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KAEHLERIAN MANIFOLDS WITH CONSTANT SCALAR CURVATURE ADMITTING A HOLOMORPHICALLY PROJECTIVE VECTOR FIELD

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To Professor C. C. Hsiung on his sixtieth birthday

1. Introduction

Let M be a connected Kaehlerian manifold of complex dimension n covered by a system of real coordinate neighborhoods $\{U; x^h\}$, where, here and in the sequel the indices $h, i, j, k, \ldots$ run over the range $\{1, 2, \ldots, 2n\}$, and let g_{ji}, F_i^h, $\{{}_j{}^h{}_i\}$, ∇_i, $K_{kji}{}^h$, K_{ji} and K be the Hermitian metric tensor, the complex structure tensor, the Christoffel symbols formed with g_{ji}, the operator of covariant differentiation with respect to $\{{}_j{}^h{}_i\}$, the curvature tensor, the Ricci tensor and the scalar curvature of M respectively.

A vector field v^h is called a *holomorphically projective* (or *H-projective*, for brevity) vector field [1], [2], [5] if it satisfies

$$(1.1)\quad \mathfrak{L}_v\{{}_j{}^h{}_i\} = \nabla_j\nabla_i v^h + v^k K_{kji}{}^h = \rho_j\delta_i^h + \rho_i\delta_j^h - \rho_s F_j^s F_i^h - \rho_s F_i^s F_j^h$$

for a certain covariant vector field ρ_j on M called the *associated* covariant vector field of v^h, where $\mathfrak{L}_v$ denotes the operator of Lie derivation with respect to v^h. In particular, if ρ_j is the zero-vector field, then v^h is called an *affine* vector field.

When we refer in the sequel to an H-projective vector field v^h, we always mean by ρ_j the associated covariant vector field appearing in (1.1).

In the present paper, we first prove a series of integral inequalities in a Kaehlerian manifold with constant scalar curvature admitting an H-projective vector field, and then find necessary and sufficient conditions for such a Kaehlerian manifold to be isometric to a complex projective space with Fubini-Study metric.

In the sequel, we need the following theorem due to Obata [4]. (See also [3].)

Theorem A. *Let M be a complete connected and simply connected Kaehlerian manifold. In order for M to admit a nontrivial solution φ of a system*

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of partial differential equations

$$\nabla_j \nabla_i \varphi_h + \frac{c}{4}(2\varphi_j g_{ih} + \varphi_i g_{jh} + \varphi_h g_{ji} - F_{ji} F_h{}^s \varphi_s - F_{jh} F_i{}^s \varphi_s) = 0 \tag{1.2}$$

with a constant $c > 0$, where $\varphi_h = \nabla_h \varphi$ and $F_{ji} = F_j{}^t g_{ti}$, it is necessary and sufficient that M be isometric to a complex projective space CP^n *with Fubini-Study metric and of constant holomorphic sectional curvature c.*

We assume in this paper that the Kaehlerian manifold under consideration is connected.

2. Preliminaries

Let M be a Kaehlerian manifold of complex dimension n. The complex structure tensor $F_i{}^h$ and the Hermitian metric tensor g_{ji} satisfy

$$F_i{}^h F_j{}^i = -\delta_j^h, \ \nabla_j F_i{}^h = 0, \ \nabla_j F_{ih} = 0, \tag{2.1}$$

$$F_j{}^s g_{si} + F_i{}^s g_{js} = 0. \tag{2.2}$$

(2.2) is equivalent to

$$g_{ji} - F_j{}^t F_i{}^s g_{ts} = 0. \tag{2.3}$$

We have [5], for the curvature tensor $K_{kji}{}^h$,

$$F_s{}^h K_{kji}{}^s - F_i{}^s K_{kjs}{}^h = 0, \tag{2.4}$$

or equivalently

$$K_{kji}{}^h + F_i{}^t F_s{}^h K_{kjt}{}^s = 0, \tag{2.5}$$

$$F_h{}^s K_{kjis} + F_i{}^s K_{kjsh} = 0, \tag{2.6}$$

or

$$K_{kjih} - F_i{}^t F_h{}^s K_{kjts} = 0, \tag{2.7}$$

where $K_{kjih} = K_{kji}{}^t g_{th}$.

