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A GENERALIZATION OF MYERS THEOREM AND AN APPLICATION TO RELATIVISTIC COSMOLOGY

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There have been several applications of Myers method to General Relativity. The earliest example of such an application which has come to our attention appears in a paper by Avez [2]. T. Frankel has used Myers theorem to obtain a bound on the size of a fluid mass in a stationary space-time universe. (See [3] for an improved version of this result which makes use of results presented here.) In [5] the present author made use of Frankel's method to obtain a *closure theorem* (i.e., a theorem which has as its conclusion the "finiteness" of the "spatial part" of a space-time obeying certain cosmological assumptions) for cosmological models more general than the classical Friedmann models.

The effort to obtain a closure result for a cosmological setting of considerably greater generality has led to a rather curious generalization of Myers theorem. We wish to present this generalization along with the closure result which follows from it.

All manifolds under consideration in this paper, whether Riemannian or pseudo-Riemannian, are assumed to be smooth.

1. A generalization of Myers theorem

Let M^n be a Riemannian manifold, and γ a geodesic joining two points of M^n. Recall (see [6]) that Myers actually shows that if along γ the Ricci curvature, Ric, satisfies

$$\mathrm{Ric}(\mathbf{T}, \mathbf{T}) \geqslant a > 0$$

and the length of γ exceeds $\pi\sqrt{n-1}\,/\sqrt{a}$, where $\mathbf{T}$ is the unit tangent to γ, then γ cannot be minimal. We have the following generalization.

Lemma 1.1. *Let M^n be a Riemannian manifold, and γ a geodesic joining two points of M^n. Let $\mathbf{T}$ be the unit tangent, and σ the arc length along γ. If*

$$\mathrm{Ric}(\mathbf{T}, \mathbf{T}) \geqslant a + \frac{df}{d\sigma} \tag{1.1}$$

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holds along γ, where a is a positive constant and f is any differentiable function of the arc length bounded in absolute value by $c \geqslant 0$, and

$$\text{(1.2)} \qquad \textit{length of } \gamma > \frac{\pi}{a}\left(c + \sqrt{c^2 + a(n-1)}\right),$$

then γ cannot be minimal.

Remark. This lemma reduces to Myers result when $f = c = 0$. Condition (1.1) does not require that the Ricci curvature be positive along γ.

Proof. The proof of Lemma 1.1 proceeds just as in the proof of Myers result until explicit use is made of (1.1).

Let γ be a geodesic of length l from point p to point q in M^n. Let σ be the arc length along γ, and let $\mathbf{T}$ be the unit tangent vector to γ. Let $\mathbf{T}, \mathbf{Y}_2, \ldots, \mathbf{Y}_n$ be orthonormal vectors at p. Parallelly displace the vectors $\mathbf{Y}_2, \ldots, \mathbf{Y}_n$ along γ so that $\mathbf{T}, \mathbf{Y}_2, \ldots, \mathbf{Y}_n$ remain orthonormal along γ. Define $n-1$ vector fields along γ by

$$\mathbf{X}_i(\sigma) = f_i(\sigma)\mathbf{Y}_i(\sigma), \; i = 2, \ldots, n,$$

where $f_2(\sigma), \ldots, f_n(\sigma)$ are $n-1$ smooth functions on $0 \leqslant \sigma \leqslant l$.

For each i, define the variation

$$(\sigma, \varepsilon_i) \xrightarrow{x_i} \exp_{\gamma(\sigma)} \varepsilon_i \mathbf{X}_i(\sigma),$$

where $\exp_{\gamma(\sigma)} \varepsilon_i \mathbf{X}_i(\sigma)$ is the point in M^n reached by traveling along the geodesic with initial tangent $\mathbf{X}_i(\sigma)$ at $\gamma(\sigma)$ for a distance $\varepsilon_i \|\mathbf{X}_i(\sigma)\|$. By construction $\mathbf{X}_i$ (for each i) is the variation vector field.

Since $\mathbf{X}_i$ and $\mathbf{T}$ are othogonal and γ is a geodesic, the formula for the first variation of the arc length gives

$$l_i'(0) = 0, \; i = 2, \ldots, n,$$

where $l_i(\varepsilon_i)$ is the length of the varied curve $\sigma \to \exp_{\gamma(\sigma)} \varepsilon_i \mathbf{X}_i(\sigma)$. Using Synge's formula for the second variation of the arc length we can obtain

$$\text{(1.4)} \qquad l_i''(0) = \int_0^l \left[\left| \frac{df_i}{d\sigma}(\sigma) \right|^2 - |f_i(\sigma)|^2 K(\mathbf{T} \wedge \mathbf{Y}_i) \right] d\sigma, \; i = 2, \ldots, n,$$

where $K(\mathbf{T} \wedge \mathbf{Y}_i)$ is the sectional curvature of the 2-plane spanned by $\mathbf{T}$ and $\mathbf{Y}_i$. Now make the special choice

$$f_i(\sigma) = \sin\left(\frac{\pi\sigma}{l}\right), \; i = 2, \ldots, n.$$

Then (1.4) becomes

$$l_i''(0) = \frac{\pi^2}{l^2} \cdot \frac{l}{2} - \int_0^l \sin^2\left(\frac{\pi\sigma}{l}\right) K(\mathbf{T} \wedge \mathbf{Y}_i) \, d\sigma.$$

