J. DIFFERENTIAL GEOMETRY
13 (1978) 11-23

# RESIDUES OF SINGULARITIES OF HOLOMORPHIC FOLIATIONS

BOHUMIL CENKL

**1.** This note contains an algorithm for the computation of the residues associated with the singularities of holomorphic foliations on compact complex analytic manifolds. We assume that the singular set is a closed holomorphic subvariety, and we drop the requirement, which is essential in [1], [3], that the dimension of the singular subvariety is one less than the dimension of the leaves.

First of all let us briefly review the known results in this direction. Let $M$ be a compact complex analytic manifold of complex dimension $n$, $T$ the holomorphic tangent bundle, and $F$ a holomorphic vector bundle of fibre dimension $k$, $1 \leq k \leq n$. Denote by $\underline{T}$ and $\underline{F}$ the sheaves of germs of holomorphic sections of $T$ and $F$ respectively. Suppose that $f\colon F \to T$ is a holomorphic vector bundle map such that: (1) the singular set $\sum$ is a closed holomorphic subvariety of $M$, (2) $f(F)\,|\,M - \sum$ is a holomorphic foliation $\mathscr{F}$ of codimension $n - k$, (3) $\dim_c \sum = k - r$, $r \geq 1$, (4) the subsheaf $f(\underline{F})$ of the sheaf $\underline{T}$ is integrable and full. See [1, p. 282]. The integrability guarantees that $\sum$ is a singularity of a foliation on $M$, and the fullness rules out any unessential singularities. Let $\phi$ be a symmetric homogeneous polynomial of degree $l$, $n - k < l \leq n$, in $n$ variables $x_1, \cdots, x_n$, and $\tilde{\phi}$ the unique polynomial in the elementary symmetric functions $\sigma_1, \cdots, \sigma_n$ of $x_1, \cdots, x_n$ such that $\tilde{\phi}(\sigma_1, \cdots, \sigma_l) = (x_1, \cdots, x_n)$. Let $\underline{Q} = \underline{T}/f(\underline{F})$, $c_j(\underline{Q})$ = the $j$th Chern class of $\underline{Q}$, and $\phi(\underline{Q}) = \tilde{\phi}(c_1(\underline{Q}), \cdots, c_l(\underline{Q}))$. Then there exists a homology class $\operatorname{Res}_\phi(\mathscr{F}, \sum) \in H_{2n-2l}(\sum; C)$ which depends only on $\phi$ and on the local behavior of $\mathscr{F}$ near $\sum$, [1]. Moreover, if $\mu_*\colon H_{2n-2l}(\sum; C) \to H^{2l}(M; C)$ is the inclusion followed by the Poincaré duality, $\mu_* \operatorname{Res}_\phi(\mathscr{F}, \sum) = \phi(\underline{Q})$, ([1] and [3] for $k = 1$). One of the basic problems is to compute this class in terms of the "local behavior" of $\mathscr{F}$ near $\sum$. All the results have been obtained ([1], [3]) under the assumption $r = 1$, i.e., $\dim_C \sum + 1$ = dimension of the leaves of $\mathscr{F}$.

For $r = 1$ and $k = 1$ we have a foliation $\mathscr{F}$ by holomorphic curves with a singularity set $\sum$ being isolated zeros of a holomorphic vector field $X_{\mathscr{F}}$ defining $\mathscr{F}$. If $\lambda_1(p), \cdots, \lambda_n(p)$ are the eigenvalues of the automorphism of $T_p$, $p \in \sum$, defined by $X_{\mathscr{F}}$, then under the obvious regularity assumptions there

Received November 14, 1975, and, in revised form, August 10, 1976. Supported in part by the National Science Foundation Grant GP-43363.

is the formula (for reference see [1])

$$\mathrm{Res}_\phi(\mathscr{F}, \textstyle\sum) = \displaystyle\sum_{q\in\Sigma} \frac{\phi(\lambda_1(p), \cdots, \lambda_n(p))}{\lambda_1(p)\cdots\lambda_n(p)} .$$

The general case, $r = 1$ and $1 \leq k \leq n - 1$, can be basically reduced to the previous one provided that $\sum$ is a closed holomorphic subvariety. Let $\{\sum_i\}_{i\in I}$ be the irreducible components of $\sum$, and $[\sum_i]$ the fundamental class of $\sum_i$. At a regular point $p^i$ of $\sum_i$ a transversal $(n - k + 1)$-dimensional disk $\Pi^i$ intersects $\mathscr{F}$ in holomorphic curves with zero at $p^i$. Therefore in $\Pi^i$ we have the previous situation. If $R_i$ denotes the residue associated to it by the above formula, then ([1], [3]) $\mathrm{Res}_\phi(\mathscr{F}, \sum) = \sum_{i\in I} R_i[\sum_i]$ for any polynomomial $\phi$ of degree $l = n - k + 1$. If $n - k + 1 < l \leq n$, then the general formula could be derived from the examples, [1].

The residue problem has been studied also for the meromorphic vector fields on compact complex analytic manifolds, [1], and many various results have been obtained also in the real case by Baum and Cheeger and for the Riemannian foliations by Lazorov and Pasternack.

In the case $r > 1$ the "transversal disk" method is not the right tool for the study of the residues. In this note we give an inductive construction which to a compact manifold $M$ of dimension $n$ with a holomorphic foliation $\mathscr{F}$ of leaf dimension $k$ with the connected regular (Definition 1) singular set $\sum$ of dimension $\dim_C \sum = k - r$ associates a compact manifold $P$, which is a projective line bundle $p\colon P \to M$ with holomorphic foliation $\mathscr{F}_p$ of the same leaf dimension as $\mathscr{F}$, with regular singular set $\sum_p = p^{-1}(\sum)$ of dimension $\dim_C \sum_p = k - r + 1$. Then the relation between the residues for the two situations $(M, \mathscr{F}, \sum)$ and $(P, \mathscr{F}_p, \sum_p)$ can be expressed in the following way: Let $\phi$ be a polynomial of degree $l$ as above, and $\mathrm{Res}_\phi(\mathscr{F}, \sum)$ the residue. The inclusion $\sum \to M$ and the Poincaré duality give the map $\zeta^*\colon H_\cdot(\sum, C) \to H^\cdot(M, C)$, and the residue defines the cohomology class $\zeta^* \mathrm{Res}_\phi(\mathscr{F}, \sum)$. With a given polynomial $\phi$ we can associate a unique polynomial $\psi$ of degree $l + 1$ (Proposition 5) and the cohomology class $\xi^* \mathrm{Res}_\psi(\mathscr{F}_p, \sum_p)$ defined by the residue $\mathrm{Res}_\psi(\mathscr{F}_p, \sum_p)$; $\xi^*\colon H_\cdot(\sum_p, C) \to H^\cdot(P, C)$ being defined by the inclusion $\sum_p \to P$ and the Poincaré duality. Denote by $c_1(K)$ the first Chern class of the holomorphic line bundle along the fibres of $p\colon P \to M$. Then the cohomology class $\xi^* \mathrm{Res}_\psi(\mathscr{F}_p, \sum_p)$ expands as a polynomial in $c_1(K)$:

$$\xi^* \mathrm{Res}_\psi(\mathscr{F}_p, \textstyle\sum_p) = \Phi^0 + p^*\xi^* \mathrm{Res}_\phi(\mathscr{F}, \textstyle\sum)\cdot c_1(K) + \displaystyle\sum_{j\geq 2} \Phi^j\cdot(c_1(K))^j .$$

