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ANTI-HOLOMORPHIC AUTOMORPHISMS OF THE EXCEPTIONAL SYMMETRIC DOMAINS

H. A. JAFFEE

Introduction

A serious fault of the theory described in [8] and called "real forms of hermitian symmetric spaces" was the lack of information about the exceptional symmetric domains. This gap has been filled, and the new results are given here.

Let me now express my thanks to Professor Kuga for having posed the problems of [7], [8], and to Professors Borel, Helgason, and Langlands for several enlightening discussions about the present work. In particular, while the idea behind the "hard part" of Lemma (2.4) is my own, Borel is to be credited for adding the necessary rigor to my original argument. Various improvements in my original paper were suggested by the referee, especially the use of Theorem (2.10) in § 4.

1. The problem

Let X denote a hermitian symmetric space of noncompact type (for short, symmetric domain), $x_0 \in X$ a base point, $\mathscr{C}$ the set of anti-holomorphic involutive automorphisms of X, and $\mathscr{C}_0 = \{\sigma \in \mathscr{C} \mid \sigma(x_0) = x_0\}$. If G^h is the group of holomorphic automorphisms of X, and K^h the isotropy group at x_0, we have $X \approx G^h/K^h$. For any $\sigma \in \mathscr{C}$, we call $X^\sigma = \{x \in X \mid \sigma(x) = x\}$ the *real form* of X *associated to* σ. G^h acts by conjugation on $\mathscr{C}$, and K^h preserves $\mathscr{C}_0$. We call the quotient $\mathscr{C}/G^h$ the set of *complex conjugations* of X. Representing a conjugation by a $\sigma \in \mathscr{C}$ (or even $\mathscr{C}_0$ by Remark 2.3) we associate a real form to each conjugation. Another representative σ' for the same conjugation is G^h-conjugate to σ, hence the real form associated to a conjugation is well determined, up to isometry.

In Theorems (4.3) and (4.4) we give $\mathscr{C}/G^h$ and the associated real forms for the two exceptional symmetric domains. The theorem in [8] on the conjugations of a symmetric domain without exceptional factor now applies with no restriction. It follows that, in general, distinct conjugations have nonisometric real forms; we know no *a priori* reason for this.

The next section (§ 2) applies more generally than just to the exceptional sym-

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metric domains. Thus it also provides a sketch of a derivation of the (major) results of [8], although these results were first obtained by much more *ad hoc* procedures.

2. Preliminaries

(2.0) In § 1, we defined the set of complex conjugations $\mathscr{C}/G^h$ of X; the conjugations *fixing the base point* x_0 are $\mathscr{C}_0/K^h$. There is a hermitian symmetric space X_u of compact type associated to X "by duality" (see [4] for details). Let G_u^h be the (compact) group of holomorphic isometries of X_u, and K_u^h the isotropy subgroup at some chosen base point of X_u $(= G_u^h/K_u^h)$, $\mathscr{C}_u$ the set of anti-holomorphic involutive isometries of X_u, $\mathscr{C}_u^*$ the subset of $\mathscr{C}_u$ with fixed points, and $\mathscr{C}_u^0$ the subset fixing the base point. By results of Harish-Chandra and Borel [4, pp. 311–322], we may assume the following has been done. The domain X is holomorphically embedded as a bounded open subset of a $\boldsymbol{C}$-linear subspace p_- in the complexification of the Lie algebra of G^h, with the point x_0 going to the origin, and so that the isometries of X fixing x_0 are restrictions to X of $\boldsymbol{R}$-linear automorphisms of p_-. (Elements of K^h are $\boldsymbol{C}$-linear, and elements of $\mathscr{C}_0$ conjugate linear.) We embed p_- as a Zariski open [2, § 4.3 (4)] in X_u, with x_0 $(=0 \in p_-)$ going to the base point of X_u, and isometries of X_u which fix x_0 preserve the embedded p_- and the domain X inside it; we thus make identifications $K^h = K_u^h$, $\mathscr{C}_0 = \mathscr{C}_u^0$.