Using (2.4) and the identity

$$K_{kji}{}^h + K_{ikj}{}^h + K_{jik}{}^h = 0,$$

we obtain

$$F_s{}^h K_i^s = g^{ut} F_s{}^h K_{iut}{}^s = F^{ts} K_{its}{}^h = \tfrac{1}{2} F^{ts}(K_{its}{}^h - K_{ist}{}^h) = -\tfrac{1}{2} F^{ts} K_{tsi}{}^h,$$

where g^{ji} are contravariant components of g_{ji} and $F^{ts} = g^{ti} F_i{}^s$, that is,

$$F_s{}^h K_i^s = -\tfrac{1}{2} F^{kj} K_{kji}{}^h, \tag{2.8}$$

from which it follows that

$$F_i{}^s K_{hs} = -\tfrac{1}{2} F^{kj} K_{kjih}. \tag{2.9}$$

For the Ricci tensor K_{ji}, from (2.8) we have

$$F_i^s K_s^h - F_s^h K_i^s = 0, \tag{2.10}$$

or equivalently

$$K_i^h + F_i^t F_s^h K_t^s = 0. \tag{2.11}$$

Similarly, from (2.9) we have

$$F_j^s K_{si} + F_i^s K_{js} = 0, \tag{2.12}$$

or equivalently

$$K_{ji} - F_j^t F_i^s K_{ts} = 0. \tag{2.13}$$

A vector field u^h on M is said to be *contravariant analytic* if

$$F_j^s \nabla_s u_i + F_i^s \nabla_j u_s = 0, \tag{2.14}$$

or equivalently

$$\nabla_j u_i - F_j^t F_i^s \nabla_t u_s = 0, \tag{2.15}$$

where $u_i = g_{ih} u^h$. Since

$$\mathfrak{L}_u F_i^h = -F_i^s \nabla_s u^h + F_s^h \nabla_i u^s = -\left(F_i^t \nabla_t u_s + F_s^t \nabla_i u_t\right) g^{sh},$$

a vector field u^h on M is contravariant analytic if and only if

$$\mathfrak{L}_u F_i^h = 0 \tag{2.16}$$

holds, where $\mathfrak{L}_u$ denotes the operator of Lie derivation with respect to u^h. It is known [5] that if M is compact, then a necessary and sufficient condition for a vector field u^h on M to be contravariant analytic is that

$$\nabla^j \nabla_j u^h + K_i^h u^i = 0 \tag{2.17}$$

holds, where $\nabla^j = g^{ji} \nabla_i$.

For an H-projective vector field v^h on M defined by (1.1), we have

$$\nabla_j \nabla_s v^s = 2(n+1)\rho_j, \tag{2.18}$$

$$\nabla^j \nabla_j v^h + K_i^h v^i = 0. \tag{2.19}$$

(2.18) shows that the associated covariant vector field ρ_j is gradient. Putting

$$\rho = \frac{1}{2(n+1)} \nabla_s v^s \tag{2.20}$$

we have

$$\rho_j = \nabla_j \rho. \tag{2.21}$$

If an H-projective vector field v^h on M is contravariant analytic, then substituting (1.1) in the well-known formula [5], [6]

$$\mathfrak{L}_v K_{kji}{}^h = \nabla_k \mathfrak{L}_v \left\{ {}_j{}^h{}_i \right\} - \nabla_j \mathfrak{L}_v \left\{ {}_k{}^h{}_i \right\}$$

and using a straightforward computation we find

$$(2.22)\qquad \mathfrak{L}_v K_{kji}{}^h = -\delta_k^h \nabla_j \rho_i + \delta_j^h \nabla_k \rho_i + \left(F_k{}^h \nabla_j \rho_s - F_j{}^h \nabla_k \rho_s\right) F_i^s + \left(F_k{}^s \nabla_j \rho_s - F_j^s \nabla_k \rho_s\right) F_i^h,$$

from which by contracting with respect to h and k we obtain

$$(2.23)\qquad \mathfrak{L}_v K_{ji} = -2n \nabla_j \rho_i - 2 F_j^t F_i^s \nabla_t \rho_s.$$

A Kaehlerian manifold M has the constant holomorphic sectional curvature k if and only if

$$(2.24)\qquad K_{kji}{}^h = \frac{k}{4}\left(\delta_k^h g_{ji} - \delta_j^h g_{ki} + F_k{}^h F_{ji} - F_j^h F_{ki} - 2F_{kj} F_i^h\right).$$