If $\sum$ is not connected, then we have to take the sum over the connected components.

**2.** Suppose that $M$ is a compact complex—analytic manifold, $\dim_C M = n$, and $\mathscr{F}$ is a holomorphic foliation with the singular set $\sum$, defined in § 1.

Furthermore we assume that $\dim_C \sum + r = \dim_{(l)} \mathscr{F}$ (the leaf dimension of $\mathscr{F}$ at a regular point), $r > 1$. Let $M' = M \backslash \sum$ and let $F = f(F) | M'$ be the holomorphic subbundle of the holomorphic tangent bundle $T' = T | M'$, and $T \oplus \bar{T} = T_R \otimes_R C$, $T_R$ being the real tangent bundle of $M$. The quotient bundle $q\colon Q' = T'/F \to M'$ is a holomorphic vector bundle with fibre dimension $n - k$, where $k$ is the fibre dimension of $F$. If $Q'^*$ is the dual of $Q'$, then

$$\bigwedge^{n-k} Q'^* \to \bigwedge^{n-k} T'^*$$

is the inclusion of the holomorphic line bundle $\bigwedge^{n-k} Q'^* \to M'$ into the holomorphic vector bundle.

Because we want an extension of the line bundle $\bigwedge^{n-k} Q'^*$ over the singular set $\sum$, some regularity assumptions on $\sum$ are needed.

**Definition 1.** A singular set $\sum$ is regular if it is a closed subvariety of an $n$-dimensional complex analytic manifold $M$ such that

(i) $\dim_C \sum \leq n - 4$,

(ii) there exists a closed subset $W$ of $M$, $\sum \subset W$ with the property

$$H^j(W; Z) \cong H^j(W \backslash \textstyle\sum; Z) , \qquad j = 1, 2 .$$

We will assume that the singular set $\sum$ is regular in the above sense. Then by Theorem 2, proved in the last section of this note, there exists a holomorphic extension $\bigwedge^{n-k} Q^* \to M$ of the holomorphic line bundle $\bigwedge^{n-k} Q'^*$ across the singular set $\sum$. Let $p\colon P = P(E) \to M$ be the projective bundle, with fibre $CP^1$, associated with the vector bundle $E = \bigwedge^{n-k} Q^* \oplus \bigwedge^{n-k} Q^*$.

**Lemma 1.** *Let $\mathscr{F}$ be a holomorphic foliation on a complex-analytic manifold $M$ with regular singular set $\sum$, and let $\dim_C M = n$, $\dim_C \sum = \dim_{(l)} \mathscr{F} - r$, $r > 1$. Then on the complex analytic manifold $P$ with $p\colon P \to M$, there is a holomorphic foliation $\mathscr{F}_p$ with the singular set $\sum_p = p^{-1}(\sum)$ such that $\dim_{(l)} \mathscr{F}_p = \dim_{(l)} \mathscr{F}$ and $\dim_C \sum_p = 1 + \dim_C \sum$.*

Before we give the proof, let us make some observations. Let $U \subset M'$ be a coordinate neighborhood with coordinates $z = (z_1, \cdots, z_n)$, and $(\omega^1, \cdots, \omega^{n-k})$ a holomorphic frame field of $Q'^*$ over $U$. The integrability of $F$ is equivalent to (on $U$)

$$\partial\omega^j = \sum_{l=1}^{n-k} \zeta_l^j \wedge \omega^l , \quad \bar{\partial}\omega^j = 0 , \quad j = 1, 2, \cdots, n - k .$$

A partial connection is a $C$-linear map

$$\nabla\colon C^\infty(Q'^* | U) \to C^\infty(F^* \oplus \bar{T}^* \otimes Q'^* | U) ,$$

such that $\nabla(f \cdot s) = \rho(df) \otimes s + f \nabla s$, where $\rho\colon T^* \to F^*$ is the projection. There is a natural partial connection on $Q^* | U$, namely,

$$\nabla_Z \omega^j = i(Z)\partial\omega^j\ , \qquad \nabla_K \omega^j = i(K)\bar{\partial}\omega^j = 0\ ,$$

for $Z \in C^\infty(F \mid U)$ and $K \in C^\infty(\bar{T} \mid U)$. An extension of this partial connection:

$$(2.1)\qquad \begin{aligned} D_Y\omega^j &= \textstyle\sum_{l=1}^{n-k} \xi_l^j(Y)\omega^l\ , & Y &\in C^\infty(T \mid U)\ ,\\ D_K\omega^j &= i(K)\bar{\partial}\omega^j = 0\ , & K &\in C^\infty(\bar{T} \mid U)\ ,\end{aligned}$$

where $\rho(\xi_l^j) = \zeta_l^j$, is a basic connection.

