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EIGENVALUES OF THE LAPLACIAN AND UNIQUENESS IN THE MINKOWSKI PROBLEM

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Introduction

Throughout this paper by a convex hypersurface in Euclidean space E^{m+1}, $m \geqslant 2$, we mean a subdomain of a closed convex three times differentiable hypersurface with strictly positive Gaussian curvature.

Let S' and S'' be two convex hypersurfaces satisfying the following conditions:

(a) they have a common spherical image ω on the unit hypersphere Σ; the boundary of ω consists of a finite number of piecewise smooth $(m-1)$-dimensional manifolds homeomorphic to the $(m-1)$-dimensional sphere;

(b) the products of the principal radii of curvature of S' and S'' have equal values at the points with the same unit exterior normal;

(c) their support functions coincide on the boundary of the spherical image.

One of the versions of the well-known uniqueness theorem in the Minkowski problem says that when S' and S'' are closed (conditions (a) and (c) in this case are omitted), they are equal up to a translation [7]. When S' and S'' are open, satisfy (a), (b), (c), and their spherical image ω is contained in a hemisphere, Alexandrov [1] proved that the hypersurfaces coincide. In the same paper he also conjectured that the uniqueness theorem fails when ω is not contained in a hemisphere.

The purpose of this paper is to present the two following theorems.

Theorem A. *Let S' and S'' be two convex hypersurfaces satisfying conditions* (a), (b), *and* (c). *Then, if ω is contained in a hemisphere or contains a hemisphere, the hypersurfaces coincide.*

Theorem B. *There exists a domain ω on a unit hypersphere (neither containing nor contained in a hemisphere), such that two convex hypersurfaces having ω as their spherical image and satisfying conditions* (a), (b) *and* (c) *may not be translation equivalent.*

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It is interesting to note that under boundary conditions different from (c), the uniqueness theorem still holds. Namely, two results are known to us on this matter. Stoker [8] considered convex surfaces of class C^3 in E^3 with a finite number of holes with the following property: each one is bounded by a convex closed plane curve having continuous third derivatives such that the plane of the curve is tangent to the surface all along the edge of the hole. He has shown that such surfaces are uniquely defined (up to a translation) by the product of the principal radii of curvature given as a function of the normal to the surface.

Hsiung [4] has proved the following result: Let S' and S'' be two convex surfaces of class C^2 with boundaries C' and C'' in E^3. Suppose that there is a differentiable homeomorphism H of the surface S' onto the surface S'' such that at corresponding points the two surfaces S' and S'' have the same unit inner normal vectors and equal Gaussian curvatures. If the homeomorphism H restricted to the boundary C' is a translation carrying the boundary C' onto the boundary C'', then the homeomorphism H is a translation carrying the whole surface S' onto the whole surface S''.

The proofs of both Theorems A and B are based on the study of the first two eigenvalues of an elliptic boundary value problem to which the uniqueness problem for open hypersurfaces can be reduced. In its turn, this boundary value problem is considered as a "perturbation" of an eigenvalue problem corresponding to the uniqueness problem for convex closed hypersurfaces. The eigenvalues of the last problem are actually known.

Finally we wish to notice that in the case where $m = 2$, Theorem A was the subject of our paper [6].

1. Preliminaries

In what follows we preserve the notation from the introduction. In addition, we use $u^1, u^2, \cdots, u^m$ to represent local coordinates of a point $n \in \Sigma$, and n also denotes the unit vector whose origin is at the center of Σ.

In this section, unless otherwise stated, it is not supposed that the hypersurfaces under consideration are convex.

Let F be a C^2 hypersurface in E^{m+1} defined as the envelope of an m-parameter family of hyperplanes with normal equations

$$rn = h(n),$$

where r is the position vector of F, and n is the unit vector of exterior normal at the point r. The function $h(n)$ determines F, and it is called a support function of F. If the Gaussian curvature of F does not vanish, then h is of class C^2 [3]. Let F be a hypersurface with nonzero Gaussian curvature, and let b_{ij} and $g_{ij}(i, j = 1, 2, \cdots, m)$ be the second and third fundamental tensors of F. The principal radii of curvature $R_1, R_2, \cdots, R_m$ are the roots of equation

$$\det(b_{ij} + Rg_{ij}) = 0,$$

and

$$R_1 R_2 \cdots R_m = \frac{b}{g},$$

where $b = \det(b_{ij})$, $g = \det(g_{ij})$.