We define tensor fields G_{ji} and $Z_{kji}{}^h$ on M by

$$(2.25)\qquad G_{ji} = K_{ji} - \frac{K}{2n} g_{ji},$$

$$(2.26)\qquad Z_{kji}{}^h = K_{kji}{}^h - \frac{K}{4n(n+1)}\left(\delta_k^h g_{ji} - \delta_j^h g_{ki} + F_k{}^h F_{ji} - F_j^h F_{ki} - 2F_{kj} F_i^h\right)$$

respectively. We then easily see that the tensor fields G_{ji} and $Z_{kji}{}^h$ satisfy

$$(2.27)\qquad G_{ji} = G_{ij},\ G_{ji} g^{ji} = 0,\ Z_{tji}{}^t = G_{ji},$$

$$(2.28)\qquad Z_{kjih} = -Z_{jkih},\ Z_{kjih} = Z_{ihkj},$$

$$(2.29)\qquad Z_{kji}{}^h + Z_{ikj}{}^h + Z_{jik}{}^h = 0,$$

where $Z_{kjih} = Z_{kji}{}^t g_{th}$. If $G_{ji} = 0$, then M is a Kaehler-Einstein manifold and K is a constant provided $n > 1$; if $Z_{kji}{}^h = 0$, then M is of constant holomorphic sectional curvature $K/n(n+1)$ provided $n > 1$.

3. Lemmas

In this section, we prove some lemmas which we need in the next section.

Lemma 1. *If an H-projective vector field v^h on a Kaehlerian manifold M of complex dimension $n > 1$ is contravariant analytic, then the associated vector field ρ^h is also contravariant analytic, and*

$$(3.1)\qquad \mathfrak{L}_v K_{ji} = -2(n+1) \nabla_j \rho_i,$$

where $\rho^h = \rho_i g^{ih}$.

Proof. Applying the operator $\mathfrak{L}_v$ of Lie derivation with respect to v^h to both sides of (2.13) and using $\mathfrak{L}_v F_i^h = 0$, we have

$$\mathfrak{L}_v K_{ji} = F_j^t F_i^s \mathfrak{L}_v K_{ts},$$

from which together with (2.23) we see that ρ^h is contravariant analytic and (3.1) holds.

Lemma 2. *If a Kaehlerian manifold M is compact, then an H-projective vector field v^h on M is contravariant analytic, and consequently $\mathfrak{L}_v F_i^h = 0$. Moreover, if $n > 1$, then the associated vector field ρ^h is contravariant analytic.*

Proof of this lemma is easy and therefore omitted.

Lemma 3. *For a contravariant analytic H-projective vector field v^h on a Kaehlerian manifold M with constant scalar curvature K of complex dimension $n > 1$, we have*

$$\mathfrak{L}_v G_{ji} = -\nabla_j w_i - \nabla_i w_j, \tag{3.2}$$

where we have put

$$w^h = (n+1)\rho^h + \frac{K}{2n} v^h, \tag{3.3}$$

and $w_i = g_{ih} w^h$.

Proof. This follows from (2.25), (3.1) and the fact that ρ_j is gradient, that is, $\rho_j = \nabla_j \rho$.

Lemma 4. *For an H-projective vector field v^h on a compact Kaehlerian manifold M, we have*

$$\int_M \rho f \, dV = -\frac{1}{2(n+1)} \int_M \mathfrak{L}_v f \, dV \tag{3.4}$$

for any real function f on M, where dV denotes the volume element of M, and ρ is the function defined by (2.20).

Proof. This follows from (2.20) and

$$0 = \int_M \nabla_i (f v^i) \, dV = \int_M f \nabla_i v^i \, dV + \int_M v^i \nabla_i f \, dV.$$

Lemma 5. *In a compact Kaehlerian manifold M, we have*

$$\begin{aligned} \int_M \mathfrak{L}_{Df} h \, dV &= \int_M \mathfrak{L}_{Dh} f \, dV = \int_M (\nabla_i f)(\nabla^i h) \, dV \\ &= -\int_M f \Delta h \, dV = -\int_M h \Delta f \, dV \end{aligned} \tag{3.5}$$

for any real functions f and h on M, where $\mathfrak{L}_{Df}$ denotes the operator of Lie derivation with respect to the vector field $\nabla^i f$, and $\Delta = g^{ji}\nabla_j\nabla_i$.

