J. DIFFERENTIAL GEOMETRY
14 (1979) 1-6

VECTOR FIELDS OF A FINITE TYPE G-STRUCTURE

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0. Introduction

Let M be a connected manifold, g a Riemannian metric on M, and $\mathcal{F}$ either the set of Killing vector fields or the set of conformal vector fields. The following theorems are known.

(0.1) Theorem. *If $U \subset M$ is open and $X, Y \in \mathcal{F}$, then $X|U = Y|U$ implies $X = Y$ on the whole of M.*

(0.2) Theorem. *If M and g are analytic, M is simply connected, and X is a Killing (resp. conformal) field on U, open subset of M, then there is a unique extension of X to an analytic Killing (resp. conformal) field defined on the whole of M.*

These theorems were proved in [4] for the Killing case and in [3] for the conformal case. The aim of this paper is to generalize them, when $\mathcal{F}$ is taken to be set of vector fields of a finite type G-structure. The precise definitions and statements of the theorems are in §2 and §3. §4 is devoted to proving some auxiliary results on fields on a parallelisable manifold. When no precision is made about the differentiability class of a manifold or map, it will be understood that the definition or result works for both the category of manifolds of class infinity and real analytic manifolds.

1. Parallelism fields

Let $m = \dim M$, and π be a parallelism on M; that is, a 1-exterior form on M with values in R^m such that for all $x \in M$, $\pi(x) : TM(x) \to R^m$ is an isomorphism. Suppose that X is a vector field on M, and $\{\psi_t : t \in R\}$ the corresponding pseudogroup of diffeomorphisms. Then we say that X is a parallelism field if for all $t \in R$, $\psi_t^* \pi = \pi$, or, equivalently, if $L_X \pi = 0$. Let $(u^1, \cdots, u^m)$ be a coordinate system on U. If X is a field on U and $c: I \to U$

Received November 22, 1976. The author would like to thank Dr. A. J. Ledger for his kindness and help in writing this paper.

is a smooth curve, I being an interval, we write

$$X = \sum X^i \frac{\partial}{\partial u^i}, \quad c^i(t) = u^i \circ c(t) \quad \text{for } t \in I, 1 \leqslant i \leqslant m,$$

$$\pi = \sum_{i,j} \left(\pi_j^i \, du^j\right) e_i,$$

where $\{e_1, \cdots, e_m\}$ is the canonical basis of R^m.

(1.1) Lemma. *If X is a parallelism field on U, then the curve $t \to (X^1 \circ c(t), \cdots, X^m \circ c(t))$ is a solution of the linear system*

$$\frac{dx^i}{dt} = \sum_j \mathcal{Q}_j^i(t) x^j \quad 1 \leqslant i \leqslant m,$$

where

$$\mathcal{Q}_j^i(t) = -\sum_{h,k} \rho_h^i(c(t)) \frac{\partial \pi_k^h(c(t))}{\partial u^j} \frac{dc^k(t)}{dt},$$

and $(\rho_j^i(x))$ is the inverse matrix of $\pi_j^i(x)$ for all $x \in U$.

Proof. If is just an easy computation, if we write the equation $L_X\pi \cdot c'(t) = 0$ in local coordinates, $c'(t) \in TM(c(t))$ being the velocity of c at the point t.

(1.2) Lemma. *If X, Y are parallelism fields on M, and $X(x_0) = Y(x_0)$ for some $x_0 \in M$, then $X = Y$ on M.*

Proof. Let x_1 be an arbitrary point of M, and c: $[0, 1] \to M$ a smooth curve such that $c(0) = x_0$, $c(1) = x_1$. We prove that $X = Y$ on $c([0, 1])$; hence $X(x_1) = Y(x_1)$. Certainly, $X(c(0)) = Y(c(0))$. The idea–quite standard–is to show that if $X = Y$ on $c([0, \tau])$, with $0 \leqslant \tau < 1$, there is $\varepsilon > 0$ such that $X = Y$ on $c([0, \tau + \varepsilon])$, and this is done by using (1.1). If $(u; U)$ is a coordinate system around $c(\tau)$, there is $\varepsilon > 0$ such that $c([\tau - \varepsilon, \tau + \varepsilon]) \subset U$, and the curves $(X^j \circ c)$ and $(Y^i \circ c)$ defined on $(\tau - \varepsilon, \tau + \varepsilon)$ are solutions of the system (1.1). Since they coincide for $t = \tau$, they are equal on their domain of definition. This proves $X = Y$ on $c([0, \tau + \varepsilon])$.