We choose a $\sigma_0 \in \mathscr{C}_0 = \mathscr{C}_u^0$; the group Gal $= \{1, \sigma_0\}$ acts by conjugation on G^h, on $K^h = K_u^h$, and on G_u^h. G^h acts by conjugation on $\mathscr{C}$, and K^h preserves $\mathscr{C}_0$; G_u^h acts by conjugation on $\mathscr{C}_u$ and preserves the subsets $\mathscr{C}_u^*$ and $\mathscr{C}_u^\emptyset \underset{\text{def'n}}{=} \mathscr{C}_u - \mathscr{C}_u^*$ of $\mathscr{C}_u$, and K_u^h preserves the subset $\mathscr{C}_u^0$. An easy translation of the definitions (see, e.g., [9, p. I–56]) gives

(2.1) Proposition. *There are canonical identifications* (*after the choice of* σ_0):

(2.1.1) $$\mathscr{C}/G^h = H^1(\mathrm{Gal}, G^h) \ ;$$

(2.1.2) $$\mathscr{C}_0/K^h = H^1(\mathrm{Gal}, K^h) \ ;$$

(2.1.3) $$\mathscr{C}_u/G_u^h = H^1(\mathrm{Gal}, G_u^h) \ ;$$

(2.1.4) $$\mathscr{C}_u^*/G_u^h = \textit{a subset } H_*^1(\mathrm{Gal}, G_u^h) \textit{ of } H^1(\mathrm{Gal}, G_u^h) \ ;$$

(2.1.5) $$\mathscr{C}_u^\emptyset/G_u^h = H_\emptyset^1(\mathrm{Gal}, G_u^h) \underset{\text{def'n}}{=} H^1(\mathrm{Gal}, G_u^h) - H_*^1(\mathrm{Gal}, G_u^h) \ .$$

We will use the abbreviations $H^1(G^h)$ for $H^1(\mathrm{Gal}, G^h)$, *etc.*

(2.2) Theorem. *The diagrams below are identical by Proposition* (2.1). *The maps* ι_1 *and* ι_2 *are bijections. Hence* $\mathscr{C}/G^h$ *is bijective to both* $\mathscr{C}_0/K^h$ *and* $\mathscr{C}_u^*/G_u^h$.

(2.2.1)
$$\begin{array}{ccc} & \overset{\iota_1}{\nearrow} & H^1_*(G^h_u) \subset H^1(G^h_u) \\ H^1(K^h) & \approx \quad \approx & \downarrow\Vert \\ & \underset{\iota_2}{\searrow} & H^1(G^h) \end{array}$$

(2.2.2)
$$\begin{array}{ccc} & \overset{\iota_1}{\nearrow} & \mathscr{C}^*_u/G^h_u \subset \mathscr{C}_u/G^h_u \\ \mathscr{C}_0/K^h & \approx \quad \approx & \downarrow\Vert \\ & \underset{\iota_2}{\searrow} & \mathscr{C}/G^h \end{array}$$

(2.3) Remark. The proof of Theorem (2.2) requires a lemma, but first we note several things. If σ has a fixed point x, and $gx = x_0$, then $g\sigma g^{-1}$ fixes x_0. The surjectivity of ι_1 and ι_2 then follows from the fact that G^h and G^h_u are transitive on X and X_u, respectively (and is a restatement of the fact that a $\sigma \in \mathscr{C}$ or $\mathscr{C}^*_u$ is G^h- or G^h_u-conjugate to a $\sigma' \in \mathscr{C}_0$).

(2.4) Lemma. *Let $\sigma \in \mathscr{C}_0$ or $\mathscr{C}^*_u$. Then the fixed point sets X^σ and X^σ_u of σ on X and X_u are connected.*

Proof. For X, let x_1 and x_2 be fixed by σ, and γ the unique geodesic segment joining x_1 and x_2. Since σ is an isometry γ must be preserved, and since the endpoints are fixed, γ is fixed pointwise.