Hence

$$\sum_{i=2}^{n} l_i''(0) = \frac{\pi^2}{l^2} \cdot \frac{l}{2}(n-1) - \int_0^l \sin^2\left(\frac{\pi\sigma}{l}\right) \mathrm{Ric}(\mathbf{T}, \mathbf{T})\, d\sigma, \tag{1.5}$$

since $\mathrm{Ric}(\mathbf{T}, \mathbf{T}) = \sum_{i=2}^{n} K(\mathbf{T} \wedge \mathbf{Y}_i)$. Now suppose condition (1.1) holds along γ. Then

$$\int_0^l \sin^2\left(\frac{\pi\sigma}{l}\right) \mathrm{Ric}(\mathbf{T}, \mathbf{T})\, d\sigma \geqslant a \cdot \frac{l}{2} + \int_0^l \sin^2\left(\frac{\pi\sigma}{l}\right) \frac{df}{d\sigma}\, d\sigma. \tag{1.6}$$

We integrate by parts to obtain an inequality for the integral on the right-hand side of (1.6),

$$\begin{aligned} \int_0^l \sin^2\left(\frac{\pi\sigma}{l}\right) \frac{df}{d\sigma}\, d\sigma &= -\frac{\pi}{l} \int_0^l f(\sigma) \sin\left(\frac{2\pi\sigma}{l}\right) d\sigma \\ &\geqslant -\frac{\pi}{l} \int_0^l \left| f(\sigma) \sin\left(\frac{2\pi\sigma}{l}\right)\right| d\sigma \\ &\geqslant -\frac{\pi}{l} \cdot cl = \pi c, \end{aligned} \tag{1.7}$$

where in the last step we have used $|f(\sigma)| \leqslant c$ along γ. Combining (1.5)–(1.7) we obtain the inequality

$$\sum_{i=2}^{n} l_i''(0) \leqslant \frac{l}{2}\left[\frac{\pi^2}{l^2}(n-1) + \frac{2\pi c}{l} - a\right]. \tag{1.8}$$

The expression in brackets is negative when

$$l < \frac{\pi}{a}\left(c + \sqrt{c^2 + a(n-1)}\right), \tag{1.9}$$

in which case

$$l_i'(0) = 0, \quad l_i''(0) < 0$$

for at least one i. Thus for such an i the varied curve $\sigma \to \exp_{\gamma(\sigma)} \varepsilon_i X_i(\sigma)$, which passes through p and q, would have length less than γ for sufficiently small ε_i. Hence the geodesic γ from p to q cannot be minimal if its length l satisfies (1.9). This concludes the proof.

As in Myers result there are two important consequences of Lemma 1.1.

Theorem 1.2. *Let M^n be a complete Riemannian manifold. Suppose there exist constants $a > 0$ and $c \geqslant 0$ such that for every pair of points in M^n and minimal geodesic γ joining those points having unit tangent* $\mathbf{T}$, *the Ricci curvature satisfies*

$$\mathrm{Ric}(\mathbf{T}, \mathbf{T}) \geqslant a + \frac{df}{d\sigma} \quad \textit{along } \gamma, \tag{1.10}$$

where f is some function of the arc length σ satisfying $|f(\sigma)| \leqslant c$ along γ. Then

M^n is compact and

$$\operatorname{diam}(M^n) \leqslant \frac{\pi}{a}\left(c + \sqrt{c^2 + a(n-1)}\right). \tag{1.11}$$

Proof. Since M^n is complete, any pair of points in M^n can be joined by a minimal geodesic. By Lemma 1.1 a minimal geodesic joining any pair of points in M^n and satisfying (1.10) must have length less than or equal to the diameter estimate in (1.11). Thus (1.11) is established, and M^n is bounded. Furthermore, since closed and bounded sets of a complete Riemannian manifold are compact, M^n is compact.