Proof. This follows from

$$0 = \int_M \nabla_i(f\nabla^i h)\, dV = \int_M (\nabla_i f)(\nabla^i h)\, dV + \int_M f\Delta h\, dV,$$

$$0 = \int_M \nabla_i(h\nabla^i f)\, dV = \int_M (\nabla_i h)(\nabla^i f)\, dV + \int_M h\Delta f\, dV.$$

Lemma 6. *If, in a compact Kaehlerian manifold M, a nonconstant function φ satisfies*

$$(3.6) \qquad \nabla_j\nabla_i\varphi_h + \frac{c}{4}(2\varphi_j g_{ih} + \varphi_i g_{jh} + \varphi_h g_{ji} - F_{ji}F_h{}^s\varphi_s - F_{jh}F_i{}^s\varphi_s) = 0,$$

where $\varphi_h = \nabla_h\varphi$, c being a real constant, then the constant c is necessarily positive.

Proof. Transvecting (3.6) with g^{ih}, we have

$$\nabla_j\Delta\varphi + (n+1)c\varphi_j = 0,$$

from which and Lemma 5 it follows that

$$c\int_M \varphi_j\varphi^j\, dV = -\frac{1}{n+1}\int_M (\nabla_j\Delta\varphi)\varphi^j\, dV = \frac{1}{n+1}\int_M (\Delta\varphi)^2\, dV,$$

where $\varphi^j = g^{jj}\varphi_i$. Since φ is a nonconstant function, two inequalities

$$\int_M \varphi_j\varphi^j\, dV > 0, \quad \int_M (\Delta\varphi)^2\, dV > 0$$

hold, and consequently C is necessarily positive.

Lemma 7. *If a Kaehlerian manifold M with constant scalar curvature K admits an H-projective vector field v^h, and the vector field w^h defined by* (3.3) *is a Killing vector field, then the associated covariant vector field ρ_j satisfies*

$$(3.7) \qquad \nabla_j\nabla_i\rho_h + \frac{K}{4n(n+1)}(2\rho_j g_{ih} + \rho_i g_{jh} + \rho_h g_{ji} - F_{ji}F_h{}^s\rho_s - F_{jh}F_i{}^s\rho_s) = 0.$$

Moreover, if M is complete and simply connected, K is positive and v^h is non-affine, then M is isometric to a complex projective space CP^n with Fubini-Study metric of constant holomorphic sectional curvature $K/n(n+1)$.

Proof. By using (1.1) we have

$$(3.8) \qquad \nabla_j(\nabla_i v_h + \nabla_h v_i) = 2\rho_j g_{ih} + \rho_i g_{jh} + \rho_h g_{ji} - F_{ji}F_h{}^s\rho_s - F_{jh}F_i{}^s\rho_s.$$

If w^h is a Killing vector field, then

$$\nabla_i w_h + \nabla_h w_i = 0$$

holds, and consequently

$$2(n+1)\nabla_i\rho_h + \frac{K}{2n}(\nabla_i v_h + \nabla_h v_i) = 0,$$

which together with (3.8) implies (3.7). The second part of the lemma follows from Theorem A.

Remark. Using Lemma 6 we see that in Lemma 7 if M is compact, then we can remove the positiveness of the scalar curvature K.

In the following Lemmas 8, . . . , 15, M is a compact Kaehlerian manifold of complex dimension $n > 1$ with constant scalar curvature K, and v^h is an H-projective vector field on M.

Lemma 8. *For a vector field v^h on M we have*

$$\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)\, dV = 2\int_M (\nabla_t w^t)^2\, dV. \tag{3.9}$$

Proof. By using a well-known integral formula [5], [6] on a compact orientable Riemannian manifold, we have

$$\int_M (\nabla^j\nabla_j w^h + K_i^h w^i) w_h\, dV - \int_M (\nabla_t w^t)^2\, dV$$
$$+ \frac{1}{2}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)\, dV = 0.$$

On the other hand, by Lemma 2 the associated vector field ρ^h is contravariant analytic and hence satisfies

$$\nabla^j\nabla_j\rho^h + K_i^h\rho^i = 0.$$

Consequently (3.9) follows immediately from (2.19) and the above relations since K is a constant.