For X_u, Remark (2.3) says that without loss of generality we may assume $\sigma \in \mathscr{C}^0_u$; as explained above, σ preserves a Zariski open set, say C, centered at x_0 and isomorphic to $\boldsymbol{C}^n$. The restriction of σ to $\boldsymbol{C}$ decomposes as a direct sum $(+1) \oplus (-1)$, where the $(+1)$-eigenspace D is isomorphic to $\boldsymbol{R}^n$, and Zariski dense in $\boldsymbol{C}$. Let x_1 be any fixed point of σ in X_u, outside of C. σ preserves another Zariski open C', centered at x_1 and isomorphic to $\boldsymbol{C}^n$. Let D' be the fixed point set of σ in C'. We have $D \cup D' \subset x^\sigma_u$, and will show that $D \cap D'$ is nonempty. But $C \cap C' = U$ is Zariski open in, say, C. Therefore $D \cap U$ is nonempty; hence $D \cap C'$ and also $D \cap D'$. This proves the lemma.

Proof of Theorem (2.2). We have already the surjectivity of ι_1 and ι_2. We shall prove the injectivity of ι_1, the proof for ι_2 being completely analogous. By [9, Cor. 1, pp. 1–65], $\ker(\iota_1)$ may be identified with the quotient of $(X_u)^{\sigma_0}$ by $(G^h_u)^{\sigma_0} =$ the centralizer of σ_0 in G^h_u. We have $x_0 \in (X_u)^{\sigma_0}$; let $x_1 \in (X_u)^{\sigma_0}$ and (using Lemma (2.4)) let γ be a geodesic (in $(X_u)^{\sigma_0}$) from x_0 to x_1. By [4, Th. 3.3, p. 173] we identify γ with a ray in the Lie algebra of $(G^h_u)^{\sigma_0}$, orthogonal to that of K^h_u, and by exponentiation obtain an element of $(G^h_u)^{\sigma_0}$ which transforms x_0 to x_1. This shows $\ker(\iota_1)$ is trivial. Now ι_1 is also the canonical map $\mathscr{C}_0/K^h \to \mathscr{C}^*_u/G^h_u$, so we may repeat the argument for any other $\sigma \in \mathscr{C}_0$ (applying Lemma (2.4) to each) to get that all fibres of ι_1 are trivial. This proves Theorem (2.2).

The following theorem of de Siebenthal implies that there are only a finite number of real forms of a symmetric domain.

(2.5) Lemma (*de Siebenthal*) [10, pp. 57–58]. *Let G be a compact Lie group, G_0 the identity component, $\sigma_0 \in G$, and T a maximal torus of the centralizer $(G_0)^{\sigma_0}$*

of σ_0 *in* G_0. *Given any* σ *in the component of* G *containing* σ_0, *there are a* $t \in T$ *and a* $g \in G_0$ *with* $\sigma = g(\sigma_0 t)g^{-1}$.

(2.6) Remark. If σ_0 and σ are involutions, then $t^2 = 1$.

(2.7) Theorem. *The set* $\mathscr{C}_u/G_u^h = H^1(G_u^h)$ *is finite.*

Proof. Let G be the compact group generated by G_u^h and σ_0. If $\sigma \in \mathscr{C}_u$ is in the same component of G as σ_0, then by Lemma (2.5) σ is G_0-conjugate, and therefore G_u^h-conjugate, to some $\sigma_0 t$, and $t^2 = 1$ by (2.6). If $l = \operatorname{rk}(T)$ with T defined in Lemma (2.5), then there are only 2^l such elements t. If G has only two components, then we would have card $(\mathscr{C}_u/G_u^h) \leq 2^l$. If not, let $\sigma_1 \in \mathscr{C}_u$ be an involution in a different component of G than σ_0. Then G is also generated by G_u^h and σ_1, and we can apply the above argument to the component of σ_1. Clearly, this process will bound card $(\mathscr{C}_u/G_u^h)$ by a (finite) sum of powers of 2, and finishes the proof.

(2.8) Corollary. *The number of complex conjugations* $c(X) =$ card $(\mathscr{C}/G^h)$ *of a symmetric domain* X *is finite.*

Proof. By Theorem (2.2), $\mathscr{C}/G^h$ is isomorphic to the subset $\mathscr{C}_u^*/G_u^h$ of $\mathscr{C}_u/G^h$. We will make extensive use of the classification of symmetric spaces, together with Theorem 2.10 below, to determine the isometry types of the real forms X^σ. Similar techniques were used in [5].

