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FINITE TYPE CONDITIONS FOR REAL HYPERSURFACES

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Introduction

An important advancement in partial differential equations and several complex variables in the 1960's was the solution of the $\bar\partial$-Neumann problem for strongly pseudo-convex manifolds. Let X be a complex manifold with boundary, and consider the Dolbeault complex

$$0 \to C^\infty(X) \xrightarrow{\bar\partial} \Lambda^{0,1}(T^*(X)) \to$$

Let $\mathfrak{D}$ be the L_2 adjoint of $\bar\partial$, and let H be the orthogonal projection onto the null space of $\mathfrak{D}\bar\partial + \bar\partial\mathfrak{D}$. The $\bar\partial$-Neumann problem is the overdetermined system of partial differential equations

$$\bar\partial f = g \quad \text{where } \bar\partial g = Hg = 0.$$

Kohn [4] has proved that regularity for the unique solution f orthogonal to the holomorphic functions follows from a subelliptic estimate of the form

$$|||u|||_\varepsilon^2 \leqslant C\left(\|\bar\partial u\|^2 + \|\mathfrak{D}u\|^2 + \|u\|^2\right),$$

whenever u is in the domain of $\mathfrak{D}$, and where $|||\ |||_\varepsilon$ denotes the tangential Sobolev norm for some ε with $0 < \varepsilon \leqslant \frac{1}{2}$. Characterizing subellipticity in terms of the geometry of $M = \operatorname{bd} X$ is an open problem. Kohn found such a characterization if $\dim X = 2$. He defined a notion of "point of finite type" in terms of iterated commutators of vector fields, and showed that if p is a point of finite type, there is a subelliptic estimate near p. Greiner [5] proved the converse. Generalizations of Kohn's definition to higher dimensions give neither the right geometric analogue nor the right condition for the estimates.

We make the following definition of point of finite type. Let M be a real hypersurface of a complex manifold X'.

Definition. $p \in M$ is a point of finite type if there is a finite algebraic obstruction to the existence of a nontrivial germ at p of a complex analytic subvariety of M.

We interpret the Levi form as a 2-jet obstruction in this sense. In case the degeneracies in the Levi form are particularly simple, such as what must

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happen if dim $X' = 2$, we show that our definition is equivalent to that of Kohn. The general case presents more difficult algebraic problems, because we must consider subvarieties with singularities at p. We prove several theorems clarifying the algebraic and geometric content of our definition, and give evidence supporting our conjecture that finite type is equivalent to subellipticity. The author would like to thank Professors J. J. Kohn and H. Hironaka for helpful conversations.

1. Real hypersurfaces

Let X' be a complex manifold of dim $n + 1$, and $T^{10}(X')$ the bundle of type $(1, 0)$ vector fields to X'. A real hypersurface M of X' is a real $(2n + 1)$-dimensional submanifold. Let $CT(M)$ be the complexified tangent bundle, and put $T^{10}(M) = CT(M) \cap T^{10}(X')$. Write $T^{01}(M)$ for the complex conjugate bundle. Then $T^{10}(M) \oplus T^{01}(M)$ is a subbundle of codimension 1 in $CT(M)$. Let η be a purely imaginary nonvanishing 1-form annihilating $T^{10}(M) \oplus T^{01}(M)$. Write $\langle\ ,\ \rangle$ for both a fixed hermitian metric on $T^{10}(M)$ and contraction of vector fields and 1-forms. For each purely imaginary vector field N with $\langle N, \eta\rangle = 1$ we can write $CT(M) = T^{10}(M) \oplus T^{01}(M) \oplus N(M)$ where $N(M)$ denotes the bundle generated by N.

Definition. Let L be the quadratic form on $T^{10}(M)$ defined by $\langle LX, Y\rangle = \langle [X, \bar{Y}], \eta\rangle$. L is called the Levi form. For each N with $\langle N, \eta\rangle = 1$ let w_N be the 1-form defined by $\langle X, w_N\rangle = \langle [N, Y - \langle X, \eta\rangle N], \eta\rangle$. It is easy to check that L is Hermitian symmetric, that w_N is a real 1-form, and that $\langle X, w_N - w_{N'}\rangle = \langle d\eta, X \wedge (N - N')\rangle$ if $\langle N', \eta\rangle = 1$. Therefore, w_N is independent of N precisely when $d\eta = 0$, which is the well known integrability criterion for flat M. M is called pseudoconvex if L is positive semi-definite.

