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COMPACT REAL HYPERSURFACES WITH CONSTANT MEAN CURVATURE OF A COMPLEX PROJECTIVE SPACE

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Introduction

The differential geometry of hypersurfaces of a Riemannian manifold of constant curvature and complex hypersurfaces of a Kaehlerian manifold has been studied for a long time. In particular, many global results have been obtained (for example [1], [3]) since the establishment of J. Simons' formula [6] for the Laplacian of the second fundamental form. However, the differential geometry of real hypersurfaces of a Kaehlerian manifold has not been explored to any great extent, even in the case where the ambient manifold is a complex projective space CP^m. One of the main reasons for us not to be able to get many results on a real hypersurface is the lack of enough "words" to describe differential geometric properties of the hypersurface. For instance, totally geodesic hypersurfaces and totally umbilical hypersurfaces characterize respectively hyperplanes and hyperspheres, when the ambient manifold is a Euclidean space, and they respectively characterize great and small spheres, when the ambient manifold is a sphere. But if the ambient manifold is a CP^m, as a consequence of Codazzi equation, we know that there exist neither totally geodesic hypersurfaces nor totally umbilical hypersurfaces (for example, [7]). One way to overcome such poverty of vocabulary has been established by H. B. Lawson [2] who introduced the notion of generalized equator $M^c_{p,q}$ of a CP^m. His idea is to construct a circle bundle over a real hypersurface, which is compatible with the Hopf fibration. Thus we can use many words to characterize remarkable classes of submanifolds of a sphere. By making use of the second fundamental form and the fundamental tensor of submersion, the present author [4] gave a condition for the circle bundle over a real hypersurface of a CP^m to be a product of two spheres.

Keeping this point of view, in this paper we study compact real hypersurfaces of a CP^m with constant mean curvature.

In § 1, we review necessary results obtained in [2] and [4] for the use in § 3. In § 2 we compute the Laplacian of the length of the second fundamental form of a real hypersurface of a CP^m.

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Arranging so nicely the terms appeared in the Laplacian that we may use the results stated in § 1 and § 3, we first prove Lawson's theorem by a different method and then prove two theorems which give sufficient conditions for a real hypersurface of a CP^m to be $M^c_{o,p}$.

1. Submersions and real hypersurfaces of a complex projective space

Let S^{n+2} be an odd-dimensional unit sphere, $CP^{(n+1)/2}$ the complex projective space, and $\tilde{\pi}$ the Riemannian submersion of S^{n+2} to $CP^{(n+1)/2}$ defined by the Hopf fibration. The almost complex structure J of $CP^{(n+1)/2}$ is nothing but the fundamental tensor of the submersion $\tilde{\pi}$, and the Riemannian metric G of $CP^{(n+1)/2}$ is induced naturally from that of S^{n+2}. With respect to (J, G), $CP^{(n+1)/2}$ is a Kaehlerian manifold of constant holomorphic sectional curvature 4. For a real hypersurface M of $CP^{(n+1)/2}$ the circle bundle $\overline{M}$ over M which is compatible with the submersion $\tilde{\pi}$ is a hypersurface of S^{n+2}. Thus we have the following commutative diagram of submersions $\tilde{\pi}$, π and imbeddings $\tilde{i}$ and i:

$$\begin{array}{ccc} \overline{M} & \xrightarrow{\tilde{i}} & S^{n+2} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\tilde{\pi}} \\ M & \xrightarrow[i]{} & CP^{(n+1)/2} \end{array}$$

The imbedding $\tilde{i}$ is an isometry on the fibres. The diagram implies that for the unit vertical vector field $\overline{V}$ of $\overline{M}$, $\tilde{i}(\overline{V})$ is also a unit vertical field of S^{n+2} and that for any tangent vector field X to M, $i(X)^L = \tilde{i}(X^L)$, where X^L denotes the horizontal lift of X. For an arbitrary point $p \in M$ we may choose a field of unit normal vectors N to M defined in a neighborhood $\mathscr{U}$ of p. Let $\bar{p}$ be an arbitrary point of the fibre over p. Then the lift N^L of N is a field of unit normal vectors to $\overline{M}$ defined in a tubular neighborhood over $\mathscr{U}$.