Let $a_1, \cdots, a_{n-k}$ be the local coordinates of a point $a \in Q'^* \mid U$, $q(a) = z$, with respect to the local frame $(\omega^1, \cdots, \omega^{n-k})$. We denote by $X$ a holomorphic section of $F \mid U$, and by $\rho = \rho_1\omega^1 + \cdots + \rho_{n-k}\omega^{n-k}$ the horizontal section of $Q'^* \mid U$ over the integral curve $\chi$ of $X$ through the point $a$. The functions $\rho_1, \cdots, \rho_{n-k}$ are defined along the holomorphic curve $\chi$ passing through $z$ and satisfy $D_X\rho = 0$. Because $D_X\rho = i(X) \sum_{i=1}^{n-k} d\rho_i \wedge \omega^i + \sum_{i,l=1}^{n-k} \rho_i\xi_l^i(X)\omega^l$, and $\xi_l^i(X) = \zeta_l^i(X)$, the equation $D_X\rho = 0$ is equivalent to the system

$$(2.2)\qquad X(\rho_i) + \sum_{j=1}^{n-k} \rho_j\zeta_i^j(X) = 0\ , \qquad i = 1, 2, \cdots, n-k\ ,$$

along the integral curve $\chi$ of $X$. Let $\tilde{X}(a)$ be a vector tangent to the section $\rho$ at $a$. Then

$$\tilde{X}(a) = X(z) + X(\rho_1)(z)\omega^1 + \cdots + X(\rho_{n-k})(z)\omega^{n-k}\ .$$

By (2.2) we have $\tilde{X}(a) = X(z) - \sum_{j-1}^{n-k} \rho_j(z)\omega^j$, and

$$(2.3)\qquad \tilde{X}(a) = X(z) - \sum_{j=1}^{n-k} a_j\omega^j(z)\ .$$

Now we associate with the holomorphic vector field $X$ the vector field $\tilde{X}$ defined by (2.3). As $X$ is holomorphic and $\omega^j$ is a holomorphic function, the vector field $\tilde{X}$ is also a holomorphic vector field on $Q^* \mid U$.

Because the curvature $K_{X,Y} = D_XD_Y - D_YD_X - D_{[X,Y]}$ is zero for any vector fields $X, Y \in C^\infty(F \mid U)$, the horizontal lift $\tilde{F} \mid U$ of the distribution $F \mid U$ is integrable and holomorphic. Finally as $Q'^* \to M'$ is holomorphic and $K_{X,Y} = 0$, $X, Y \in C^\infty(F)$, is a global condition we conclude that $\tilde{F}$ is a holomorphic distribution on $Q'^* \to M'$, $q_*\tilde{F} = F$. This proves

**Proposition 1.** *On the holomorphic vector bundle $q\colon Q'^* \to M'$ there is a holomorphic $k$-dimensional integrable distribution $\tilde{F}$ such that $q_*(\tilde{F}) = F$.*

Next step in the proof of Lemma 1 is

**Proposition 2.** *On the holomorphic line bundle $q\colon \bigwedge^{n-k} Q'^* \to M'$ and the holomorphic vector bundle $E \mid M'$, there is a holomorphic integrable distribution which projects onto $F$.*

*Proof.* The basic connection $D\colon C^\infty(Q'^*) \to C^\infty(T^* \oplus \bar{T}^* \otimes Q'^*)$ extends as a derivation and defines a connection $D^\wedge$ on the holomorphic line bundle $\overset{n-k}{\bigwedge} Q'^* \to M'$

$$D^\wedge(s_1 \wedge \cdots \wedge s_{n-k}) = \sum_{i=1}^{n-k} (-1)^{i+1} s_i \wedge \cdots \wedge Ds_i \wedge \cdots \wedge s_{n-k} ,$$

for any $s_1 \wedge \cdots \wedge s_{n-k} \in C^\infty(\overset{n-k}{\bigwedge} Q'^*)$. The curvature satisfies $K^\wedge_{X,Y} = 0$ for $X, Y \in C^\infty(F)$, and $D^\wedge$ gives a unique connection $D^0\colon C^\infty(E \mid M') \to C^\infty(T^* \oplus \bar{T}^* \otimes E \mid M')$, where again $K^0{}_{X,Y} = 0$ for $X, Y \in C^\infty(F)$. The rest of the argument is exactly the same as in the proof of Proposition 1.

Now let us return to the situation from the proof of Proposition 1. Let $a, b \in Q'^* \mid U$, and $q(a) = q(b) = z_0 \in U$, such that for some $c \in C$, $b_i = a_i \cdot c$, $i = 1, 2, \cdots, n-k$. Let $\rho$, $\sigma$ be the horizontal sections over the curve $\chi$ passing through $a$, $b$ respectively. The components $\rho_i$ and $\sigma_i$ of $\rho$ and $\sigma$ satisfy (2.2) and the initial conditions $\rho_i(z_0) = a_i$, $\sigma_i(z_0) = b_i$. The functions $c \cdot \rho_i(z)$ satisfy (2.2) also, and $c \cdot \rho_i(z_0) = \sigma_i(z_0) = b_i$. From the uniqueness of the solution of (2.2) it follows that $c \cdot \rho_i(z) = \sigma_i(z)$. Hence if a curve $\rho(z)$, $z \in \chi$, is on the leaf of the foliation $\tilde{\mathscr{F}}$ given by $\tilde{F}$, so is the curve $c \cdot \rho(z)$.

If we apply this to the bundle $E \mid M'$ we can have

**Proposition 3.** *On the projective bundle $P \mid M'$ there is a holomorphic foliation $\mathscr{F}_p$ of the same leaf dimension $k$ as the foliation $\mathscr{F}$ on $M'$.*

Lemma 1 follows from these two propositions.

**3.** We succeeded in replacing the compact manifold $M$ with a foliation $\mathscr{F}$ and the singular set $\sum$ such that $\dim_{(l)} \mathscr{F} - \dim_C \sum = r$ by another compact manifold $P$ and a foliation $\mathscr{F}_P$ with singular set $\sum_P$ such that $\dim_{(l)} \mathscr{F}_P - \dim_C \sum_P = r - 1$. If this procedure is repeated $(r-1)$-times we end up with a compact complex analytic manifold with a holomorphic foliation whose singular set is a closed regular subvariety of complex dimension one less than the leaf dimension of the foliation. That is the situation where the residue can be explicitly computed, [1], [3].

Let us recall [1] the definition of the residue for a foliation $\mathscr{F}$ on $M$ with a connected regular singular set $\sum$. Let $U$ be an open subset of $M$ such that $\sum \subset U$ and $\sum$ is a deformation retract of $U$. Recall that the quotient bundle $q\colon Q' \to M'$ is a holomorphic vector bundle which is equivalent to a normal bundle of $\mathscr{F}$ on $M'$. Let $E_s, E_{s-1}, \cdots, E_0$ be $C^\infty$ complex vector bundles over $U$ such that over $U' = U \setminus \sum$ there is an exact sequence of vector bundles

$$0 \longrightarrow E_s \mid U' \xrightarrow{\mu_s} E_{s-1} \mid U' \xrightarrow{\mu_{s-1}} \cdots \xrightarrow{\mu_1} E_0 \mid U' \xrightarrow{\mu_0} Q' \mid U' \longrightarrow 0 , \tag{3.1}$$

$\mu_j$'s being vector bundle maps. (3.1) is a resolution of $Q'$ over $U'$. Furthermore, suppose that there are $C^\infty$ connections

$$(3.2) \qquad \nabla_i\colon C^\infty(E_i) \to C^\infty(T_C^*(U) \otimes E_i)\,, \qquad i = 0, 1, \cdots, s\,,$$