(1.3) Lemma. *Let M be analytic, and $(u; U)$ a coordinate system such that $u(U) \subset R^m$ is convex. Then any parallelism field X defined on an open connected subset V of U can be extended to a unique parallelism field Y on U.*

Proof. The uniqueness of the extension follows from (1.2) or, more easily, from the fact that if two analytic vector fields coincide on V, they must coincide in the connected component of V in the domain of definition.

Choose $x_0 \in V$. Define c_x: $[0, 1] \to U$ for $x \in U$ as the curve determined by the condition $u(c_x(t)) = (1 - t)u(x_0) + tu(x)$. The map $U \times [0, 1] \to U$, $(x, t) \to c_x(t)$ is analytic. Clearly c_x is a curve joining x_0 and x. Substitute c for c_x in the formula for $\mathcal{Q}_j^i(t)$ in (1.1). One gets a family of analytic maps

$\mathcal{Q}^i_j(x, t)$, and we have a linear system S_x of equations depending on a parameter x. For each $x \in U$ the solution (α^i_x) with initial condition $\alpha^i_x(0) = u^i(x_0)$ is defined for $t = 1$. We define $Y = \Sigma\, Y^i \partial/\partial u^i$ by the formula $Y^i(x) = \alpha^i_x(1)$, and show that Y is the required extension.

There is a neighborhood W of x_0 with the following property: If $x \in W$, then $c_x(t) \in V$ for all $t \in [0, 1]$. Using (1.1) and the uniqueness of the solution we get for $x \in W$: $X^i(x) = X^i(c_x(1)) = \alpha^i_x(1) = Y^i(x)$. Therefore $X|W = Y|W$, and this implies, since our fields are analytic, that $X = Y|V$. The field Y is a parallelism field because $L_Y\pi|_V = L_X\pi = 0$ implies, using analyticity once more, that $L_Y\pi = 0$ on U.

2. The uniqueness theorem

Let p: $\mathcal{Q} \to M$ be a G-structure, and P the corresponding pseudogroup of transformations. By definition a diffeomorphism f: $U \to V$; U, V open subsets of M, is in P if and only if the natural lift f_* to the frame bundle sends $\mathcal{Q}|U$ into $\mathcal{Q}|V$. If $f \in P$, we denote this natural restriction of f_* by f_0, and we still call it the natural lift. If X is a vector field on M, and $\{\psi_t;\ t \in R\}$ the corresponding pseudogroup induced by X, we say that X is an $\mathcal{Q}$-field if for all $t \in R$, $\psi_t \in P$. If this is the case, X has a natural lift to a field X_0 on $\mathcal{Q}$ which projects on X. The pseudogroup determining X_0 is just $\{(\psi_t)_0;\ t \in R\}$. We denote the set of $\mathcal{Q}$-fields by $\mathcal{F}$. If $U \subset M$ is open, then $\mathcal{F}_U$ will denote the set of $\mathcal{Q}|_U$-fields. Let θ be the canonical 1-form on $\mathcal{Q}$ with values in R^m. It is well known that $f_0^*\theta = \theta$ for $f \in P$, and $L_{X_0}\theta = 0$ for $X \in \mathcal{F}$.

We now quote some facts about Sternberg prolongations. The reader interested in details should go to [1], whose notation we keep as much as possible. If $\mathcal{G}$ is the Lie algebra of G, we denote by $\mathcal{G}_k$ the kth prolongation of $\mathcal{G}$, and write $E_k = R^m \oplus \mathcal{G} \oplus \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_k$.

We collect the necessary facts in the following theorem:

(2.1) Theorem. *There is a sequence of manifolds* $\mathcal{Q}_k$ $(k \geqslant 0)$, *maps* p_k: $\mathcal{Q}_k \to \mathcal{Q}_{k-1}$ $(k \geqslant 1)$ *and groups* G_k $(k \geqslant 0)$ *such that the following hold*:

(a) $\mathcal{Q}_0 = \mathcal{Q}$, $G_0 = G$, *and* G_k *is isomorphic to the vector group* $\mathcal{G}_k$ *for* $k \geqslant 1$.

(b) p_k: $\mathcal{Q}_k \to \mathcal{Q}_{k-1}$ *is a* G_k-*structure. All the maps* p_k *admit global sections*; *hence, these principal bundles are trivial.*

(c) *If* θ_k *is the canonical* 1-*form on* $\mathcal{Q}_k$, *then* θ_k *takes values in* E_{k-1}.

(d) *If* $X \in \mathcal{F}$, *one can define inductively a lift* X_k *of* X *to a field in* $\mathcal{Q}_k$ *for each* $k \geqslant 0$. X_0 *is defined as in the paragraph above, and* $X_k = (X_{k-1})_0$ *for* $k \geqslant 1$.