Let $\overline{D}$ and D be the Riemannian connections of S^{n+2} and $CP^{(n+1)/2}$ respectively. Then the respective Riemannian connections $\overline{\nabla}$, ∇ and the second fundamental forms $\bar{h}$, h of $\overline{M}$ and M are given by

$$\text{(1.1)} \quad \overline{D}_{\tilde{i}(\overline{X})}\tilde{i}(\overline{Y}) = \tilde{i}(\overline{\nabla}_{\overline{X}}\overline{Y}) + \bar{h}(\overline{X}, \overline{Y}) , \qquad D_{i(X)}i(Y) = i(\nabla_X Y) + h(X, Y) .$$

We denote the Weingarten maps corresponding to $\bar{h}$ and h by $\overline{H}$ and H respectively, that is, $\bar{h}(\overline{X}, \overline{Y}) = \bar{g}(\overline{H}\,\overline{X}, \overline{Y})N^L$, $h(X, Y) = g(HX, Y)N$. On the other hand, the fundamental equations of submersions $\tilde{\pi}$ and π are given by

$$\text{(1.2)} \quad \begin{aligned} \overline{D}_{Y'^L}X'^L &= (D_{Y'}X')^L + G(JY', X')^L\tilde{i}(\overline{V}) , \\ \overline{\nabla}_{Y^L}X^L &= (\nabla_Y X)^L + g(FY, X)^L\overline{V} , \end{aligned}$$

where F is the fundamental tensor of the submersion π.

For the second fundamental tensors of $\overline{M}$ and M, we have the following identities [4]:

$$\bar{g}(\overline{H}X^L, Y^L) = g(HX, Y)^L , \tag{1.3}$$

$$(Ji(X))^L = \tilde{i}(FX)^L - \bar{g}(\overline{H}\overline{V}, X^L)N^L , \tag{1.4}$$

$$\overline{H}X^L = (HX)^L + \bar{g}(\overline{H}X^L, \overline{V})\overline{V} , \tag{1.5}$$

$$\text{trace } \overline{H} = (\text{trace } H)^L , \tag{1.6}$$

$$\bar{g}(\overline{H}\overline{V}, \overline{V}) = 0 . \tag{1.7}$$

Furthermore in [4] we proved

Lemma 1.1. *In order that the Weingarten map $\overline{H}$ of $\overline{M}$ be covariant constant, it is necessary and sufficient that the Weingarten map H of M commutes with the fundamental tensor F of π.*

Now we consider the transforms $Ji(X)$ and JN of $i(X)$ and N by J at a point $p \in M$. Then from the skew symmetric property of J and (1.4), we may put

$$Ji(X) = i(FX) + u(X)N , \qquad JN = -i(U) ,$$

for some $U \in T_p(M)$. Using (1.4) again, we get

$$\bar{g}(\overline{H}\overline{V}, X^L) = -g(U, X)^L , \tag{1.8}$$

$$u(X) = g(U, X) . \tag{1.9}$$

Making use of (1.5) and (1.8), we can easily prove

Lemma 1.2. *Let X be an eigenvector of H corresponding to an eigenvalue λ. If X is perpendicular to U, X^L is an eigenvector of $\overline{H}$ corresponding to the eigenvalue λ.*

By iterating the operator J on $i(X)$ and N, we obtain

$$F^2X = -X + u(X)U , \tag{1.10}$$

$$FU = 0 , \tag{1.11}$$

$$g(U, U) = 1 . \tag{1.12}$$

As to the covariant derivatives of F and U, we have

$$(\nabla_Y F)X = u(X)HY - g(HX, Y)U , \tag{1.13}$$

$$\nabla_X U = FHX , \tag{1.14}$$

because of the fact that J is a covariant constant.