Theorem 1.3. *Let M^n be a complete Riemannian manifold. Suppose there exist constants $a > 0$ and $c \geqslant 0$ such that for every pair of points in M^n (not necessarily distinct) and geodesic γ joining these points with unit tangent* $\mathbf{T}$, *the Ricci curvature satisfies* (1.10) *where f is some function of the arc length σ satisfying $|f(\sigma)| \leqslant c$ along γ. Then the universal covering manifold of M^n is compact, with diameter bound as in* (1.11), *and hence the fundamental group of M^n is finite.*

Remark. The curvature condition (1.10) is required to hold for all geodesics in M^n, not just minimal geodesics. Thus the hypotheses of Theorem 1.3 are stronger than that of Theorem 1.2. Theorem 1.3 is false if (1.10) is required to hold for minimal geodesics only. For example, the flat torus satisfies the hypotheses of Theorem 1.2 but, of course, does not have finite fundamental group.

Proof of Theorem 1.3. Let $\tilde{M}^n$ be the universal covering manifold of M^n. By insisting that the universal covering map $\psi: \tilde{M}^n \to M^n$ be a local isometry, $\tilde{M}^n$ is furnished with the same local differential geometry as M^n.

Now $\tilde{M}^n$ is complete since M^n is. Let $\tilde{\gamma}: [0, l] \to \tilde{M}^n$ be a minimal geodesic in $\tilde{M}^n$, parameterized by the arc length σ, joining $p = \gamma(0)$ and $q = \gamma(l)$. Since ψ is a local isometry, the curve $\gamma = \psi \circ \tilde{\gamma}: [0, l] \to M^n$ is a geodesic (not necessarily minimal) in M^n *also parameterized by the arc length.* Let $\tilde{\mathbf{T}}$ be the unit tangent to $\tilde{\gamma}$, and $\mathbf{T} = \psi_* \tilde{\mathbf{T}}$ be the unit tangent to γ, where we let ψ_* represent the differential of ψ.

According to the hypotheses there is a function $f: [0, l] \to R$ such that (1.10) holds and $|f(\sigma)| \leqslant c$ for $\sigma \in [0, l]$. But since $\tilde{M}^n$ and M^n have the same local differential geometry,

$$\operatorname{Ric}_{\tilde{M}^n}(\tilde{\mathbf{T}}, \tilde{\mathbf{T}}) = \operatorname{Ric}_{M^n}(\mathbf{T}, \mathbf{T}).$$

Hence

$$\operatorname{Ric}_{\tilde{M}^n}(\tilde{\mathbf{T}}, \tilde{\mathbf{T}}) \geqslant a + \frac{df}{d\sigma} \quad \text{along } \tilde{\gamma},$$

where f is a function of the arc length in $\tilde{M}^n$ satisfying $|f(\sigma)| \leqslant c$ along $\tilde{\gamma}$.

Thus $\tilde{M}^n$ satisfies the hypotheses of Theorem 1.2 and, consequently, is compact, with diameter bound as in (1.11). Hence $\tilde{M}^n$ must be a *finite-sheeted* cover of M^n, and $\pi_1(M^n)$ is a finite group.

2. The closure result of Friedmann cosmology

By a *space-time* we mean a smooth 4-dimensional manifold M^4 furnished with a pseudo-Riemannian metric

$$ds^2 = \sum_{i,j=0}^{3} g_{ij}\, dx^i\, dx^j$$

with signature $-+++$. Let $\langle\ ,\ \rangle$ denote the metric defined by the line element ds^2. A vector $\mathbf{V}$ is said to be time-like, light-like or space-like according as $\langle \mathbf{V}, \mathbf{V}\rangle < 0$, $\langle \mathbf{V}, \mathbf{V}\rangle = 0$ or $\langle \mathbf{V}, \mathbf{V}\rangle > 0$. The arc length along a time-like curve (i.e., a curve with time-like tangent) is called *proper time*.

In general relativity it is postulated that the metric components g_{ij} of a space-time obey the tensor equation

$$R_{ij} - \frac{1}{2} Rg + dij = 8\pi\kappa T_{ij}, \tag{2.1}$$

where R_{ij} is the Ricci tensor, R is the scalar curvature, κ is the universal gravitational constant, and T_{ij} is the energy-momentum tensor which characterizes the energy-momentum content of the space-time model.

The simplest cosmological models based on Einstein's general theory of relativity are the classical Friedmann models. These models have space-time topology $M^4 = R \times V^3$ and are *dust* filled, i.e., filled with a "collisionless" fluid characterized by a mass density ρ and a unit, time-like velocity field $\mathbf{u}$ which is orthogonal to each "spatial section" $V_t^3 \equiv \{t\} \times V^3$. The Friedmann models are characterized by stringent symmetry conditions; all observations made by an observer comoving with the cosmic fluid are assumed to be *locally* isotropic. As a consequence, the spatial sections V_t^3 are locally isotropic spaces in the induced metric and scalar fields such as the density ρ, and Hubble expansion parameter $h = \frac{1}{3}$ div $\mathbf{u}$ are constant on any section.