$T_C^* = T^* \oplus \bar{T}$, and a basic connection $\nabla_{-1}$ on $Q'$ over $U'$ such that for some compact subset $U_0$ of $U$, with $\sum$ contained in the interior of $U_0$, the exact sequence of vector bundles (3.1) is compatible with these connections over $U \setminus U_0$. Let $K_0, K_1, \cdots, K_s$ be the curvatures of the connections

$$\nabla_0, \nabla_1, \cdots, \nabla_s, K_i \in \bigwedge^2 T_C^* \otimes \operatorname{End} E_i\,, \qquad i = 0, 1, \cdots, s\,.$$

Now we define the differential forms $\kappa_1, \cdots, \kappa_n, \kappa_i$ being a $2i$-form on $U$, by

$$\prod_{i=0}^{s} (\det (I + K_i))^{\varepsilon(i)} = 1 + \kappa_1 + \cdots + \kappa_n\,, \qquad \varepsilon(i) = (-1)^i\,.$$

Suppose that $\phi$ is a symmetric homogeneous polynomial of degree $l$, $n - k < l \leq n$, and $\tilde{\phi}$ is the associated polynomial as in § 1. Then the closed differential $2l$-form $(\frac{1}{2}\sqrt{-1}/\pi)^l\, \tilde{\phi}(\kappa_1, \cdots, \kappa_l)$ with support $U_0$ defines a cohomology class $(\frac{1}{2}\sqrt{-1}/\pi)^l[\tilde{\phi}(\kappa_1, \cdots, \kappa_l)]$ in the compatibly supported cohomology $H_c^{2l}(U, C)$. The Poincaré dual of this class is a class in the homology group $H_{2n-2l}(\sum, C)$, as $\sum$ is the deformation retract of $U$ by assumption. It is called the residue and denoted by

$$(3.3) \qquad \operatorname{Res}_\phi (\mathscr{F}, \textstyle\sum) = D\{(\frac{1}{2}\sqrt{-1}/\pi)^l[\tilde{\phi}(\kappa_1, \cdots, \kappa_l)]\}\,.$$

It was proved that this homology class depends only on $\phi$, $\sum$ and $\mathscr{F}$ in a neighborhood of $\sum$ on $M$.

Because we want to relate the residues corresponding to the situation given by $\sum$, $\mathscr{F}$ on $M$ and $\sum_P$, $\mathscr{F}_P$ on $P$ for various polynomials $\phi$, we must construct a resolution of the normal bundle of $\sum_P$, analogous to (3.1), and also a sequence of connections compatible with that resolution as closely related to the $\nabla_i$'s (3.2) as possible.

Let $V = p^{-1}(U)$, and $V' = V \setminus \sum_P$, let $K \to V \subset P$ be the bundle of vectors tangent along the fibre, and let $\tilde{E}_i = p^{-1}E_i$, $i = 1, 2, \cdots, s$, and $\tilde{E}_0 = p^{-1}E_0 \oplus K$. If $\tilde{F}$ is the distribution tangent to the holomorphic foliation $\mathscr{F}_P$, then the bundle normal to $\mathscr{F}_P$ is equivalent to $\tilde{Q}' = T(P')/\tilde{F}$, and $\tilde{Q}'$ is equivalent to $q^{-1}Q' \oplus K$. Hence the exact sequence (3.1) gives immediately the exactness of the sequence

$$(3.4) \quad 0 \longrightarrow \tilde{E}_s | V' \xrightarrow{\lambda_s} E_{s-1} | V' \xrightarrow{\lambda_{s-1}} \cdots \xrightarrow{\lambda_1} \tilde{E}_0 | V' \xrightarrow{\lambda_0} \tilde{Q} | V' \longrightarrow 0\,,$$

where $\lambda_i = \tilde{\mu}_i$, the lift of $\mu_i$, $i = 1, \cdots, s$, and $\lambda_0$ will be defined below.

Now we define the connections $\tilde{\nabla}_i$, $i = -1, 0, 1, \cdots, s$, which are compatible with the bundle maps in (3.4), and such that $\tilde{\nabla}_{-1}$ is a basic connection on $\tilde{Q}'$. First, let us start with the construction of $\tilde{\nabla}_{-1}$. The injection $\kappa\colon K \to T(P)$ together with the projection $\tau: T(P') \to \tilde{Q}' = T(P')/\tilde{F}$ on the quotient defines a subbundle $\tilde{K} = \tau \cdot \kappa(K)$ of $\tilde{Q}'$. In fact, $\tilde{Q}'/\tilde{K}$ is equivalent to $p^{-1}Q'$, and if we identify $\tilde{Q}^1$ with $p^{-1}Q' \oplus K$ then

$$\lambda_0 : \tilde{E}_0 = p^{-1}E_0 \oplus K \to \tilde{Q}' = p^{-1}Q' \oplus K$$

is a bundle map defined by $\lambda_0 = \tilde{\mu}_0 \oplus 1$, $\tilde{\mu}_0 = p^{-1}\mu_0 p$. Note that on each pull back $p^{-1}E_i$ and on $p^{-1}Q'$ there is a natural pull back of connections $\bar{\nabla}_i = (p^{-1} \otimes p^*) \circ \nabla_i \circ p$, $i = -1, 0, 1, \cdots, s$.

**Proposition 4.** *On the quotient bundle $\tilde{Q}'$ there is a basic connection $\tilde{\nabla}_{-1}$ such that it maps the subbundle $\tilde{K}$ of $\tilde{Q}'$ into itself and on the quotient bundle it is the pull back $\bar{\nabla}_{-1}$ of $\nabla_{-1}$ via the isomorphism $\varepsilon: \tilde{Q}'/\tilde{K} \to p^{-1}Q'$.*

*Proof.* Let $\nabla: C^\infty(\tilde{Q}') \to C^\infty(\tilde{F}^* \oplus \bar{T}^*(P') \otimes \tilde{Q}')$ be the partial connection associated with the foliation $\mathscr{F}_p$ on $P$. It is characterized by the following two properties:

(1) $i(X)\nabla(\tau(Y)) = \tau([X, Y])$, $\quad X \in C^\infty(\tilde{F})$, $Y \in C^\infty(T(P))$,

(2) $\nabla$ is of type $(1, 0)$.

