This imbedding is not proper, i.e., $M$ need not lie in the interior of our $H$-convex surface.*

*Proof.* Let $M \subset W$. As constructed in the last section, our imbedding is $C^\infty$ for $K > 0$ and $C^2$ for $K \geqslant 0$. Since $M$ is compact, there is a compact $V \subset W$ with $M \subset V$. Let $d$ be the elementary length of $V$, and let $u$ be the smallest parameter value such that any geodesic tangent to $M$ at $p_0$ lies to the outside of $B_{r_0}(p_0)$ for all parameter values less than $u$. Now let $\delta = \min(u, r_0, d)$. Choose a parametrization $\chi$: $[0, 1] \to \partial M$ by arc length.

About $p_0$ choose the ball $B_{\delta/2}(p_0)$ and assume that $\chi(0) = p_0$. Let $v = \dot{\chi}(0)$. By convexity, the minimal geodesic $\alpha$, with $\alpha(0) = p_0$ and $\dot{\alpha}(0) = v$, lies to the outside of $M$ for all $t \in [0, \delta/2]$. Let $\alpha(\delta/2) = q \in \partial B_{\delta/2}(p_0)$, and let $t_1$ be the point such that $t_1 > 0$ and $\chi(t_1) \cap \overline{B_{\delta/2}(p_0)} \neq \varnothing$. Put $\chi(t_1) = s$, and choose points $t_0 \in [0, t_1]$ and $r_1 \in M$ as follows.

(a) *Point of type* **1.** If the interval $[t_0, t_1]$ is such that $k_{|[t_0,t_1]} = 0$ and $k_{|[t_0-\varepsilon,t_1]} \neq 0$ for all $\varepsilon > 0$, then let $r_1 = \chi(t_0)$ be called of type 1.

(b) *Point of type* **2.** If $k_{\psi(t_1)} < 0$, let $r_1 = s = \psi(t_1)$ be called of type 2.

Let $\beta$ be the unique minimal geodesic from $q$ to $r_1$. By the triangle inequality $\beta$ exists and lies in $B_\delta(p_0)$.

Now we have the triangle $(p_0, q, r_1)$. If $\beta$ is already tangent to $\chi$, denote by $\Delta p_0$ the triangle and its interior. If not, depending upon the orientation, either $\dot{\chi}(t_1)$ or $-\dot{\chi}(t_1)$ is interior to $(p_0, q, r_1)$. Denote that interior vector $v$. By the parametrization of $\chi$, $v$ is of unit length. Let $\gamma(t) = \exp_{r_1} tv$, $t \in [0, w]$. This geodesic will have to leave the triangle sooner or later. It cannot leave by crossing $\chi$, since $\chi$ is not a geodesic by choice of $p_0$ and in our neighborhood convexity forces $\gamma$ away from $\chi$. If $\gamma$ leaves by crossing $\beta$, we would have a conjugate point on the minimal geodesic $\beta$ with respect to $r_1$, contradicting our choice of $\delta$.

So $\gamma$ leaves the triangle by crossing $\alpha$ and making a convex angle at the point of intersection. Denote by $\Delta_{p_0}$ this triangle (which is contained in $B\delta(p_0)$) and its interior. If $r_1$ is of type 1, we continue along $\psi$ until we find a point $p_1$ where $k_{p_1} < 0$ and start again. If there is no such point we are done with this part of the construction.

If $r_1$ is of type 2, we start the process over again at $r_1$, choosing another ball and continuing as before. By the compactness of $\partial M$, we can continue around $\psi$ coming up with $\tilde{M} = M \cup_{i \in I} \Delta p_i$, $I$ a finite set. $\tilde{M}$ is an $H$-convex extension of $M$. q.e.d.