By solving the Einstein equations exactly we find that the curvature of the section V_t^3 has the same sign as the empirical quantity $8\pi\kappa\rho - 3h^2$. If

$$8\pi\kappa\rho - 3h^2 > 0 \tag{2.2}$$

holds on V_t^3, then V_t^3 is a space of constant *positive* curvature and, in fact, is covered by the 3-sphere if we assume that V_t^3 is complete. Thus V_t^3 is compact with finite diameter, i.e., the (spatial) universe is "finite." This is the closure result of Friedmann cosmology.

The geometric result to be presented in the next section, when interpreted cosmologically, represents a closure theorem analogous to the closure result of Friedmann cosmology for a vastly more general cosmological setting.

3. A general closure theorem

Let $\mathbf{u}$ be a smooth unit time-like vector field on a space-time M^4. The existence of such a vector field implies, in particular, that M^4 is time-orientable with $\mathbf{u}$ "pointing into the future." (When interpreting physically we should view M^4 as a model of the actual space-time universe, and $\mathbf{u}$ as the velocity field describing the average flow of matter (galaxies) in the universe.)

We say that $\mathbf{u}$ is *irrotational* if the covector ν, defined by $\nu(\mathbf{X}) = \langle \mathbf{u}, \mathbf{X} \rangle$ for all $\mathbf{X}$, has exterior derivative $d\nu$ which vanishes on all vectors orthogonal to $\mathbf{u}$. By using the formula

$$d\nu(\mathbf{X}, \mathbf{Y}) = \mathbf{X}\nu(\mathbf{Y}) - \mathbf{Y}\nu(\mathbf{X}) - \nu([\mathbf{X}, \mathbf{Y}]),$$

it follows immediately that $d\nu$ vanishes on all vectors orthogonal to $\mathbf{u}$ if and only if the bracket of two vectors orthogonal to $\mathbf{u}$ is still orthogonal to $\mathbf{u}$.

Thus by well-known theorems (see [7], for example), if $\mathbf{u}$ is irrotational, one can pass through any point of M^4 a maximal connected 3-dimensional submanifold V^3 orthogonal to $\mathbf{u}$. The conclusions of the closure theorem shall refer to such a V^3 (which physically, we may interpret as the "spatial universe at some moment in time"). No assumption of homogeneity or isotropy of the hypersurface V^3 in the induced *Riemannian* metric is imposed.

Let $\mathbf{X}$ be a vector field defined along a flow line generated by $\mathbf{u}$, and suppose that $\mathbf{X}$ is invariant under the flow generated by $\mathbf{u}$,

$$[\mathbf{X}, \mathbf{u}] = \nabla_{\mathbf{X}}\mathbf{u} - \nabla_{\mathbf{u}}\mathbf{X} = 0, \tag{3.1}$$

where [,] is the Lie bracket, and ∇ is the connection associated with the space-time metric. Let $\mathbf{X}^{\perp}$ be the projection of $\mathbf{X}$ onto the subspace orthogonal to $\mathbf{u}$, i.e.,

$$\mathbf{X}^{\perp} = \mathbf{X} + \langle \mathbf{X}, \mathbf{u} \rangle \mathbf{u}.$$

The proper time derivative $(d/ds)\|\mathbf{X}^{\perp}\|$, where $\|\mathbf{X}^{\perp}\|$ is the length of the projected vector $\mathbf{X}^{\perp}$, measures the rate at which "nearby" flow lines in the direction of $\mathbf{X}^{\perp}$ recede from the flow line along which $\mathbf{X}$ is defined. A positive derivative, $(d/ds)\|\mathbf{X}^{\perp}\| > 0$, indicates a recession of "nearby" flow lines in the direction of $\mathbf{X}^{\perp}$, and a negative second derivative, $(d^2/ds^2)\|\mathbf{X}^{\perp}\| < 0$, indicates a deceleration of the recession in the direction of $\mathbf{X}^{\perp}$. For a more thorough discussion of the kinematics of fluids, see [5] and references cited therein.

We are now in a position to state the closure theorem.

Theorem 3.1. *Let* $\mathbf{u}$ *be a smooth unit time-like vector field on a space-time* M^4. *Furthermore, suppose* $\mathbf{u}$ *is irrotational so that through any point of* M^4 *one can pass a maximal connected* 3*-dimensional submanifold of* M^4 *orthogonal to* $\mathbf{u}$. *Let* V^3 *be such a manifold, and suppose* V^3 *is complete in the induced Riemannian metric. Suppose, in addition, the following conditions hold on* V^3.