For any $Y \in C^\infty$ $(K')$, $K' = K|P'$, and $X \in C^\infty(\tilde{F})$ the bracket $[X, Y] \in C^\infty(K' \oplus \tilde{F})$ as $K' \oplus \tilde{F}$ is an integrable distribution defining the foliation $p^{-1}(\mathscr{F})$ of codimension $k$ on $P'$. This shows that the partial connection induces a partial connection $\nabla'$ on the subbundle $\tilde{K}$ of $\tilde{Q}'$. It is flat along the leaves of $\mathscr{F}_p$.

It remains to show that the partial connection $\nabla$ can be extended to a basic connection $D_{-1}$ with the same properties as $\nabla'$ on $\tilde{K}$ and at the same time on the quotient $\tilde{Q}'/\tilde{K}$ it should be $\bar{\nabla}_{-1}$. This is done as follows: The partial connection $\nabla'$ on $K'$ extends to a connection $\tilde{\nabla}': C^\infty(\tilde{K}) \to C^\infty(T^*(P') \oplus \bar{T}^*(P') \otimes \tilde{K})$. $\tilde{\nabla}'$ is again flat along $\mathscr{F}_p$. Then the connection

$$\tilde{\nabla}_{-1} = (\bar{\nabla}_{-1} \circ \varepsilon \oplus \tilde{\nabla}') \cdot \iota\,, \tag{3.5}$$

where $\iota: \tilde{Q}' \to \tilde{Q}'/\tilde{K} \oplus \tilde{K}$ is an isomorphism, has the required properties as both defining connections are flat along $\mathscr{F}_p$.

The isomorphism of vector bundles $\tau \cdot \kappa: K \to \tilde{K}$ over $P'$ and the connection $\tilde{\nabla}'$ on $\tilde{K}$ over $P'$ define a connection on $K$ over $P'$. Let $V_0 = p^{-1}U_0$ be the compact subset of $P$, $V_0 \subset V$, containing $\sum_P$ in its interior. As $K$ is a vector bundle defined over the whole $P$ we can choose extension $\tilde{\nabla}$ of $\tilde{\nabla}'$ such that $\tilde{\nabla}$ restricted to $P \setminus V_0$ is $\tilde{\nabla}$. Then define the connections

$$\tilde{\nabla}_i: C^\infty(\tilde{E}_i) \to C^\infty(T^*_C \otimes \tilde{E}_i)\,,$$

$\tilde{\nabla}_i = \bar{\nabla}_i$, $i = 1, 2, \cdots, s$, $\tilde{\nabla}_0 = \bar{\nabla}_0 \oplus \tilde{\nabla}$.

**Proposition 5.** *The exact sequence of vector bundles* (3.4) *is compatible with the connection $\tilde{\nabla}_i$, $i = -1, 0, 1, \cdots, s$* over $V \setminus V_0$.

*Proof.* In order to systematize the notation we denote $Q'$ by $\tilde{E}_{-1}$ and $\bar{\nabla}_{-1} \circ \varepsilon$ simply by $\tilde{\nabla}_{-1}$. Then the problem is to show that over $V \setminus V_0$ for $i = 0, 1, \cdots, s$ the following diagram commutes:

$$\begin{tikzcd} C^\infty(\tilde{E}_i) \arrow[r, "\lambda_i"] \arrow[d, "\tilde{\nabla}_i"'] & C^\infty(\tilde{E}_{i-1}) \arrow[d, "\tilde{\nabla}_{i-1}"] \\ C^\infty(T^*_C \otimes \tilde{E}_i) \arrow[r, "\lambda_i"] & C^\infty(T^*_C \otimes \tilde{E}_{i-1}) . \end{tikzcd}$$

Let $a \in C^\infty(\tilde{E}_i \mid V\setminus V_0)$, $\tilde{E}_i = p^{-1}E_i \oplus K$, and $a = a_E + a_K$. For $i = 1, 2, \cdots, s$ the proposition follows from the compatibility of (3.1) with the $\nabla_i$'s, and for $i = 0$ we have

$$\lambda_0(a) = (\tilde{\mu}_0 \oplus 1)(a) = \tilde{\mu}_0(a_E) + a_k .$$

From the compatibility of (3.1) with $\nabla_0$ over $U\setminus U_0$ we get

$$\begin{aligned} \tilde{\nabla}_{-1}\lambda_0(a) &= (\bar{\nabla}_{-1} \oplus \tilde{\nabla})(\tilde{\mu}_0(a_E) + a_K) \\ &= \bar{\nabla}_{-1}\tilde{\mu}_i(a_E) + \tilde{\nabla}(a_K) = \tilde{\mu}_0\bar{\nabla}_0(a_E) + \tilde{\nabla}(a_K) \\ &= (\tilde{\mu}_0 \oplus 1)(\bar{\nabla}_0 \oplus \tilde{\nabla})(a_E + a_K) = \lambda_0\tilde{\nabla}_0(a) . \end{aligned}$$

Now we can compute the formula relating the residue $\operatorname{Res}_{\phi_l}(\mathscr{F}, \sum)$, where $\phi_l$ is a symmetric homogeneous polynomial of degree $l$, $n - k < l \leq n$, with the residue $\operatorname{Res}_{\psi_t}(\mathscr{F}_P, \sum_P)$, where $\psi_t$ is a symmetric homogeneous polynomial of degree $t$, $n - k + 1 < t \leq n + 1$. The connections $\tilde{\nabla}_i$, $i = 0, 1, \cdots, s$ can be extended to the connections over the whole $P$. We keep the same notation for the extensions. The curvature from $\Omega_0$ of the connection $\tilde{\nabla}_0$ is the matrix-valued form on $P$

$$\Omega_0 = \begin{pmatrix} \bar{K}_0 & 0 \\ 0 & R \end{pmatrix} ,$$

where $\bar{K}_0 = p^*K_0$, and $R$ is the curvature 2-form of the connection $\tilde{\nabla}$ defined on the holomorphic line bundle $K$. Hence we have

$$\begin{aligned} \det(I + \Omega_0) &= \det(I + \bar{K}_0)\cdot\det(I + R) \\ &= \det(I + \bar{K}_0)\cdot(1 + R) , \end{aligned}$$

where 1 is the identity in End $(K)$, and for $i = 1, 2, \cdots, s$, $\Omega_i = p^*K_i$ is the curvature of the connection $\tilde{\nabla}_i$. The product

$$\begin{aligned} \prod_{i=0}^{s} ((\det(I + \Omega_i))^{\varepsilon(i)} &= (1 + \tilde{\kappa}_1 + \cdots + \tilde{\kappa}_n)(1 + R) \\ &= 1 + \omega_1 + \omega_2 + \cdots + \omega_{n+1} , \qquad \varepsilon(i) = (-1)^i , \end{aligned} \tag{3.6}$$

where $\tilde{\kappa}_i = p^*\kappa_i$, $i = 1, 2, \cdots, n$, are $2i$-forms on $P$.