For now, let X denote a hermitian symmetric space which is purely non-Euclidean, σ an isometry, X^σ the set of fixed points of σ, and let $x_0 \in X^\sigma$. Let $G(X)$ be the identity component of the isometry group of X, $K(X) \subset G(X)$ the isotropy subgroup at x_0, and $G(X)^\sigma$ and $K(X)^\sigma$ the respective centralizers of σ. Let $G(X^\sigma)$ and $K(X^\sigma)$ be the full isometry and isotropy groups for X^σ, with base point x_0. Denote by $\mathfrak{G}(X)$, $\mathfrak{K}(X)$, *etc.* the Lie algebras of $G(X)$, $K(X)$, *etc.*

If $g \in G(X)^\sigma$ and $y \in X^\sigma$, then $g(y) \in X^\sigma$. By restriction, we associate an isometry $\rho(g)$ of X^σ to g. Denote by $\rho\colon G(X)^\sigma \to G(X^\sigma)$ and $d\rho\colon \mathfrak{G}(X)^\sigma \to \mathfrak{G}(X^\sigma)$ the homomorphisms thus defined. If $s \in G(X)$ is the symmetry in X at a point $y \in X^\sigma$, then $\sigma s \sigma^{-1}$ has differential -1 at y; thus $s \in G(X)^\sigma$, and $\rho(s)$ is a geodesic reflection for X^σ at y. This shows that X^σ itself is a globally symmetric space, and a symmetric subspace of X.

(2.9) Proposition. *With the above notation, the pair* $(d\rho(\mathfrak{G}(X)^\sigma), d\rho(\mathfrak{K}(X)^\sigma))$ *is an orthogonal involutive Lie subalgebra* (*in the sense of* [11, p. 235]) *of the orthogonal involutive Lie algebra of the component of* X^σ *containing* x_0.

Proof. The point is that the orbit of x_0 under $G(X)^\sigma$ is the entire component of X^σ containing x_0. This follows from [4, Theorem 3.3] as in the proof of Theorem (2.2). If X is of noncompact type, this fact is [12, Theorem 2.4.1]; the author is thankful to the referee for pointing this out.

Now if σ is an antiholomorphic involution, X^σ is connected by Lemma (2.4). Moreover $\dim_{\mathbf{R}} X^\sigma = \dim_{\mathbf{C}} X$ since multiplication by the complex structure J of X, along X^σ, defines an isomorphism between the tangent and normal bundles to X^σ. For the same reason, a holomorphic isometry of X is the identity if its restriction to X^σ is the identity. This implies that the maps ρ and $d\rho$ above are injective, as far as we are concerned in this paper.

(2.10) Theorem. *Let X be a hermitian symmetric space as above, and σ an antiholomorphic involutive isometry. If $\mathfrak{G}(X)^\sigma$ is semi-simple, or of the form $\boldsymbol{R}^1 \times$ semi-simple, then the orthogonal Lie algebra of X^σ is isomorphic to $(\mathfrak{G}(X)^\sigma, \mathfrak{K}(X)^\sigma)$.*

Proof. By injectivity of $d\rho$, the subalgebra in Proposition (2.9) is isomorphic to $(\mathfrak{G}(X)^\sigma, \mathfrak{K}(X)^\sigma)$. In the semi-simple case, [11, Lemma 8.2.3] says that this subalgebra is maximal, which gives the required statement. In the other case, the classification of orthogonal Lie algebras [11, Theorem 8.2.4], and of the Euclidean ones [11. Theorem 8.2.10], gives the same thing.

3. Descriptions of EIII and EVII

There are two irreducible hermitian symmetric spaces with exceptional isometry groups; they are the Riemannian symmetric spaces EIII and EVII of (complex) dimensions 16 and 27. Their Lie algebras are given [4, p. 354] in the noncompact form as $(e_{6(-14)}, so(10) + \boldsymbol{R})$ and $(e_{7(-25)}, e_6 + \boldsymbol{R})$, and the respective ranks are 2 and 3. The groups K^h and G^h are connected in each case [3, Théorème H].