Lemma 9. *For a vector field v^h on M we have*

$$\int_M G_{ji}\rho^j w^i\, dV = \frac{1}{4(n+1)}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)\, dV. \tag{3.10}$$

Proof. From Lemma 2, the associated vector field ρ^h is contravariant analytic and hence satisfies

$$\nabla^j\nabla_j\rho^i + K_j^i\rho^j = 0,$$

from which and the equality

$$\nabla_i\nabla_t\rho^t = \nabla^t\nabla_t\rho_i - K_{ji}\rho^j$$

we find

$$\nabla_i\nabla_t\rho^t = -2K_{ji}\rho^j.$$

Using the above equation, (2.18), (2.25), (3.3) and Lemma 8, we have

$$\int_M G_{ji}\rho^j w^i\, dV = -\frac{1}{2}\int_M (\nabla_i\nabla_t\rho^t)w^i\, dV - \frac{K}{4n(n+1)}\int_M (\nabla_i\nabla_t v^t)w^i\, dV$$

$$= -\frac{1}{2(n+1)}\int_M (\nabla_i\nabla_t w^t)w^i\, dV = \frac{1}{2(n+1)}\int_M (\nabla_t w^t)^2\, dV$$

$$= \frac{1}{4(n+1)}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)\, dV.$$

Lemma 10. *For a vector field v^h on M we have*

$$(3.11)\quad \int_M G_{ji}\rho^j\rho^i\, dV + \frac{K}{8n(n+1)^2}\int_M \mathfrak{L}_v\big[(\mathfrak{L}_v G_{ji})g^{ji}\big]\, dV$$

$$= \frac{1}{4(n+1)^2}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)\, dV.$$

Proof. From (2.25) and (3.3), we have

$$(3.12)\quad \int_M G_{ji}\rho^j w^i\, dV = (n+1)\int_M G_{ji}\rho^j\rho^i\, dV + \frac{K}{2n}\int_M G_{ji}\rho^j v^i\, dV.$$

On the other hand, using the identities $G_{ji}g^{ji} = 0$ and

$$(3.13)\qquad \nabla^j G_{ji} = \frac{n-1}{2n}\nabla_i K = 0,$$

and integrating

$$\nabla^j(\rho G_{ji}v^i) = G_{ji}\rho^j v^i + \frac{1}{2}\rho G_{ji}(\nabla^j v^i + \nabla^i v^j)$$

$$= G_{ji}\rho + ujv^i - \frac{1}{2}\rho G_{ji}\mathfrak{L}_v g^{ji}$$

$$= G_{ji}\rho^j v^i + \frac{1}{2}\rho(\mathfrak{L}_v G_{ji})g^{ji}$$

over M, we find

$$\int_M G_{ji}\rho^j v^i\, dV = -\frac{1}{2}\int_M \rho(\mathfrak{L}_v G_{ji})g^{ji}\, dV,$$

which implies, in consequence of Lemma 4,

$$(3.14)\qquad \int_M G_{ji}\rho^j v^i\, dV = \frac{1}{4(n+1)}\int_M \mathfrak{L}_v\big[(\mathfrak{L}_v G_{ji})g^{ji}\big]\, dV.$$

By (3.10), (3.12) and (3.14), we readily obtain (3.11).

Lemma 11. *For a vector field v^h on M we have*

$$(3.15)\quad \int_M (\nabla^j\mathfrak{L}_v G_{ji})w^i\, dV = \frac{1}{2}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)\, dV.$$

Proof. Integrating

$$\nabla^j\left[(\pounds_v G_{ji})w^i\right] = \left(\nabla^j \pounds_v G_{ji}\right)w^i + \frac{1}{2}(\pounds_v G_{ji})(\nabla^j w^i + \nabla^i w^j)$$

over M and using (3.2), we obtain (3.15).