Suppose $p \in M$, and near p, $M = r^{-1}(0)$ for some smooth real valued fn. r on X' with $dr(q) \neq 0$ for $q \in M$. By the implicit function theorem we can choose coordinates in a small neighborhood of p so that p is the origin and $r(z) = 2\,\mathrm{Re}\, z_0 + f(z_1, \cdots, z_n, \mathrm{Im}\, z_0)$. Formulas for the above notions in terms of derivatives of r appear in [2].

Suppose V is a complex analytic submanifold of M, and $Y \in T(V) \cap T^{10}(M)$. Then $[Y, \bar{Y}]$ must lie in the space generated by Y and $\bar{Y}$, so $\langle [Y, \bar{Y}], \eta\rangle_V = 0$. It is natural to consider the following concept.

Definition. Let $Y \in T^{10}(M)$. We say $t_p Y = m$ if

$$m = \min\{k\colon \langle [[Y, \bar{Y}], \cdots, Y], \eta\rangle_p \neq 0\}$$

for some combination of k brackets.

Remark. The necessary and sufficient condition for subellipticity mentioned in the introduction for $M \subset C^2$ is that $t_p Y < \infty$ for all $Y \in T^{10}(M)$

with $Y_p \neq 0$. The following example shows that this definition is not the "right" one in higher dimensions.

Example. [1]

$$r(z) = 2 \operatorname{Re} z_0 + |z_1^2 - z_2^3|^2 \cdot p = \text{origin}.$$

Then $t_p Y = 3$ or $t_p Y = 5$ if $Y_p \neq 0$, $Y \in T^{10}(M)$. Notice that V is an analytic subvariety of M, where

$$V = \{z \in C^3 \colon z_0 = 0\ z_1^2 = z_2^3\}.$$

Any vector field tangent to V must be singular at p, since p is a singular point of V. Therefore this type condition does not prevent analytic subvarieties from lying in M. It is also not an open condition, and therefore cannot be the right condition for the estimates. One can also verify that local regularity for $\bar\partial$ (a corollary of a subelliptic estimate) fails near V.

2. Points of finite type

Suppose V is a complex analytic subvariety of X', and $p \in V$. Then there are a $\delta > 0$ and a holomorphic map $z\colon \{t \in C\colon |t| < \delta\} \to V$. If $V \subset M$, and r is a defining function for M, then $z^*r = r(z(t)) = 0$. Put

$$c_{ab}f = \left(\frac{d}{dt}\right)^a \left(\frac{d}{d\bar t}\right)^b f(t)\Big|_{t=0}$$

for f any smooth function on C. Then formally

$$z^*r = \sum_{a,b=0}^{\infty} c_{ab} z^* r \frac{t^a\, \bar t^b}{a!\, b!}.$$

Fix the defining function. Consider the vanishing of $c_{ab}z^*r$ as algebraic equations for the unknown Taylor coefficients for z. We formalize this as follows.

Let $C((t))$ = ring of formal power series in t,

$$A_p = \{z \in C((t))^{n+1} : z(0) = p\},$$

$$S_p^k(M) = \{z \in A_p : c_{ab}z^*r = 0\ \forall a, b \leqslant k\},$$

$$p_p^k(M) = \{\text{polynomial equations defining } s_p^k(M)\}.$$

If we write $z(t) = \sum_{m=0}^{\infty} z^{(m)} t^m/m!$ where $z^{(m)} \in C^{n+1}$, and let $j_k f$ denote the k jet of any smooth functon f, then j_k is a linear projection on A_p. We write $\mathfrak{N}^k$ and $\mathfrak{R}^k$ for the null space and range of j_k respectively. Let $(D_{ab}r)$: $\mathfrak{R}^a \times \overline{\mathfrak{R}}^b \to C$ be the multilinear map defined by the a-th holomorphic and b-th anti-holomorphic derivatives of r. Thus

$$(D_{10}r)(z^{(1)}) = \sum r_{z_i} z_i^{(1)},$$

$$(D_{11}r)(z^{(1)}, \bar{z}^{(1)}) = \sum r_{z_i \bar{z}_j} z_i^{(1)} \bar{z}_j^{(1)}.$$

$p_k(M)$ is defined by setting these two expressions equal to 0. Using the chain rule it is easy to compute more of these equations.

Definition. $P_p^k(M)$ is a finite obstruction to the existence of complex analytic data passing through p and lying in M if the following holds: Suppose $j_k r' = j_k r$. Then the equations $z^* r' = 0$ and $z(0) = p$ imply that z is a constant map.

Definition. $p \in M$ is a point of finite type if there is an integer k so that $p_p^k(M)$ is such an obstruction.