The space EVII is described in [1, pp. 525–527] as a symmetric subspace of $\mathfrak{h}_{28}$, the Siegel space of genus 28. The action of the group G^h is given there. The same picture is given in [5], where a bounded version (cf. § 2.0 above) is also given. We will denote this bounded symmetric domain by Z. The group K^h for Z is described using Jordan algebras. If $\mathfrak{a}$ is the exceptional central simple reduced (nondivision) Jordan algebra ([6, p. 80] for the definition) of dimension 27 over $\boldsymbol{R}$, Aut $(\mathfrak{a})$ is the compact Lie group F_4. Then K^h is a subgroup of $GL(\mathfrak{a} \otimes \boldsymbol{C})$: $K^h = \{\exp(iL(x)) \cdot V \mid V \in \text{Aut}(\mathfrak{a})$ and $L(x)$ is left multiplication by an $x \in \mathfrak{a}\}$. The center T^1 of K^h is $\{\exp(iL(\xi 1)) \mid 1 = \text{identity of } \mathfrak{a}, \xi \in \boldsymbol{R}\}$, and $K^h = T^1 \cdot E_6$ (see [11, p. 315]); the intersection $T^1 \cap E_6 = (\varepsilon)$ is the center of E_6, which is cyclic of order 3 [11, Cor. 8.9.28] generated by, say, ε.

The space EIII is a holomorphic subdomain of Z. Let, for example, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ be a (primitive) idempotent of $\mathfrak{a}$ (see [5], [6] for details). The Peirce-decomposition $\mathfrak{a} = \mathfrak{a}_0 + \mathfrak{a}_1 + \mathfrak{a}_{1/2}$ is associated, which is the eigenspace decomposition for the idempotent. Define $a \in F_4 \subset K^h$ as

$$+1 \text{ on } \mathfrak{a}_0 + \mathfrak{a}_1 , \qquad \text{and } -1 \text{ on } \mathfrak{a}_{1/2} .$$

Let ι denote the geodesic reflection of Z at $0 \in \mathfrak{a} \otimes \boldsymbol{C}$; $\iota(z) = -z$. The space Y of fixed points of $\iota \circ a$ on Z is shown in [5] to be isomorphic to the hermitian symmetric space EIII.

For determining $\mathscr{C}/G^h$ and the associated real forms for the two symmetric domains described above, we shall use an explicitly constructed element $\sigma_0 \in \mathscr{C}(Z)$. Define σ_0 by $z \mapsto -\bar{z}$ in the nonbounded versions of [1] and [5] and by

$z \mapsto \bar{z}$ in the bounded version Z. If $0 \in Z$ denotes the base point referred to above, we have $\sigma_0(0) = 0$. In the notation of [1], σ_0 is represented by the matrix $\left(\begin{array}{c|c} 1_{28} & 0 \\ \hline 0 & -1_{28} \end{array}\right)$. Conjugation by σ_0 on K^h is the involutive automorphism $\pi(\sigma_0)$:

$$\exp(iL(x)) \cdot V \mapsto \exp(-iL(x)) \cdot V ,$$

so the centralizer $(K^h)^{\sigma_0}$ has identity component F_4.

4. Computations of $\mathscr{C}/G^h$ and the real forms

We first compute the set of complex conjugations $\mathscr{C}/G^h$ for the space Z (cf. § 3) of type EVII. By (2.2.2), this set is bijective to $\mathscr{C}_0/K^h$. $\mathscr{C}_0$ is the set of elements of order 2 in the component $K^h \cdot \sigma_0$ of $K = K^h \cup K^h \cdot \sigma_0$, where $\sigma_0 \in \mathscr{C}_0$ is defined at the end of § 3. $\mathscr{C}_0/K^h$ is thus the set of K^h-conjugacy classes of elements of order 2 in $K^h \cdot \sigma_0$. The automorphism $\pi(\sigma_0)$ of $K^h = T^1 \cdot E_6$ leaves the "factors" invariant. Thus the subset $\tilde{A} = E_6 \cup E_6 \cdot \sigma_0$ of K is a subgroup. Define $\pi: \tilde{A} \to \mathrm{Aut}(E_6)$ through the adjoint action (hence the symbolism $\pi(\sigma_0)$). $\mathrm{Aut}(E_6)/\mathrm{Inn}(E_6) \approx \boldsymbol{Z}/2\boldsymbol{Z}$ by [11, Cor. 8.11.3], and $\pi(\sigma_0)$ fixes a subgroup (F_4) of nonmaximal rank in E_6, so $\pi(\sigma_0)$ is an "outer" automorphism. Hence π is surjective, with $\ker(\pi) = (\varepsilon)$.