The total Chern class of the virtual bundle $\tilde{E} = \sum_{i=0}^{s}(-1)^i\tilde{E}_i$ is defined by

$$c(\tilde{E}) = \prod_{i=0}^{s} (c(\tilde{E}_i))^{\varepsilon(i)} ,$$

and the Chern classes $c_1(\tilde{E}), \cdots, c_{n+1}(\tilde{E})$ are determined by

$$c(\tilde{E}) = 1 + c_1(\tilde{E}) + \cdots + c_{n+1}(\tilde{E}) .$$

The $2i$-form $\omega_i$ is closed and represents the $i$-th Chern class $c_i(\tilde{E})$:

$$[\omega_i] = (2\pi/\sqrt{-1})^i c_i(\tilde{E}) .$$

As above let $\psi_t$, $n - k + 1 < t \leq n + 1$, be symmetric and homogeneous polynomial, and let $\tilde{\psi}_t$ be the associated polynomial in the elementary symmetric functions. Then $\psi_t(\tilde{E})$ is defined by

$$\psi_t(\tilde{E}) = \tilde{\psi}_t(c_1(\tilde{E}), \cdots, c_t(\tilde{E})) , \tag{3.7}$$

and [1, Theorem 2] implies that

$$\psi_t(\tilde{E}) = \sum \xi^* \operatorname{Res}_{\psi_t}(\mathscr{F}_P, \textstyle\sum_P^r) , \tag{3.8}$$

where the summation is over the connected components $\sum_P^r$ of $\sum_P$, and $\xi^*$ is the composition of $i_*\colon H.(\sum_P, C) \to H.(P, C)$ induced by the inclusion with the Poincaré duality. Similarly if $E = \sum_{i=0}^{s} (-1)^i E_i$ is the virtual bundle over $M$, and $c_i(E)$ is its $i$-th Chern class, then for any symmetric and homogeneous polynomial $\phi_l$ of degree $l$, $n - k < l \leq n$, we define

$$\phi_l(E) = \tilde{\phi}_l(c_1(E), \cdots, c_l(E)) , \tag{3.9}$$

and [1, Theorem 2] gives the relation

$$\phi_l(E) = \sum \xi^* \operatorname{Res}_{\phi_l}(\mathscr{F}, \textstyle\sum^r) , \tag{3.10}$$

where the summation is over the connected components of $\sum$, and $\zeta^*$ is the composition of $i_*\colon H.(\sum, C) \to H.(M, C)$ induced by the inclusion $i\colon \sum \to M$ composed with the Poincaré duality.

The cohomology class $\psi_t(\tilde{E})$ is represented by the $2t$-form $\tilde{\psi}_t(\omega_1, \cdots, \omega_t)$, where

$$\begin{aligned} \omega_1 &= \tilde{\kappa}_1 + R , \quad \omega_2 = \tilde{\kappa}_2 + \tilde{\kappa}_1 \wedge R, \cdots , \\ \omega_n &= \tilde{\kappa}_n + \tilde{\kappa}_{n-1} \wedge R , \quad \omega_{n+1} = \tilde{\kappa}_n \wedge R . \end{aligned} \tag{3.11}$$

Before we proceed any further, few algebraic observations are needed. Let $\sigma_1, \cdots, \sigma_l$ be the elementary symmetric functions in the $n$ variables $x_1, \cdots, x_n$, $l \leq n$, and let $\rho_1, \cdots, \rho_l$ be the elementary symmetric functions in the $n + 1$ variables $x_1, \cdots, x_n, y$. These two sets of the elementary symmetric functions are related by the formulas

$$(3.12) \qquad \rho_1 = \sigma_1 + y\,, \quad \rho_j = \sigma_j + \sigma_{j-1}y\,, \quad j = 2, 3, \cdots, l\,.$$

Then for any $l$-tuple of nonnegative integers $\alpha_1, \alpha_2, \cdots, \alpha_l$ such that $\alpha_1 + 2\alpha_2 + \cdots + l\alpha_l = l$ we define the homogeneous polynomials $p^j_{\alpha_1\cdots\alpha_l}$, $j = 0, 1, \cdots$, in the elementary symmetric functions $\sigma_1, \cdots, \sigma_l$ by the formula

$$(3.13) \qquad \rho_1^{\alpha_1+1}\rho_2^{\alpha_2} \cdots \rho_l^{\alpha_l} = p^0_{\alpha_1\cdots\alpha_l} + p^1_{\alpha_1\cdots\alpha_l}y + (*)\,,$$

where $(*)$ stands for the terms of higher orders in $y$. Let us denote $p^1_{\alpha_1\cdots\alpha_l}$ simply by $p_{\alpha_1\cdots\alpha_l}$.

**Proposition 6.** *Let $\beta_1, \beta_2, \cdots, \beta_l$ be a sequence of nonnegative integers such that $\beta_1 + 2\beta_2 + \cdots + l\beta_l = l$. Then the monomial $\sigma_1^{\beta_1}\sigma_2^{\beta_2} \cdots \sigma_l^{\beta_l}$ can be written as a polynomial, with rational coefficients, in the polynomials $p_{\alpha_1\cdots\alpha_l}$ for various sequences of nonnegative integers $\alpha_1, \cdots, \alpha_l$ satisfying $\alpha_1 + 2\alpha_2 + \cdots + l\alpha_l = l$, where $l$ is a fixed positive integer.*

*Proof.* For a fixed sequence $\alpha_1, \cdots, \alpha_l$ we rearrange the terms in the polynomial $p_{\alpha_1\cdots\alpha_l}$. First start with the monomials involving $\sigma_j$ with the highest index $j$, and order these monomials with the decreasing powers of $\sigma_j$. Therefore the term involving the $\sigma_j$ with the highest index $j$ and in the highest power will be first. We call this term the leading term of $p_{\alpha_1\cdots\alpha_l}$. After all terms with $\sigma_j$, $j$ being the highest index, are exhausted, repeat the same procedure with the $\sigma_k$'s where $k$ is the next highest integer, etc. If there are two monomials with the same $\sigma_j$ with the same highest index and having the same powers, then look at the next highest index of $\sigma_k$ in those monomials. The one whose next highest index is higher comes first.