2. Laplacian for the length of the second fundamental form

Let M be a real hypersurface of $CP^{(n+1)/2}$ with constant mean curvature, that is, (trace $H)/n$ is constant. If trace H vanishes identically on M, M is said to be *minimal*. Since the curvature tensor of $CP^{(n+1)/2}$ is given by

$$\begin{aligned} R'(X', Y')Z' = {} & G(Y', Z')X' - G(X', Z')Y' + G(JY', Z')JX' \\ & -G(JX', Z')JY' - 2G(JX', Y')JZ' , \end{aligned} \tag{2.1}$$

where X', Y', Z' are tangent vector fields on $CP^{(n+1)/2}$, the Gauss equation for the curvature tensor R of M and the Codazzi equation become, respectively,

$$\begin{aligned} R(X, Y)Z = {} & g(Y, Z)X - g(X, Z)Y + g(FY, Z)FX - g(FX, Z)FY \\ & -2g(FX, Y)FZ + g(HY, Z)HX - g(HX, Z)HY , \end{aligned} \tag{2.2}$$

$$(\nabla_X H)Y - (\nabla_Y H)X = u(X)FY - u(Y)FX - 2g(FX, Y)U . \tag{2.3}$$

Now we consider the function $f =$ trace H^2, which is globally defined on M, and will compute its Laplacian Δf. Denoting the restricted Laplacian for H by $\Delta' H$, we have

$$\tfrac{1}{2}\Delta f = g(\Delta' H, H) + g(\nabla H, \nabla H) , \tag{2.4}$$

where we have extended the metric g to the tensor space in the standard fashion. Since the mean curvature of M is constant, making use of (1.11), (1.12), (1.13), (1.14), (2.3) and computing in entirely the same way as in [3], we obtain

$$\begin{aligned} \Delta' H = {} & \sum_{i=1}^{n} [R(E_i, X), H]E_i \\ & + 3FHFX + 3(\text{trace } H)u(X)U - 3u(HX)U , \end{aligned} \tag{2.5}$$

where $\{E_1, \cdots, E_n\}$ is an orthonormal frame at a point $p \in M$.

Substituting (2.2) in the right-hand side of (2.5) gives

$$\begin{aligned} (\Delta' H)X_{(p)} = {} & \sum_{i=1}^{n} \{R(E_i, X)HE_i - HR(E_i, X)E_i\} \\ & + 3FHFX + 3(\text{trace } H)u(X)U - 3u(HX)U \\ = {} & 6FHFX + 3(\text{trace } H)u(X)U - 3u(HX)U + (n + 3)HX \\ & - (\text{trace } H)X - (\text{trace } H^2)HX - 3u(X)HU + (\text{trace } H)H^2X . \end{aligned}$$

Thus it follows that

$$\begin{aligned} g(\Delta' H, H) = {} & \text{trace } (\Delta' H)H \\ = {} & 6 \text{ trace } (FH)^2 + 3(\text{trace } H)u(HU) - 6u(H^2U) \end{aligned}$$

$$+ (n+3) \text{ trace } H^2 - (\text{trace } H)^2 - (\text{trace } H^2)^2 + (\text{trace } H)(\text{trace } H^3) \,,$$

from which we obtain

$$\begin{aligned} \tfrac{1}{2}\Delta f = {} & 6 \text{ trace } (FH)^2 + 3(\text{trace } H)u(HU) - 6u(H^2U) \\ & + (n+3) \text{ trace } H^2 - (\text{trace } H)^2 - (\text{trace } H^2)^2 \\ & + (\text{trace } H)(\text{trace } H^3) + g(\nabla H, \nabla H) \,. \end{aligned} \tag{2.6}$$

In order to translate the conditions imposed on M into those on $\overline{M}$ we have to change (2.6) to a more favorable form. For this purpose we calculate the length of $HF - FH$. Since $HF - FH$ is a symmetric linear transformation on $T_p(M)$, it follows that

$$\begin{aligned} g(HF - FH, HF - FH) &= \text{trace } (HF - FH)^2 \\ &= 2 \text{ trace } (HF)^2 - 2 \text{ trace } H^2F^2 \\ &= 2 \text{ trace } (HF)^2 + 2 \text{ trace } H^2 - 2u(H^2U) \,, \end{aligned} \tag{2.7}$$

because of (1.10). From the last two equations, we have

$$\begin{aligned} \tfrac{1}{2}\Delta f = {} & 3g(HF - FH, HF - FH) + (n-3)(\text{trace } H^2) \\ & + 3(\text{trace } H)u(HU) - (\text{trace } H)^2 - (\text{trace } H^2)^2 \\ & + (\text{trace } H)(\text{trace } H^3) + 2(n-1) + g(\overset{*}{\nabla} H, \overset{*}{\nabla} H) \,, \end{aligned} \tag{2.8}$$