We give two more elementary lemmas; the first is a simple exercise, the second is in [2, VI.2.1].

(2.2) Lemma. *Let $q\colon X \to Y$ be a quotient map of topological spaces (this is the case if q is continuous, open and onto). If $q^{-1}(y)$ is connected for all $y \in Y$, and $Z \subset Y$ is open and connected, then $q^{-1}Z$ is connected.*

(2.3) Lemma. *If X_0 is the natural lift of $X \in \mathcal{F}$, then it has the following properties:*

(a) *For all $g \in G$, $g_* X_0 = X_0$,*

(b) $L_{X_0}\theta = 0$,

(c) *X_0 projects on X.*

Conversely, if Y is a field on an open subset U of $\mathcal{Q}$, satisfying (a) *and* (b), *then Y is projectable onto a field X on pU and $Y = X_0$ on U.*

We get from this lemma that if $X \in \mathcal{F}$, then $L_{X_k}\theta_k = 0$ for all $k \geqslant 0$. We make from now on the hypothesis that $\mathcal{G}$ is of finite type; hence there is $k \geqslant 1$ such that $\mathcal{G}_{k-1} \neq 0$ and $\mathcal{G}_k = 0$. In this case θ_k is a parallelism on $\mathcal{Q}_k$, and X_k is a parallelism field for θ_k if $X \in \mathcal{F}$.

(2.4) Proposition. *If M is connected, and X, $Y \in \mathcal{F}$ are such that for some $a_k \in \mathcal{Q}_k$, $X_k(a_k) = Y_k(a_k)$, then $X = Y$.*

Proof. We get from (2.1)(a), (2.1)(b) and (2.2) that the connected components of $\mathcal{Q}_k$ are the sets $(p_1 \circ \cdots \circ p_k)^{-1}C = C_k$ where C is a component of $\mathcal{Q}$. If $a_k \in C_k$, then $X_k = Y_k$ on C_k by (1.2). Since these fields project on X and Y and $(p \circ p_1 \circ \cdots \circ p_k)C_k = M$, we get $X = Y$.

(2.5) Theorem. *If M is connected, and X, $Y \in \mathcal{F}$ are such that $X|U = Y|U$ for some open $U \subset M$, then $X = Y$.*

Proof. By definition of a k-lift, if $X|U = Y|U$ then $X_k = Y_k$ on $(p \circ \cdots \circ p_k)^{-1}U \subset \mathcal{Q}_k$, and the theorem follows from (2.4).

This generalizes (0.1) since the Lie algebras of the orthogonal group and the conformal group are of finite type [1, I.2].

3. The extension theorem

(3.1) Proposition. *Let the structure $\mathcal{Q}$ be analytic, and Z a vector field on an open connected subset W of $\mathcal{Q}_k$. Let $V \subset M$ be open, and $X \in \mathcal{F}_V$ such that $X_k = Z$ on $W \cap (p \circ \cdots \circ p_k)^{-1}V$. Then Z is projectable on a field $Y \in \mathcal{F}_U$, with $U = (p \circ \cdots p_k)W$, $Y = X$ on $U \cap V$, and $Y_k|W = Z$.*

Proof. Consider the 1-forms $L_Z\theta_k$ and $L_{X_k}\theta_k$; they coincide on $W \cap (p \circ \cdots \circ p_k)^{-1}V$, and by (2.3)(b) the second form is 0 there. Thus the analytic form $L_Z\theta_k = 0$ on W which is connected. Analogously one proves that for all $g_k \in G_k$, $(g_k)_*Z - Z = 0$ on W. Using (2.3) once more, we get that Z projects on a field Z_1 defined on $p_k(W) \subset \mathcal{Q}_{k-1}$, and $(Z_1)_0 = Z$ on W. It is easy to construct with these ideas a sequence of fields Z_h defined on $(p_{k-h+1} \circ \cdots \circ p_k)W \subset \mathcal{Q}_{k-h}$ which coincide with X_{k-h} on

$(p_{k-h+1} \circ \cdots \circ p_k)W \cap (p \circ \cdots p_{k-h})^{-1}V$, and on the common domain of definition Z_h equals the p-lift of Z_{h+p}. This construction can be carried down to M with the convention $\mathcal{Q}_{-1} = M$ and $p_0 = p$. It is immediate that the field $Y = Z_{k+1}$ has the required properties.