Since the components of the second fundamental tensor can be expressed as $-b_{ij} = \nabla_{ij}h + g_{ij}h$, where $\nabla_{ij}h$ are the second covariant derivatives of h in metric g_{ij} on Σ we have

$$R_1 R_2 \cdots R_m = (-1)^m \frac{1}{g} \det(\nabla_{ij}h + g_{ij}h).$$

Proposition 1. *Let h', b'_{ij} and h'', b''_{ij} be the support functions and the second fundamental tensors of hypersurfaces F' and F'' of class C^3. Suppose that F' and F'' have a common spherical image ω on the unit hypersphere Σ, and their products of the principal radii of curvature have equal values at the points with the same unit normal. Then the difference $h = h'' - h'$ satisfies a linear formally self-adjoint differential equation*

$$(1) \qquad (\Delta + Q)h \equiv \frac{1}{\sqrt{g}} \sum_{i=1}^{m} \frac{\partial}{\partial u^i} \left(\sum_{j=1}^{m} \frac{\tilde{b}^{ij}}{\sqrt{g}} \frac{\partial h(n)}{\partial u^j} \right) + \frac{1}{g} \sum_{i,j=1}^{m} \tilde{b}^{ij} g_{ij} h(n) = 0,$$

$$n \in \omega,$$

where

$$\tilde{b}^{ij} = \int_0^1 b^{ij}(t)\, dt,$$

and $b^{ij}(t)$ is the cofactor of the element

$$b_{ij}(t) = (1 - t)b'_{ij} + t, b''_{ij}, t \in [0, 1].$$

Proof. At first we show that h satisfies a linear differential equation

$$(2) \qquad \frac{1}{g} \sum_{i,j=1}^{n} \tilde{b}^{ij}(\nabla_{ij}h + g_{ij}h) = 0.$$

In fact, this equation is implicitly contained in Alexandrov's paper [2], where an analogous equation is derived under more general circumstances. However, the particular form of this equation is of special importance to us. Set

$$\phi(t) = \frac{1}{g} \det(b_{ij}(t)).$$

Since F' and F'' have equal products of the principal radii of curvature at the points with the same unit normal, we have $\phi(1) - \phi(0) = 0$, and therefore

$$\int_0^1 \frac{d\phi(t)}{dt}\, dt = \phi(1) - \phi(0) = 0.$$

On the other hand,

$$\frac{d\phi(t)}{dt} = \frac{1}{g} \sum_{i,j=1}^{n} b^{ij}(t)(b''_{ij} - b'_{ij}).$$

Finally, putting

$$\tilde{b}^{ij} = \int_0^1 b^{ij}(t)\, dt,$$

and noting that

$$b''_{ij} - b'_{ij} = -(\nabla_{ij}h + g_{ij}h),$$

we obtain (2).

To show that (2) can be presented in the form (1), it is sufficient to show that

$$\frac{1}{\sqrt{g}} \sum_{i=1}^{n} \frac{\partial}{\partial u^i}\left(\sum_{j=1}^{n} \frac{b^{ij}(t)}{\sqrt{g}} \frac{\partial h}{\partial u^j} \right) = \sum_{i,j=1}^{n} \frac{b^{ij}(t)}{g} \nabla_{ij}h.$$

This follows from simple computations in combination with the Codazzi equations. Hence the proposition is proved.

Proposition 2. *Let S' and S'' be two hypersurfaces as described in Proposition 1. If, in addition, S' and S'' are convex hypersurfaces, then the operator $\Delta + Q$ is defined and uniformly elliptic on the entire hypersphere Σ.*

That $\Delta + Q$ is defined on Σ follows from the fact that S' and S'' are subdomains of closed convex hypersurfaces. The other part of the proposition is a well-known theorem, whose proof can be found, for example, in [2].

2. Proofs of Theorems A and B

In this section S' and S'' are supposed to be convex hypersurfaces satisfying conditions (a), (b), and (c) in the Introduction. Under these hypotheses the question of whether S' and S'' are translation equivalent or not, reduces, by virtue of Proposition 1, to the question of uniqueness for the following problem:

$$\Delta h + Qh = 0, \quad n \in \omega,$$
$$h|_{\partial\omega} = 0.$$

It is easy to see that two hypersurfaces with the same support function always coincide except for two cases: when the spherical image is a hemisphere or when it is a whole hypersphere. In the second case the hypersurfaces may differ by a translation, and in the first case this translation is possible only in the direction perpendicular to the hyperplane of the equator bounding the hemisphere.