Lemma 12. *For a vector field v^h on M we have*

$$\int_M g^{kj}(\pounds_v \nabla_k G_{ji})w^i\, dV = \frac{n}{2(n+1)}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)\, dV. \tag{3.16}$$

Proof. Substituting (1.1) in the well-known formula [5], [6]

$$\pounds_v \nabla_k G_{ji} = \nabla_k \pounds_v G_{ji} - G_{si}\pounds_v\{{}_k{}^s{}_j\} - G_{js}\pounds_v\{{}_k{}^s{}_i\}$$

and using $F_{kj}G^{kj} = 0$ and

$$F_k{}^s G_{sj} + F_j{}^s G_{ks} = 0,$$

which follows from (2.2), (2.12) and (2.25), we have

$$g^{kj}\pounds_v \nabla_k G_{ji} = g^{kj}\nabla_k \pounds_v G_{ji} - 2G_{ji}\rho^j,$$

and therefore

$$\int_M g^{kj}(\pounds_v \nabla_k G_{ji})w^i\, dV = \int_M \left(\nabla^j \pounds_v G_{ji}\right)w^i\, dV - 2\int_M G_{ji}\rho^j w^i\, dV.$$

(3.16) follows from (3.10), (3.15) and the above relation.

Lemma 13. *For a vector field v^h on M we have*

$$\int_M \pounds_v\left[(\pounds_v G_{ji})G^{ji}\right] dV = -\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)\, dV. \tag{3.17}$$

Proof. Using (3.2) and (3.13) we have

$$\nabla^j\left(\rho G_{ji} w^i\right) = G_{ji}\rho^j w^i - \frac{1}{2}\rho(\pounds_v G_{ji})G^{ji}.$$

Integrating this over M and using Lemmas 4 and 9, we arrive at (3.17) immediately.

Lemma 14. *For a contravariant analytic vector field v^h on M we have*

$$\left(\pounds_v Z_{kji}{}^h\right)g^{ji} = -\frac{1}{n+1}(\nabla_k w^h + \nabla^h w_k) - \frac{1}{n+1}\delta_k^h \nabla_t w^t, \tag{3.18}$$

$$\left(\pounds_v Z_{kji}{}^h\right)Z^{kji}{}_h = \frac{4}{n+1}(\pounds_v G_{ji})G^{ji}. \tag{3.19}$$

Proof. Using (2.16), (2.22) and (2.26), we have

$$\begin{aligned}\mathfrak{L}_v Z_{kji}{}^h = & -\delta_k^h \nabla_j \rho_i + \delta_j^h \nabla_k \rho_i + F_k{}^h (\nabla_j \rho_s) F_i^s \\ & - F_j^h (\nabla_k \rho_s) F_i^s + F_k^s (\nabla_j \rho_s) F_i^h - F_j^s (\nabla_k \rho_s) F_i^h \\ & - \frac{K}{4n(n+1)} \big[\delta_k^h \mathfrak{L}_v g_{ji} - \delta_j^h \mathfrak{L}_v g_{ki} + F_k{}^h F_j{}^s \mathfrak{L}_v g_{si} \\ & \qquad - F_j^h F_k{}^s \mathfrak{L}_v g_{si} - 2 F_k{}^s (\mathfrak{L}_v g_{sj}) F_i^h \big].\end{aligned}$$

Using this relation, (2.1), $\cdots$, (2.13), (2.25), (2.26), Lemma 3 and contravariant analyticity of v^h and ρ^h, we obtain (3.18) and (3.19) by a straightforward computation.

Lemma 15. *For a vector field v^h on M we have*

$$\begin{aligned}&\int_M \mathfrak{L}_v \big[(\mathfrak{L}_v Z_{kji}{}^h) Z^{kji}{}_h \big] \, dV \\ &= - \frac{4}{n+1} \int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j) \, dV.\end{aligned} \tag{3.20}$$

Proof. This follows from (3.17) and (3.19).

4. Propositions

In this section, we prove a series of integral inequalities and obtain necessary and sufficient conditions for a Kaehlerian manifold to be isometric to a complex projective space.

Proposition 1. *A complete simply connected Kaehlerian manifold M of complex dimension $n > 1$ with positive constant scalar curvature K admits a nonaffine and contravariant analytic H-projective vector field v^h such that*

$$\mathfrak{L}_v G_{ji} = 0, \tag{4.1}$$

if and only if M is isometric to a complex projective space CP^n *with Fubini-Study metric and of constant holomorphic sectional curvature $K/n(n+1)$.*

Proof. This follows from Lemmas 3 and 7.

Remark. In Proposition 1 if M is further compact, then by Lemmas 2 and 6 we can remove the contravariant analyticity of H-projective vector field v^h and the positiveness of scalar curvature K. The same remark applies to the following Proposition 2.