where we put

$$(\overset{*}{\nabla}_Y H)X = (\nabla_Y H)X + u(X)FY + g(FY, X)U \,.$$

3. Theorems on compact real hypersurfaces of $CP^{(n+1)/2}$

Before we state our results we should explain the models which will appear in our theorems. In S^{n+2} we have the family of generalized Clifford surfaces $M_{p,q} = S^p \times S^q$, where $p + q = n + 1$. By choosing the spheres to lie in complex subspaces we get fibrations $S^1 \to M_{2p+1,2q+1} \to M^c_{p,q}$ compatible with the Hopf fibration, where $p + q = n$. In the special case $p = 0$, this surface is diffeomorphic to the sphere.

Remark. In [1] and [2], $M_{p,q}$ always means $S^p \times S^q$ which is immersed in S^{n+1} minimally, and so the radius of S^p and S^q are respectively $p/(n+1)$ and $q/(n+1)$. But here we do not necessarily need the condition that M is minimal.

We begin with

Theorem 3.1 (*H. B. Lawson*). *Let M be a compact n-dimensional real minimal hypersurface of $CP^{(n+1)/2}$ over which the second fundamental form satisfies the inequality*

$$\text{trace } H^2 \leq n - 1 .$$

Then trace $H^2 = n - 1$ *and* $M = M^c_{p,q}$ *for some p and q.*

Proof. The right hand side of (2.8) becomes

$$3g(HF - FH, HF - FH) + \{(\text{trace } H^2) + 2\}\{(n-1) - \text{trace } H^2\} \\ + (\text{trace } H)\{(3u(HU) - (\text{trace } H) + (\text{trace } H^2)\} + g(\overset{*}{\nabla} H, \overset{*}{\nabla} H) .$$

Thus, if M is minimal and trace $H^2 \leq n - 1$, from Bochner's lemma $\Delta f = 0$ and consequently trace $H^2 = n - 1$ it follows that $HF = FH$. Hence, because of Lemma 1.1, $\bar{M}$ has parallel second fundamental form. This and a result of Ryan [5] show that $\bar{M}$ is a sphere S^{n+1} or a product of two spheres. Since the fibration $\pi: \bar{M} \to M$ is compatible with the Hopf fibration, we have $M = M^c_{p,q}$ for some p, q. This completes the proof.

In order to get further results, we need

Lemma 3.2. *On a real hypersurface M of* $CP^{(n+1)/2}$ *the inequality*

$$(\text{trace } H)^2 \leq (n-1)(\text{trace } H^2) + 2u(HU)(\text{trace } H) \tag{3.1}$$

holds.

Proof. For any $X \in T_p(M)$, set

$$KX = HX + \frac{1}{n-1}(\text{trace } H)u(X)U . \tag{3.2}$$

Since K is a symmetric linear transformation on $T_p(M)$, we have

$$n \text{ trace } K^2 \geq (\text{trace } K)^2 ,$$

which implies (3.1).

Theorem 3.3. *Let M be a compact real hypersurface of* $CP^{(n+1)/2}$ *with constant mean curvature on which the second fundamental form is semidefinite. If* trace $H^2 \leq n - 1$, *then* trace $H^2 = n - 1$ *and* $M = M^c_{p,o}$.

Proof. By means of Lemma 3.2, (2.8) becomes

$$\begin{aligned} \tfrac{1}{2}\Delta f \geq\ & 3g(HF - FH, HF - FH) + (\text{trace } H)(\text{trace } H^3) - (\text{trace } H^2)^2 \\ & + (\text{trace } H)u(HU) + 2\{(n-1) - (\text{trace } H^2)\} + g(\overset{*}{\nabla} H \cdot \overset{*}{\nabla} H) . \end{aligned} \tag{3.3}$$