(4.1) Proposition. *Any element $\lambda e \cdot \sigma_0 \in K^h \cdot \sigma_0$ ($\lambda \in T^1$, $e \in E_6$) is K^h-conjugate to $e \cdot \sigma_0$.*

Proof. Conjugate $\lambda e \cdot \sigma_0$ by any $\sqrt{\lambda}^{-1}$ in T^1. Then use $\sigma_0\sqrt{\lambda} = \sqrt{\lambda}^{-1}\sigma_0$.

(4.2) Proposition. *The elements $\sigma_1 = e_1 \cdot \sigma_0$ and $\sigma_2 = e_2 \cdot \sigma_0$ are K^h-conjugate if $\pi(\sigma_1)$ and $\pi(\sigma_2)$ are* ad (E_6)*-conjugate in* Aut (E_6).

Proof. Suppose $\exists a \in \mathrm{ad}(E_6)$ with $\pi(\sigma_1) = a\pi(\sigma_2)a^{-1}$. Lift a to any $\tilde{a} \in \pi^{-1}(a)$ and obtain $\sigma_1\varepsilon^i = \tilde{a}\sigma_2\tilde{a}^{-1}$ for some $\varepsilon^i \in (\varepsilon)$. Now apply Proposition (4.1) to $\sigma_1\varepsilon^i = \varepsilon^{-i}\sigma_1$ with $\lambda = \varepsilon^{-i}$.

Computing $\mathscr{C}/G^h$ now reduces to finding the ad (E_6)-conjugacy classes of outer involutive automorphisms of E_6. These are determined in [11, p. 288] as part of the classification of symmetric spaces of E_6. Each such automorphism is ad (E_6)-conjugate to either $\pi(\sigma_0)$ or another automorphism, say $\pi(\sigma_1)$, and the Lie algebras of the centralizers in E_6 of σ_0 and σ_1 are[1] respectively f_4 and $sp(4)$. By Proposition (4.2), σ_0 and σ_1 represent all conjugations of Z; strictly speaking as yet they might be K^h-conjugate. We will find that the associated real forms are nonisometric, hence σ_0 and σ_1 are not G^h-conjugate.

As in § 2, extend each of σ_0 and σ_1 to the compact dual Z_u of Z. Let $Z_u^{\sigma_i}$ ($i = 0, 1$) be the respective fixed point sets. The centralizers $G(Z_u)^{\sigma_i}$ are such that the quotients $G(Z_u)/G(Z_u)^{\sigma_i}$ are symmetric spaces of $G(Z_u) = \mathrm{ad}(E_7)$. By classification of these [11, table, p. 285] the only possibilities for $\mathfrak{G}(Z_u)^{\sigma_i}$ are

[1]The hidden fact here is that $\exists\sigma_1 \in \pi^{-1}(\pi(\sigma_1))$ with $\sigma_1^2 = 1$. $F_4 \subset \tilde{A}$, and $F_4 \subset$ centralizer of σ_0, so $\pi|_{F_4}$ is an isomorphism. By [11, p. 288], $\pi(\sigma_1) = \pi(\varphi) \circ \pi(\sigma_0)$ with $\pi(\varphi) \in \pi(F_4)$ and $\pi(\varphi)^2 = 1$. The unique lift $\varphi \in \pi^{-1}(\pi(\varphi))$ then satisfies $\varphi^2 = 1$, $\varphi\sigma_0 = \sigma_0\varphi$. Now $\sigma_1 = \varphi \circ \sigma_0$ is as required.

(1) $\boldsymbol{R} \times e_6$, (2) $su(8)$, (3) $so(12) \times su(2)$.
By Theorem (2.10), the orthogonal Lie algebra $\mathfrak{G}(Z_u^{\sigma_i})$ is isomorphic to one of these. Since $\dim Z_u^{\sigma_i} = 27$, $\mathrm{rk}\,(Z_u^{\sigma_i}) \leq 3$, and $\mathfrak{R}(Z_u^{\sigma_i}) \approx f_4$ or $sp(4)$ $(i = 0, 1)$, we see that the classification of symmetric spaces with Lie algebras (1), (2), or (3) uniquely determines $\mathfrak{G}(Z_u^{\sigma_i})$. Namely, $\mathfrak{G}(Z_u^{\sigma_0}) = (\boldsymbol{R} \times e_6, f_4)$, $\mathfrak{G}(Z_u^{\sigma_1}) = (su(8), sp(4))$. By duality, we get the real forms Z^{σ_0} and Z^{σ_1}.