(I) *At each point* p *of* V^3

(i) *the flow generated by* $\mathbf{u}$ *is expanding in all directions, i.e.,*

$$\frac{d}{ds}\Big|_p \|\mathbf{X}^\perp\| > 0$$

for all vector fields X *defined along the flow line through* p *which satisfy* (3.1),

(ii) *the rate of expansion is decreasing in all directions, i.e.,*

$$\frac{d^2}{ds^2}\Big|_p \|\mathbf{X}^\perp\| \leqslant 0$$

for all vector fields $\mathbf{X}$ *as in* (i).

(II) *The Ricci curvature* Ric *of* M^4 *satisfies*

$$\inf_{V^3}\left(\mathrm{Ric}(\xi, \xi) - 3h^2\right) = \lambda > 0,$$

$$\|\xi\| = 1, \ \langle \xi, u\rangle = 0,$$

where $h = \frac{1}{3}\operatorname{div}\mathbf{u}$.

(III) *The flow lines are of bounded geodesic curvature on* V^3, *i.e.,*

$$\sup_{V^3}\|\nabla_{\mathbf{u}}\mathbf{u}\| = \mu < \infty.$$

Then V^3 *is compact and, in fact,*

$$\mathrm{diam}(V^3) \leqslant \frac{\pi}{\lambda}\left(\mu + \sqrt{\mu^2 + 2\lambda}\right). \tag{3.1}$$

Remarks. Conditions (I)(i) and (I)(ii) are certainly satisfied in the expansion phase of every Friedmann model, where, in fact, the expansion is isotropic. We emphasize, however, that conditions (I)(i) and (I)(ii) do not require that the expansion or rate of expansion be isotropic. If $\mathbf{u}$ is interpreted physically as the velocity field describing the average motion of galaxies in the universe, then according to (I)(i) and (I)(ii) there must be galaxy recession and deceleration of the recession in all directions about each point in the spatial universe. Experimental evidence suggests the occurence of this type of expansion behavior from our vantage point in the universe. Also, condition (I)(i) is equivalent to the assumption that the second fundamental form B on V^3, defined by

$$B(\mathbf{X}, \mathbf{Y}) = -\langle \nabla_{\mathbf{X}}\mathbf{u}, \mathbf{Y}\rangle$$

for vectors $\mathbf{X}$, $\mathbf{Y}$ tangent to V^3, be negative semi-definite.

It is worth pointing out that the statement and proof of Theorem 3.1 do not make any use of the Einstein equations; the theorem may be viewed as a purely geometric result. If, however, it is assumed that the metric tensor satisfies the Einstein equations (2.1), then the Ricci tensor may be expressed in terms of the energy-momentum tensor as

$$\operatorname{Ric}(\xi, \xi) = 8\pi\kappa\left(T(\xi, \xi) - \frac{1}{2}T\right), \tag{3.2}$$

for unit vectors ξ orthogonal to $\mathbf{u}$, where T is the trace of the energy-momentum tensor. In the case of a dust, characterized by a velocity field $\mathbf{u}$ and mass density ρ, the form of the energy-momentum tensor is well known, and the right-hand side of (3.2) equals $4\pi\kappa\rho$, which is half of the density term appearing in the crucial Friedmann condition (2.2). Thus condition (II) is a slightly strengthened analogue of (2.2).

In relativity, the covariant derivative $\nabla_{\mathbf{u}}\mathbf{u}$ is sometimes called the 4-acceleration. Condition (III) that the 4-acceleration be bounded on V^3 (but not necessarily on all of M^4) seems to be a physically reasonable condition. In the case of a dust the flow lines generated by $\mathbf{u}$ are necessarily geodesics and, hence, $\nabla_{\mathbf{u}}\mathbf{u} = 0$ on M^4 so that condition (III) is automatically satisfied.

4. Proof of the closure theorem

About each point of V^3 there exist a local coordinate neighborhood U^3 and a local coordinate t with values in the interval $(-\varepsilon, \varepsilon)$ such that

$$W = (-\varepsilon, \varepsilon) \times U^3,$$

where W is an open neighborhood of M^4, and $U_t^3 \equiv \{t\} \times U^3$ is orthogonal to $\mathbf{u}$. The neighborhood U_t^3 is obtained by following the flow generated by $\mathbf{u}$ through U^3 for coordinate time t. This existence of the coordinate t, which we shall refer to as a *synchronous* time coordinate, is guaranteed by the Frobenius theorem since $\mathbf{u}$ is assumed to be irrotational.

In W, the space-time metric can be expressed in the form

$$ds^2 = -\varphi^2 dt^2 + \sum_{\alpha,\beta=1}^{3} g_{\alpha\beta}(x, t)\, dx^\alpha\, dx^\beta, \tag{4.1}$$

where x^α are local coordinates introduced in U^3 and $\mathbf{u} = (1/\varphi)\partial/\partial t$.