Since the second fundamental form is semidefinite, it follows that

$$HF = FH , \tag{3.4}$$

$$(\text{trace } H)(\text{trace } H^3) = (\text{trace } H^2)^2 , \tag{3.5}$$

$$u(HU) = 0 , \tag{3.6}$$

(3.7) $$\operatorname{trace} H^2 = n - 1 \,.$$

(3.8) $$(\nabla_Y H)X = -u(X)FY - g(FY, X)U \,.$$

Let $a_1, \cdots, a_n$ be eigenvalues of H. Then (3.5) becomes

$$\sum_{i<j} a_i a_j (a_i - a_j)^2 = 0 \,,$$

which, together with the fact that $CP^{(n+1)/2}$ has no totally umbilical real hypersurfaces, implies that H has exactly two eigenvalues and one of them must be zero. Moreover (3.4) and (3.5) show that U is one of the eigenvectors corresponding to zero, i.e., $HU = 0$. Differentiating covariantly this equation and making use of (1.11), (1.14) and (3.8), we obtain

$$H^2X - X + u(X)U = 0 \,,$$

so that only the vectors in the direction of U correspond to eigenvalue zero of H. Thus from (3.7) it follows that with repect to the orthonormal frame formed by the eigenvectors, H takes one of the following forms:

$$(H) = \begin{pmatrix} 0 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix} \quad \text{or} \quad (H) = \begin{pmatrix} 0 & & & & 0 \\ & -1 & & & \\ & & -1 & & \\ & 0 & & \ddots & \\ & & & & -1 \end{pmatrix},$$

Hence, because of (1.6) and Lemma 1.2, it follows that with respect to a suitable orthonormal frame $\bar{H}$ of $\bar{M}$ takes one of the following forms:

$$(\bar{H}) = \begin{pmatrix} \alpha & & & & 0 \\ & -\alpha & & & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix} \quad \text{or} \quad (\bar{H}) = \begin{pmatrix} \alpha & & & & 0 \\ & -\alpha & & & \\ & & -1 & & \\ & 0 & & \ddots & \\ & & & & -1 \end{pmatrix}.$$

Since the respective unit eigenvectors $\bar{X}$, $\bar{Y}$ of $\bar{H}$ corresponding to α and $-\alpha$ can be written in the form

$$\bar{X} = \bar{V}\cos\theta + U^L \sin\theta \,, \qquad Y = -\bar{V}\sin\theta + U^L\cos\theta \,,$$

we have

$$\bar{H}\bar{X} = \bar{H}\bar{V}\cos\theta + \bar{H}U^L\sin\theta = \alpha\bar{V}\cos\theta + \alpha U^L\sin\theta \,,$$
$$\bar{H}\bar{Y} = -\bar{H}\bar{V}\sin\theta + \bar{H}U^L\cos\theta = \alpha\bar{V}\sin\theta - \alpha U^L\cos\theta \,.$$

Computing the inner product $\bar{g}(\bar{H}\bar{X}, \bar{V})$ and making use of (1.7), (1.8), we get $\alpha = -\tan\theta$. On the other hand computing $\bar{g}(\bar{H}\bar{X}, U^L)$ gives $\alpha = -\cot\theta$. Thus $\alpha = \pm 1$ and

$$(\bar{H}) = \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ 0 & & \ddots & \\ & & & 1 \end{bmatrix} \quad \text{or} \quad (\bar{H}) = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ 0 & & \ddots & \\ & & & -1 \end{bmatrix}.$$

In both cases we have $\bar{M} = S^1 \times S^n$ and consequently $M = M^c_{o,p}$, $p = \frac{1}{2}(n-1)$. This completes the proof.

Theorem 3.4. *Let M be a compact real hypersurface of $CP^{(n+1)/2}$ with constant mean curvature such that the second fundamental form is semidefinite. If* (trace $H)^2 \le (n-1)^2$, *then* $M = M^c_{o,p}$, $p = \frac{1}{2}(n-1)$.

Proof. From (2.8) and Lemma 3.2, we have

$$\tfrac{1}{2}\Delta f \ge 3g(HF - FH, HF - FH) + (\mathrm{trace}\, H)(\mathrm{trace}\, H^3) - (\mathrm{trace}\, H^2)^2 + \frac{n+3}{n-1}(\mathrm{trace}\, H)u(HU) + \frac{2}{n-1}\{(n-1)^2 - (\mathrm{trace}\, H)^2\}.$$

If $(\mathrm{trace}\, H)^2 \le (n-1)^2$, we get (3.4), (3.5), (3.6), (3.8) and $(\mathrm{trace}\, H)^2 = (n-1)^2$. Thus we can prove the theorem in entirely the same way as we proved Theorem 3.3.

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