(4.3) Theorem. *There are* 2 *complex conjugations of the symmetric domain* $Z = (e_{7(-25)}, e_6 + \boldsymbol{R})$. *The associated real forms are* $\boldsymbol{R} \times (e_{6(-26)}, f_4)$, *and* $(su^*(8), sp(4))$ *both of rank* 3.

We next compute the set $\mathscr{C}/G^h$ for the domain Y (see § 3) of type EIII. Let Y_u be the compact dual. By (2.2.2) we have $\mathscr{C}/G^h \approx \mathscr{C}_u^*/G_u^h$. It is shown in [11, p. 316] that every isometry of Y_u has a fixed point; thus $\mathscr{C}_u^* = \mathscr{C}_u$. Now $\mathscr{C}/G^h \approx \mathscr{C}_u/G_u^h$. The identity component Is_0 of the isometry group Is of Y_u is isomorphic to ad (E_6) by [11, Theorem 8.7.9]. $Is(Y_u)$ has 2 components [3, Theorem H] and some element of $Is - Is_0$ gives rise to an outer automorphism of E_6 [11, p. 316]. Hence there is an isomorphism $\psi\colon \mathrm{Aut}\,(E_6) \rightleftarrows Is(Y_u)$. Now ψ defines an isomorphism between $\mathscr{C}_u/G_u^h$ and the ad (E_6)-conjugacy classes of outer involutive automorphisms of E_6. As quoted before, the latter are represented by $\pi(\sigma_0)$ and $\pi(\sigma_1)$, so that $\tau_0 = \psi(\pi(\sigma_0))$ and $\tau_1 = \psi(\pi(\sigma_1))$ represent the complex conjugations of Y under $\mathscr{C}_u/G_u^h \rightleftarrows \mathscr{C}/G^h$.

We will find the associated real forms on Y by working on Y_u and then dualizing. The centralizers $\mathfrak{G}(Y_u)^{\tau_i}$ $(i = 0, 1)$ are isomorphic to f_4 and $sp(4)$. Hence by Theorem (2.10) the orthogonal Lie algebra $\mathfrak{G}(Y_u^{\tau_i})$ is isomorphic to f_4 and $sp(4)$ for $i = 0, 1$. The classification of these orthogonal involutive Lie algebras, together with $\dim Y_u^{\tau_i} = 16$, $\mathrm{rk}\,(Y_u^{\tau_i}) \leq 2$, again uniquely determines $\mathfrak{G}(Y_u^{\tau_i})$. Namely, $\mathfrak{G}(Y_u^{\tau_0}) = (f_4, so(9))$, $\mathfrak{G}(Y_u^{\tau_1}) = (sp(4), sp(2) \times sp(2))$. By duality, we have

(4.4) Theorem. *There are* 2 *complex conjugations of the symmetric domain* $Y = (e_{6(-14)}, so(10) + \boldsymbol{R})$. *The associated real forms are* $(f_{4(-20)}, so(9))$ *of rank* 1, *and* $(sp(2, 2), sp(2) \times sp(2))$ of rank 2.

5. Retrospections

One of the conjugations of Z is represented by the involution σ_0 which was defined explicitly in § 3. We would like explicit definitions also for σ_1, τ_0, τ_1 whose real forms are given in Theorems (4.3) and (4.4). For Z, F_4 is a subgroup of K^h. An involution $\beta \in F_4$ (the "quaternion" case) is defined in [6, Theorem 13]; β and σ_0 commute. The author suspects that $\beta \circ \sigma_0$ represents the same conjugation of Z as σ_1.

σ_0 commutes with the defining isometry $\iota \circ \alpha$ of Y, hence leaves Y invariant. It is not difficult to show using [6] that the restriction of σ_0 to Y represents the same conjugation as τ_0. The author suspects that $\beta \circ \sigma_0$ also leaves Y invariant, and that the restriction is conjugate to τ_1.

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