(3.2) Proposition. *Suppose that $\mathcal{Q}$ is analytic of finite type k. Let $(u; U)$ be a chart such that $u(U) \subset R^m$ is convex. If $V \subset U$ is open and connected, then any $X \in \mathcal{F}_V$ has a unique extension to a field $Y \in \mathcal{F}_U$.*

Proof. The uniqueness is clear from analyticity or (2.5). We prove the existence, assuming first that G is connected. Take a chart $(u'; U')$ on G such that $u'(U')$ is convex. We get easily from (2.1)(a) and (2.1)(b) that there is an open set $W \subset \mathcal{Q}_k$ diffeomorphic to the convex set $u(U) \times u'(U') \times \mathcal{G}_1 \times \cdots \times \mathcal{G}_k$ which projects onto U. On the other hand $(p \circ \cdots \circ p_k)^{-1}V \subset \mathcal{Q}_k$ is connected by (2.2), since G is connected. Now by applying (1.3) we obtain a field Z on W equal to X_k on $W \cap (p \circ \cdots \circ p_k)^{-1}V$, which is connected. By (3.1) Z projects on Y defined on U, and so is the required extension.

If G is arbitrary, let $C \subset \mathcal{Q}$ be a connected component of $\mathcal{Q}$, and let H be the connected component of the identity in G. It is clear that C is an H-structure on M of finite type k. If $\mathcal{F}'$ is the set of C-fields, we proved in the preceding paragraph that there is a field $Y \in \mathcal{F}'_U$ which extends X. We only need to show that $Y \in \mathcal{F}_U$. Let $\{\psi_t; t \in R\}$ be pseudogroup of Y. Then $Y \in \mathcal{F}_U$ if for all $t \in R$ and $a \in \mathcal{Q}$, $(\psi_t)_0 a \in \mathcal{Q}$. Writing $a = cg$ with $c \in C$ and $g \in G$ we get $(\psi_t)_0 a = (\psi_t)_0(cg) = ((\psi_t)_0 c)g \in Cg \subset \mathcal{Q}$. This ends the proof.

(3.3) Theorem. (*Generalization of* (0.2)). *Let M be a connected simply connected manifold, and $\mathcal{Q}$ an analytic G-structure on M of finite type. If U is an open connected subset of M and $X \in \mathcal{F}_U$, then X has a unique extension to a field $Y \in \mathcal{F}$.*

Proof. The uniqueness of the extension follows from analyticity or (2.5). The idea for proving the existence of the extension is a standard one in algebraic topology, and therefore we just give a sketch of the proof. Fix $x_0 \in U$. For each $x_1 \in M$ choose a continuous curve c: $[0, 1] \to M$ with $c(0) = x_0$ and $c(1) = x_1$. One shows: (a) There are a neighborhood N of $c([0, 1])$ and a field $Z \in \mathcal{F}_N$ which coincides with X in a neighborhood of x_0. (b) If c_0, c_1 are curves joining x_0 and x_1 and if Z_0, Z_1 are fields constructed as in (a), then $Z_0 = Z_1$ on a neighborhood of x_1. It follows from (a) and (b) that if we define the field Y on M by $Y(x_1) = Z_0(x_1) = Z_1(x_1)$, then Y is well defined and $Y \in \mathcal{F}$.

To prove (a) one considers the set S of $s \in [0, 1]$ such that there are a neighborhood M of $c([0, s])$ and a field $Z \in \mathcal{F}_N$ which coincides with X in a neighborhood of x_0. We want to show that $S = [0, 1]$. This follows from the fact that $0 \in S$, S is an interval open in $[0, 1]$, and $\sup S \in S$ by (3.2).

The proof of (b) is analogous. If $(s, t) \to c_s(t)$ is a homotopy between c_0 and c_1, consider S, the set of $s \in [0, 1]$ such that there are a neighborhood N of $\{c_r(t)\colon 0 \leqslant r \leqslant s, 0 \leqslant t \leqslant 1\}$ and a field $Z \in \mathcal{F}_N$ which coincides with X on a neighborhood of x_0. One shows that $0 \in S$, S is an interval open in $[0, 1]$ and $\sup S \in S$. This last fact requires (3.2) for its proof. It follows then that $S = [0, 1]$ proving (b).

Remark. Our main results (2.5) and (3.3) are also valid when M is the family of infinitesimal transformations of a linear connection ω on a G-structure A. If θ is the fundamental form on A, then $\pi = \theta \oplus \omega$ is a parallelism on A with values on $R^m \oplus \mathcal{G}$, and the natural lift of X is a field of the parallelism π [1]. The reader may check easily that the methods of proof of (2.5) and (3.3) work in this new situation.

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