In order to make use of Theorem 1.2 it is necessary to compute the Ricci curvature along geodesics in the hypersurface V^3 orthogonal to $\mathbf{u}$.

Lemma 4.1. *Let γ be a geodesic in V^3 with unit tangent ξ, and let σ be the arc length along γ. Extend ξ along the flow lines through γ by making it* *invariant under the flow generated by* $\mathbf{u}$. *Let* U^3 *be a coordinate neighborhood of* V^3 *which intersects the geodesic* γ, *and* t *be a synchronous time coordinate in the neighborhood* $(-\varepsilon, \varepsilon) \times U^3$ *of* M^4 *so that the space-time metric takes the form* (4.1). *Then the Ricci curvature of* V^3, Ric_V, *is related, along* γ *in* U^3, *to the Ricci curvature* Ric *of space-time by the equation*

$$\begin{aligned}\mathrm{Ric}_V(\xi, \xi) &= \mathrm{Ric}(\xi, \xi) - \frac{d^2}{ds^2}\|\xi^\perp\| + \left(\frac{d}{ds}\|\xi^\perp\|\right)^2 - \theta\frac{d}{ds}\|\xi^\perp\| \\ &\quad + \frac{d}{d\sigma}\langle \xi, \nabla_{\mathbf{u}}\mathbf{u}\rangle + \left(\frac{1}{\varphi}\frac{d\varphi}{d\sigma}\right)^2 + 2B^2(\xi, \mathbf{e}_2) + 2B^2(\xi, \mathbf{e}_3),\end{aligned} \tag{4.2}$$

where $\theta = \operatorname{div}\mathbf{u}$, B *is the second fundamental form on* V^3, s *is proper time, and* ξ, $\mathbf{e}_2$, $\mathbf{e}_3$ *are orthonormal vectors.*

The proof of Lemma 4.1 is computational in nature, and an outline of it is given in §5.

Now the goal is to show that the hypotheses of Theorem 1.2 hold on V^3 with $a = \lambda$ and $c = \mu$. Since V^3 is a complete Riemannian manifold, every pair of points can be joined by a minimal geodesic. Let γ: $[0, l] \to V^3$ be a minimal geodesic joining points $p = \gamma(0)$ and $q = \gamma(l)$, parameterized by the arc length σ. Let ξ be the unit tangent along γ, and ξ, ξ_2, ξ_3 be orthonormal vectors at p. Parallelly transport ξ_2, ξ_3 along γ so that ξ, ξ_2, ξ_3 remain orthonormal along γ. Extend ξ, ξ_2, ξ_3 along the flow in the usual way be making them invariant under the flow generated by $\mathbf{u}$. Let $\mathbf{e}_1 = \xi^\perp$ and $\mathbf{e}_i = \xi_i^\perp$, $i = 2, 3$.

Introduce the notation

$$\|\mathbf{e}_\alpha\|^{\boldsymbol{\cdot}} = \frac{d}{ds}\|\mathbf{e}_\alpha\|, \quad \alpha = 1, 2, 3.$$

Then

$$\begin{aligned}\theta = \operatorname{div}\mathbf{u} &= \sum_{\alpha=1}^{3}\langle \nabla_{\mathbf{e}_\alpha}\mathbf{u}, \mathbf{e}_\alpha\rangle - \langle \nabla_{\mathbf{u}}\mathbf{u}, \mathbf{u}\rangle = \sum_{\alpha=1}^{3}\langle \nabla_{\mathbf{e}_\alpha}\mathbf{u}, \mathbf{e}_\alpha\rangle \\ &= \sum_{\alpha=1}^{3}\|\mathbf{e}_\alpha\|^{\boldsymbol{\cdot}} \quad \text{along } \gamma,\end{aligned} \tag{4.3}$$

since it can be shown that $\|\mathbf{e}_\alpha\|^{\boldsymbol{\cdot}} = \langle \nabla_{\mathbf{e}_\alpha}\mathbf{u}, \mathbf{e}_\alpha\rangle$ along γ. Thus,

$$\begin{aligned}-\left[\frac{d}{ds}\|\xi^\perp\|^2 - \theta\frac{d}{ds}\|\xi^\perp\|\right] &= -(\|\mathbf{e}_1\|^{\boldsymbol{\cdot}})^2 + \theta\|\mathbf{e}_1\|^{\boldsymbol{\cdot}} \\ &= \|\mathbf{e}_1\|^{\boldsymbol{\cdot}}\|\mathbf{e}_2\|^{\boldsymbol{\cdot}} + \|\mathbf{e}_1\|^{\boldsymbol{\cdot}}\|\mathbf{e}_3\|^{\boldsymbol{\cdot}} \\ &\leqslant \sum_{\alpha<\beta}\|\mathbf{e}_\alpha\|^{\boldsymbol{\cdot}}\|\mathbf{e}_\beta\|^{\boldsymbol{\cdot}},\end{aligned} \tag{4.4}$$