Theorem 1. *Suppose $S_p^i \subset \mathfrak{N}^i$ for $i = 1$ or $i = 2$. Then $S_p^{ki} \subset \mathfrak{N}^k$ $\forall k$, and p is a point of finite type. On the other hand, if* $\dim M = 2n + 1 \geqslant 5$, *the containment $S_p^m \subset \mathfrak{N}^1$ for $m \geqslant 3$ is not sufficient to guarantee that p is a point of finite type.*

Corollary. *If M is strongly pseudoconvex at p, p is a point of finite type with* $k = 1$.

Proof of Corollary. M is strongly pseudoconvex at p if and only if

$$\sum r_{z_i \bar{z}_j} z_i^{(1)} z_j^{(1)} > 0 \quad \text{whenever } \sum r_{z_i} z_i^{(1)} = 0 \text{ and } z^{(1)} \neq 0.$$

Therefore, if M is strongly pseudoconvex at p, then $S_p^1(M) \subset \mathfrak{N}^1$.

Proof of theorem. The second statement follows from the following example. Put $r(z) = 2 \operatorname{Re} z_0 + |z_1^{m-1} - z_2^m|^2$ where p is the origin. Here $S_p^m \subset \mathfrak{N}^1$ if $m \geqslant 3$, but $z^* r = 0$ for $z(t) = (0, t^m, t^{m-1})$. We prove the first statement by induction on k, assuming $i = 2$, as the case $i = 1$ is similar and easier. Suppose $S^{2k} \subset \mathfrak{N}^k$. We must show that $z \in S^{2k+2} \Rightarrow j_{k+1} z = 0$. Let $f \in P^{2k+2}$. Then $f(z)$ has the following form:

$$f(z) = \sum_{a=0}^{2k+2} \sum_{b=0}^{d} h_{abdk} (D_{ab} r)(z^{(q_1)}, , z^{(q_a)}, \bar{z}^{(m_1)}, , \bar{z}^{(m_b)}),$$

where h_{abdk} are constant positive integers, $q_i, m_i \geqslant 1$ and $\sum q_i = 2k + 2$ and $\sum m_i = d$. If $z \in S^{2k+2}$, then $z \in S^{2k}$, so that $j_k z = 0$. Depending on d, after substituting 0 for $z^{(1)}$ through $z^{(k)}$, we see that $f(z) = 0$ reduces to one of the following equations

$$(D_{10}r)(u) = 0,$$

$$(D_{11}r)(u, \bar{u}) = 0,$$

$$(D_{10}r)(v) + h(D_{20}r)(u, u) = 0,$$

$$(D_{11}r)(v, \bar{u}) + h(D_{21}r)(u, u, \bar{u}) = 0,$$

$$(D_{11}r)(v, \bar{v}) + h\, 2 \operatorname{Re}(D_{21}r)(u, u, \bar{v}) + h(D_{22}r)(u, u, \bar{u}, \bar{u}) = 0,$$

where $u = z^{(k+1)}$ and $v = z^{(2k+2)}$. We see that these are precisely the equations defining S^2, i.e., $S^{2k+2}/\mathfrak{N}^k \approx S^2$ under the correspondence $u \to z^{(1)}$, $v \to h\, z^{(2)}$. Since $S^2 \subset \mathfrak{N}^1$, we get that $u = z^{(k+1)} = 0$. Therefore $S^{2k+2} \subset \mathfrak{N}^{k+1}$, completing the induction. Therefore the only solution to $z^*r = 0$ with $z(0) = p$ is a constant.

Things are much easier in C^2.

Theorem 2. *Let $M \subset C^2$ be a pseudoconvex hypersurface, and $p \in M$. The following are equivalent*:

(1) *p is a point of finite type.*

(2) *The $\bar{\partial}$-Neumann problem is ε subelliptic at p with $\varepsilon = 1/(k+1)$.*

(3) *$t_pY = 2k - 1\ \forall Y \in T^{10}(M)$ with $Y_p \neq 0$.*

(4) *$S^k \subset \mathfrak{N}^1$, where k is the smallest such integer.*

Proof. The equivalent of (2) and (3) follows from the work of Kohn [6] and Greiner [5]. We will demonstrate the equivalence of (1), (3) and (4). We may assume p is the origin, $r(z) = 2\,\mathrm{Re}\, z_0 + f(z, \mathrm{Im}\, z_0)$, and the Taylor series for f up to terms of order $<2k + 2$ contains no holomorphic or anti-holomorphic terms. Then

$$r(z(t)) = z_0(t) + \overline{z_0(t)} + 0(|t|^2) + 0(|t|^{k+1}).$$

Suppose $c_{a0}z^*r = 0$, $a \leqslant k$. Then $z^{(a)} = 0$, $\forall a \leqslant k$. We may therefore assume $z(t) = (0, t)$. This implies that (1) and (4) are equivalent. If $M \subset C^2$, $T^{10}(M)$ is 1-dimensional, locally spanned by

$$L_1 = \frac{\partial}{\partial z_1} - r_{z_1}(r_{z_0})^{-1}\frac{\partial}{\partial z_0}.$$