Next we wish to make the extension in Lemma 3.1 a proper one. Let us look at three successive geodesics of $\partial\tilde{M}$, $\alpha_1$, $\alpha_2$, and $\alpha_3$. Denote by $\theta_i$ the convex angle where $\alpha_i$ meets $\alpha_{i+1}$. Then pick outward unit vectors $X_i$ at $\Theta_i$'s vertex $p_i$ such that $\sphericalangle(X_i, -\dot{\alpha}_i(p_1)) = \sphericalangle(X_i, \dot{\alpha}_{i+1}(p_i)) = (2\pi - \Theta_i)/2$, $i = 1, 2$. If $p_1$ and $p_2$ are not conjugate to each other, there exists a Jacobi field $J$ along $\alpha_2$ with $J(p_i) = X_i$. $J \neq 0$ along $\alpha_2$ as long as there are no conjugate points at all along $\alpha_2$. If all the geodesic segments of our $H$-convex $\tilde{H}$ are minimal, then there is no problem with conjugate points. Certainly, all those geodesics added on to $M$ are minimal by construction. However, as in the case of points of type 1, they may hook up with a geodesic which is part of the original boundary of $M$. If the sum of the lengths of these geodesics is too long, conjugate points may occur. So we must make sure that any geodesic segment of $\partial M$ has, first, no conjugate points to begin with, and secondly, when hooked up with added segments of geodesics of the construction, no conjugate points will occur. It is clear that having no conjugate points on our original boundary is sufficient by that boundary being compact.

This nonvanishing Jacobi field $J$ generates a variation through geodesics, so, by going outward a parameter of one, for instance, we get the geodesic $\alpha_2'$. We continue this process at all corners to get a proper $H$-extension with convex corners which we can easily smooth out keeping convexity.

So we get as final product a smooth convex $M'$ properly containing $M$ where the extension is $C^\infty$ if $K > 0$ and $C^2$ if $K \geqslant 0$. We have thus proved

**Theorem 3.2.** *Let $M$ be a convex surface of positive (respectively, nonnegative) curvature with a point $p$ such that $k_p < 0$. If there are no conjugate points along $\partial M$, then $M$ can be properly $C^\infty$ (respectively, $C^2$) imbedded in a convex surface of positive (respectively, nonnegative) curvature.*

**Corollary 3.3.** *Let $M$ be a convex surface of nonnegative curvature. If there is a point $p \in \partial M$ such that $k_p < 0$, and there are no conjugate points on $\partial M$, then $M$ can be $C^2$ imbedded in a complete open surface, and also in a compact surface (without boundary), both of nonnegative curvature.*

**Corollary 3.4.** *Let $M$ be as above. If there exist conjugate points on $\partial M$, there does not exist any proper convex extension of $M$ arbitrarily close to $M$.*

*Proof.* The existence of conjugate points implies that locally, geodesics intersect.

Of course, Corollary 3.4 is not true for sufficiently large proper, or sufficiently small nonproper extensions, as is the necessity of the conjugate point hypothesis in Corollary 3.3 not true in general. Examples are convex sets in a paraboloid of revolution whose boundary contains a sufficiently large piece of a meridian.

So one question still remains: If $M$ is a convex surface of nonnegative curvature with conjugate points on its boundary, what, if any, are the conditions guaranteeing the imbedding of $M$ in a complete open surface of nonnegative curvature?

By the Gauss-Bonnet theorem we have that if $M$ is convex, $\int_M K\,dM \leqslant 2\pi$. So if $K > 0$, and $\partial M$ is a geodesic, then it is impossible to imbed $M$ in any larger convex $M'$ of nonnegative curvature. In fact, it may not even be imbeddible in something complete even if $K \geqslant 0$ unless $K_{|\partial M} = 0$. In the case of the hemisphere we see that we would have to add on a flat cylinder to the boundary, and by the curvature difference, this imbedding would not even be $C^2$. However, in this maximal case, $M$ can always be $C^2$ imbedded in the compact surface homeomorphic to $S^2$ obtained by taking the double of $M$.

Further questions in higher dimensions will be taken up in a future paper.

## References

[1] R. L. Bishop & R. J. Crittenden, *Geometry of manifolds*, Pure and Applied Math., Vol. 15, Academic Press, New York, 1964.

[2] J. Cheeger & D. G. Ebin, *Comparison theorems in Riemannian geometry*, North Holland, Amsterdam, 1975.

[3] J. Cheeger & D. Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. **96** (1972) 413–443.

[4] D. Gromoll, W. Klingenberg & W. Meyer, *Riemannsche Geometrie im Grossen*, Lecture Notes in Math. Vol. 55, Springer, Berlin, 1968.

[5] S. Kronwith, *Convex Riemannian manifolds of nonnegative curvature*, Thesis, State University of New York at Stony Brook, 1977.

St. Johns University, New York
State University of New York, Stony Brook