since, by (I)(i), $\|\mathbf{e}_\alpha\|^{\cdot} \geqslant 0$. Now, by the Schwartz inequality,

$$\sum_{\alpha<\beta} \|\mathbf{e}_\alpha\|^{\cdot}\|\mathbf{e}_\beta\|^{\cdot} = \frac{1}{2}\left[\left(\sum_\alpha \|\mathbf{e}_\alpha\|^{\cdot}\right)^2 - \sum_\alpha (\|\mathbf{e}_\alpha\|^{\cdot})^2\right]$$

$$\leqslant \frac{1}{2}\left[\left(\sum_\alpha \|\mathbf{e}_\alpha\|^{\cdot}\right)^2 - \frac{1}{3}\left(\sum_\alpha \|\mathbf{e}_\alpha\|^{\cdot}\right)^2\right] \tag{4.5}$$

$$= \frac{1}{3}\left(\sum_\alpha \|\mathbf{e}_\alpha\|^{\cdot}\right)^2 = \frac{1}{3}\theta^2 = 3h^2.$$

The inequalities (4.4) and (4.5) may be combined to give

$$\left(\frac{d}{ds}\|\xi^\perp\|\right)^2 - \theta\frac{d}{ds}\|\xi^\perp\| \geqslant -3h^2. \tag{4.6}$$

Furthermore, by assumption (I) (ii) the inequality

$$\frac{d^2}{ds^2}\|\xi^\perp\| \leqslant 0 \tag{4.7}$$

holds.

It now follows from (4.3), (4.6) and (4.7) that

$$\mathrm{Ric}_V(\xi, \xi) \geqslant \mathrm{Ric}(\xi, \xi) - 3h^2 + \frac{df}{d\sigma}, \tag{4.8}$$

where, for each $\sigma_0 \in [0, l]$,

$$f(\sigma) = \langle \xi, \nabla_{\mathbf{u}}\mathbf{u}\rangle(\sigma).$$

Since, by assumption III,

$$|\langle \xi, \nabla_{\mathbf{u}}\mathbf{u}\rangle| \leqslant \|\nabla_{\mathbf{u}}\mathbf{u}\| \leqslant \mu,$$

we have

$$|f(\sigma)| \leqslant \mu \quad \text{for } 0 \leqslant \sigma \leqslant l. \tag{4.9}$$

Using assumption (II) we reduce (4.8) to

$$\mathrm{Ric}_V(\xi, \xi) \geqslant \lambda + \frac{df}{d\sigma} \quad \text{along } \gamma, \tag{4.10}$$

where f satisfies (4.9). Theorem 3.1 now follows from Theorem 1.2.

Remark. The derivation of (4.10) with f satisfying (4.9) makes use of the fact that γ is a geodesic but not that it is minimal, i.e., inequality (4.10) holds along *all* geodesics. This observation together with Theorem 1.3 establishes the following.

Theorem 4.1. *If the hypotheses of Theorem* 3.1 *are satisfied, then the fundamental group of V^3 is finite.*

Remark. As a final remark we note that it would be desirable to obtain a generalization of the closure result presented here which does not require the irrotation assumption. If the approach of this paper is to be used, this problem amounts to finding the appropriate generalization of (4.2).

5. Outline of the proof of Lemma 4.2

The proof makes use of the following known result, a proof of which can be found in [4].

Lemma 5.1. *The* 4-*acceleration* $\nabla_{\mathbf{u}}\mathbf{u}$ *is related to the function* φ *(appearing in* (4.1)) *by the equation*

$$\nabla_{\mathbf{u}}\mathbf{u} = \operatorname{grad}_V \log \varphi,$$

where grad_V *is the gradient operator on* V^3 *in the induced metric* $g_{\alpha\beta}$.

The setting is as in Lemma 4.2. Let $\mathbf{e}_1 = \xi^{\perp}$. Then a computation, using Lemma 5.1, shows that $\mathbf{e}_1$ is invariant under the flow generated by $\partial/\partial t$. Let $\mathbf{e}_2$, $\mathbf{e}_3$, together with $\mathbf{e}_1$, form an orthonormal triad of vectors along γ and tangent to V^3. Denote by $K(\mathbf{e}_i \wedge \mathbf{e}_j)$ and $K_V(\mathbf{e}_i \wedge \mathbf{e}_j)$ the sectional curvatures of M^4 and V^3 respectively associated with the plane spanned by $\mathbf{e}_i$ and $\mathbf{e}_j$.