Suppose $Y \in T^{10}(M)$ satisfies $Y_p \neq 0$. Then $t_pY = t_pL_1$. Since the only curve in consideration is $z(t) = (0, t)$, the conditions that $S_p^k \subset \mathfrak{N}^1$ and that $t_pL_1 = 2k - 1$ are easily seen to be equivalent. This shows that (1), (2) and (4) are all equivalent to subellipticity in C^2.

We now consider an illuminating special case. If f_j are germs of holomorphic functions at 0 in C^n, we write

$$V(f) = \{z \in C^n : f_j(z) = 0\ \forall j\}$$

and (f) for the ideal in $\mathcal{O}_n$ generated by the f_j.

Theorem 3. *Suppose $r(z) = 2\,\mathrm{Re}\, z_0 + \Sigma|f_j(z)|^2$ where $f_j \in \mathcal{O}_n$ and $0 \in V(f)$. The following are equivalent*

(1) 0 *is a point of finite type.*

(2) *There is a neighborhood U of* 0 *so that $V(f) \cap U = \{0\}$.*

(3) *$\mathcal{O}_n/(f)$ is a finite dimensional complex vector space.*

(4) *$\forall i$, $\exists p_i$ with $z_i^{p_i} \in (f)$.*

(5) *There is a constant K so that for z near* 0,

$$\sum |z_i|^{2p_i} \leqslant K \sum |f_j(z)|^2.$$

Proof. (2) and (4) are equivalent by Hilbert's Nullstellensatz. (3) is another equivalent formulation whose proof follows from Nakayama's lemma. (5) obviously implies (2). Conversely, since each z_i vanishes on $V(f)$, by Lojasiewicz's inequality there are integers p_i and a constant K so that (5) holds. It remains to demonstrate the equivalence of these statements with our notion of finite type. If (2) were false, then $\dim V(f) \geqslant 1$. By Oka's normalization theorem, there is a curve $w(t)$ whose image lies in $V(f)$. Put $z(t) = (0, w(t))$. Then $z^*r = 0$, so p is not a point of finite type. Conversely suppose (5) holds. We may assume $z(t) = 0$. If $z^*r = 0$, then by (5) we see that $z_i(t) = 0 \ \forall i$. To verify finite type we must show that this information depends upon only finitely many derivatives of r. The existence of some constant K so that (5) holds in some neighborhood of 0 depends upon some finite jet of r by Taylor's theorem. This jet then defines a finite obstruction to finding a holomorphic curve through 0.

The above example shows how the positivity condition guaranteed by pseudoconvexity shows up. Suppose in general that r is a real analytic function. Then there are holomorphic functions h_a so that $r(z) = 2 \operatorname{Re} z_0 + \sum h_a(z)\bar{z}^a$. (Here a is a multi-index.) When r has the form in Theorem 3, we claim $V(f) = V(h)$. Notice that $V(h) \subset V(f)$ is obvious, and in this case we see that each h_a is a linear combination of the f_j, so that $V(f) \supset V(h)$. Without pseudoconvexity, $V(h) = \{0\}$ does not prevent holomorphic curves from lying in $r^{-1}(0)$.

Example. $r(z) = 2 \operatorname{Re} z_0 + |z_1|^2 - |z_2|^2$. Here $z^*r = 0$ for $z(t) = (0, t, t)$, but $V(h)$ is clearly $\{0\}$.

Lemma. *Suppose g is a real analytic subharmonic function defined near* 0 *in* $\mathbf{C}$. *Put*

$$g(t) = \sum c_{ab} \frac{t^a \bar{t}^b}{a!\, b!}.$$

Suppose g contains no pure terms, i.e., $c_{a0} = c_{0a} = 0 \ \forall a$. Suppose that $c_{ab} \neq 0$ for minimum $a + b$. Then $a + b = 2k$ is even, and $c_{kk} \neq 0$.