Now we order all the polynomials $p_{\alpha_1\cdots\alpha_l}$ for various sequences $\alpha_1, \cdots, \alpha_l$ satisfying $\alpha_1 + 2\alpha_2 + \cdots + l\alpha_l = l$ for fixed $l$, ordered in the above manner, according to the leading terms as follows: Start with the monomial $p \cdots$ which has the leading term containing $\sigma_1$ in the highest power. Then order all those polynomials $p \cdots$ whose leading term contains $\sigma_1$ according to the decreasing powers, and continue in the same manner with those polynomials $p \cdots$ whose leading term contains $\sigma_2$, etc. If two different polynomials have the leading terms involving the $\sigma_j$ with the same highest index $j$ in the same highest power, then look at the $\sigma_k$'s in those leading monomials with the next highest indices. If they are different, then the polynomial with the leading term involving $\sigma_k$ in the higher power will be first, etc.

Let, for example, $c_1, \cdots, c_r$, $r \leq l$, be a sequence such that $c_1 + 2c_2 + \cdots + rc_r = l$. The leading term of $p_{c_1\cdots c_r0\cdots0}$ is equal to $(c_1 + 1)\sigma_1^{c_1} \cdots \sigma_r^{c_r}$. The other monomials in $p_{c_1\cdots c_r0\cdots0}$ are constant multiples of $\sigma_1^{b_1} \cdots \sigma_r^{b_r}$, $b_1 + 2b_2 + \cdots + rb_r = l$, $b_1 > c_1$. But these terms, possibly with different constant coefficients, already occured as leading terms in the previous polynomials of our ordering. Hence we have a system of $N$ linear equations with $N$ unknowns, where $N$ is the number of partitions $\alpha_1, \cdots, \alpha_l$ such that $\alpha_1 = 2\alpha_2 + \cdots + l\alpha_l = l$. The above ordering puts the matrix of the coefficients of this system into the triangular form, with zeros above the second diagonal and with non-zero elements along the diagonal. In fact the elements along the diagonal are precisely the coefficients of the leading terms. Therefore we can solve this regular system for the leading terms. It remains to be mentioned that each monomial $\sigma_1^{\beta_1} \cdots \sigma_l^{\beta_l}$, $\beta_1 + 2\beta_2 + \cdots + l\beta_l = l$, actually appears as the leading term of the polynomial

$$\frac{1}{\beta_1 + 1} p_{\beta_1 \cdots \beta_l} .$$

**Corollary.** *Let $\sigma_1, \cdots, \sigma_l$ be the elementary symmetric functions in $x_1, \cdots, x_n$, and let $\tilde{\phi}_l$ be the polynomial associated with symmetric homogeneous polynomial $\phi_l$ of degree $l$ so that $\phi_l(x_1, \cdots, x_n) = \tilde{\phi}_l(\sigma_1, \cdots, \sigma_l)$. Then with any polynomial $\tilde{\phi}_l$ can be associated a polynomial $\tilde{\psi}_{l+1}(\rho_1, \cdots, \rho_l)$ of degree $l + 1$ in the variables $x_1, \cdots, x_n, y$ with $\rho_i$ given by* (3.12) *such that*

$$\tilde{\psi}_{l+1}(\rho_1, \cdots, \rho_l) = \tilde{\phi}_l^0(\sigma_1, \cdots, \sigma_l) + \tilde{\phi}_l(\sigma_1, \cdots, \sigma_l)y + \sum_{j \geq 2} \tilde{\phi}_l^j(\sigma_1, \cdots, \sigma_l)y^j , \tag{3.14}$$

*where $\tilde{\phi}_l^0$ has degree $l + 1$, and $\tilde{\phi}_l^j$ has degree $l - j + 1$.*

Now we are ready to prove

**Theorem 1.** *Let $\phi_l$ be a symmetric homogeneous polynomial of degree $l$, $n - k < l \leq n$, and $\psi_{l+1}$ the associated polynomial. Then the residue cohomology class corresponding to the situation $(P, \mathscr{F}_P, \sum_P)$ has an expansion as a polynomial in $c_1(\mathscr{K})$:*

$$\begin{aligned} &\textstyle\sum \xi^* \operatorname{Res}_{\psi_{l+1}} (\mathscr{F}_P, \sum_P^r) \\ &\quad = \Phi_l^0 + p^* \textstyle\sum \xi^* \operatorname{Res}_{\phi_l} (\mathscr{F}, \sum^r) \cdot c_1(K) + \displaystyle\sum_{j \geq 2} \Phi_l^j \cdot (c_1(K))^j . \end{aligned} \tag{3.15}$$

*Proof.* If $\tilde{\phi}_l$ is the polynomial associated with $\phi_l$, then the cohomology class $\phi_l$ $(E)$ given by (3.9) is represented by the de Rham cocycle

$$(\tfrac{1}{2}\sqrt{-1}/\pi)^l \tilde{\phi}_l(\kappa_1, \cdots, \kappa_l) , \tag{3.16}$$

where $\kappa_1, \cdots, \kappa_l$ are the forms on $M$ defined by the connections $\nabla_i$ given by (3.2). From the vanishing theorem [1] it follows that the support of (3.16) is in the compact subset $U_0$ of $U$. Hence $p^*\tilde{\phi}_l(\tilde{\kappa}_1, \cdots, \kappa_l) = \tilde{\phi}_l(\kappa_1, \cdots, \tilde{\kappa}_l)$ has support in $V_0$. From (3.11) and (3.14) it follows that there exists a symmetric homogeneous polynomial $\tilde{\psi}_{l+1}$ such that

$$\tilde{\psi}_{l+1}(\omega_1, \cdots, \omega_l) = \tilde{\phi}_l^0(\tilde{\kappa}_1, \cdots, \tilde{\kappa}_l) + \tilde{\phi}_l(\tilde{\kappa}_1, \cdots, \tilde{\kappa}_l) \wedge R + \sum_{j \geq 2} \tilde{\phi}_l^j(\tilde{\kappa}_1, \cdots, \tilde{\kappa}_l) \wedge R^j , \tag{3.17}$$

where $R^j = R \wedge \cdots \wedge R$ ($j$-times), and $\tilde{\phi}_l^j, j = 0, 2, \cdots$, are polynomials of degree $l-j+1$. Since $\tilde{\phi}_l(\tilde{\kappa}_1, \cdots, \tilde{\kappa}_l)$ has support in $V_0$, from the vanishing theorem [1] it follows that also $\tilde{\psi}_{l+1}(\omega_1, \cdots, \omega_l)$ has support in $V_0$.