The Ricci curvatures may be expressed as sums of sectional curvatures, namely,

$$\operatorname{Ric}(\xi, \xi) = \sum_{j=0}^{3} K(\mathbf{e}_1 \wedge \mathbf{e}_j), \tag{5.1}$$

$$\operatorname{Ric}_V(\xi, \xi) = \sum_{j=1}^{3} K_V(\mathbf{e}_1 \wedge \mathbf{e}_j), \tag{5.2}$$

where by definition $K(\mathbf{e}_1 \wedge \mathbf{e}_1) = K_V(\mathbf{e}_1 \wedge \mathbf{e}_1) = 0$ and $\mathbf{e}_0 = \mathbf{u}$. By using the Gauss equations

$$K(\mathbf{e}_1 \wedge \mathbf{e}_j) = K_V(\mathbf{e}_1 \wedge \mathbf{e}_j) - B^2(\mathbf{e}_1, \mathbf{e}_j) + B(\mathbf{e}_1, \mathbf{e}_1)B(\mathbf{e}_j, \mathbf{e}_j), i = 1, 2,$$

and the identity

$$\theta = \sum_{j=1}^{3} B(\mathbf{e}_j, \mathbf{e}_j),$$

we can combine (5.1) and (5.2) to give

$$\operatorname{Ric}_V(\xi, \xi) = \operatorname{Ric}(\xi, \xi) - K(\mathbf{e}_1 \wedge \mathbf{u}) + \theta B(\mathbf{e}_1, \mathbf{e}_1) + \sum_{j=1}^{3} B^2(\mathbf{e}_1, \mathbf{e}_j). \tag{5.3}$$

Now using the definition of sectional curvature we have

$$\begin{aligned} K(\mathbf{e}_1 \wedge \mathbf{u}) &= -\langle R(\mathbf{e}_1, \mathbf{u})\mathbf{u}, \mathbf{e}_1\rangle \\ &= -\langle \nabla_{\mathbf{e}_1}\nabla_{\mathbf{u}}\mathbf{u}, \mathbf{e}_1\rangle + \langle \nabla_{\mathbf{u}}\nabla_{\mathbf{e}_1}\mathbf{u}, \mathbf{e}_1\rangle + \langle \nabla_{[\mathbf{e}_1,\mathbf{u}]}\mathbf{u}, \mathbf{e}_1\rangle. \end{aligned}$$

Calculations, together with the use of Lemma 5.1, yield the equalities

$$\langle \nabla_{\mathbf{e}_1} \nabla_{\mathbf{u}} \mathbf{u}, \mathbf{e}_1 \rangle = \frac{d}{d\sigma} \langle \xi, \nabla_{\mathbf{u}} \mathbf{u} \rangle,$$

$$\langle \nabla_{\mathbf{u}} \nabla_{\mathbf{e}_1} \mathbf{u}, \mathbf{e}_1 \rangle = \frac{1}{2} \frac{d^2}{d\sigma^2} \|\mathbf{e}_1\|^2 - \sum_{j=1}^{3} B^2(\mathbf{e}_1, \mathbf{e}_j),$$

$$\langle \nabla_{[\mathbf{e}_1, \mathbf{u}]} \mathbf{u}, \mathbf{e}_1 \rangle = -\left(\frac{1}{\varphi} \frac{d\varphi}{ds} \right)^2$$

along γ. Thus

$$K(\mathbf{e}_1 \wedge \mathbf{u}) = \frac{1}{2} \frac{d^2}{ds^2} \|\mathbf{e}_1\|^2 - \frac{d}{d\sigma} \langle \xi, \nabla_{\mathbf{u}} \mathbf{u} \rangle - \left(\frac{1}{\varphi} \frac{d}{ds} \right)^2 + \sum_{j=1}^{3} B^2(\mathbf{e}_1, \mathbf{e}_j). \tag{5.4}$$

By substituting (5.4) into (5.3) and making use of the identities

$$\frac{d^2}{ds^2} \|\mathbf{e}_1\|^2 = 2 \left[\left(\frac{d}{ds} \|\mathbf{e}_1\| \right)^2 + \frac{d^2}{ds^2} \|\mathbf{e}_1\| \right],$$

$$B(\mathbf{e}_1, \mathbf{e}_1) = -\frac{d}{ds} \|\mathbf{e}_1\| \quad \text{along } \gamma,$$

we obtain the desired result.

Remark. Recently we were made aware of a paper of W. Ambrose [1] in which a *qualitative* generalization of Myers theorem is obtained. Although this result may be used (via Lemma 4.1) to prove the compactness of V^3, it does not give a bound on the diameter.

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