Proof. Write $t = |t|e^{i\theta}$. Since g is subharmonic,

$$\Delta g = \sum c_{ab} \frac{e^{i\theta(a-b)}|t|^{a+b-2}}{(a-1)!\,(b-1)!} \geqslant 0.$$

For $|t|$ sufficiently small we must then have

$$\sum_{a+b=2k} \frac{c_{ab} e^{i\theta(a-b)}}{(a-1)!\,(b-1)!} \geqslant 0.$$

By Bochner's theorem we must have $c_{ab} = 0\ \forall a, b$ or else $c_{kk} > 0$.

Lemma 2. *Put* $z_0 = x + iy$. *Suppose* r *is real analytic near* 0, *defines a pseudoconvex hypersurface near* 0, *and has the following form*:

$$r(z, z_0) = 2x + p(z) + yg(z) + y^2h(z) + 0(y^3).$$

Then the following matrix is positive semi-definite near 0 *in* C^n:

$$\left(\left(1 + \frac{1}{4}g^2\right)\right)(p_{z_k\bar{z}_j}) + \frac{1}{2}h(p_{z_k}p_{\bar{z}_j}) + \frac{1}{2}i\left(\left(1 - \frac{1}{2}ig\right)g_{z_k}p_{\bar{z}_j}\right) - \left(\left(1 + \frac{1}{2}ig\right)p_{z_k}g_{\bar{z}_j}\right),$$

where all the functions are evaluated at z.

Proof. Use the formula for the Levi form in terms of the derivatives of r, and then evaluate at $y = 0$.

Theorem 4. *Suppose* r *is as above, and* 0 *is a point of finite type. Suppose that* $z \neq 0$, *but* $c_{a0}z^*r = 0$ *for all* a. *Then there is an integer* k *such that* $c_{ab}z^*r = 0$ *whenever* $a + b < 2k$, *and* $c_{kk}z^*r > 0$. (*This shows that the obstructions are given by positivity conditions.*)

Proof. We may assume without loss of generality that the Taylor expansion for $r(z, z_0) - 2\,\mathrm{Re}\, z_0$ contains no pure terms, and that $z_0(t) = 0$. (See the proof of Theorem 2). Therefore the functions p, g and h of Lemma 2 have no pure terms, and when we pull back that formula via z, the last three terms vanish to higher order than the first. This shows that z^*r is subharmonic. By Lemma 1 we can conclude the existence of such an integer k.

3. Remarks and open questions

As stated in the introduction, a very important question is characterizing subellipticity. Subellipticity holds with $\varepsilon = \frac{1}{2}$ precisely when M is strongly pseudoconvex [4]. It is natural to conjecture that subellipticity holds for some ε with $0 < \varepsilon \leqslant \frac{1}{2}$ whenever the (possibly degenerate) Levi form behaves qualitatively as if it were definite. Kohn has proved for example, that subellipticity holds when the Levi form is smoothly diagonalizable by vector fields L_i with $t_pL_i < \infty$, or when M is real analytic and the Levi form has isolated degeneracies [6], [7], the notion of finite type presented here makes clear what it means for the Levi form to behave as if it were nondegenerate. In case M contains an analytic subvariety, generalizations of the proof for flat M [6] indicate that subellipticity will fail. Therefore we conjecture that finite type is precisely equivalent to the subelliptic estimates.

There are also important geometric and algebraic questions. One important such question is whether or not the assignment of an integer (or ∞) to each point of M given by $t(p) = \min\{k\colon P_p^k(M)$ is an obstruction$\}$ is an upper semi-continuous function on M. Also, does $t(p)$ have any relation to ε? A second question is whether the complex analytic methods used in Theorem 3 have a generalization to more general plurisubharmonic functions than $F(z) = \sum|f^j(z)|^2$, f^j holomorphic. A third question is what happens when $r(z)$ has zeroes of infinite order, but $r^{-1}(0)$ contains no complex analytic varieties. Here there will be no subelliptic estimate, but $\bar\partial$ should still have good properties.

Finally we mention the Bergman kernel function $K(z, w)$ in case M is the boundary of an open subset of C^{n+1}. $K(z, w)$ is C^∞ off the boundary diagonal whenever there is a subelliptic estimate. Calculations in [2] show that the singularity of $K(z, z)$ is closely related to the degeneracy in $\det \Lambda$. It would be very important to relate more precisely the Bergman kernel, subellipticity, and the notion of point of finite type.

Since the writing of this article, two very important papers have appeared concerning real analytic hypersurfaces. Using a result of Diederich-Fornaess [3] and the theory of ideals of real analytic functions, Kohn [8] has proved our conjecture about subellipticity; namely, the subelliptic estimates hold for (p, q) forms near a point if and only if there are no q-dimensional complex analytic subvarieties of M passing through that point. Diederich-Fornaess have also shown that this holds whenever M is compact.

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