Proposition 2. *A complete simply connected Kaehlerian manifold M of complex dimension $n > 1$ with positive constant scalar curvature K admits a nonaffine and contravariant analytic H-projective vector field v^h such that*

$$\mathfrak{L}_v Z_{kji}{}^h = 0, \tag{4.2}$$

if and only if M is isometric to a complex projective space CP^n *with Fubini-Study metric and of constant holomorphic sectional curvature* $K/n(n+1)$.

Proof. If (4.2) holds, then from (3.18) we have $\nabla_t w^t = 0$ and hence w^h is a Killing vector field. Consequently the proposition follows from Lemma 7.

Remark. In Proposition 2, (4.2) can be replaced by

$$(\pounds_v Z_{kji}{}^h) g^{ji} = 0. \tag{4.3}$$

In the following Propositions 3, · · · , 8, we suppose that a compact Kaehlerian manifold M of complex dimension $n > 1$ with constant scalar curvature K admits an H-projective vector field v^h.

Proposition 3. *For M we have*

$$\int_M G_{ji}\rho^j w^i \, dV \geqslant 0, \tag{4.4}$$

where w^i *is defined by* (3.3). *Assume moreover that M is simply connected and* v^h *is nonaffine, then the equality in* (4.4) *holds if and only if M is isometric to a complex projective space* CP^n *with Fubini-Study metric and of constant holomorphic sectional curvature* $K/n(n+1)$.

Proof. This follows from Lemmas 7 and 9.

Proposition 4. *For M we have*

$$\int_M G_{ji}\rho^j\rho^i \, dV + \frac{K}{8n(n+1)^2}\int_M \pounds_v\left[(\pounds_v G_{ji})g^{ji}\right] dV \geqslant 0. \tag{4.5}$$

Assume moreover that M is simply connected and v^h *is nonaffine, then the equality in* (4.5) *holds if and only if M is isometric to a complex projective space* CP^n *with Fubini-Study metric and of constant holomorphic sectional curvature* $K/n(n+1)$.

Proof. This is an immediate consequence of Lemmas 7 and 10.

Proposition 5. *For M we have*

$$\int_M (\nabla^j \pounds_v G_{ji}) w^i \, dV \geqslant 0, \tag{4.6}$$

where w^i *is defined by* (3.3). *Assume moreover that M is simply connected and* v^h *is nonaffine, then the equality in* (4.6) *holds if and only if M is isometric to a complex projective space* CP^n *with Fubini-Study metric and of constant holomorphic sectional curvature* $K/n(n+1)$.

Proof. This follows from Lemmas 7 and 11.

Proposition 6. *For M we have*

$$\int_M g^{kj}(\pounds_v \nabla_k G_{ji}) w^i \, dV \geqslant 0, \tag{4.7}$$

where w^i *is defined by* (3.3). *Assume moreover that* M *is simply connected and* v^h *is nonaffine, then the equality in* (4.7) *holds if and only if* M *is isometric to a complex projective space* CP^n *with Fubini-Study metric and of constant holomorphic sectional curvature* $K/n(n+1)$.

Proof. This is an immediate consequence of Lemmas 7 and 12.

Proposition 7. *For* M *we have*

$$\int_M \mathfrak{L}_v\{(\mathfrak{L}_v G_{ji})G^{ji}\}\, dV \leqslant 0. \tag{4.8}$$

Assume moreover that M *is simply connected and* v^h *is nonaffine, then the equality in* (4.8) *holds if and only if* M *is isometric to a complex projective space* CP^n *with Fubini-Study metric and of constant holomorphic sectional curvature* $K/n(n+1)$.

Proof. This is an immediate consequence of Lemmas 7 and 13.

Proposition 8. *For* M *we have*

$$\int_M \mathfrak{L}_v\{(\mathfrak{L}_v Z_{kji}{}^h)Z^{kji}{}_h\}\, dV \leqslant 0. \tag{4.9}$$

Assume moreover that M *is simply connected and* v^h *is nonaffine, then the equality in* (4.9) *holds if and only if* M *is isometric to a complex projective space* CP^n *with Fubini-Study metric and of constant holomorphic sectional curvature* $K/n(n+1)$.

Proof. This follows from Lemmas 7 and 15.

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