Now multiply (3.17) by $(\frac{1}{2}\sqrt{-1}/\pi)^{l+1}$ and observe that $(\frac{1}{2}\sqrt{-1}/\pi)[R] = c_1(K)$. We get

$$\begin{aligned}
&(\tfrac{1}{2}\sqrt{-1}/\pi)^{l+1}[\tilde{\psi}_{l+1}(\omega_1, \cdots, \omega_l)] \\
&\quad = (\tfrac{1}{2}\sqrt{-1}/\pi)^{l+1}[\tilde{\phi}_l^0(\tilde{\kappa}_1, \cdots, \tilde{\kappa}_l)] \\
&\qquad + (\tfrac{1}{2}\sqrt{-1}/\pi)^{l}[\tilde{\phi}_l(\tilde{\kappa}_1, \cdots, \tilde{\kappa}_l)]\cdot c_1(K) \\
&\qquad + \sum_{j\geq 2} (\tfrac{1}{2}\sqrt{-1}/\pi)^{l-j+1}[\tilde{\phi}_l^j(\tilde{\kappa}_1, \cdots, \tilde{\kappa}_l)]\cdot c_1(K))^j ,
\end{aligned}$$

where • stands for the cup product, and $c_1{}^j = c_1 \bullet \cdots \bullet c_1$ ($j$-times). Hence we have

$$\psi_{l+1}(\tilde{E}) = p^*\phi_l^0(E) + p^*\phi_l(E)\cdot c_1(K) + \sum_{j\geq 2} p^*\phi_l^j(E)\cdot(c_1(K))^j .$$

Finally if we denote $\Phi_l^0 = p^*\phi_l^0(E)$ and $\Phi_l^j = p^*\phi_l^j(E)$, then the relation (3.15) follows from (3.8) and (3.10).

**Remark.** Similar construction and formulas can be obtained in a real situation for differentiable foliations with singularities on a compact differentiable manifold.

In order to illustrate this situation we give an example of a differentiable foliation of an $n$-sphere $S^n$ by $(n-4)$-dimensional manifolds with singularity $S^{n-8}$ for $n \geq 8$.

Let $\pi: S^7 \to S^4$ be the Hopf fibration of the 7-sphere by the 3-spheres $S^3$, and let $S^7$ be the unit sphere in $R^8$. If we join each point of a fibre $S^3$ on $S^7$ with the origin $0 \in R^8$ we get a foliation of the disk $D^8$ by 4-dimensional submanifolds with singularity 0. Glue two copies of this foliated disk $D^8$ along the boundaries $S^7$ in such a way that the foliations smoothly match. This gives a differentiable foliation of $S^8$ by 4-dimensional submanifolds which has two singular points. Let these points be the antipodal points on the unit sphere in $R^9$. Now join each point of a leaf of the foliation of $S^8$ by segment with $0 \in R^9$. We get the foliation of $D^9$ by 5-dimensional submanifolds with the singularity being 1-dimensional segment. Gluing two copies of $D^9$ along $S^8$ so that the match of the foliations gives the foliation of $S^9$ by 5-dimensional submanifolds with singularity $S^1$. The inductive procedure produces the above example.

If we look at $S^7$ as at a unit sphere in $C^4$, then one can show, using quaternions, that the foliation of $C^4$ by cones constructed by projecting the leaves $S^3$ on $S^7$ from $0 \in C^4$ is a foliation by complex analytic submanifolds of complex dimension 2, which is real analytic in the direction transversal to the leaves.

**4.** In this last section we prove the theorem used in part 2 of this paper in the construction of the projective bundle $P$. The theorem is actually a corollary of the theorem proved by Sheja [2].

**Theorem 2.** *Let $M$ be a complex analytic $n$-dimensional manifold, $\mathcal{O}$ the structure sheaf (sheaf of germs of holomorphic functions), and $\sum$ a closed subvariety of $M$ which is regular (in the sense of Definition 1). Then there is an isomorphism*

$$H^1(M, \mathcal{O}^*) \to H^1(M \setminus \sum, \mathcal{O}^*) .$$

In other words; any holomorphic line bundle $L \to M \setminus \sum$ can be holomorphically extended to a line bundle $K \to M$ in such a way that $K | (M \setminus \sum)$ is holomorphically equivalent to $L$.

*Proof.* Using the Čech cohomology with coefficients in a coherent sheaf, Sheja [2] proved that if $\dim \sum \leq n - 4$ then we get a bijection

$$H^j(M, \mathcal{O}) \to H^j(M \setminus \sum, \mathcal{O}) , \qquad j = 1, 2 .$$

From the excision theorem we get the isomorphism

$$H^*(M - U, (M \setminus \sum) - U; Z) \xrightarrow{\sim} H^*(M, M \setminus \sum; Z) ,$$

where $U = M - W$, $W$ being a closed subset of $M$ from the definition of regularity of $\sum$, and the comparison of the cohomology sequences for the pairs $(W, W \setminus \sum)$ and $(M, M \setminus \sum)$ respectively shows that from (ii) in Definition 1 follows the isomorphism

$$H^j(M, Z) \to H^j(M \setminus \sum, Z) , \qquad j = 1, 2 .$$

The exact sequence $0 \to Z \to \mathcal{O} \to \mathcal{O}^* \to 0$ leads to the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^1(M,Z) \arrow[r] \arrow[d, "i_1"'] & H^1(M,\mathcal{O}) \arrow[r] \arrow[d, "i_2"'] & H^1(M,\mathcal{O}^*) \arrow[r] \arrow[d, "\iota"'] & H^2(M,Z) \arrow[r] \arrow[d, "i_3"'] & H^2(M,\mathcal{O}) \arrow[r] \arrow[d, "i_4"'] & \cdots \\
\cdots \arrow[r] & H^1(M\setminus\sum,Z) \arrow[r] & H^1(M\setminus\sum,\mathcal{O}) \arrow[r] & H^1(M\setminus\sum,\mathcal{O}^*) \arrow[r] & H^2(M\setminus\sum,Z) \arrow[r] & H^2(M\setminus\sum,\mathcal{O}) \arrow[r] & \cdots
\end{tikzcd}$$

with exact rows and isomorphisms $i_1$, $i_2$, $i_3$, $i_4$. Hence by the five lemma $\iota$ is an isomorphism.

## References


[1] P. Baum & R. Bott, *Singularities of holomorphic foliations,* J. Differential Geometry **7** (1972) 279–342.

[2] G. Scheja, *Riemannsche Hebbarkeitssätze für Cohomologieklassen,* Math. Ann. **144** (1961) 345–360.

[3] S. M. Visik, *Singularities of analytic foliations and characteristic classes,* Functional Anal. Appl. **7** (1973) 1–15.


NORTHEASTERN UNIVERSITY, BOSTON