Because of Proposition 2, $\Delta + Q$ is defined everywhere on Σ, and we can consider the following eigenvalue problems with respect to λ:

(3) $$\Delta h + \lambda Q h = 0, \quad n \in \Sigma;$$

(4) $$\Delta h + \lambda Q h = 0, \quad n \in \omega,$$

(5) $$h|_{\partial\omega} = 0;$$

(6) $$\Delta h + \lambda Q h = 0, \quad n \in \Sigma^+,$$

(7) $$h|_{\partial\Sigma^+} = 0,$$

where Σ^+ is a hemisphere of Σ.

In the case where $\lambda = 1$, problem (3) corresponds to the classical Minkowski problem for closed convex hypersurfaces, and it is known (see, e.g., [7]) that $\lambda = 1$ is an eigenvalue of multiplicity $m + 1$. The appropriate normalized eigenfunctions are the components of the unit vector n. Obviously, $\lambda = 0$ is an eigenvalue of (3) of multiplicity 1 with eigenfunction $h \equiv 1$. We interpolate linearly between (3) and the equation $\nabla^2 h + \lambda m h = 0$, $n \in \Sigma$, where ∇^2 is the Laplacian on Σ. Since eigenvalues depend continuously on the parameter of interpolation, it follows that $\lambda = 0$ and $\lambda = 1$ are the first and the second eigenvalues throughout the interpolation. Now we observe that $\lambda = 1$ is an eigenvalue for the problem (6), (7), and the corresponding eigenfunction is positive. (In a suitably chosen spherical coordinates $\theta_1, \theta_2, \cdots, \theta_m, -\frac{\pi}{2} \leqslant \theta_k \leqslant \frac{\pi}{2}$, $k \leqslant m - 1$, $0 \leqslant \theta_m \leqslant 2\pi$, the eigenfunction is $\sin \theta_1$.) From this follows [5] that $\lambda = 1$ is the first eigenvalue, and a simple one. If $\omega \subset \Sigma^+$, then, because of continuous monotonicity of the eigenvalues of a self-adjoint elliptic operator [9], the first eigenvalue λ of the problem (4), (5) is greater than 1. For the same reason, when $\omega \supset \Sigma^+$, $\omega \neq \Sigma$, the first eigenvalue of (4), (5) is greater than zero but less than 1, and Theorem A follows.

Now let N and S be the endpoints of a diameter of the hypersphere Σ, and let B_N^ε and B_S^δ be open m-balls in Σ with centers N and S and radii ε and δ respectively. Let $\omega_\delta = \Sigma \setminus (B_N^\varepsilon \cup B_S^\delta)$, and let λ_1 be the first eigenvalue of the problem (4), (5) for $\omega = \omega_\delta$. Since the first eigenvalue for problem (3) is zero, ε and δ can be selected sufficiently small so that $\lambda_1 < 1$. By increasing δ until B_S^δ becomes a hemisphere we obtain $\lambda_1 > 1$ (Theorem A). Therefore there exists a δ such that $\lambda_1 = 1$ when $\omega = \omega_\delta$. The corresponding eigenfunction does not represent a parallel translation, since a function $h = cn$, where c is a constant vector $\neq 0$, cannot be a solution of (3), (4) in ω. Thus we obtain Theorem B.

3. Conclusion remarks

1. As it can be seen from the proof of Theorem B, the domain ω is of a "belt" shape and contains the equator of the hypersphere. Note that similar to the proof of Theorem A one establishes uniqueness in the domains either containing or contained in the domain ω from Theorem B.

2. In Theorem B we were unable to indicate the domain ω explicitly; only the existence has been proved. However, it can be shown, as in [10], that when the operator $\Delta \equiv \nabla^2$ and $Q \equiv m$, the boundary of ω is defined by the equation (in the spherical coordinates introduced in the Proof of Theorem A),

$$\sin \theta_1 \int \cos^{1-m}\theta_1 \sin^{-2}\theta_1 \, d\theta_1 = 0. \tag{8}$$

In the particular case, where $m = 2$, this equation becomes

$$\sin \theta_1 \ln \tan\left(\frac{\pi}{4} - \frac{\theta_1}{2}\right) = -1.$$

The hypothesis is that in the general case of operator $\Delta + Q$ the domain ω is defined by the same equation (8).

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